On the Lehmer–Serré conjecture

by

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To the memory Andrej Novikov

The main purpose of this paper is to prove the Lehmer famous conjecture that $\tau(n)$ is not zero for all n , where the Ramanujan function $\tau(n)$ is the n^{th} Fourier coefficient of the cusp form $\Delta(z)$ of weight 12 on $PSL(2,\mathbb{Z})$.

More generally, we will show: for any even weight **k** for which the space of cusp forms of weight k and level 1 is non–zero, and for any $n \geq 1$, there exists at last one cusp form of weight k and level 1 (in the respect to the full modular group) whose n^{th} Fourier coefficient is not zero.

And furthermore, the surprising theorem will be established here: if q is a prime and $q \longrightarrow +\infty$ then $\tau^2(q) >> q^{9-\alpha}$ for any fixed positive α .

So the Serré conjecture is true for the Ramanujan function τ .

This last assertion is the result of the work what was beginning together with A. Novikov. So it happened that this deal was not finished in his life.

This work has a long history and it finished only in time of the inviting visit to Max–Planck–Institut für Mathematik; I am grateful to professor F. Hirzebruch for this invitation.

§ 1. Introduction.

Let $G = PSL(2,\mathbb{Z})$ be the full modular group which acts on the upper half–plane $\mathbb{H}$, $\mathbb{H} = \{z = x+iy \mid x,y \in \mathbb{R}, y > 0\}$, by $z \longmapsto gz = \frac{az+b}{cz+d}$, a, b, c, d are rational integers with $ad-bc = 1$.

We write $j(g,z) = cz+d$ if a transformation g is defined by a matrix with second line (c,d) .

A holomorphic function f on $\mathbb{H}$ is called a cusp form of the weight k with respect to the full modular group G if

i) for $g \in G$

$$f(gz) = j^k(g,z)f(z) \ ,$$

ii) the G–invariant function $y^{k/2}|f(z)|$ is bounded on $\mathbb{H}$,

iii) the 0^{th} Fourier coefficient $\int_0^1 f(z)dx$ is 0 .

We denote by $\mathcal{M}_k$ the space of cusp forms of the weight k ; we assume that k is an even integer and $k \geq 12$.

It is well known that $\mathcal{M}_k$ has finite dimension; namely (see [1]):

$$(1.1) \qquad \dim \mathcal{M}_k = [\tfrac{k}{12}] \ , \text{ if } k \not\equiv 2 \pmod{12} ,$$
$$= [\tfrac{k}{12}]-1 \ , \text{ when } k \equiv 2 \pmod{12}$$

where [x] is the integral part of x .

The space $\mathcal{M}_k$ is generated by the Poincaré series $P_n(z;k)$; by the definition,

$$(1.2) \qquad P_n(z;k) = \frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \sum_{g \in G_\infty \backslash G} j^{-k}(g,z)e(ngz), \ e(z) = e^{2\pi i z} \ ,$$

where G_∞ is the cyclic subgroup of G generated by the transformation $z \longmapsto k+1$.

All these series are identically zero if k = 4,6,8,10 or 14 , because $\dim \mathcal{M}_k = 0$ for these cases. At the same time it is well known that

(1.3) $$P_n(z;k) \not\equiv 0$$

for $1 \leq n \leq \dim \mathcal{M}_k$ if $\dim \mathcal{M}_k \geq 1$.

In general the problem of the identical vanishing for these series was unsolved up to the present day.

The main result of this work is the following

Theorem 1. *Let* $\dim \mathcal{M}_k \geq 1$. *Then for any* $n \geq 1$ *the Poincaré series* $P_n(z;k)$ *is not identically zero.*

There are other forms of this assertion. Let us denote by (f_1,f_2) the Petersson inner product for $f_1,f_2 \in \mathcal{M}_k$:

(1.4) $$(f_1,f_2) = \int_{G\backslash H} f_1 \bar{f}_2\, y^k d\mu(z)\ ,\ \ d\mu(z) = \frac{dxdy}{y^2}\ .$$

Then we have the Petersson formula for arbitrary $f \in \mathcal{M}_k$

(1.5) $$(f,P_n) = a_f(n)\ ,$$

where $a_f(n)$ is the n–th Fourier coefficient of the expansion

(1.6) $$f(z) = \sum_{n\geq 1} a_f(n) e(nz)\ ,\ \ e(z) = e^{2\pi i z}\ .$$

If $f_1,...,f_\nu$, $\nu = \dim \mathcal{M}_k$, is an orthonormal basis of $\mathcal{M}_k$ then from (1.5) we conclude that

(1.7) $$P_n(z;k) = \sum_{j=1}^{\nu} a_j(n) f_j(z) \ , \ \ a_j(n) = a_{f_j}(n) \ .$$

So $P_n \equiv 0$ is equivalent to $a_j(n) = 0$ for all $j = 1,\dots,\nu$. For this reason we have

Corollary 1. *Let* $\nu = \dim \mathcal{M}_k \geq 1$ *and* $a_1(n),\dots,a_\nu(n)$ *are* n^{th} *Fourier coefficients of the base functions; then for any* $n \geq 1$

(1.8) $$\sum_{j=1}^{\nu} |a_j(n)|^2 \neq 0 \ .$$

The special case $k = 12$ (when $\dim \mathcal{M}_{12} = 1$) gives

Corollary 2. *Let* $\tau(n)$, $n = 1,2,\dots,$ *be the Ramanujan function which is defined by the expansion*

(1.9) $$q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n=1}^{\infty} q^n \tau(n) \ , \quad |q| < 1 \ .$$

Then for all $n \geq 1$ *we have* $\tau(n) \neq 0$ *and hence*

(1.10) $$|\tau(n)| \geq 1 \ .$$

This corollary of the theorem 1 means that the Lehmer conjecture is true.

D. Lehmer conjectured that $\tau(n) \neq 0$ ([2]) on the basis of a computer experiment; up to now it was known that $\tau(n) \neq 0$ for all $n \leq 214\,928\,639$. J.–P. Serré has given a deeper basis for this conjecture in his excellent work [3].

The outstanding achievement of P. Deligne is the estimate

(1.11) $$|\tau(q)| \leq 2q^{11/2}$$

for a prime q (more generally, for the Fourier coefficients of a cusp form of the weight k what are the eigenfunctions of the Hecke operators Deligne proved the inequality

(1.12) $$|a(n)| \leq n^{\frac{k-1}{2}} d(n)|a(1)|$$

where d(n) is the number of a divisor of the natural n). It is known that for a positive proportion of the primes (Rum Murty, [4])

(1.13) $$|\tau(q)| \geq (1.189.....)q^{11/2} .$$

Our theorem 1 gives the more weak inequality (1.10) but for any individual prime; the following additional result will be proved at the end of this paper.

Theorem 2. *Let* q *be a prime and* $q \longrightarrow +\infty$. *Then for any positive* $\epsilon > 0$ we have

(1.14) $$\tau^2(q) >> q^{9-\epsilon} .$$

§ 2. Preliminaries.

2.1. *The forms of the weight zero.*

Let us denote by $E(z,s)$ the Eisenstein–Maass series which is defined for $\mathrm{Re}\, s > 1$ and $z \in \mathbb{H}$ by the infinite sum

(2.1)
$$E(z,s) = \sum_{g \in G_\infty \backslash G} (\mathrm{Im}\, gz)^s$$

$$= \sum_{g \in G_\infty \backslash G} |j(g,z)|^{-2s} y^s$$

We have the well known way to give the analytic continuation of this series by using the Fourier expansion

(2.2)
$$E(z,s) = y^s + \frac{\xi(1-s)}{\xi(s)} y^{1-s} +$$

$$+ \frac{2}{\xi(s)} \sum_{n \neq 0} \tau_s(n) e(nx) \sqrt{y}\, K_{s-1/2}(2\pi |n| y) \ ,$$

where $K_{s-1/2}(\cdot)$ is the modified Bessel function of the order $s-1/2$, with the usual designation for the Riemann zeta–function and gamma–function

(2.3)
$$\xi(s) = \pi^{-s} \Gamma(s) \zeta(2s) \ ,$$

and

$$\tau_s(n) = |n|^{s-1/2}\sigma_{1-2s}(n) = \sum_{\substack{d\,|\,n \\ d>0}} \left[\frac{|n|}{d^2}\right]^{s-1/2} . \tag{2.4}$$

May be it would useful to note: the modified Bessel function is exponentially decreasing when an argument is large,

$$K_{s-1/2}(y) = \sqrt{\frac{\pi}{2y}}\, e^{-y}(1 + 0(\tfrac{1}{y})),\ \ y \longrightarrow +\infty\ . \tag{2.5}$$

The main characteristic property of the Fourier coefficients $\tau_s(n)$ of the Eisenstein–Maass series for the full modular group is the well known Ramanujan identity: for Re s > 1 we have

$$\tau_s(n) = n^{s-1/2}\zeta(2s) \sum_{c\geq 1} \frac{S(0,n;c)}{c^{2s}},\ \ n \geq 1\ , \tag{2.5}$$

where S is the Ramanujan sum; this one is the special case of the Kloosterman sum S(n,m;c) ,

$$S(n,m;c) \quad = \sum_{\substack{(d\,,\,c)=1,\ dd'\equiv 1 (\mathrm{mod}\ c) \\ d(\mathrm{mod}\ \ c)\ ,}} e\left[\frac{nd}{c} + \frac{md'}{c}\right] . \tag{2.6}$$

The Eisenstein–Maass series E(z,s) for Re s = 1/2 is the eigenfunction of the continuous spectrum of the automorphic Laplacian.

There exists the infinite sequence

$$0 = \lambda_0 < \lambda_1 < \lambda_2 \leq .....$$

so that for each λ_j we have the non–zero G–automorphic solution u_j of the equation

$$(2.7) \qquad -y^2\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]u_j=\lambda_j u_j$$

with the condition

$$(2.8) \qquad (u_j,u_j)=\int_{g\backslash H}|u_j|^2 d\mu(z)<\infty \ .$$

It is convenient to assume that the functons u_j are choosen real and these ones are the eigenfunctions of all Hecke operators $T(n)$ and the reflection operator T_{-1} which are defined by the equalities

$$(2.9) \qquad (T(n)f)(z)=\frac{1}{\sqrt{n}}\sum_{\substack{ad=n\\ d>0}}\ \sum_{b(\mathrm{mod}\ d)} f\left[\frac{az+b}{d}\right] ,$$

$$(2.10) \qquad (T_{-1}f)(z)=f(-\bar{z}) \ .$$

For this choice we have $\rho_j(\pm 1)\neq 0$ for all $j\geq 1$ and for any $n\geq 1$ the real quantities

$$(2.11) \qquad t_j(n)=(\rho_j(1))^{-1}\rho_j(n)=(\rho_j(-1))^{-1}\rho_j(-n)$$

are the eigenvalues of the Hecke operator $T(n)$. As a consequence we have for all integers $n,m\geq 1$

(2.12) $$t_j(n)t_j(m) = \sum_{d\,|\,(n,m)} t_j\left[\frac{nm}{d^2}\right] .$$

The similar notations we shall use for the eigenvalues of the Hecke operators in the space of a cusp form of the integer even weight ℓ .

If the base functions $f_1,\dots,f_{\nu_\ell}$, $\nu_\ell = \dim \mathcal{M}_\ell$, are the Hecke basis then their Fourier coeffcients $a_{j,\ell}(n)$ are connected with the eigenvalues $t_{j,\ell}(n)$ of the n–th Hecke operator by the relations

(2.13) $$t_{j,\ell}(n) = (a_{j,\ell}(1))^{-1}\, n^{-\frac{\ell-1}{2}}\, a_{j,\ell}(n) .$$

2.2. *The trace formulas.*

Now the bilinear form of the eigenvalues of the Hecke operators is expressed in terms of the sum of Kloosterman sums; it is essential for our proof. We have the following identity ([5] , [6] , [7] ; sometimes it is called "the Kuznetsov trace formula").

<u>Theorem 3</u> (1). *Let* $\varphi \in C^3(0,\infty)$, $\varphi(0) = \varphi'(0) = 0$ *and* $|\varphi(x)| + |\varphi'(x)| + |\varphi''(x)| = =(x^{-B})$ *for some positive* $B > 2$ *if* $x \longrightarrow +\infty$. *Let* $S(n,m;c)$ *be the Kloosterman sum*

(2.14) $$S(n,m;c) = \sum_{\substack{ad\equiv 1 (\mathrm{mod}\ c)\\ d(\mathrm{mod}\ c)}} \ell\left[\frac{na+md}{c}\right] ,\ c \geq 1 .$$

Then for all $n,m \geq 1$ *we have*

(2.15)
$$\sum_{c\geq 1} \frac{1}{c} S(n,m;c)\varphi\left[\frac{4\pi\sqrt{nm}}{c}\right] =$$

$$= \sum_{j\geq 1} \alpha_j t_j(n)t_j(m)h(\kappa_j) + \frac{1}{\pi}\int_{-\infty}^{\infty} \tau_{1/2+ir}(n)\tau_{1/2+ir}(m) \frac{h(r)}{|\zeta(1+2ir)|^2} dr$$

$$+ \sum_{\substack{\ell\geq 12 \\ \ell \equiv 0 (\mathrm{mod}\ 2)}} h_\ell \sum_{j=1}^{\nu_\ell} \alpha_{j,\ell} t_{j,\ell}(n)t_{j,\ell}(m) ,$$

where the normalizing coefficients α_j, $\alpha_{j,\ell}$ *are defined by the equalities*

(2.16)
$$\alpha_j = (\mathrm{ch}(\pi x_j))^{-1} |\rho_j(1)|^2 , \quad \kappa_j = \sqrt{\lambda_j - 1/4}$$

(2.17)
$$\alpha_{j,\ell} = \frac{\Gamma(\ell)}{(4\pi)^\ell} |a_{j,\ell}(1)|^2, \quad 1 \leq j \leq \nu_\ell = \dim \mathscr{M}_\ell ,$$

and the function h(r) *and the coefficients* h_ℓ *are defined by the integral transforms* (*with the standard notations for the Bessel functions*)

(2.18)
$$h(r) = \frac{i\pi}{2\mathrm{sh}(\pi r)} \int_0^\infty (J_{2ir}(x) - J_{-2ir}(x))\varphi(x) \frac{dx}{x} ,$$

(2.19)
$$h_\ell = i^\ell \int_0^\infty J_{\ell-1}(x)\varphi(x)\frac{dx}{x}$$

It will be useful later to reformulate this theorem when the weight function in the biliner form of the quantities $t_j(n)$ is considered as a given.

Theorem 3 (2). *Let* h(r) *be a regular even function in the strip* $|\mathrm{Im}\, r| \le \Delta$ *for some* $\Delta > 1/2$ *and* $|h(r)| << |r|^{-B}$ *for some* $B > 2$ *when* $r \longrightarrow \infty$ *in this strip. Then for any* $n,m \ge 1$ *we have*

$$\sum_{j\ge 1} \alpha_j t_j(n) t_j(m) h(\kappa_j) + \tag{2.20}$$

$$+ \frac{1}{\pi} \int_{-\infty}^{\infty} \tau_{1/2+ir}(n) \tau_{1/2+ir}(m) \frac{h(r)}{|\zeta(1+2ir)|^2}\, dr =$$

$$= \frac{\delta_{n,m}}{\pi^2} \int_{-\infty}^{\infty} r\, \mathrm{th}(\pi r) h(r) dr + \sum_{c\ge 1} \frac{1}{c} S(n,m;c) \varphi\left[\frac{4\pi\sqrt{nm}}{c}\right]$$

where for $x > 0$

$$\varphi(x) = \frac{i}{\pi} \int_{-\infty}^{\infty} (J_{2ir}(x) - J_{-2ir}(x)) \frac{r h(r)}{\mathrm{ch}(\pi r)}\, dr \ . \tag{2.21}$$

§ 3. The initial identity.

Now we shall consider the following integral of the Rankin type in which three automorphic functions occur:

$$A_f(s,N) = \int_{G\backslash N} y^k f(z)\, \overline{P_N(z,k)}\, E(z,s) d\mu(z) \ , \tag{3.1}$$

where $f \in \mathscr{M}_k$, $k \geq 4$ is the fixed integer and N is a positive integer; G is written instead of $\Gamma_0(1)$.

Because each cusp form decreases exponentially when $\mathrm{Im}\, z \longrightarrow +\infty$ and

$$|E(z,s)| << y^{\max(\mathrm{Re}\, s, 1-\mathrm{Re}\, s)} ,$$

the integral (3.1) converges for all fixed $s \in \mathbb{C}$.

There are two different ways to calculate this integral since both P_N and E have a representation as a sum over the group.

If the definition (1.2) will be used then one can go to the next expression for A_f:

$$\text{(3.2)} \qquad \frac{\Gamma(k-1)}{(4\pi N)^{k-1}} A_f(s,N) = \sum_{g \in G_\infty \backslash G} \int_{G\backslash \mathbb{H}} y^k f(z) \overline{(j^{-k}(g,z) e(Ngz))} E(z,s) d\mu$$

(because of the absolute convergence in (1.2) for $k \geq 4$)

$$= \sum_{g \in G_\infty \backslash G} \int_{g(G\backslash \mathbb{H})} y^k f(z)\, \overline{e(Nz)}\, E(z,s)\, d\mu(z)$$

(since $f(gz) = j^k(g,z) f(z)$, $E(gz,s) = E(z,s)$ and $d\mu(z)$ is G–invariant measure)

$$= \int_0^\infty \int_0^1 y^k f(z)\, \overline{e(Nz)}\, E(z,s) \frac{dxdy}{y^2}$$

(it is a consequence of the fact that $\bigcup_{g \in G_\infty \backslash G} g(G\backslash \mathbb{H})$ is a fundamental domain of G_∞ which may be choosen as the strip $0 \leq x < 1$, $y > 0$).

After replacing f and E by their Fourier expansions we obtain in this way the first expression for $A_f(s,N)$, containing the integer N as an additive variable,

$$(3.3)\qquad A_f(s,N) = \frac{1}{\Gamma(k-1)\xi(s)}\left\{\left[\frac{\Gamma(k-1+s)}{(4\pi N)^s}\,\xi(s) + \frac{\Gamma(k-s)}{(4\pi N)^{1-s}}\,\xi(1-s)\right]a(N) + \right.$$

$$\left. + \frac{2^k N^{k-1}}{\sqrt{2\pi}} \sum_{n\neq N} a(n)\tau_s(n-N)\int_0^\infty e^{-(n+N)y}K_{s-1/2}(|n-N|y)y^{k-3/2}dy\right\} .$$

This equality holds for all $s \in \mathbb{C}$ for which the series on the right side is convergent. Another way is to do the same using (2.1) instead of (1.2) and replacing P_N by the corresponding expansion (1.9) over the base functions in $\mathcal{M}_k$.

In this way one gets the equality

$$(3.4)\qquad A_f(s,N) = \frac{\Gamma(k-1+s)}{(4\pi)^{k-1+5}} \sum_{j=1}^{\nu_k} |a_{j,k}(1)|^2 t_{j,k}(N)R_{j,k}(s)$$

where for $\operatorname{Re} s > 1$

$$(3.5)\qquad R_{j,k}(s) = \sum_{n=1}^{\infty} n^{-s}t(n)t_{j,k}(n)$$

and for all $s \in \mathbb{C}$ we have for these Rankin series

$$(3.6)\qquad R_{j,k}(s) = \frac{(4\pi)^{k-1+s}}{\Gamma(k-1+s)}\,|a_{j,k}(1)|^{-2}\int_{G\backslash H} y^k f\,\bar{f}_j\,E(z,s)d\mu(z) .$$

Now in (3.4) the integer N is the argument of a multiplicative function only. One can look in the tables of the integral transforms ([8] , ch. 10, (23); this equality may be checked easily by the comparison of the differential equations for the left and right sides) to see that for $\mathrm{Re}(\alpha+1) > 0$

$$(3.7) \qquad \int_0^\infty e^{-\alpha y} K_{s-1/2}(y) y^{k-3/2} dy =$$

$$= \left[\frac{\pi}{2}\right]^{1/2} \frac{2^s}{(\alpha+1)^{k-1+s}} \frac{\Gamma(k-1+s)\Gamma(k-s)}{\Gamma(k)} F(k-1+s,s;k;\tfrac{\alpha-1}{\alpha+1}) ,$$

where F is the Gauss hypergeometric function.

It follows from this equality that the series on the right side (3.3) converges absolutely when

$$(3.8) \qquad 3/2 - k/2 < \mathrm{Re}\, s < k/2 - 1/2$$

(this strip is not empty if $k > 2$).

In connection with (3.7) it will be useful later to introduce the function

$$(3.9) \qquad \psi(\xi,s) = (\mathrm{th}\ \xi/2)^{k-1/2} (\mathrm{ch}\ \xi/2)^{1-2s} F(k-1+s,s;k;\mathrm{th}^2\ \xi/2)$$

and take into account that for

$$\xi = \xi_n(N) = \log \frac{\sqrt{n} + \sqrt{N}}{|\sqrt{n} - \sqrt{N}|}$$

we have

$$\mathrm{th}^2\,\xi/2 = \frac{n+N-|n-N|}{n+N+|n-N|}\ .$$

Then the result of our calculations of the integral $A_f(s,N)$ on two ways is the following initial identity.

Lemma 3.1. *Let* k *be an even integer with* $k \geq 4$, N *is a positive integer. Then for any* $f \in \mathcal{M}_k$ *and* $s \in \mathbb{C}$ *satisfying* $|\mathrm{Re}\, s - 1/2| < k/2-1$ *we have*

$$(3.10)\qquad \sum_{\substack{n=1\\ n\neq N}}^{\infty} (nN)^{-1/4} t(n)\tau_s(n-N)\psi(\xi_n(N),s) =$$

$$= -\Gamma(k)t(N)\left\{\frac{\xi(s)}{(4\pi N)^s\Gamma(k-s)} + \frac{\xi(1-s)}{(4\pi N)^{1-s}\Gamma(k-1+s)}\right\} +$$

$$+ \frac{\Gamma(k)\Gamma(k-1)\,\xi(s)}{(4\pi)^{k-1+s}\Gamma(k-s)} \sum_{j=1}^{\nu_k} t_j(N)\,|a_{j,k}(1)|^2 R_{j,k}(s)$$

where

$$(3.11)\qquad t(n) = n^{-\frac{k-1}{2}} a(n)\,|a(1)|^{-1},\quad t_{j,k}(n) = n^{-\frac{1-k}{2}} a_{j,k}(n)\,|a_{j,k}(1)|^{-1}$$

and $\xi(s)$, $R_{j,k}(s)$ *are defined by* (2.3), (3.5).

Now the following simple philosophy underlies of this work. If we want to know whether the quantities $t_j(q)$ are zero for a fixed q, then we can use the fact that there are two free variables in the identity of Lemma 3.1.

Let us consider the simplest case when $\nu_k = 1$ ($k = 12$, for example) so there is one arithmetical quantity $t(N)$ in the right side of (3.10). We are writing mq instead of N in (3.10); then $t(N) = t(n)t(q)$ for m which will be coprime with q . After this we shall construct some average over two free variables s and m $((m,q) = 1)$.

A common principle is "an average can be estimated more precisely than an individual".

For this reason we can hope that the result of a reasonable averaging in the left side will be near to the true order. At the same time the average in the right side is proportional to the quantity $t(q)$.

Now we have the inequality

(3.12) (average on the left side) $<<$ $|t(q)|$ · (average on the right side) .

If the estimate on the left is non–trivial (and non–zero for this reason) then we immediately have $t(q) \neq 0$.

If we have also a reasonable estimate on the right then we shall be able to give a non–trivial lower estimate for the quantity $|t(q)|$ of the kind $|t(q)| \geq t_0(q)$ with some increasing function t_0 .

§ 4. The preparation to the averaging.

To avoid some difficulties for the terms with large "n" in the left side (3.10) we shall rewrite this identity. The aim is to replace the sum over $n >> N$ by the linear form of the eigenvalues of the N–th Hecke operator.

4.1. *The certain Rankin series.*

To express the result we shall introduce some new notations. First of all we shall consider the family of the Rankin series

$$R_j(s) = \sum_{n=1}^{\infty} n^{-s} t(n) t_j(n) \,, \quad \operatorname{Re} s > 1 \,, \tag{4.1}$$

where $t(n)$, $n = 1,2,\ldots$, are the same normalized Fourier coefficients of the fixed (non–zero) cusp form as in (3.10) and $t_j(n)$ are connected with the Fourier coefficients of j–th eingenfunction by (2.14).

Further let $R_{j,\ell}$ be the similar Rankin series

$$R_{j,\ell}(s) = \sum_{n=1}^{\infty} n^{-s} t(n) t_{j,\ell}(n) \,, \quad \operatorname{Re} s > 1 \tag{4.2}$$

which is associated with a regular cusp form of the weight ℓ.

The same symbols R_j, $R_{j,\ell}$ we shall use for the analytical continuation of these functions in the half plane $\operatorname{Re} s \leq 1$.

To do the analytical continuation we introduce the Eisenstein–Maass series with the Hecke character; for $\operatorname{Re} s > 1$ and $z \in \mathbf{H}$ it is the sum

$$E_m(z,s) = \sum_{g \in G_\infty \backslash G} e^{-im \arg j(g,z)} (\operatorname{Im} gz)^s \tag{4.3}$$

$$= y^s \sum_{g \in G_\infty \backslash G} j^{-m}(g,z) \, |j(g,z)|^{-2s+m}$$

We shall assume that m is an even integer; then we have the Fourier expansion

$$\pi^{-s}\zeta(2s)\Gamma(s+\tfrac{m}{2})E_m(z,s) = \tag{4.4}$$

$$= \pi^{-s}\zeta(2s)\Gamma(s+\tfrac{m}{2})y^s + \pi^{-1+s}\zeta(2-2s)\Gamma(1-s+\tfrac{m}{2})y^{1-s} +$$

$$+ i^{-m}\sum_{n=1}^{\infty}\frac{\tau_s(n)}{\sqrt{n}}e(nx)W_{m/2,1/2-s}(4\pi ny) +$$

$$+ i^{-m}\sum_{n=1}^{\infty}\frac{\tau_s(n)}{\sqrt{n}}e(-nx)\frac{\Gamma(s+\frac{m}{2})}{\Gamma(s-\frac{m}{2})}W_{-m/2,1/2-s}(4\pi ny)$$

where $W_{\mu,\nu}$ is the Whittaker function. This Fourier expansion gives the meromorphic continuation for E_m on the whole s–plane; furthermore, as the consequence of the Kummer relation

$$W_{m,1/2-s}(y) = W_{m,s-1/2}(y) \quad , \tag{4.5}$$

we have the functional equation

$$E_m^*(z,s) = E_m^*(z,1-s), \;\; m \in \mathbb{Z}, \;\; m \equiv 0 (\mathrm{mod}\ 2) \quad , \tag{4.6}$$

for the function

$$E_m^*(z,s) = \pi^{-s}\zeta(2s)\Gamma(s+\tfrac{m}{2})E_m(z,s) \quad . \tag{4.7}$$

Now for the first Rankin series (4.1) we have the representation

$$(4.8)\qquad \frac{\Gamma(\frac{k-1}{2}+s+\kappa_j)\Gamma(\frac{k-1}{2}+s-i\kappa_y)}{2^{k-1}\pi^{k/2-1}\Gamma(k/2+s)}(2\pi)^{-2s}\rho_j(1)\,\overline{a(1)}\,R_j(s) =$$

$$= \int_{G\backslash H} u_j \cdot \overline{y^{k/2}f(z)} \cdot E_k(z,s)d\mu(z) \ ,$$

where $f \in \mathscr{M}_k$. Finally it follows from (4.8) that $R_j(s)$ has a meromorphic continuation on the whole s–plane and further: the function

$$(4.9)\qquad R_j^*(s) = (2\pi)^{-2s}\Gamma(\tfrac{k-1}{2}+s+i\kappa_j)\Gamma(\tfrac{k-1}{2}+s-i\kappa_j)\zeta(2s)R_j(s)$$

satisfies to the functional equation

$$(4.10)\qquad R_j^*(s) = R_j^*(1-s) \ .$$

The similar functional equation we have for the Rankin series (4.2); for this case for $f_j \in \mathscr{M}_\ell$, $f \in \mathscr{M}_k$ we have

$$(4.11)\qquad (4\pi)^{-\frac{\ell+k}{2}+1-s}\Gamma(\tfrac{\ell+k}{2}+s-1)\,\overline{a_{j,\ell}(1)}\,a(1)\,R_{j,\ell}(s) =$$

$$= \int_{G\backslash H} \overline{y^{\ell/2}f_j} \cdot y^{k/2}f \cdot E_{\ell-k}(z,s)d\mu(z) \ .$$

Hence the functional equation is satisfied

(4.12) $$R^*_{j,\ell}(s) = R^*_{j,\ell}(1-s) \ ,$$

where

(4.13) $$R^*_{j,\ell}(s) = (2\pi)^{-2s}\Gamma(\frac{|\ell-k|}{2}+s)\Gamma(\frac{\ell+k}{2}-1+s)\zeta(2s)R_{j,\ell}(s) \ .$$

The functional equations (4.10) and (4.12) are well known for the Rankin series so there is no need to add the details.

4.2. *The Hecke series.*

In the special case when the quantities $t_j(n)$ in the definition (2.7) are replaced by the Fourier coefficients of the Eisenstein series this Rankin series may be expressed as the product two Hecke series. Namely, we have

(4.14) $$\sum_{n=1}^{\infty} \frac{\tau_\nu(n)t(n)}{n^s} = \frac{1}{\zeta(2s)} H(s+\nu-1/2)H(s-\nu+1/2)$$

if $\operatorname{Re} s > 1 + |\operatorname{Re}(\nu-1/2)|$. Here $H(s) = \sum_{n=1}^{\infty} n^{-s}t(n)$; this identity is another form of the multiplicative relations

(4.15) $$t(n)t(m) = \sum_{d|(n,m)} t\left[\frac{nm}{d^2}\right] \ .$$

Note that $H(s)$ is the entire function in s .

4.3. *One convolution formula.*

Theorem 4. *Let* $\Phi \in C^{\infty}(0,\infty)$ *with a bounded support and* $\Phi(x) \equiv 0$ *for* $x \leq 1+\delta$ *for some* $\delta > 0$. *Then for every* $N \geq 1$ *and* $\mathrm{Re}\, s = 1/2$ *we have*

$$(4.16) \qquad \frac{1}{\sqrt{N}} \sum_{n \geq N} t(n)\tau_s(n-N)\Phi(\tfrac{n}{N}) =$$

$$= \zeta(2s)\Big\{ \sum_{j\geq 1} \alpha_j t_j(N) R_j(s) h(\kappa_j,s) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{H(s+ir)H(s-ir)}{|\zeta(1+2ir)|^2}\, \tau_{1/2+ir}(N) h(r,s)\,dr +$$

$$+ \sum_{\substack{\ell > 12 \\ \ell \equiv 0 \ (\mathrm{mod}\ 2)}} h_\ell(\delta) \sum_{j=1}^{\nu_\ell} \alpha_{j,\ell} t_{j,\ell}(N) R_{j,\ell}(s) \Big\}$$

where R_j and $R_{j,\ell}$ are defined by (4.9) and (4.13), the function h is defined by the integral transform

$$(4.17) \quad h(r,s) = \tfrac{1}{2}\sin(\pi s)(2\pi)^{1-2s} \int_1^{\infty} y^{-1/4}\Phi(y) \Big\{ \frac{A(s,r)}{A(1-s,r)}\, v(y;s,r) + v(y;1-s,r) \Big\}\, dy$$

with

$$(4.18) \quad v(y;s,r) = y^{k/2-1/4}(y-1)^{s-1/2}\Gamma(1-2s) F(\tfrac{k-1}{2}+s+ir, \tfrac{k-1}{2}+s-ir; 2s; 1-y) \ ,$$

$$(4.19) \qquad A(s,r) = \Gamma(\tfrac{k-1}{2}+s+ir)\Gamma(\tfrac{k-1}{2}+s-ir) \ .$$

The coefficients $h_\ell(s)$ are defined by the integrals

$$(4.20)\quad h_\ell(s) = \frac{1}{2}\sin(\pi s)\cdot(2\pi)^{1-2s}\cdot\frac{\Gamma(\frac{\ell+k}{2}-1+s)\Gamma(\frac{\ell-k}{2}+s)}{\Gamma(\ell)}\int_1^\infty y^{-1/4}\Phi(y)v_\ell(y,s)dy$$

where

$$(4.21)\qquad v_\ell(y,s) = y^{-\ell/2+1/4}(1-\frac{1}{y})^{s-1/2}F(\frac{\ell+k}{2}-1+s,\frac{\ell-k}{2}+s;\ell;\frac{1}{y}) .$$

4.4. *The proof of the convolution formula.*

To prove the identity (4.16) we shall use the Ramanujan identity (2.5) (note again – this representation for the Fourier coefficients of the Eisenstein series is a pecularity of the full modular group) and the following summation formula from [9].

Theorem 5. *Let* $\varphi : [0,\infty) \longrightarrow \mathbb{C}$ *be a* C^∞*–function with a bounded support and* $t(n)\cdot n^{\frac{k-1}{2}}$, $n = 1,2,\dots$, *are the Fourier coefficients of a cusp form from* $\mathcal{M}_k$. *Then for every integer* $c \geq 1$ *and any* a *which is relatively prime to* c *we have*

$$(4.22)\qquad \sum_{n=1}^{\infty} t(n)e(\frac{na}{c})\varphi(n) = \frac{4\pi i^k}{c}\sum_{n=1}^{\infty} t(n)e(-\frac{na'}{c})\Phi(\frac{4\pi\sqrt{n}}{c})$$

where a' is defined by the congruence

$$aa' \equiv 1 \pmod c$$

and for $x > 0$

(4.23)
$$\Phi(x) = \int_0^\infty J_{k-1}(xy)\varphi(y^2)y\,dy\ .$$

It is useful to note an immediate consequence of (4.22). There is the possibility to express the sum of the Ramanujan sums in terms of the Kloosterman sums:

(4.24)
$$\sum_{n=1}^\infty t(n)S(0,n-N;c)\varphi(n) = \frac{4\pi i^k}{c} \sum_{n=1}^\infty t(n)S(n,N;c)\Phi\left(\frac{4\pi\sqrt{n}}{c}\right)\ .$$

The conditions of Theorem 5 are not necessary, of course; for the practical using (4.22) it is sufficient to know that both series on the left and right sides are convergent absolutely. Now for an arbitrary "good" function Φ we have for $\mathrm{Re}\ s > 1$:

(4.25)
$$\sum_{n=1}^\infty t(n)\tau_s(n-N)\Phi(\tfrac{n}{N}) =$$

$$= \zeta(2s) \sum_{c\geq 1} \frac{1}{c^{2s}} \Big(\sum_{n\geq 1} t(n)S(0,n-N;c)(n-N)^{s-1/2}\Phi(\tfrac{n}{N})\Big)\ .$$

Using (4.24) for the inner sum we come to the double series with the terms

(4.26)
$$i^k\cdot(4\pi)^{1-2s}\cdot \frac{\sqrt{N}}{n^s}\cdot t(n)\ \cdot\ \frac{S(n,N;c)}{c}\ \cdot$$

$$\cdot\ x^{2s}\int_1^\infty J_{k-1}(x\sqrt{y})(y-1)^{s-1/2}\Phi(y)dy,\ \ x = \frac{4\pi\sqrt{nN}}{c}\ .$$

For any fixed $B \geq 2$ the last integral is $0(x^{-B})$ when $x \longmapsto +\infty$ (it is the result of the multiple integration by parts because it was assumed $\Phi \equiv 0$ in some neighbourhood of 1). Further, if for y large we have $\Phi(y) << y^{-B_1}$ with a sufficiently large B_1 then this integral is $0(x^{B_2})$, $B_2 = \min(2B_1-3/2, k-1+2\mathrm{Re}\, s)$ when $x \longrightarrow 0$.

So the double series is absolutely convergent and, furthermore, for $k \geq 4$ and $\mathrm{Re}\, s > 1$ the conditions of the theorem 3(1) are fullfilled for the function

$$\tilde{\Phi}(x) = x^{2s} \int_1^\infty J_{k-1}(x\sqrt{y})(y-1)^{s-1/2}\Phi(y)dy \ . \tag{4.27}$$

For this reason all series on the right side (2.18) (with the function (4.27) instead of φ) will be convergent for $\mathrm{Re}\, s > 1$.

So for $\mathrm{Re}\, s > 1$ we can write the expression on the right side (4.16) in the form

$$\zeta(2s)(4\pi)^{1-2s}i^k \sum_{n=1}^\infty \frac{t(n)}{n^s} \Big\{ \sum_{j\geq 1} \alpha_j t_j(n) t_j(N) \tilde{h}(\kappa_j, s) + \tag{4.28}$$

$$+ \frac{1}{4\pi} \int_{-\infty}^\infty \tau_{1/2+ir}(n)\tau_{1/2+ir}(N) \frac{\tilde{h}(r,s)}{|\zeta(1+2ir)|^2} dr +$$

$$+ \frac{4}{\pi} \sum_{\substack{\ell \geq 12 \\ \ell \equiv 0 \pmod 2}} \tilde{h}_\ell(s) \sum_{j=1}^{\nu_\ell} \alpha_{j,\ell} t_{j,\ell}(n) t_{j,\ell}(N) \Big\}$$

where $\tilde{h}(r,s)$ and $\tilde{h}_\ell(s)$ are defined by the integral transformations (2.21) and (2.22) with φ replaced by the function $\tilde{\varphi}$ from (4.27).

The result of the summation over $n \geq 1$ for $\mathrm{Re}\, s > 1$ is expressed in the terms of the corresponding Rankin series (the definitions (4.1), (4.2) and the identity (4.14)).

It rests to calculate the coefficients $\tilde{h}(r,s)$ and $\tilde{h}_\ell(s)$ and to see that our identity may be continued in the strip $0 < \mathrm{Re}\, s \leq 1$.

What will be the result of the integral transformation

$$\tilde{h}(r,s) = \frac{i\pi}{2\mathrm{sh}(\pi r)} \int_0^\infty (J_{2ir}(x) - J_{-2ir}(x)) \tilde{\Phi}(x) \frac{dx}{x}$$

we can obtain by the following manner.

For the case when $\Phi \in C^\infty(0,\infty)$ has a bounded support the function $\tilde{h}(\cdot;s)$ is the regular function of s in the half–plane $1 - \frac{k}{2} < \mathrm{Re}\, s$.

There is the strip $1 - \frac{k}{2} < \mathrm{Re}\, s < \frac{1}{2}$ where we can integrate over x under the sign of the integration over y (after the replacement of $\tilde{\phi}$ by its definition (4.27)).

The inner integral is the well known Weber–Schafheitlin integral; for any positive numbers $0 < a < b$ and for $-1 < \mathrm{Re}\, \rho < \mathrm{Re}(\nu+\mu+1)$ we have

$$(4.29) \qquad \int_0^\infty J_\mu(ax) J_\nu(bx) x^{-\rho} dx =$$

$$= \frac{a^\mu}{2^\rho b^{\mu-\rho+1}} \cdot \frac{\Gamma(A)}{\Gamma(C)\Gamma(1-B)} F(A,B;C;\frac{a^2}{b^2})$$

with $C = \mu+1$, $A = \frac{1}{2}(1+\nu+\mu-\rho)$, $B = \frac{1}{2}(1+\mu-\nu-\rho)$.

As a special case we have now

$$(4.30) \qquad h(r,s) = i^k (4\pi)^{1-2s} \tilde{h}(r,s) =$$

$$= \frac{\pi i^{k+1}}{4\mathrm{sh}(\pi r)} \cdot (2\pi)^{1-2s} \cdot \frac{\Gamma(\frac{k-1}{2}+s+ir)}{\Gamma(1+2ir)\Gamma(\frac{k+1}{2}-s-ir)} \cdot$$

$$\cdot \int_1^\infty y^{-1/2-ir}(1-\tfrac{1}{y})^{s-1/2}F(\tfrac{k-1}{2}+s+ir,\tfrac{1-k}{2}+s+ir;1+2ir;\tfrac{1}{y}) \cdot$$

$$\cdot\ \Phi(y)dy + \{\text{the same with } r \longmapsto -r\}\ .$$

By the same way we come to the formula (4.20) for the coefficients in the sum over the regular cusp forms.

From the representations (4.30) and (4.20) it is clear that $h(\cdot,s)$ and $h_\ell(s)$ are regular in $\mathrm{Re}\, s \geq 0$.

Furthermore, we can integrate any times by parts; it gives the estimate $|h(r,s)| << |r|^{-B}$ for any fixed B when $r \longrightarrow \pm\infty$ and s is a fixed with $\mathrm{Re}\, s \geq 0$. So the sums over the discrete and the continuous spectrum are convergent.

The same is true for the sum over cusp forms because h_ℓ are exponentially small when $\ell \longrightarrow \infty$ since $\phi(y) \equiv 0$ for $y \leq 1+\delta$ with a positive δ .

It is convenient to have two representations for the coefficients $h(r,s)$. The first is (4.30) and the second one is given in (4.17). It follows from (4.30) with the help of the Kummer relations ([8], the relation between u_1, u_2 and u_6 is the subsection 2.9) and the simple relation $F(A,B;C;z) = (1-z)^{-A}F(A,C-B;C;\frac{z}{z-1})$ which follows from the Gauss integral representation for the hypergeometric function.

4.5. *The truncation on the left side of the initial identity.*

For many cases it will be convenient to use the smoothed (by a special manner) characteristic function of a given interval.

Let a and b be real numbers and $a < b$. We define the function $\omega_{a,b}(x) \in C^{\infty}(-\infty,\infty)$ supposing

$$\omega_{a,b}(x) = 0, \ -\infty < x \le a \ , \quad = 1, \ a + \frac{b-a}{4} \le x \le b - \frac{b-a}{4} \ , \quad = 0, \ x \ge b \ . \tag{4.31}$$

We assume $\omega_{a,b}$ be a monotonic in $(-\infty, \frac{a+b}{2})$ and in $(\frac{a+b}{2}, \infty)$ and for all x $0 \le \omega_{a,b}(x) \le 1$.

Further, for a given $\omega_{a,b}$ we define

$$\eta_{a,b}(x) = c_{a,b} \int_{-\infty}^{x} \omega_{a,b}(y)dy \ , \ c_{a,b} = \left(\int_{-\infty}^{\infty} \omega_{a,b}(y)dy\right)^{-1} \ , \tag{4.32}$$

so this function from $C^{\infty}(-\infty,\infty)$ is a smoothed "step": $\eta_{a,b}(x) \equiv 0$ for $x \le a$ and $\eta_{a,b}(x) \equiv 1$ if $x \ge b$.

Let a, b with $0 < a < b$ be the fixed positive numbers and T is a large parameter (the suitable value for T will be choosen later as a result of the certain estimates).

We write the sum on the left side (3.10) in the form

$$\sum_{\substack{n \ge 1 \\ n \ne N}} (nN)^{-1/4}(1-\eta_{a,b}(\tfrac{n}{NT}))t(n)\tau_s(n-N)\psi(\xi_n(N),s) + Z_{N,T}(s) \tag{4.33}$$

with

$$(4.34) \qquad Z_{N,T}(s) = N^{-1/2} \sum_{n\geq 1} \eta_{a,b}(\tfrac{n}{NT})(\tfrac{n}{N})^{-1/4} t(n)\tau_s(n-N)\psi(\xi_n(N),s) \ .$$

The first sum in (4.33) is the truncated initial sum; for any given $T > 1$ it is finite. The second sum $Z_{N,T}$ will be expressed with the help of the convolution formula (4.16) in the form which will be convenient for the estimates.

Lemma 4.1. *For any positive* $a < b$ *and for* T *with* $T > 1$ *we have for* $\mathrm{Re}\, s = 1/2$

$$(4.35) \qquad Z_{N,T}(s) = \sum_{j\geq 1} \alpha_j t_j(N) R_j(s)\zeta(2s) h_T(\kappa_j,s) +$$

$$+ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\zeta(2s)H_j(s+ir)H_j(s-ir)}{|\zeta(1+2ir)|^2} \tau_{1/2+ir}(N) h_T(r,s)dr +$$

$$\sum_{\substack{\ell > 12 \\ \ell\equiv 0 \pmod 2}} h_{T,\ell}(s) \sum_{j=1}^{\nu_\ell} \alpha_{j,\ell} t_{j,\ell}(N)\zeta(2s)R_{j,\ell}(s)$$

where the same notations are used as in (4.16) and $h_T(r,s)$, $h_{T,\ell}(s)$ are the coefficients (4.17), (4.20) with Φ replaced by the function

$$(4.36) \qquad \Phi(x;s) = x^{-1/4}\eta_{a,b}(\tfrac{x}{T})\psi(\log \frac{\sqrt{x}+1}{\sqrt{x}-1},s) \ .$$

Later we shall write $\eta(x)$ instead of $\eta_{a,b}$; the parameter T will be choosen closed to L .

§ 5. The identity with an arbitrary function.

What is a "fitting" averaging? We can integrate (3.10) over s after multiplication by an arbitrary function. This function we choose as the Fourier transformation in the following sense.

Let $\varphi \in C^{\infty}(0,\infty)$ be a function with a bounded support; then we define $\hat{\varphi}$ by the equality

$$(5.1)\qquad \hat{\varphi}(s) = \int_0^{\infty} \psi(y,s)\varphi(y)dy$$

($\psi(\cdot,s)$ is the same what occurs in (3.10)).

Indeed $\psi(\cdot,s)$ is the eigenfunction of the singular boundary problem

$$(5.2)\qquad -\frac{d^2\psi}{d\xi^2} + \frac{(k-1)^2-1/4}{\mathrm{sh}^2\xi}\,\psi = -(s-1/2)^2\psi$$

with the condition

$$(5.3)\qquad \psi(\xi,s) = \left(\frac{\xi}{2}\right)^{k-1/2}(1+0(\xi^2))$$

when $\xi \longrightarrow 0$. When $\xi \longrightarrow +\infty$, Re $s = 1/2$, then

$$(5.4)\qquad \psi(\xi,s) = 2\Gamma(k)\mathrm{Re}\left\{\frac{2^{2s-1}\Gamma(1-2s)}{\Gamma(1-s)\Gamma(k-s)}\, e^{(1/2-s)\xi}(1+O(e^{-\xi}))\right\}\ .$$

Of course both equalities (5.3) and (5.4) are the consequence of the explicit form

(3.9): the first is evident and the second follows after using the Kummer relation which links hypergeometric function of argument z with the functions of argument 1–z .

Now for any arbitrary $\varphi \in C^{\infty}(0,\infty)$ with bounded support the following expansion holds – and this fact is the true reason for considering the transformation (5.1) –

$$(5.5) \qquad \varphi(x) = \int_{(1/2)} \psi(x,s)\hat{\varphi}(s)d\chi(s)$$

where $\int_{(\alpha)}$ denotes the integral over the line $\mathrm{Re}\, s = \alpha$ and the spectral measure $d\chi$ is

$$(5.6) \qquad d\chi(s) = \frac{s-1/2}{2i\pi\Gamma^2(k)} \cos(\pi s)\Gamma(k+s-1)\Gamma(k-s)ds$$

so that

$$(5.7) \qquad d\chi(1/2+it) = \frac{t\ th(\pi t)}{2\Gamma^2(k)} \left\{\prod_{j=1}^{k-1} ((k-1/2-j)^2+t^2)\right\}dt) \ .$$

We do not need to prove (5.5) because this equality is a slightly modified form of the known theorem (see [10], ch. 4, section 4.16).

To integrate the terms of the sum on the left side of (3.10) we have to calculate the integral

$$(5.8) \qquad u(\xi,\tau;\varphi) = \int_{(1/2)} \psi(\xi,s)e^{(1/2-s)\tau}\hat{\varphi}(s)d\chi(s) \ .$$

This integral arosed because

(5.9) $$\tau_s(n) = \sum_{d\,|\,n} e^{(s-1/2)\tau_{n,d}}, \quad \tau_{n,d} = \log\left[\frac{|n|}{d^2}\right] .$$

Note that

(5.10) $$\frac{d\chi}{ds}(s) = \frac{d\chi}{ds}(1-s), \quad \psi(\xi,s) = \psi(\xi,1-s) ,$$

so that we have together with (5.8)

(5.11) $$u(\xi,\tau;\varphi) = \int_{(1/2)} \psi(\xi,s)\mathrm{ch}((s-1/2)\tau)\hat{\varphi}(s)d\chi(s) .$$

Taking in account the differential equation for ψ and the representation (5.11) we see that u is the solution of the Cauchy problem

(5.12) $$\frac{\partial^2 u}{\partial \xi^2} - \frac{(k-1)^2-1/4}{\mathrm{sh}^2\xi}u = \frac{\partial^2 u}{\partial \tau^2}, \quad \xi > 0, \quad \tau \geq 0 ,$$

with the initial conditions

(5.13) $$u(\xi,0;\varphi) = \varphi(\xi) , \quad \frac{\partial u}{\partial \tau}(\xi,0;\varphi) = 0 .$$

This observation is essential because there is a second way to find the same solution besides the representation (5.8). It is sufficient for this to know the Riemann function for the problem (5.12) (in other words – to know the fundamental solution or the Green function).

It is possible to find the explicit form of the Riemann function for our Cauchy problem. Let us define

(5.14) $$z(\xi,x;\tau) = \frac{\mathrm{ch}\ \tau - \mathrm{ch}(x-\xi)}{2\ \mathrm{sh}x\ \mathrm{sh}\xi} .$$

Then we have (it is the result of the direct differentiation):

Proposition 5.1.

(5.15) $$\left[\frac{\partial z}{\partial \xi}\right]^2 - \left[\frac{\partial z}{\partial \tau}\right]^2 = -\frac{z(1-z)}{\mathrm{sh}^2\xi}$$

(5.16) $$\frac{\partial^2 z}{\partial \xi^2} - \frac{\partial^2 z}{\partial \tau^2} = \frac{2z-1}{\mathrm{sh}^2\xi} .$$

Consequently there is a solution of the equation (5.12) in the form

$$u = W(z(\xi,x;\tau))$$

since the same denominator $\mathrm{sh}^2\xi$ will occur for all terms.

Proposition 5.2. *The differential identity holds:*

(5.17) $$\frac{\partial^2 W}{\partial \xi^2} - \frac{(k-1)^2-1/4}{\mathrm{sh}^2\xi} W - \frac{\partial^2 W}{\partial \tau^2} =$$

$$= \left[\left[\frac{\partial z}{\partial \xi}\right]^2 - \left[\frac{\partial z}{\partial \tau}\right]^2\right] W'' + \left[\frac{\partial^2 z}{\partial \xi^2} - \frac{\partial^2 z}{\partial \tau^2}\right] W' - \frac{(k-1)^2-1/4}{\mathrm{sh}^2\xi} W$$

$$= -\frac{1}{\mathrm{sh}^2\xi}\left\{z(1-z)W'' + (1-2z)W' + ((k-1)^2-1/4)W\right\} ,\quad ' = \frac{d}{dz} .$$

In other words, by our substitution (5.14) we have reduced the partial differential equation (5.12) to an ordinary one.

In our case when k is the even integer we shall define for all $z \geq 0$

$$(5.18) \qquad W(z) = \frac{1}{2} \lim_{\epsilon \to 0} \left\{ F(3/2-k, k-1/2; 1; z+i\epsilon) + F(3/2-k, k-1/2; 1; z-i\epsilon) \right\} .$$

Since $F(a,b;c;z)$ is the solution of the differential equation

$$(5.19) \qquad z(1-z)F'' + (c-(a+b+1)z)F' - abF = 0$$

the function $W(z)$ turns the differential expression (5.12) into zero.

For $z > 1$ as the result of an analytical continuation to both sides of the cut $(1,+\infty)$ we have

Proposition 5.3. For an positive $z > 1$

$$(5.20) \qquad W(z) = -\frac{\Gamma^2(k-1/2)}{\pi\Gamma(2k-1)} z^{-k+1/2} F(k-1/2, k-1/2; 2k-1; \tfrac{1}{z}) .$$

Finally at $z = 1$ the function W has the logarithmical singularity,

$$(5.21) \qquad W(z) = +\frac{1}{\pi} \log|1-z| + \frac{2}{\pi} \left[\frac{\Gamma'}{\Gamma}(k-1/2) - \frac{\Gamma'}{\Gamma}(1) \right] +$$

$$+ 0(|z-1| \log \frac{1}{|z-1|}) .$$

Note that $z = 1$ corresponds to the lines

$$x + \xi \pm \tau = 0$$

on the plane (ξ,τ) and $z = 0$ if

$$x - (\xi \pm \tau) = 0 \ .$$

Now with this function $W(z(\xi,x,\tau))$ we have

Lemma 5.1. *The solution of the Cauchy problem* (5.12)–(5.13) *is given by the formulae*

$$u(\xi,\tau;\varphi) = \frac{1}{2}\varphi(\xi+\tau) + \frac{1}{2}\varphi(\xi-\tau) + \tag{5.22}$$

$$+ \frac{1}{4}\int_{\xi-\tau}^{\xi+\tau} W'(z(\xi,x;\tau)) \frac{sh\tau}{sh\xi \ shx} \varphi(x)dx \quad \textit{if} \ \xi \geq \tau \geq 0 \ ,$$

$$u(\xi,\tau;\varphi) = \frac{1}{2}\varphi(\xi+\tau) + \tag{5.23}$$

$$+ \frac{1}{4} \fint_{0}^{\xi+\tau} W'(z(\xi,x;\tau)) \frac{sh\tau}{sh\xi \ shx} \varphi(x)dx \quad \textit{if} \ \tau \geq \xi > 0 \ ,$$

where $\fint$ *is Cauchy's principal value,*

$$\fint_0^{\xi+\tau} = \lim_{\epsilon \to 0+} \left[\int_0^{\tau-\xi-\epsilon} + \int_{\tau-\xi-\epsilon}^{\xi+\tau} \right] ,$$

and $z(\xi,x;\tau)$ *is defined by* (5.14).

We have two representations for the same solution of the Cauchy problem; using the first (it is averaging over s) we have the following identity with an arbitrary function.

Lemma 5.2. *Let* $\varphi \in C^{\infty}(0,\infty)$ *be an arbitrary function with a bounded support which is separated from zero. Then for any integer* $N \geq 1$ *and for sufficiently large* $T > 1$ *we have*

$$(5.24) \qquad \sum_{\substack{n=1\\ n\neq N}}^{\infty} (nN)^{-1/4}(1-\eta(\tfrac{n}{NT}))t(n) \sum_{d\,|\,(n-N)} u(\xi_n(N),\tau_{n-N,d};\varphi) +$$

$$+ \int_{(1/2)} Z_{N,T}(s)\hat{\varphi}(s)d\chi(s) =$$

$$= -\Gamma(k)t(N) \int_{(1/2)} \left[\frac{\Gamma(s)\zeta(2s)}{(2\pi\sqrt{N})^{2s}\Gamma(k-s)} + \frac{\Gamma(1-s)\zeta(2-2s)}{(2\pi\sqrt{N})^{2-2s}\Gamma(k-1+s)}\right] \hat{\varphi}(s)d\chi(s) +$$

$$+ 4\pi\Gamma(k-1) \sum_{j=1}^{\nu_k} \alpha_{j,k}t_{j,k}(N) \int_{(1/2)} (2\pi)^{-2s} \frac{\Gamma(s)\zeta(2s)}{\Gamma(k-s)} R_{j,k}(s)\hat{\varphi}(s)d\chi(s) \ ,$$

where $\hat{\varphi}$ *is defined by* (5.1), $\xi_n(N)$ *and* $\tau_{m,d}$ *are defined by the equalities*

$$\xi_n(N) = \log \frac{\sqrt{n}+\sqrt{N}}{|\sqrt{n}-\sqrt{N}|}, \quad \tau_{m,d} = \log \frac{|m|}{d^2}$$

and other designations as in Lemma 3.1.

§ 6. The averaging over two free integer variables.

6.1. *Why there is a need for a new averaging.*

First of all we shall consider the following idea (which is ideal but idle).

Let φ_ϵ be a model of the Dirac δ–function: the support of φ_ϵ is $(x_0-\epsilon, x_0+\epsilon)$ and $\int \varphi_\epsilon(x)dx = 1$ for all $\epsilon > 0$. When $\epsilon \longrightarrow 0$ we have $\lim_{\epsilon\to 0} \epsilon\, \varphi_\epsilon(x) = 0$ if $x \neq x_0$ and $\lim_{\epsilon\to 0} \epsilon\, \varphi_\epsilon(x_0) = C \neq 0_\epsilon$ where C is the normalizing constant. For this reason for $-\xi + \tau \neq x_0$ we have

$$\lim_{\epsilon\to 0} \epsilon\, u(\xi,\tau;\varphi_\epsilon) = 0$$

if $\xi \pm \tau \neq x_0$ and

$$\lim_{\epsilon\to 0} \epsilon\, u(\xi,\tau;\varphi_\epsilon) = \tfrac{1}{2} C$$

if $\xi + \tau = x_0$ or $\xi - \tau = x_0$.

Suppose now that we could do the passage to the limit $\epsilon \to 0$ under the summation sign (namely, this possibility is an idle idea). Then we have

$$\text{(6.1)} \qquad \lim_{\epsilon\to 0} \epsilon \sum_{\substack{n=1\\ n\neq N}}^{\infty} \frac{t(n)}{n^{1/4}} \sum_{d\,|\,(n-N)} u(\xi_n(N), \tau_{n-N,d};\varphi_\epsilon) =$$

$$= \tfrac{1}{2} C \sum_{\substack{\xi_n(N)\pm\tau_{n-N,d}=x_0\\ d\,|\,(n-N),\ n\neq N}} n^{-1/4} t(n)$$

Let us choose

$$x_0 = \xi_L(N) + \tau_{N-L,1}$$

where $L > N$ and NL is not a perfect square. For this case the equation

$$\xi_n(N) + \tau_{n-N,d} = x_0$$

which is the same as

$$\log\left[\frac{\sqrt{n}+\sqrt{N}}{|\sqrt{n}-\sqrt{N}|} \cdot \frac{|n-N|}{d^2}\right] = 2\log\frac{\sqrt{n}+\sqrt{N}}{d} = 2\log(\sqrt{L}+\sqrt{N})$$

or

$$\sqrt{n} = d\sqrt{L} + (d-1)\sqrt{N}$$

has the unique integer solution

$$(n,d) = (L,1) \ ,$$

(because this equation means that $n = (\text{integer}) + 2d(d-1)\sqrt{NL}$ and LN is not a full square). By the same manner we have

$$\xi_n(N) - \tau_{n-N,d} = x_0$$

if and only if $(n,d) = (L, |N-L|)$. At the same time the equation

$$\tau_{n-N,d} - \xi_n(N) = x_0$$

or, equivalently,

$$\sqrt{n} = d\sqrt{L} + (d+1)\sqrt{N} \ ,$$

has no integer solution (n,d) .

The sum on the left side (5.24) is a multiple to $t(N)$ if $\dim \mathscr{M}_k = 1$ (see (5.24) with $T = \infty$); if the quantity $t(N)$ is zero then the result of the passage to the limit $\epsilon \longrightarrow 0$ will be zero also. So we would have $t(L) = 0$ if NL is not a perfect square. If N itself is not a perfect square (if N is prime, for example) then this conclusion is the contradiction.

There is no need to consider the case when N is a full square because the idea to do the passage under the summation sign fails. The reason – the nonuniform convergence of the series when ϵ tends to zero.

To avoid this difficulty we need stronger methods. We shall see that for the case of the full modular group the double averaging will be sufficient.

6.2. *The Averaging over two free integer variables.*

Let $q > 1$ be a fixed integer number.

To prove the Poincaré series $P_q(z;k)$ is not the identical zero we shall average the identity (5.24) over two integer variables by the following manner.

We suppose $N = mq$ with $(m,q) = 1$ in (5.24) and we take in this identity the function φ as a model of the Dirac delta–function with the small support $(x_0-\epsilon, x_0+\epsilon)$

where for the new integer variable ν the point x_0 is defined by the equality

$$(6.2)\qquad x_0 = x_0(mq,\nu) = \xi_\nu(mq) + \tau_{\nu-mq,1} = 2\log(\sqrt{\nu}+\sqrt{mq}) \ .$$

For the more definiteness we assume in (5.24) $\varphi(x) = \varphi_\epsilon(x-x_0)$ where

$$(6.3)\qquad \begin{aligned} \varphi_\epsilon(y) &= \frac{C}{\epsilon}\exp\left[-\frac{\epsilon^2}{\epsilon^2-y^2}\right] , && |y| < \epsilon \ , \\ &= 0 \ , && |y| \geq \epsilon \ , \end{aligned}$$

with the normalizing constant C from the condition $\int \varphi_\epsilon(y)dy = 1$, so $C = \left(\int_{-1}^{1} \exp(-(1-y^2)^{-1})dy\right)^{-1}$.

Furthermore, let $\omega_{a,b}$ be the same function what was used for the definition of the function $Z_{N,T}$ in (4.34).

Assuming in (5.24)

$$N = mq,\ \ \varphi(x) = \varphi_\epsilon(x-x_0(mq,\nu)),\ \ x_0 = 2\log(\sqrt{\nu}+\sqrt{mq}) \ ,$$

we shall consider the double average of both sides

$$(6.4)\ \sum_{(m,q)=1} \omega_{a_1,b_1}\left[\frac{mq}{M}\right] \sum_\nu \left[\frac{L}{\nu}\right]^{1/4} \omega_{a_2,b_2}\left[\frac{(\sqrt{\nu}+\sqrt{mq})^2}{L}\right] t(\nu)\{\text{the left side of (5.24)}\} =$$

$$= \{\text{the same double average on the right side (5.24)}\} \ .$$

Here M and L are the suitable large parameters which will be choosen as certain functions in ϵ, a_1, b_1, a_2, b_2 are certain fixed positive numbers. The main conditions for the parameter M, L are

$$(6.5)\qquad L, M \longrightarrow \infty\,,\ M = o(L)\,,\ \epsilon^2 L \longrightarrow 0$$

For more definiteness we fixe the small positive numbers $\delta_2 > \delta_1 > 0$ and define

$$(6.6)\qquad L = \epsilon^{-2+\delta_1}\,,\ M = \epsilon^{-2+\delta_2}$$

The parameter $Q = (\epsilon^2 L)^{-1}$, $Q \longrightarrow +\infty$, will be connected with q later; it will be assumed $q^{4+\alpha} << Q$ for any fixed positive α.

The multiplier $t(\nu)$ is introduced in (6.4) to increase the separating effect which occurs initially by the specialization of the function φ_ϵ (the coefficient before $t(n)$ is large if φ is replaced by φ_ϵ from (6.3) and n is near to ν). After the summation over ν in the long interval this effect will be better because of the essentially different estimates (when $L \longrightarrow +\infty$) for the sums

$$\sum_\nu t^2(\nu)\tilde{\omega}(\nu) \ \text{ and } \ \sum t(\nu)t(\nu+\nu_0)\tilde{\omega}(\nu)\,,\ \nu_0 \neq 0\,,$$

with a certain smoothed characteristic function of the interval $(L,2L)$.

Unfortunately I know only one way to show that the left side in (6.4) is not zero. This way is to give a kind of an asymptotic formula for this quantity when ϵ tends to zero and at the same time L, M tend to ∞ in the manner specified above.

We must remember that function φ in our identity depends on m and ν; to express this fact we set for $\varphi = \varphi_\epsilon(x - 2\log(\sqrt{\nu} + \sqrt{mq}))$

$$(6.7)\qquad V_\epsilon(n,d;mq,\nu) = u\left(\log \frac{\sqrt{n}+\sqrt{mq}}{|\sqrt{n}-\sqrt{mq}|}\ ,\ \left|\log \frac{|n-mq|}{d^2}\right|;\varphi_\epsilon\right) .$$

Furthermore $u(\xi,\tau;\varphi)$ is the even function of τ. For this reason the sum over divisors of $(n-mq)$ may be written as

$$(6.8)\qquad \sum_{d\,|\,(n-mq)} V_\epsilon(n,d;mq,\nu) = 2 \sum_{\substack{d\,|\,(n-mq)\\ 1\le d\le \sqrt{|n-mq|}}}{}' V_\epsilon(n,d;mq,\nu)$$

where $\sum'$ means that the term with $d=\sqrt{|n-mq|}$ is to be counted with multiplicity 1/2 because V_ϵ is invariant under $d\longrightarrow \frac{|n-mq|}{d}$.

Now we can write the sum on the left side of (6.4) in the form

$$(6.9)\qquad \sum\nolimits_1 + \int_{(1/2)} \sum\nolimits_2(s)d\chi(s)$$

where with the notation

$$(6.10)\qquad \Omega_{M,L}(n,N,\nu) = \left[\frac{L}{\nu n N}\right]^{1/4}(1-\eta(\tfrac{n}{NT}))\omega_{a_1,b_1}(\tfrac{N}{M})\omega_{a_2,b_2}\left[\frac{(\sqrt{\nu}+\sqrt{N})^2}{L}\right]$$

the sum $\sum_1$ is equal to

$$(6.11)\qquad \sum\nolimits_1 = 2\sum_{(m,q)=1}\ \sum_\nu \sum_n \Omega_{M,L}(n,mq,\nu)t(\nu)t(n) \sum_{\substack{d\,|\,(n-mq)\\ 1\le d\le\sqrt{|n-mq|}}} V_\epsilon(n,d;mq,\nu)$$

and where $\sum_2$ is the same average of the function $\hat{\varphi}\cdot Z_{mq}$, (6.12) $$\sum\nolimits_2(s) =$$

$$= \sum_{(m,q)=1} \sum_{\nu} \omega_{a_1,b_1}\left[\frac{mq}{M}\right]\left[\frac{L}{\nu}\right]^{1/4} \omega_{a_2,b_2}\left[\frac{(\sqrt{\nu}+\sqrt{mq})^2}{L}\right] t(\nu) Z_{mq}(s) \hat{\varphi}_\epsilon(s;x_0(mq,\nu)) .$$

In the last equality $\hat{\varphi}_\epsilon$ is the Fourier transform of our function φ_ϵ,

(6.13) $$\hat{\varphi}_\epsilon(s;x_0) = \int_0^\infty \psi(y,s)\varphi_\epsilon(y-x_0)dy, \ \ x_0 = 2\log(\sqrt{\nu} + \sqrt{mq}) .$$

§ 7. <u>The main part of the sum</u> $\sum_1$.

7.1. *The non–zero terms in the sum.*

The presence of the multiplier $\Omega_{M,L}$ in the terms of the sum $\sum_1$ means these terms are not zeroes only for

(7.1) $$mq \asymp M , \ \ \nu \asymp L , \ \ n << MT .$$

Let us write, for the brevity, ξ and τ instead of $\xi_n(N)$ and $\tau_{n-N,d}$. As it follows from the representations (5.22) and (5.23) in the case where $\varphi = \varphi_\epsilon(x-x_0)$ we have

(7.2) $$V_\epsilon(n,d;mq,\nu) = 0$$

if $\xi + \tau \leq x_0 - \epsilon = 2\log(\sqrt{\nu} + \sqrt{N}) - \epsilon$. So the terms of our sum are not zeroes only for

$$\sqrt{n} + \sqrt{N} \geq (\sqrt{\nu} + \sqrt{N})e^{-\epsilon/2}d \quad ; \tag{7.3}$$

here d is a divisor of $|n-N|$. Since $N = o(L)$ we have for non–zero terms

$$n \geq \nu\, d^2(1+o(1)) \geq a_1 d^2 L(1+o(1)) \quad . \tag{7.4}$$

In particular, for non–zero terms of our sum we have

$$d^2 << \frac{TM}{L} \quad . \tag{7.5}$$

For non–zero terms of the sum $\sum_1$ we have

$$\xi_n(N) = 2\sqrt{\frac{N}{n}}\,(1+o(1)) << \sqrt{\frac{M}{L}}$$

and at the same time

$$\tau_{n-N,d} = \log\frac{|n-N|}{d^2} >> \log L \quad .$$

So ξ is a small quantity in the comparison with τ and the representation (5.23) must be used.

7.2. The main part of the sum $\sum_1$.

In the accordance with the formula (5.23) for u we subdivide the sum $\sum_1$ onto two subsums

$$\sum_1 = \sum_{1,0} + \sum_{1,1} \tag{7.6}$$

where $\sum_{1,0}$ contains the terms with $\varphi_\epsilon(\xi+\tau-x_0)$ for $d = 1$ and $\sum_{1,1}$ contains the rest (so $\sum_{1,1}$ contains the terms with φ_ϵ for $d \geq 2$ also).

For the sum $\sum_{1,0}$ we have

$$\xi = \xi_n(mq) = \log \frac{\sqrt{n} + \sqrt{mq}}{|\sqrt{n} - \sqrt{mq}|} , \quad \tau = \tau_{n-mq,1} = \log|n-mq| \ .$$

Since $\varphi_\epsilon(x) \neq 0$ only for $|x| \leq \epsilon$ the region of the summation for this sum $\sum_{1,0}$ is defined by the condition

$$e^{-\epsilon/2}(\sqrt{\nu} + \sqrt{mq}) \leq \sqrt{n} + \sqrt{mq} \leq e^{\epsilon/2}(\sqrt{\nu} + \sqrt{mq}) \ . \tag{7.7}$$

It means that for these terms

$$|\nu-n| << \epsilon\nu << \epsilon L \ . \tag{7.8}$$

Now we replace n by $\nu+n$ where the new variable n is an integer with $|n| << \epsilon L$ and after this the sum $\sum_{1,0}$ has the form

$$(7.9)\qquad \sum_{1,0} = \varphi_\epsilon(0) \sum_{(m,q)=1} \sum_{\nu} \Omega_{M,L}(\nu, mq, \nu) t^2(\nu) +$$

$$+ \sum_{(m,q)=1} \sum_{\substack{|n| \le 4\epsilon L \\ n \ne 0}} \sum_{\nu} \Omega_{M,L}(n+\nu, mq, \nu) t(\nu) t(\nu+n) \varphi_\epsilon\left(2 \log \frac{\sqrt{\nu+n} + \sqrt{mq}}{\sqrt{\nu} + \sqrt{mq}}\right)$$

$$= \sum_{1,0,0} + \sum_{1,0,1}$$

where $\sum_{1,0,0}$ ($\sum_{1,0,1}$) is the first (the second) sum on the right side. For the first sum the following asymptotic formula holds.

Lemma 7.1. *Under our assumptions for the parameters we have the asymptotic equality*

$$(7.10)\qquad \sum_{1,0,0} = C_{k,q} \frac{(ML)^{3/4}}{\epsilon} + 0\left[\frac{M^{3/4} L^{1/4}}{\epsilon} \exp(-c_0 (\log L)^{3/5})\right]$$

where with the notation r_k *for the residue at* $s = 1$ *of the Rankin series* $R(s) = \sum_{n \ge 1} n^{-s} t^2(n)$, *with* C *from* (6.3) *we have*

$$(7.11)\quad C_{k,q} = \frac{C}{q} r_k \left[\prod_{p|q} (1 - \frac{1}{p})\right] \int_0^\infty \omega_{a_1,b_1}(x) x^{-1/4} dx \cdot \int_0^\infty \omega_{a_2,b_2}(y) y^{-1/2} dy\ ,$$

and c_0 *is a fixed positive constant.*

Firstly we write the Mellin integral

$$\Omega_{M,L}(\nu,N,\nu) = \frac{1}{2\pi i} \int_{(\alpha)} \tilde{\Omega}(\nu,\rho) N^{-\rho} d\rho\ , \quad \alpha > 1\ , \tag{7.12}$$

where $\int_{(\alpha)}$ denotes the integral over the line $\mathrm{Re}\,\rho = \alpha$ and

$$\tilde{\Omega}(\nu,\rho) = \int_0^{\infty} \Omega_{M,L}(\nu,N,\nu) N^{\rho-1} dN\ . \tag{7.13}$$

For $\nu \asymp L$ the function $\tilde{\Omega}(\nu,\rho)$ is an entire function in ρ and for any fixed value $\mathrm{Re}\,\rho = \alpha$ and for any fixed $B > 2$ we have

$$|\tilde{\Omega}(\nu,\rho)| << M^{\alpha-1/4} |\rho|^{-B} \tag{7.14}$$

when $|\rho| \longrightarrow \infty$, $\mathrm{Re}\,\rho = \alpha$.

Since

$$\sum_{(m,q)=1} \frac{1}{(mq)^{\rho}} = \frac{1}{q^{\rho}} \prod_{p|q} (1 - \frac{1}{p^{\rho}}) \zeta(\rho)$$

and there is the unique pole of this function at $\rho = 1$ with the residue $\frac{1}{q} \prod_{p|q} (1 - \frac{1}{p})$ we have for any $B > 2$

$$\sum_{(m,q)=1} \Omega_{M,L}(\nu,mq,\nu) = \frac{1}{q} \prod_{p|q} (1 - \frac{1}{p}) \tilde{\Omega}(\nu,1) + 0\left(\left[\frac{q^2}{M}\right]^B M^{-1/4}\right) \tag{7.15}$$

(it is obvious that $\prod_{p|q} p \le q$).

After this we write

$$\tilde{\Omega}(\nu,1) = \frac{1}{2\pi i} \int_{(\alpha)} \hat{\Omega}(s)\nu^{-s}ds$$

and

(7.16)
$$\sum_{\nu} t^2(\nu)\tilde{\Omega}(\nu,1) = \frac{1}{2\pi i} \int_{(\alpha)} \hat{\Omega}(s)R(s)ds\ , \quad \alpha > 1\ .$$

Here $\hat{\Omega}(s)$ is the entire function in s, for any $B > 2$ we have

(7.17)
$$|\hat{\Omega}(s)| << L^{\alpha-1/4}|s|^{-B}$$

when $|s| \longrightarrow \infty$ and $\mathrm{Re}\, s = \alpha$. Further, $R(s)$ has the simple pole at $s = 1$ (with the residue r_k), the function $(s-1)\zeta(2s)R(s)$ is the regular one for $\mathrm{Re}\, s > 0$. It is known that $\zeta(s)$ has no zeroes for $\mathrm{Re}\, s > 1/2$ and $|\mathrm{Im}\, s| \le 13 \cdot 10^5$ and for $|\mathrm{Im}\, s| \ge 13 \cdot 10^5$ there is no zeroes in $\sigma \ge 1 - c_0(\log|t|)^{-2/3}$, $s = \sigma + it$, for some constant c_0. Since for $|\hat{\Omega}(s)|$ we have the estimate (7.17) our assertion (7.10) follows.

7.3. *The convolution of the Fourier coefficients of a cusp form*

Let $f \in \mathcal{M}_k$ be the cusp form of the even integer weight $k \ge 12$ in the respect to the full modular group and $a(\nu)$, $\nu = 1,2,\ldots$, are the Fourier coefficients of this cups form.

We consider the series (it is some kind of the convolution)

(7.18)
$$\sum_{\nu=1}^{\infty} t(\nu)\,\overline{t(\nu+n)}\,\phi(\nu)$$

where $t(\nu) = \nu^{-(k-1)/2}a(\nu)$ and Φ is an arbitrary "good" function; the explicit form will be given for this convolution. Slightly other way for the estimates of the similar sums was proposed by A. Good [11].

To formulate the result we shall introduce some notations. Let u_j be j–th eigenfunction of the automorphic Laplacian (as in the subsection 2.1); for $j \geq 1$ we define the quantities γ_j by the equality

(7.19)
$$\gamma_j = \int_{g\backslash \mathbb{H}} y^k\,|f(z)|^2 u_j(z)d\mu(z) \ .$$

Further, for the given $\Phi \in C^{\infty}(0,\infty)$ and for any positive integer $n \geq 1$ we define the integral transform $\Phi \longmapsto h_n$:

(7.20)
$$h_n(r) = \int_{s}^{\infty} \Phi(n\mathrm{sh}^2(\xi/2))(W(\xi,r) + W(\xi,-r))\sqrt{\mathrm{sh}\,\xi}\,d\xi \ ,$$

where with the standard notation for the Gauss hypergeometric function the kernel is defined by the equality

(7.21)
$$W(\xi,r) = \frac{2^{-1/2}\Gamma(1/2+ir)\Gamma(3/2-k+ir)}{i\ \mathrm{th}(\pi r)\ \Gamma(1+2ir)}$$

$$(\mathrm{th}\ \xi/2)^{3/2-k}(\mathrm{ch}\ \xi/2)^{-2ir}F(1/2+ir,3/2-k+ir;1+2ir;(\mathrm{ch}\ \xi/2)^{-2}) \ .$$

Furthermore, let $R(s)$ be the Rankin series for our cusp form f ,

$$R(s) = \sum_{n=1}^{\infty} n^{-s} |t^2(n)| \ , \ \text{Re } s > 1 \ ,$$

and

$$(7.22) \qquad \mathscr{E}(r) = \frac{\Gamma(k-1/2+ir)R(1/2+ir)}{\pi^{2ir}\Gamma(1/2-ir)\zeta(1-2ir)} \ .$$

Theorem 6. *Let* $\Phi \in C^{\infty}(0,\infty)$ *be a function with a bounded support. Then for any positive integer* $n \geq 1$

$$(7.23) \qquad \sum_{\nu=1}^{\infty} t(\nu)\,\overline{t(\nu+n)}\,\Phi(\nu) =$$

$$= (4\pi)^{k-1} \sum_{j\geq 1} \gamma_j \sqrt{n}\,\overline{\rho_j(n)}\,h_n(\kappa_j) + 2\int_{-\infty}^{\infty} \sqrt{n}\,\tau_{1/2+ir}(n)\,\mathscr{E}(r) h_n(r) dr \ .$$

The proof is not long. Let $U_n(z,s)$ denotes the Poincaré–Selberg series,

$$(7.24) \qquad U_n(z,s) = \sum_{g \in G_{\infty}\backslash G} e(ng\,z)(\text{Im } gz)^s \ , \ \text{Re } s > 1 \ , \ n \geq 1 \ .$$

Then we have the obvious identity for $\text{Re } s > 1$:

$$(7.25)\qquad \frac{\Gamma(k-1+s)}{(4\pi)^{k-1+s}}\sum_{\nu=1}^{\infty}\frac{a(\nu)\overline{a(\nu+n)}}{(\nu+n)^{k-1}}\cdot\frac{1}{(\nu+n)^{s}}=\int_0^{\infty}\int_0^{1} y^k|f(z)|^2 e(nz)d\mu(z)$$

$$=\int_{g\backslash\mathbb{H}} y^k|f(z)|^2\, U_n(z,s)d\mu(z)\ .$$

Using the Parseval identity for the full system of the eigenfunctions of the automorphic Laplacian we can rewrite the last integral as the sum over discret and continuous spectrum. Then it equals to

$$(7.26)\qquad \sum_{j\geq 1}\Big(\int_{G\backslash\mathbb{H}} y^k|f|^2 u_j d\mu(z)\Big)\Big(\int_{G\backslash\mathbb{H}} U_n(z,s)\overline{u_j}\, d\mu(z)\Big)+$$

$$+\frac{1}{4\pi}\int_{-\infty}^{\infty}\Big(\int_{G\backslash\mathbb{H}} y^k|f|^2 E(z,1/2+ir)d\mu(z)\Big)\Big(\int_{G\backslash\mathbb{H}} U_n(z,s)\,\overline{E(z,1/2+ir)}\, d\mu(z)\Big)dr\ .$$

For the inner products we have the well known Selberg formulae:

$$(7.27)\qquad (U_n,u_j)=\frac{1}{2}(4\pi)^{1-s}\sqrt{n}\,\overline{\rho_j(n)}\cdot\frac{\Gamma(s-1/2+i\kappa_j)\Gamma(s-1/2-i\kappa_j)}{n^s\Gamma(s)}\ ,$$

$$(7.28)\quad (U_n,E(\cdot,1/2+ir))=\frac{\sqrt{n}\ \tau_{1/2+ir}(n)}{\xi(1/2-ir)}\cdot(4\pi)^{1-s}\frac{\Gamma(s-1/2+ir)\Gamma(s-1/2-ir)}{n^s\Gamma(s)}$$

(here $\xi(\cdot)$ is the same what was in (2.2)).

Now for the given Φ let us define for $x>n$

$$g(x)=\left[\frac{x}{x-n}\right]^{(k-1)/2}\Phi(x-n)\ ,$$

so that for $\nu > 0$ we have

$$\left[\frac{\nu}{\nu+n}\right]^{(k-1)/2} g(\nu+n) = \Phi(\nu) \ .$$

Let $\hat{g}(s)$ be the Mellin transform of g. We shall integrate both sides of (7.27) over the line $\operatorname{Re} s = 1+\delta$ with some fixed $\delta > 0$ with the multiplier

$$(4\pi)^{-k+1-s}(\Gamma(k-1+s))^{-1}\,\hat{g}(s) \ .$$

On the left side we come to the series

$$\sum_{\nu=1}^{\infty} t(\nu)\,\overline{t(\nu+n)}\left[\frac{\nu}{\nu+n}\right]^{(k-1)/2} g(\nu+n) = \sum_{\nu=1}^{\infty} t(\nu)\,\overline{t(\nu+n)}\,\Phi(\nu) \ . \tag{7.29}$$

On the other side we have the integral

$$\frac{1}{2\pi i}\int_{(1+\delta)} \hat{g}(s)\,\frac{\Gamma(s-1/2+ir)\Gamma(s-1/2-ir)}{\Gamma(s)\Gamma(k-1+s)}\,n^{-s}ds \ , \tag{7.30}$$

or (because g has a bounded support)

$$\int_0^{\infty} \frac{g(x)}{x}\left\{\frac{1}{2\pi i}\int_{(1+\delta)} \left(\frac{x}{n}\right)^{s}\frac{\Gamma(s-1/2+ir)\Gamma(s-1/2-ir)}{\Gamma(s)\Gamma(k-1+s)}\,ds\right\}dx \ . \tag{7.31}$$

Here the inner integral is zero for $x \le n$ (it follows after the translation of the path of the integration to the right). If $x > n$ then the same integral is the sum of the residues at $s_m = 1/2-ir-m$ and $\tilde{s}_m = 1/2+ir-m$ with $m = 0,1,2,\ldots$. Now we have

(7.32)
$$\operatorname*{Res}_{s=s_m} \left\{ \left(\frac{x}{n}\right)^s \frac{\Gamma(s-1/2+ir)\Gamma(s-1/2-ir)}{\Gamma(s)\Gamma(k-1+s)} \right\} =$$

$$= \left(\frac{x}{n}\right)^{1/2-ir-m} \frac{(-1)^m}{m!} \cdot \frac{\Gamma(-2ir-m)}{\Gamma(\frac{1}{2}-ir-m)\Gamma(k-\frac{1}{2}-ir-m)}$$

and it equals to

(7.33)
$$\left(\frac{x}{n}\right)^{1/2-ir-m} \frac{(-1)^{k-1}}{2\pi i} \frac{ch(\pi r)}{sh(\pi r)} \cdot \frac{\Gamma(3/2-k+ir+m)\Gamma(1/2+ir+m)}{\Gamma(1+2ir+m)\, m!} \left(\frac{x}{n}\right)^{-m} ,$$

so the corresponding sum over $m \geq 0$ is

(7.34)
$$\left(\frac{x}{n}\right)^{1/2-ir} \cdot \frac{(-1)^{k-1}}{2\pi i} \frac{ch(\pi r)}{sh(\pi r)} \cdot \frac{\Gamma(3/2-k+ir)\Gamma(1/2+ir)}{\Gamma(1+2ir)} F(1/2+ir, 3/2-k+ir; 1+2ir; \tfrac{x}{n})$$

It gives the integral transform (7.20) after the change of the variable $x \longmapsto n\, ch^2\, \xi/2$ and the using of the duplication formula for the gamma–function.

In the continuous spectrum one can note that for $n = 0$ the identity (7.25) gives the integral representation for the Rankin series for $\mathrm{Re}\, s > 1$ and it holds on the half–plane $\mathrm{Re}\, s \geq 1/2$. Now (7.23) follows.

It is useful to note that for an even integer $k \geq 2$

(7.35)
$$W(\xi,r)+W(\xi,-r) = \frac{2^{k-1}}{\Gamma(k)} (sh\, \xi)^{k-1/2} F(k-1/2+ir, k-1/2-ir; k; -sh^2 \xi/2) ,$$

so there is no singularity at $\xi = 0$ in the transformation (7.20). At the same time we have

(it is the consequence of the Kummer quadratic relations)

(7.36) $$F(1/2+ir,3/2-k+ir;1+2ir;(ch\,3/2)^{-2}) =$$

$$= (1+e^{-\xi})^{3-2k+2ir} F(3/2-k+ir,3/2-k;1+ir;e^{-2\xi}) \ .$$

This equality gives the uniform (over $r \in \mathbb{R}$) asymptotic expansion for the case $\xi \longrightarrow +\infty$.

It is known that for T large and $n << T^{4-\delta}$ for any $\delta > 0$ we have (see [5])

$$\sum_{\kappa_j \leq T} \frac{1}{ch(\pi\kappa_j)} |\rho_j(n)|^2 \sim \frac{1}{\pi^2} T^2$$

so the quantities $|\rho_j(n)|$ are exponentially large in the average. This growth would be compensated by the corresponding decrease of the coefficients γ_j .

Lemma 7.2. *Let* γ_j *be defined by* (7.19) *for* $j \geq 1$ *and for a fixed cusp form from* $\mathcal{M}_k$, $k \equiv 0 (\mathrm{mod}\ 2)$, $k \geq 12$. *Then for* T *large*

(7.37) $$\sum_{\kappa_j \leq T} \exp(+\pi\kappa_j)\gamma_j^2\, \kappa_j^{-2k} << T^2 \log T$$

so the quantities $\exp(\frac{\pi\kappa_j}{2})|\gamma_j|$ *are* $0(\kappa_j^k\sqrt{\log \kappa_j})$ *in the average.*

The remark. The estimate (7.37) is sufficient for the purposes of this paper but it is far from the true order of the coefficients γ_j. I assume that it is possible replace (7.37) by the more strong assertion

$$\sum_{\kappa_j \le T} \exp(\pi\kappa_j)\gamma_j^2\, \kappa_j^{2-2k} << T^2 \ . \tag{7.38}$$

To give (7.37) we use the following simple fact: the first Poincaré series $P_1,\ldots,P_{\nu_k}$, $\nu_k = \dim \mathscr{M}_k$, are the base in $\mathscr{M}_k$. So there are the constants $c_1,\ldots,c_{\nu_k}$ such that

$$f = \sum_{\ell=1}^{\nu_k} c_\ell\, P_\ell(z;k) \ . \tag{7.39}$$

For this reason

$$\gamma_j = \sum_{\ell=1}^{\nu_k} c_\ell \gamma_{j,\ell},\ \ \gamma_{j,\ell} = \int_{G\backslash\mathbb{H}} y^k f(z)\, \overline{P_\ell(z;k)}\, u_j(z) d\mu(z) \ , \tag{7.40}$$

and it is sufficient to estimate the integrals $\gamma_{j,\ell}$, $1 \le \ell \le \dim \mathscr{M}_k$. Of course the last integral is equal to

$$\frac{(4\pi\ell)^{k-1}}{\Gamma(k-1)} \int_0^\infty \int_0^1 y^k f(z)\, \overline{e(\ell z)}\, u_j(z) d\mu(z) = \tag{7.41}$$

$$= \frac{(2\ell)^{k-1}}{2\sqrt{2\pi}\ \Gamma(k-1)} \sum_{\substack{\nu=1 \\ \nu\ne\ell}}^\infty a(\nu)\rho_j(|\nu-\ell|) \int_0^\infty e^{-(\nu+\ell)y} K_{i\kappa_j}(|\nu-\ell|y) y^{k-3/2} dy \ .$$

The explicit form for these integrals may be taken from the tables and we have the representation

$$(7.42)\quad \gamma_{j,\ell} = \frac{(2\pi\sqrt{\ell})^{k-1}}{\Gamma(k)\Gamma(k-1)}\Gamma(k-1/2+i\kappa_j)\Gamma(k-1/2-i\kappa_j)\sum_{\substack{\nu=1\\ \nu\neq\ell}}^{\infty}\frac{t(\nu)\rho_j(\nu-\ell)}{(\nu\ell)^{1/4}}\,v\Big(\frac{4\nu\ell}{(\ell+\nu)^2},x_j\Big)$$

where v is expressed in the terms of the Gauss hypergeometric function,

$$(7.43)\qquad v(x,r) = x^{k/2-1/4}(1-x)^{ir}\,F\Big(\frac{k+ir}{2}+\frac{1}{4},\frac{k+ir}{2}-\frac{1}{4};k;x\Big)\,,\quad 0\leq x<1\ .$$

This normalization is accepted here to simplify the corresponding differential equation; namely, the function

$$(7.44)\qquad \tilde{v} = v(\mathrm{th}^2\tfrac{\xi}{2},r)$$

is the solution of the equation

$$(7.45)\qquad \frac{d^2\tilde{v}}{d\xi^2}+\Big[\frac{r^2}{4}-\frac{(k-1)^2}{4\,\mathrm{sh}^2\frac{\xi}{2}}+\frac{1}{4\,\mathrm{sh}^2\xi}+\frac{1}{16\,\mathrm{ch}^2\frac{\xi}{2}}\Big]\tilde{v}=0\ .$$

This solution is

$$(\xi/2)^{k-1/2}$$

when $\xi \longrightarrow 0$ and it is easy to give the asymptotic formula

$$(7.46)\qquad \tilde{v}(\xi,r)\cong\frac{\Gamma(k)2^{k-1}}{r^{k-1}}\cdot\sqrt{\xi/2}\,J_{k-1}\Big(\frac{r\xi}{2}\Big)$$

for r large, uniformly over $\xi \geq 0$. It follows from (7.46)

$$|\tilde{v}(\xi,r)| << \min(r^{-k+1/2}, \xi^{k-1/2}) \tag{7.47}$$

and we have from (7.42)

$$\kappa_j^{-2k} e^{\pi\kappa_j} \gamma_{j,\ell}^2 << e^{-\pi\kappa_j} \kappa_j^{-3} \Big(\sum_{\nu \leq \kappa_j^2} \nu^{-1/4} |t(\nu)\rho_j|\nu-\ell)| \Big)^2 + \tag{7.48}$$

$$+ e^{-\pi\kappa_j} \kappa_j^{2k-4} \Big(\sum_{\nu \geq \kappa_j^2} \nu^{-k/2} |t(\nu)\rho_j(\nu-\ell)| \Big)^2$$

For the case $A >> \kappa_j^2$ we have

$$\sum_{A \leq \nu \leq 2A} |\rho_j(\nu)|^2 e^{-\pi\kappa_j} << A \tag{7.49}$$

so the second term on the right side (7.48) gives $0(1)$. It means

$$\sum_{\kappa_j \leq T} \kappa_j^{-2k} e^{\pi\kappa_j} \gamma_{j,\ell}^2 << \sum_{\nu \leq T^2} \sum_{\sqrt{\nu} << \kappa_j \leq T} \frac{|\rho_j(\nu-\ell)|^2 e^{-\pi\kappa_j}}{\kappa_j^2} + T^2 \tag{7.50}$$

$$<< T^2 \log T$$

and the rough estimate (7.37) holds.

Of course, there is a more reasonable way; one can write

$$(7.51)\ \kappa_j^{2-2k} e^{\pi\kappa_j} \gamma_{j,\ell}^2 << \sum_\nu \sum_\mu \frac{t(\nu)\overline{t(\mu)}}{(\nu\mu)^{1/4}} \frac{\rho_j(\nu-\ell)\overline{\rho_j(\mu-\ell)}}{\mathrm{ch}(\pi\kappa_j)} \sqrt{\xi_\nu \xi_\mu}\, J_{k-1}\left(\frac{\xi_\nu \kappa_j}{2}\right) J_{k-1}\left(\frac{\xi_\mu \kappa_j}{2}\right)$$

with $\mathrm{th}^2(\frac{\xi_\nu}{2}) = 4\nu\ell \cdot (\ell+\nu)^{-2}$. Now the sum

$$(7.52)\qquad \sum_j \exp(-\epsilon\kappa_j^2)\kappa_j^{2-2k} e^{\pi\kappa_j} \gamma_{j,\ell}^2 ,\quad \epsilon > 0 ,$$

is reduced to the sum of the Kloosterman sums; the estimate (7.38) would be ensured after the using of the sum formula (2.20).

7.5. *The asymptotic formula for the sum* $\sum_{1,0,1}$

7.5.1. The first summation.

Let us write for an integer $n \geq 1$

$$(7.53)\ \Phi(\nu;n,N) = 2\varphi_\epsilon\left(2 \log \frac{\sqrt{\nu+n} + \sqrt{n}}{\sqrt{\nu} + \sqrt{N}}\right)\left(\Omega_{M,L}(\nu+N,N,\nu) + \Omega_{M,L}(\nu,N,\nu+n)\right)$$

(we replace ν by $\nu+|n|$ in the terms of the sum $\sum_{1,0,1}$ with $n \leq -1$); then our sum is the triple one

$$\sum_{\nu,n,N} \Phi(\nu;n,N)t(\nu)t(\nu+n) .$$

The "convolution formula" (7.23) will be used now; but before it is convenient to do the summation over N (note that $N = mq$ with $(m,q) = 1$). We write

$$\Phi(\nu;n,N) = \frac{1}{2\pi i}\int \tilde{\phi}(\nu;n,\rho)N^{-\rho}d\rho$$

where $\tilde{\phi}$ is the corresponding Mellin transformation. For this reason we have

$$(7.54)\qquad \sum_{(m,q)=1} \Phi(\nu;n,mq) = \frac{1}{2\pi i}\int \tilde{\phi}(\nu;n,\rho)\frac{1}{q^{\rho}}\prod_{p\mid q}(1-\frac{1}{p^{\rho}})\cdot\zeta(\rho)d\rho$$

$$= \tilde{\Phi}(\nu;n,1)\cdot\frac{1}{q}\prod_{p\mid q}(1-\frac{1}{\rho}) + \frac{1}{2\pi i}\int_{-B-i\infty}^{-B+i\infty}\tilde{\Phi}(\nu;n,\rho)\frac{\zeta(\rho)}{q^{\rho}}\prod_{p\mid q}(1-\frac{1}{p^{\rho}})d\rho$$

where we can take any positive B . Since there is the condition $\frac{M}{q^2} >> M^{\delta}$ for some positive δ the second term on the right side (7.54) may be rejected without making worse the remainder term. So it will be sufficient consider the double sum

$$(7.55)\qquad \tilde{\sum}_{1,0,1} = \mathscr{A}_q\sum_{n,\nu} t(\nu)t(\nu+n)\int_0^{\infty}\Phi(\nu;n,N)dN\ ,\quad \mathscr{A}_q = \frac{1}{q}\prod_{p\mid q}(1-\frac{1}{p})\ .$$

7.5.2. The Mellin integrals.

The sum (7.55) equals to

$$(7.56)\qquad \mathscr{A}_q\sum_n\left\{(4\pi)^{k-1}\sum_{j\geq 1}\gamma_j\overline{\rho_j(n)}\,h(x_j,n) + 2\int_{-\infty}^{\infty}\tau_{1/2+ir}(n)\ \mathscr{E}(r)h(r,n)dr\right\}$$

where

(7.57) $$h(r,n) = \sqrt{n}\int_0^\infty \sqrt{\operatorname{sh}\xi}\, W(\xi,r) + W(\xi,-r))\int_0^\infty \Phi(n\,\operatorname{sh}^2\xi/2;n,N)dN\,d\xi$$

and all others as in the identity (7.23).

It is convenient to rewrite (7.56) in the form

(7.58) $$\mathscr{A}_q \frac{1}{2\pi i}\int_{(\alpha)} \Big\{(4\pi)^{k-1}\sum_{j\geq 1} \gamma_j\,\overline{\rho_j(1)}\,\mathscr{H}_j(s)\,\hat{h}(\kappa_j,s)ds +$$

$$+ 2\int_{-\infty}^{\infty} \mathscr{E}(r)\zeta(s+ir)\zeta(s-ir)\hat{h}(r,s)dr\Big\}\,ds, \quad \alpha = \operatorname{Re} s > 1 \ ,$$

where $\mathscr{H}_j(s)$ denotes the Hecke series and $\hat{h}(r,s)$ is the Mellin transform of the function $h(r,n)$,

(7.59) $$\hat{h}(r;s) = \int_0^\infty h(r,n)n^{s-1}dn \ .$$

Note that Φ in the integrand on the right side (7.57) is not zero only under the conditions

(7.60) $$N \asymp M,\ \sqrt{n}\,\operatorname{sh}\xi/2 + \sqrt{N} \asymp \sqrt{L} \ \text{ or } \ \sqrt{n}\,\operatorname{ch}\xi/2 + \sqrt{N} \asymp \sqrt{L}$$

and

(7.61) $$\sqrt{n}\,\operatorname{ch}\xi/2 + \sqrt{N} \leq \exp(\tfrac{\epsilon}{2})(\sqrt{n}\,\operatorname{sh}\xi/2 + \sqrt{N}) \ .$$

Here $M = o(L)$ and for this reason we have $ne^{\xi} \asymp L$; together with (7.61) it gives $e^{\xi} >> 1/\epsilon$. So we can introduce the multiplier $\eta(\epsilon e^{\xi})$ in the integrand on the right side (7.57). The value of the integral will be the same if we take the suitable numbers a, b in the definition (4.32) of the truncating function η .

Now we consider the explicit form of the integral (7.59) to see the location of the singularities of this function.

Let $\hat{\omega}_1, \hat{\omega}_2$ be the Mellin transforms for ω_1, ω_2 respectively and $\Phi_1(x)$ denotes the Fourier cosine–transformation of φ_1 ,

$$\Phi_1(x) = \int_{-\infty}^{\infty} \varphi_1(y)\cos(xy)dy \ .$$

With these notations we have the first integral representation.

Proposition 7.1. For any s with $\operatorname{Re} s > 0$

$$(7.62)\ \hat{h}(r;s) = \frac{4\sqrt{2}}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \Phi_1(i\epsilon u) \int_{(\alpha)} \hat{\omega}_1(v)\hat{\omega}_2(s-v+3/4)L^{s-v+1}M^{v} \cdot \frac{\Gamma(3/2-2v)\Gamma(2s)}{\Gamma(2s+\frac{3}{2}-2v)} \cdot$$

$$\cdot\ (\hat{\eta}(r;s,u,v) + \hat{\eta}(-r;s,u,v))dvdu$$

where $\alpha = \operatorname{Re} v < 3/4$ and

$$(7.63) \qquad \hat{\eta}(r;s,u,v) = \int_0^{\infty} \eta(\epsilon e^{\xi})(\text{th } \xi/2)^{2u}(\text{ch } \xi/2)^{-2s}F(-2u,3/2-2v;2s-2v+3/2; -\frac{2e^{-\xi}}{1-e^{-\xi}}) \cdot W(\xi,r)d\xi$$

After the change of the variable $N \longrightarrow nx^2$ we come to the inner integral

$$(7.64) \qquad \frac{4\sqrt{2}}{\sqrt{\text{sh } \xi}} \int \varphi_{\epsilon}(2 \log \frac{x+\text{ch } \xi/2}{x+\text{sh } \xi/2}) \sqrt{x} \int \omega_1(\frac{n x^2}{M})(\omega_2(\frac{n}{L}(x+\text{ch } \xi/2)^2) + + \omega_2(\frac{n}{L}(x+\text{sh } \xi/2)^2) \cdot n^{s-1/4}dn$$

Now we can replace the functions φ_{ϵ} and ω_1 by their Mellin transforms. Since

$$\int_0^{\infty} \varphi_{\epsilon}(\log Y)Y^{u-1}dY = \int_{-\infty}^{\infty} \varphi_{\epsilon}(Y)\text{ch}(yu)dY = \Phi_1(i\epsilon u)$$

the Mellin transform of the function $\varphi_{\epsilon}(\log Y)$ is the Fourier cosine–transformation. After this integral over x may be expressed in the terms of the Guass hypergeometric function and the equality (7.62) follows.

7.5.3. The poles and the residues.

The integrand in (7.62) contains the multiplier $(\frac{M}{L})^v L^s$ and for this reason it will be useful to move the path of the integration over v to the right and in the integral over s – to the left.

The first move gives for the inner integral the series (we take the residues at $v = 3/4 + m/2$, $m = 0,1,\ldots$)

(7.65) $$\frac{1}{2\pi i}\int_{(\alpha)}(\dots)dv =$$

$$= \sum_{m\geq 0}\frac{(-1)^m}{2m!}\left(\frac{M}{L}\right)^{3/4+m/2}L^{s+1}\frac{\Gamma(2s)}{\Gamma(2s-m)}\hat{\omega}_1(3/4+m/2)\hat{\omega}_2(3/4+m/2)\hat{\omega}_2(s-m/2)\cdot$$

$$\cdot\left(\hat{\eta}(r;s,u,3/4+m/2)+\hat{\eta}(-r;s,u,3/4+m/2)\right)$$

Here $M = o(L)$ so the series is convergent and the term with $m = 0$ is the leading term.

The poles of $\hat{\eta}(r;s,u,3/4+m/2)$ are located at the points $s = -ir,\ -ir-1,\dots$. Indeed, we have in (7.63) after the replacement W by the expressions (7.21) and (7.36) and the change of the variable $e^{\xi} = \frac{x}{\epsilon}$

(7.66) $$\hat{\eta}(r;s,u,3/4+m/2) = \gamma_k(r)\int_0^{\infty}\eta(x)f_m\left(\frac{\epsilon}{x};r,s,u\right)x^{-s-ir-1}dx\cdot\epsilon^{s+ir}$$

where, for the brevity, we write

(7.67) $$\gamma_k(r) = \frac{1}{i\sqrt{2}}\cdot\frac{\Gamma(1/2+ir)\Gamma(3/2-k+ir)}{\mathrm{th}(\pi r)\Gamma(1+2ir)}\ ,$$

(7.68) $$f_m(z;r,s,u) = 2^{2s+2ir}(1-z)^{2u+3/2-k}(1+z)^{-2u-2s+3/2-k}\ .$$

$$\cdot\ F(3/2-k+ir,3/2-k;1+ir;z^2)F\left(-2u,-m;2s-m;-\frac{2z}{1-z}\right)\ .$$

Now the repeating integration by parts gives on the first step

$$\int_0^\infty \eta(x)f_m(\tfrac{\epsilon}{x},\ldots)a^{-s-ir-1}dx = \frac{1}{s+ir}\int_0^\infty x^{-s-ir}\frac{\partial}{\partial x}\eta(x)f_m(\tfrac{\epsilon}{x},\ldots)dx\,. \tag{7.69}$$

It means this integral has the pole at $s = -ir$. The residue at this pole equals to

$$\int_0^\infty \frac{\partial}{\partial x}(\eta\, f_m)dx = \eta(x)f_m(\tfrac{\epsilon}{x},\ldots)\Big|_0^\infty = 2^{2s+2ir}\,. \tag{7.70}$$

Furthermore, the integarl

$$\int_0^\infty \eta'(x)f_m(\tfrac{\epsilon}{x},\ldots)x^{-s-ir}dx \tag{7.71}$$

is the entire function in s and in the integral with f'_m we can do the new integration by parts:

$$-\epsilon\int_0^\infty x^{-s-ir-2}\eta(x)f'_m(\tfrac{\epsilon}{x},\ldots)dx = \tag{7.71}$$

$$= -\frac{\epsilon}{s+ir+1}\int_0^\infty x^{-s-ir-1}\frac{\partial}{\partial x}(\eta(x)f'_m(\tfrac{\epsilon}{x},\ldots))dx\,.$$

So we have the pole at $s = -ir-1$ with the residue

$$-\epsilon\, f'_m(0)$$

and we can repeat the same operations to see the poles at $-ir-2$, $-ir-3,\dots$.

The integrand in (7.58) in the case of the continuous spectrum contains two additional poles at $s = 1+ir$ and $s = 1-ir$ with the residues

$$\hat{h}(r,1+ir)\,\gamma(r)\,\zeta(1+2ir)\,,\quad \hat{h}(r,1-ir)\,\gamma(r)\,\zeta(1-2ir)\ . \tag{7.72}$$

Now we are ready to give the main terms for the contribution of the continuous spectrum in (7.58).

Proposition 7.2. Under our assumptions for the parameters

$$2\int_{-\infty}^{\infty} \gamma(r)(\zeta(1+2ir)\hat{h}(r,1+ir) + \zeta(1-2ir)\hat{h}(v,1-ir))dr = \tag{7.73}$$

$$= c_0(\epsilon^2 L)^{3/2} \cdot \frac{(ML)^{3/4}}{\epsilon} \cdot (1+0(\sqrt{\tfrac{M}{N}} + \epsilon^2 L))\ ,$$

where c_0 is the positive constant,

$$c_0 = \frac{16\pi\sqrt{2}}{\zeta(2)}\,\zeta(3)R(3/2)\int_0^{\infty} \eta(x)\varphi_1(\tfrac{4}{x})x^{-4}dx\ \hat{\omega}_1(3/4)\hat{\omega}_2(2) \tag{7.74}$$

Our integrand contains the following terms

$$(7.75)\qquad \mathscr{E}(r)\int_0^\infty \eta(x)\Big\{\gamma_k(t)(\zeta(1+2ir)f_m(\tfrac{\epsilon}{x};r,1+ir)x^{-2-2ir} +$$

$$+\ \zeta(1-2ir)f_m(\tfrac{\epsilon}{x};r,1-ir)x^{-2}) +$$

$$+\ \gamma_k(-r)(\zeta(1+2ir)f_m(\tfrac{\epsilon}{x};-r,1+ir)x^{-2} + \zeta(1-2ir)f_m(\tfrac{\epsilon}{x};-r,1-ir)x^{-2+2ir}\Big\}\frac{dx}{x^2}$$

(here the variable u is not written since it is the same for all terms).

The first observation is: there are no poles at $r=0$ and $ir = \pm 1/2$; it follows from the explicit form for $\mathscr{E}$, γ_k and f_m (note that the function $R(1/2+ir)$ has the simple pole at $ir = +1/2$, has no pole at $ir = -1/2$ and $R(1/2) = 0$).

Furthermore, we have the poles at $ir = \pm 1, \pm 2, \dots$ (due to $sh(\pi r)$ in the denominator) and $f_m(\frac{\epsilon}{x};r,1+ir)$ contains the multiplier $\epsilon L^2\cdot(\epsilon^2 L)^{ir}$. So it would be prefarable to move the path of the integration to the right (note that $\epsilon^2 L \longrightarrow 0$ when $\epsilon \longrightarrow 0$) . The path of the integration will be moved to the left in the integral with $f_m(\frac{\epsilon}{x};-r,1-ir)$ for the same reason. The sum of the residues (for the first integral at $ir = +1$ and for the second one at $ir = -1$) gives the main terms of the asymptotic.

Now the integrals with $f_m(\frac{\epsilon}{x};\pm r,1\mp ir)$ contains only $L^{\mp ir}$ (without the multiplier $\epsilon^{\mp 2ir}$); so the same operation gives the estimate $O(\epsilon L)$ for the residues and $O(L^{1/4}M^{3/4})$ for the final result. For this reason these terms may be rejected and (7.73) follows.

For the terms in the sum over the discret spectrum we have the poles of $\hat{h}(\kappa_j,s)$ at $s = \pm i\kappa_j$. So the contribution of the sum over this spectrum is smaller; namely, we have

Proposition 7.3. Under our assumptions for the parameters

$$(7.76)\qquad \sum_{j\geq 1} \gamma_j \overline{\rho_j(1)} \int \mathscr{H}_j(s)\hat{h}(\kappa_j,s)ds << \frac{L^{1/4}\ M^{3/4}}{\epsilon}\ .$$

Now as the immediate consequence from the preceding estimates we have the asymptotic formula for the sum $\sum_{1,0,1}$.

Lemma 7.3. With the positive constant c_0 from (7.74)

$$(7.77)\qquad \sum_{1,0,1} = c_0 Q^{-3/2} \cdot A_q \cdot \frac{(ML)^{3/4}}{\epsilon} (1+0(Q^{-1/2}+\sqrt{\tfrac{M}{L}})),\ \ Q = (\epsilon^2 L)^{-1},$$
$$A_q = \frac{1}{q} \prod_{p|q} (1-\frac{1}{p})\ .$$

§ 8. The sum $\sum_{1,1}$.

8.1. *The additional representations for* V_ϵ .

If the quantity ξ will be small enough then both terms ($\frac{1}{2}\varphi_\epsilon$ and integral) on the right side (5.23) must be of the same order. For this reason in the integral one integration by parts would be advisable. As a result we obtain the following representation for the function V_ϵ .

Proposition 8.1. Let $\xi = \log \frac{\sqrt{n}+\sqrt{N}}{\sqrt{n}\ -\sqrt{N}}$, $\tau = \log \frac{n-N}{d^2}$. Then for all non–zero terms of the sum $\sum_{1,1}$ with $d \geq 2$ we have

$$(8.1)\ V_\epsilon(n,d;N,\nu) = \int_0^\infty W(z) \frac{sh\,\xi}{ch\,\xi+(2z-1)sh\,\xi} \varphi_\epsilon'(\log \frac{e^{\tau-x_0}}{ch\,\xi+(2z-1)sh\,\xi})(1-\beta)dz + O(\frac{\log L}{L^2})$$

where $x_0 = 2\log(\sqrt{\nu}+\sqrt{N})$ and

$$\beta = 2\,\frac{e^{-2x}+e^{-2\tau}-2e^{-x-\tau}\operatorname{ch}\xi}{(1+e^{-2x})(1+e^{-2\tau})-4e^{-x-\tau}\operatorname{ch}\xi} \tag{8.2}$$

with

$$e^{x} = \frac{\operatorname{ch}\tau+\sqrt{\operatorname{sh}^2\tau+4z(z-1)\operatorname{sh}^2\xi}}{\operatorname{ch}\xi+(2z-1)\operatorname{sh}\xi} \ . \tag{8.3}$$

Firstly we change the variable of the integration; we define $x = x(z;\xi,\tau)$ by the equation

$$\frac{\operatorname{ch}\tau-\operatorname{ch}(x-\xi)}{2\operatorname{sh}x\ \operatorname{sh}\xi} = z \ . \tag{8.4}$$

Then this function is given in the explicit form by the equality (8.3); besides

$$\frac{\partial z}{\partial x} = \frac{\operatorname{ch}x\operatorname{ch}\tau - \operatorname{ch}\xi}{2\operatorname{sh}^2x\ \operatorname{sh}\xi} \tag{8.5}$$

and we have together with (5.23) (for $\varphi(x) = \varphi_\epsilon(x-x_0)$)

$$\overset{=}{U}(\xi,\tau;\varphi_\epsilon) = \frac{1}{2}\,\varphi_\epsilon(\xi+\tau) + \frac{1}{2}\int_0^\infty W'(z)\,\frac{\operatorname{sh}x\ \operatorname{sh}\tau}{\operatorname{ch}x\operatorname{ch}\tau-\operatorname{ch}\xi}\,\varphi_\epsilon(x(z)-x_0)dz \ . \tag{8.6}$$

Now it follows from (8.3)

(8.7) $$x(z)-\tau = -\log(\operatorname{ch}\xi + (2z-1)\operatorname{sh}\xi) + \Delta \ ,$$

where Δ is the small quantity,

(8.8) $$\Delta = \log\Big(1 + \frac{4z(z-1)\operatorname{sh}^2\xi\, e^{-\tau}}{\sqrt{\operatorname{sh}^2\tau+4z(z-1)\operatorname{sh}^2\xi}+\operatorname{sh}\tau}\Big) .$$

Let $\tilde{x} = \tau-\log(\operatorname{ch}\xi+(2z-1)\operatorname{sh}\xi)-x_0$; then $x(z)-x_0 = \tilde{x} + \Delta$ and we want reject Δ out of the argument.

Note that for z large enough we have $W'(z) = O(z^{-1/2-k})$. To the integral over the interval $(z_0,+\infty)$ is $O(\epsilon^{-1}z_0^{1/2-k}) = O(\sqrt{L}\cdot L^{\delta_1}z_0^{1/2-k}$. The last quantity is $O(L^{-2})$ for $z_0 >> L^{\alpha_1}$, $\alpha_1 = \frac{5+2\delta_1}{2k-1} \le \frac{5+2\delta_1}{23} < \frac{1}{4}$. As $e^{2\tau} >> L^2$ we have for $z \lesssim z_0$ $|z(z-1)e^{-2\tau}| << L^{-3/2}$. It means we can write Δ as the power series in Δ_1, $\Delta_1 = \frac{4z(z-1)\operatorname{sh}^2\xi\, e^{-\tau}}{\sqrt{\operatorname{sh}^2\tau+4z(z-1)\operatorname{sh}^2\xi}+\operatorname{sh}\tau}$. Now we replace the difference $\varphi_\epsilon(\tilde{x}+\Delta)-\varphi_\epsilon(\tilde{x})$ by the integral

$$\Delta\int_0^1 \varphi'_\epsilon(\tilde{x}+t\Delta)dt$$

and integrate by parts over z . It gives the integral without derivative of φ_ϵ and W because

(8.9) $$(z(z-1)W')' = ((k-1)^2-1/4)W$$

Furthermore,

$$\int_0^\infty |W(z)|\,|\varphi_\epsilon(\tilde{x}+t\Delta)|\,dz << \frac{1}{\xi}\log\frac{1}{\epsilon} << \frac{\log L}{\xi} \tag{8.10}$$

and it gives the first assertion: with the remainder term $O(\frac{\log L}{L^2})$ we can replace the argument of φ_ϵ in (8.6) by $\tilde{x}$.

After that we integrate by parts:

$$\int_0^\infty W'(z)\,\frac{\mathrm{sh}x\ \mathrm{sh}\tau}{\mathrm{ch}x\mathrm{ch}\tau-\mathrm{ch}\xi}\,\varphi_\epsilon(\tilde{x})dz = -W(0)\,\frac{\mathrm{sh}(\xi+\tau)\mathrm{sh}\tau}{\mathrm{ch}(\xi+\tau)\mathrm{ch}\tau-\mathrm{ch}\xi}\,\varphi_\epsilon(\xi+\tau) - \tag{8.11}$$

$$-\int_0^\infty W(z)\,\frac{\partial}{\partial z}\left\{\frac{\mathrm{sh}x\ \mathrm{sh}\tau}{\mathrm{ch}x\mathrm{ch}\tau-\mathrm{ch}\xi}\,\varphi_\epsilon(\tilde{x})\right\}dz\ .$$

Since

$$\mathrm{ch}(\xi+\tau)\mathrm{ch}\tau-\mathrm{ch}\xi = \mathrm{ch}(\xi+\tau)\mathrm{ch}\tau - \mathrm{ch}(\xi+\tau-\tau) = \mathrm{sh}(\xi+\tau)\mathrm{sh}\tau$$

and $W(0) = 1$, the integrated term equal to $(-\varphi_\epsilon(\xi+\tau))$. So this term cancelled with the first term on the right side (8.6). The integral with $W\,\varphi_\epsilon$ gives $O\left[\frac{\log L}{L^2}\right]$ again and the equality (8.1) follows.

8.2. *The explicit form for the terms and the amplification of the parameters.*

Now it is time to choose the parameter T in the definition (6.10); we define

$$T = L \tag{8.12}$$

Now let the new parameter Q be introduced instead of ϵ^{δ_1} in the definition (6.6):

$$\epsilon^2 L = Q^{-1} \quad . \tag{8.13}$$

Here and later we assume

$$M \le Q^{-4} L \quad . \tag{8.14}$$

We subdivide the sum $\sum_{1,1}$ onto two subsums, say $\sum^{(0)}$ and $\sum^{(1)}$, where $\sum^{(0)}$ contains the terms with $d \ge 2$ and the terms with $d = 1$ are taken in $\sum^{(1)}$.

For the sum $\sum^{(0)}$ we replace the condition $n \equiv N(\mathrm{mod}\ d)$ by the multiplier

$$\frac{1}{d} \sum_{c|d} \sum_{(a,c)=1} e\left[\frac{(n-N)a}{c}\right] = \begin{cases} 1, & n \equiv N(\mathrm{mod}\ d) \ , \\ 0, & n \not\equiv N)\mathrm{mod}\ d) \ . \end{cases}$$

As it follows from (8.1) we have for the sum $\sum^{(0)}$:

$$\sum^{(0)} = 2 \sum_{d\ge 2} \frac{1}{d} \sum_{c|d} \sum_{\substack{N \\ N=mq,\,(m,q)=1}} \sum_{n} \sum_{\nu} t(n)t(\nu)e\left[\frac{(n-N)a}{c}\right] v_{\epsilon}(n,\nu,N,d) + \tag{8.15}$$

$$+ O(L^{-1/4} M^{7/4} \log^2 L)$$

where with $Y = \frac{\sqrt{n}+\sqrt{N}}{\sqrt{\nu}+\sqrt{N}} \cdot \frac{1}{d} \cdot \left(1 + \frac{4z\sqrt{nN}}{(\sqrt{n}-\sqrt{N})^2}\right)^{-1/2}$

$$(8.16)\qquad v_\epsilon(n,\nu,N,d) = (1-\eta(\tfrac{n}{LN}))\omega_{a_1,b_1}(\tfrac{N}{M})\omega_{a_2,b_2}(\tfrac{(\sqrt{\nu}+\sqrt{N})^2}{L})\,(\tfrac{L}{\nu nN})^{1/4}$$

$$\cdot\int_0^\infty W(z)\cdot\frac{2\sqrt{nN}}{(\sqrt{n}-\sqrt{N})^2+4z\sqrt{nN}}(1-\beta)\varphi'_\epsilon(2\log Y)dz\ .$$

The terms of the sum $\sum^{(1)}$ we write in slightly different form. To avoid some difficulties what are concerned with the singularity of $W'(z)$ at $z=1$ we introduce in the integrand the function

$$\tilde\varphi_\delta(z-1)+(1-\tilde\varphi_\delta(z-1))\equiv 1,\ \ \tilde\varphi_\delta(z) = \delta\,\varphi_\delta(z)\ ,$$

with a suitable positive small δ and integrate by parts in the integral with $\tilde\varphi_\delta$.

It gives the representation

$$(8.17)\quad \sum^{(1)} = 2\sum_{\substack{N\\ N=mq,\,(m,q)=1}}\ \sum_N\sum_\nu t(n)t(\nu)\tilde v_\epsilon(n,\nu,N) + O(L^{-1/4}M^{7/4}\log^2 L)$$

where with $\omega_i = \omega_{a_i,b_i}$

$$(8.18)\qquad \tilde v_\epsilon(n,\nu,N) = (1-\eta(\tfrac{n}{LN}))\omega_1(\tfrac{N}{M})\omega_2(\tfrac{(\sqrt{\nu}+\sqrt{N})^2}{L})(\tfrac{L}{\nu nN})^{1/4}$$

$$\cdot\Big\{\int_0^\infty W'(z)(1-\tilde\varphi_\delta(z-1))\varphi_\epsilon(2\log Y)(1-\beta)dz - \int_0^\infty W(z)\frac{\partial}{\partial z}(\tilde\varphi_\delta(z-1)\varphi_\epsilon(2\log Y)(1-\beta))dz_j$$

In this representation

$$Y = \frac{\sqrt{n}+\sqrt{N}}{\sqrt{\nu}+\sqrt{N}} \cdot \left(1 + \frac{4z\sqrt{nN}}{(\sqrt{n}-\sqrt{N})^2}\right)^{-1/2},$$

the parameter δ will be chosen later and β denotes the same function what was in (8.2).

8.3. *The treatment of the sum* $\sum^{(0)}$.

The real our intend in this section is to obtain a certain asymptotic formula for the sum $\sum^{(0)}$. The most difficulty is the understanding that this sum $\sum^{(0)}$ and the integral with the sum $\sum_2(s)$ taken separately are not small enough; only the sum of these quantities give the desired estimate.

It is a very inquisitive process how to immense sums become more and more simple and give at the end the asymptotic formula which contains only the Fourier coefficients of the initial non–zero cusp form (the final result is given in § 10, Lemma 10).

The possibility to obtain the asymptotic expression for the sum $\sum^{(0)}$ is based on the sum formula (4.22) and on the similar formulae for the Fourier coefficients of the eigenfunctions of the automorphic Laplacian.

After the first using (4.22) it will be sufficient to remain $O(Q^{1+\delta_0})$ integrals for any $\delta_0 > 0$. The second summation (over n's) with using the same sum formulae gives only one integral which with a sufficient accuracy approximates this sum; so the full number is $O(Q^{1+\delta_0})$ again.

After the second summation the sum of the Kloosterman sums will be arosed. This sum will be expressed in the terms of $t_j(N)$. It allows us to carry out the summation over N's in a very explicit form: instead of this sum we obtain a finite number of integrals.

Now the corresponding function h (we will use the formula (2.15) for our sum of the Kloosterman sums) with large accuracy equals to

(8.19) $$\frac{i\pi}{2\mathrm{sh}(\pi r)}\int_0^\infty (J_{2ir}(\xi)-J_{-2ir}(\xi))J_{k-1}(\xi)\ \ (1-\eta\left[\frac{\xi_0^2}{\xi^2}\right])d\xi\ ;$$

it is a consequence of a certain good luck – the explicit form for the integrals

(8.20) $$\int_0^\infty {\cos \atop \sin}((2z-1)\xi)W(z)dz$$

what are expressed in terms $\xi^{-1/2}J_{k-1}(\xi)$.

By the very difficult way from the integral $\int\sum_2(s)d\chi(s)$ will be appeared the same sums over the spectrum and over the regular cusp forms, but with the taste function

(8.20) $$\frac{i\pi}{2\mathrm{sh}(\pi r)}\int_0^\infty (J_{2ir}(\xi)-J_{-2ir}(\xi))J_{k-1}(\xi)\eta\left[\frac{\xi_0^2}{\xi^2}\right]\frac{d\xi}{\xi}\ .$$

The sum (8.19) and (8.20) is zero, because of the orthogonality of the Bessel function of odd order to $J_{2ir}-J_{-2ir}$. The similar expressions will be for the coefficients in the sum over cusp forms. Here the main term of the sum two expressions (one from $\sum^{(0)}$ and the second one form $\int\sum_2(s)d\chi(s))$ is equal to

(8.21) $$\int_0^\infty J_{\ell-1}(\xi)J_{k-1}(\xi)\frac{d\xi}{\xi}$$

what is not zero only in the case $\ell=k$.

At this case the result follows from the asymptotical formulae for the Gauss hypergeometric functions and the explicite form the the shortened functional equation for the Rankin series.

On the choosen way we must keep an eye on the sign and the explicit coefficient before main terms; furthermore, a certain time we will remain slightly more terms in an asymptotic expansions than it is needed for the final expression.

8.4. *The first summation.*

Firstly we consider the inner sums over ν's in (8.15) and (8.17). We have

$$\sum_{\nu} t(\nu)v_{\epsilon}(n,\nu,N,d) = 4\pi i^{k} \sum_{\nu=1}^{\infty} t(\nu) \int_{0}^{\infty} J_{k-1}(4\pi x\sqrt{\nu})v_{\epsilon}(n,x^2;n,d)x\,dx\,. \tag{8.22}$$

Using the integral representation (8.16) we write

$$\int_{0}^{\infty} J_{k-1}(4\pi x\sqrt{\nu})v_{\epsilon}(n,x^2,\ldots)x\,dx = P_1 + P_2 + P_3 \tag{8.23}$$

where, if we write for the brevity

$$\lambda(n,N,z) = \frac{2(nNl)^{1/4}}{(\sqrt{n}-\sqrt{N})^2+4z\sqrt{nN}} \cdot (1-\eta(\tfrac{n}{LN}))\omega_1(\tfrac{N}{M}) \tag{8.24}$$

the function P_j is defined by the integral

$$(8.25)\ P_j(\nu,n,N,d)=\int_0^\infty W(z)\eta_j(z)\int_0^\infty J_{k-1}(4\pi x\sqrt{\nu})\omega_2\Big(\frac{(x+\sqrt{N})^2}{L}\Big)(1-\beta)\varphi'_\epsilon(2\log Y)\sqrt{x}\,dx\,dz$$

In this integral we treat by the different ways the cases $0 \le z \le Q$, $Q \le z \le \sqrt{\frac{n}{N}}$ and $\sqrt{\frac{n}{N}} \le z$. To do this by the "smooth" manner we introduce the expansion

$$(8.26)\qquad 1 \equiv (1-\eta(\tfrac{z}{Q})) + \eta(\tfrac{z}{Q})(1-\eta(z\sqrt{\tfrac{N}{n}})) + \eta(z\sqrt{\tfrac{N}{n}}) \equiv \eta_1(z) + \eta_2(z) + \eta_3(z)$$

with the clear notations.

Our nearest purpose is to prove the following assertion.

Proposition 8.2. For any fixed $\alpha > 1$ we have

$$(8.27)\qquad \sum_\nu t(\nu)v_\epsilon(n,\nu,N,d) = 4\pi i^k \sum_{\nu\le Q^\alpha} t(\nu)P_1(\nu,n,N,d) + O(L^{-2})$$

In the integral (8.25) we have

$$Y = \frac{\sqrt{n}+\sqrt{N}}{x+\sqrt{N}}\cdot\frac{1}{d}\cdot\Big(1+\frac{4z\sqrt{nN}}{(\sqrt{n}-\sqrt{N})^2}\Big)^{-1/2}$$

It is convenient to do the change of the variable and rewrite the inner integral in the form

$$(8.28)\qquad \int_{-\epsilon}^{\epsilon} \varphi'_\epsilon(2\log\tfrac{1}{1+u})J_{k-1}(4\pi\, x(u)\sqrt{\nu})\omega_2\Big(\frac{(x(u)+\sqrt{N})^2}{L}\Big)(1-\beta)\sqrt{x(u)}\,x'(u)du$$

with

$$(8.29) \qquad x(u) \equiv x(u,z) = -\sqrt{N} + \frac{\sqrt{n}+\sqrt{N}}{d} \cdot \left(1 + \frac{4z\sqrt{nN}}{(\sqrt{n}-\sqrt{N})^2}\right)^{-1/2} \cdot (1+u)$$

Of course, we can use the asymptotic expansion for the Bessel function since $x(u) >> \sqrt{M}$; this expansion may be written in the form

$$(8.30) \qquad J_{k-1}(x) = i^k \mathrm{Re}\left\{ \left[\frac{2}{\pi x}\right]^{1/2} e^{ix+i\pi/4}\left(1 + \frac{c_1}{x} + \frac{c_2}{x} + \ldots\right)\right\} \quad (k \text{ is even}) .$$

It is sufficient to take four terms and reject the terms with $x^{-5}, x^{-6}, \ldots$. Now for z large enough we have the additional resource which concerned with a possibility to integrate firstly over z .

<u>Proposition 8.3.</u>

$$(8.31) \qquad P_3 << \frac{1}{\epsilon}\left[\frac{M}{\nu}\right]^{1/4} \cdot d^{-k}(\nu L)^{3/2-k}$$

and for any positive R

$$(8.32) \qquad P_2 << \frac{1}{\epsilon} \cdot \frac{1}{d^{3/2}} \left[\frac{M}{\nu}\right]^{1/4} (Q\sqrt{\nu})^{-R} .$$

For z large we have in the exponent

$$A(1 + z/b)^{-1/2} \simeq Ab^{1/2}z^{-1/2}$$

with $A = \frac{4\pi(\sqrt{n}+\sqrt{N})\sqrt{\nu}}{d} \cdot >> \sqrt{\nu L}$ and $b \approx \frac{4\sqrt{n}}{\sqrt{N}} >> d \cdot \sqrt{\frac{L}{M}}$; in the integral P_3 we have $z >> b$. The change of the variable

$$\left[1 + \frac{z}{b}\right]^{-1/2} = z_1 b^{1/2}, \quad z = \frac{1}{z_1^2} - b$$

gives the integral

$$(8.33) \quad \int_0^\infty \eta\left[\frac{1}{z_1^2 b} - 1\right] W\left[\frac{1}{z_1^2} - b\right] \exp(i\, A b^{1/2} z_1)\,\omega_2\left[\frac{(x(u,z)+\sqrt{N})^2}{L}\right] \sqrt{x}\, \frac{\partial x}{\partial u}\, \frac{dz_1}{z_1^3}$$

Here $z << 1/\sqrt{b}$ and for W we have the power series with the main term z_1^{2k-1}. The standard asymptotic formulae from [13] for the integrals of this kind give (8.31).

The second estimate may be obtained by the same way after the change of the variable

$$(1+z/b)^{-1/2} = 1 - z_2/2b, \quad z = z_2 \cdot \frac{1-z_2/4b}{(1-z_2/2b)^2}$$

Now in the integral we have

$$\exp(-i\, A_1 z_2), \quad A_1 = \frac{(\sqrt{n}+\sqrt{N})\sqrt{N\nu}}{d\sqrt{n}} >> \frac{\sqrt{\nu M}}{d}.$$

As a consequence of the choice (8.12) we have $d << \sqrt{M}$ (see (7.5)); so $A_1 >> \sqrt{\nu}$ and we can integrate by parts any times. It gives (8.32). The quantities on the right side (8.31) and (8.32) are smaller than $L^{-2}\nu^{-2}$. In the first case it follows since $k \geq 12$ and for the second case we can take the suitable R because Q is some positive power of L .

It rests to consider the integrals P_1 for ν large, $\nu \geq Q^{\alpha}$ with $\alpha > 1$. Here $z << Q$ and $z\sqrt{\frac{N}{n}}$ is small. Now

$$\sqrt{\nu}\,\frac{\partial x}{\partial u} >> \sqrt{\nu L}$$

and for any R

$$\left[\frac{\partial}{\partial u}\right]^{R} \varphi'_{\epsilon}(2\log\frac{1}{1+u}) << \frac{1}{\epsilon^{R+2}} \ .$$

It means after R–multiple integartion by parts we obtain the additional multiplier in the integrand $(\nu\epsilon^2 L)^{-R/2} = \left[\frac{Q}{\nu}\right]^{R/2}$. So for $\nu \geq Q^{\alpha}$, $\alpha > 1$, we can take the finite R so that our integrals are estimated as $O(L^{-2}\nu^{-2})$. It gives (8.27).

In the conclusion of this subsection we rewrite the integral representation for P_1 interating by parts over u ; now we can omit index 1 in this function.

We have

$$\varphi'_{\epsilon}(2\log\frac{1}{1+u}) = -\frac{1}{2}(1+u)\,\frac{\partial}{\partial u}\,\varphi_{\epsilon}(2\log\frac{1}{1+u})$$

and

$$J'_{k-1}(x) = -J_k(x) + \frac{k-1}{x}\,J_{k-1}(x) \ .$$

So together with (8.27) we have

Proposition 8.4. For any fixed $\alpha > 1$ we have

$$(8.34)\qquad \sum_{\nu} t(\nu)v_{\epsilon}(n,\nu,N,d) = \sum_{1\le\nu\le Q^{\alpha}} t(\nu)P(\nu,n,N,d) + O(L^{-2})$$

where P is defined by the integral

$$(8.35)\quad P(\nu,n,N,d) = -4\pi i^{k}\int_{-\epsilon}^{\epsilon} \varphi_{\epsilon}(2\log\tfrac{1}{1+u})\int_{0}^{\infty} W(z)(1-\eta(\tfrac{z}{Q}))\Phi(n,\nu,N,d;u,z)dz\,du$$

where with $x(u)$ from (8.29) and $\lambda(n,N,z)$ from (8.24)

$$(8.36)\quad \Phi = \lambda(n,N,z)\Big\{2\pi\sqrt{\nu}(1+u)\sqrt{x(u)}\Big[\frac{\partial x}{\partial u}\Big]^{2}\omega_{2}(\frac{(x+\sqrt{N})^{2}}{L})(1-\beta)J_{k}(4\pi x(u)\sqrt{\nu}) -$$

$$- J_{k-1}(4\pi x(u)\sqrt{\nu})\Big[\frac{(k-1)(1+u)}{2\sqrt{x(u)}}\Big[\frac{\partial x}{\partial u}\Big]^{2}\omega_{2}(\frac{(x+\sqrt{N})^{2}}{L})(1-\beta) -$$

$$-\frac{1}{2}\frac{\partial}{\partial u}((1+u)\sqrt{x}\,\frac{\partial x}{\partial u}(1-\beta)\omega_{2}(\frac{(x+\sqrt{N})^{2}}{L}))\Big\} .$$

8.5. *The second summation.*

Proposition 8.5.

$$(8.37)\qquad \sum_{(a,c)=1}\ \sum_{n} t(n)e(\frac{(n-N)a}{c}) \sum_{1\le\nu\le Q^{\alpha}} t(\nu)P(\nu,n,N,d) =$$

$$= \frac{4\pi i^{k}}{c}\sum_{\frac{d^{2}}{c^{2}}\nu\le Q^{\alpha}} S(\nu,N;c)(t(\nu)t(\nu\frac{d^{2}}{c^{2}})\int_{0}^{\infty} P(\nu\frac{d^{2}}{c^{2}}y^{2},N,d)J_{k-1}(\frac{4\pi\sqrt{\nu}}{c})y\,dy + O(L^{-1}) .$$

Of course the assertion means: after using the sum formula (4.22) only one integral will be survived and all others give the remainder term which is small enough.

To see this we use the integral representation (8.35) and the asymptotic expansions for the Bessel functions. Then we come to the integrals (the change $y \longmapsto y\sqrt{N}$ is done)

$$(8.38)\quad \int \exp\left[\pm i\, \frac{4\pi\sqrt{\nu N}}{d} \cdot (y+1)(1+u)\left(1+\frac{4zy}{(y-1)^2}\right)^{-1/2} \pm 4\pi i\, \frac{\sqrt{nN}}{c}\, y\right] \mathcal{F}(y,\dots)dy$$

where $\mathcal{F}(y,\dots)$ is an infinitely smooth function, which is not zero only for

$$y \asymp d \cdot \sqrt{\frac{L}{M}}\ .$$

If we have $\frac{d}{c}\sqrt{n} \geq \sqrt{2\nu}$ then each integration by parts gives the additional multiplier $O(\frac{c}{\sqrt{nN}} \cdot \frac{1}{y}) = O(\frac{1}{\sqrt{nL}})$. So after five integrations we can reject these integrals.

If $\frac{d}{c}\sqrt{n}$ is near to $\sqrt{\nu}$ but $\nu \neq \left[\frac{d}{c}\right]^2 n$ then

$$|\sqrt{\nu} - \frac{d}{c}\sqrt{n}| = \left| \frac{\nu - \frac{d^2}{c^2}\, n}{\sqrt{\nu} + \frac{d}{c}\sqrt{n}} \right| >> Q^{-\alpha/2}\ .$$

At the same time $\frac{z}{y} << \frac{Q}{d} \cdot \frac{1}{Q^4}$ so the derivative of the function in the exponent is not zero for this case and its absolute value larger than $d^{-1}Q^{-\alpha/2}$. It means we can integrate by parts any times and it gives our assertion.

8.6. *The Fourier transform of the function* $W(z)$.

In the obtained expression for the sum $\sum^{(0)}$ we change d by cd (note $c|d$) and change y by $y\sqrt{N}$. Now we have

$$\sum^{(0)} = \tag{8.39}$$

$$=2\sum_N\sum_d\sum_{d^2\nu\le Q^\alpha} t(\nu)t(\nu d^2)\sum_{cd\ge 2}\frac{4\pi i^k N}{c^2}S(\nu,N;c)\int_0^\infty P(\nu d^2,y^2N,N,cd)J_{k-1}(\frac{4\pi\sqrt{\nu N}}{c}y)ydy +$$

$$+ O(M^{3/2}) .$$

At this moment we can reject some small terms in the integral representation for P and simplify the main terms. To do this the following nice result is very useful.

Lemma 8.1. Let $W(z)$ be defined by the equality (5.18) with the even integer $k \ge 4$. Then for all $\xi > 0$

$$\int_0^\infty \cos((2z-1)\xi)W(z)dz = -i^k\sqrt{\frac{\pi}{\xi}}J_{k-1}(\xi) , \tag{8.40}$$

$$\int_0^\infty \sin((2z-1)\xi)W(z)dz = i^k\sqrt{\frac{\pi}{\xi}}J_{k-1}(\xi) . \tag{8.41}$$

These equalities are a consequence of the differential equation for W , but before we note that for the usual Fourier transformation it follows from (8.40) and (8.41):

(8.42) $$\int_0^\infty \cos(\xi z)W(z)dz = -i^k \sqrt{\frac{2\pi}{\xi}}\, J_{k-1}(\xi/2)(\cos \xi/2 + \sin \xi/2) \ ,$$

(8.43) $$\int_0^\infty \sin(\xi z)W(z)dz = i^k \sqrt{\frac{2\pi}{\xi}}\, J_{k-1}(\xi/2)(\cos \xi/2 - \sin \xi/2) \ .$$

Now let us write

(8.44) $$f(\xi) = \int_0^\infty e^{i\xi(2z-1)}W(z)dz \ .$$

Since $W(z) << z^{1/2-k}$ for z large and $k \geq 4$ we can differentiate two times under the sign of the integration. It follows from the equation for W :

(8.45) $$(-(k-1)^2+1/4)f = \int_0^\infty (z(1-z)W')'e^{i\xi(2z-1)}dz$$

$$= \int_0^\infty W(z)(z(1-z)(e^{i\xi(2z-1)})')'dz$$

$$= \int_0^\infty W(z)(-4\xi^2 z(1-z)-2i\xi(2z-1))e^{i\xi(2z-1)}dz$$

$$= -\xi^2 f - \xi^2 f'' - 2\xi f' \ ,$$

since all integrated terms at $z=1$, $z=0$ amd $z=\infty$ are disappearing. If we denote ξf

by Υ then for this last function we have the differential equation

(8.46) $$\Upsilon'' + \frac{(k-1)^2-1/4}{\xi^2}\Upsilon = 0$$

It means $\Upsilon = c_1\sqrt{\xi}\,J_{k-1}(\xi) + c_2\sqrt{\xi}\,Y_{k-1}(\xi)$; but the function ξf is bounded at $\xi = 0$ and $\sqrt{\xi}\,Y_{k-1} >> \xi^{3/2-k}$ when $\xi \longrightarrow 0$. So $c_2 = 0$. The constant c_1 may be easily defined by the comparison of the asymptotic behavior for both sides on (8.40) and (8.41) at $\xi = \infty$.

8.7. *The main terms for the integrals in* (8.39) *after the summation over* N's .

We make ready to use the identity (2.15) for the transformation the sum over c's in (8.39) so it is convenient to introduce the special notation for the combination $\xi = \frac{4\pi\sqrt{\nu N}}{c}$. Now for the integrals in (8.39) we have the representation

(8.47) $$\frac{4\pi i^k N}{c}\int_0^\infty P(\nu d^2,y^2N,N,cd)J_{k-1}(\xi y)y\,dy = \frac{i^k\sqrt{N}}{\xi\sqrt{\nu}}\int_0^\infty P(\nu d^2,y^2\frac{N}{\xi^2},N,cd)J_{k-1}(y)y\,dy$$

$$= \int_{-\epsilon}^{2}\varphi_\epsilon(2\log\frac{1}{1+u})\int_0^\infty W(z)(1-\eta(\tfrac{z}{Q}))\int_0^\infty \Psi_\nu(y,\xi,N,d)\,dy\,dz\,du\ .$$

Here in the accordance with (8.35)–(8.36) the function Ψ_ν is

(8.48) $$\Psi_\nu(y,\xi,N,d) = -J_{k-1}(y)J_k(\hat{y})A + J_{k-1}(y)J_{k-1}(\hat{y})B$$

with

(8.49) $$\hat{y} = (y+\xi)(1+u)\left(1 + \frac{4zy\xi}{(y-\xi)^2}\right)^{-1/2} - \xi\, cd$$

and with the following coefficients A, B :

(8.50) $$A = \frac{1+u}{(4\pi)^2}\cdot\left[\frac{y}{\nu d^2}\right]^{3/2}\left[\frac{y+\xi}{y-\xi}\right]^2\left[1+\frac{4zy\xi}{(y-\xi)^2}\right]^{-2}\sqrt{y\xi}\cdot(1-\beta)\left(1-\eta\left(\frac{y^2}{L\xi^2}\right)\right)\cdot\omega_1\left(\frac{N}{M}\right)\cdot L^{1/4}$$
$$\cdot\ \omega_2\left[\frac{(\hat{y}+\sqrt{N})^2}{4\pi^2\nu L d^2}\right]$$

(8.51) $$B = \frac{(k-1)A}{y\sqrt{\nu}} - \frac{4\pi d}{y+\xi}\cdot\sqrt{1+\frac{4zy\xi}{(y-\xi)^2}}\,\frac{\partial A}{\partial u}\ .$$

Here $y, \hat{y}$ are large enough, $y, \hat{y} \asymp \sqrt{\nu L d^2}$ and ξ is small in the comparison with y . So the Bessel functions may be replaced by their asymptotic expansions and in (8.39) we have the following sum over N's

(8.52) $$\sum_{\substack{N \\ n=mq,\,(m,q)=1}} S(\nu,N;c)e^{\pm i\hat{y}(\xi)}\omega\left(\frac{N}{M}\right)g(N)\ , \quad \xi = \frac{4\pi\sqrt{\nu N}}{c}\ ,$$

where ω is written instead of ω_{a_1,b_1} , $g(N)$ is an infinitely smooth function for which with any fixed $R \geq 1$ we have

$$\left(\frac{\partial}{\partial N}\right)^R g(N) << M^{-R} \max_{n\asymp M} |g(N)|\ .$$

Note that in this sum we have

$$-\frac{\partial \hat{y}}{\partial \xi} = cd - \frac{1+u}{\sqrt{1+\frac{4z\xi}{(y-\xi)^2}}} + \frac{1}{2}\cdot\left[\frac{y+\xi}{y-\xi}\right]^2 \cdot \frac{1+u}{\left(1+\frac{4z\xi}{(y-\xi)^2}\right)^{3/2}} > cd-1 \geq 1$$

since $cd \geq 2$. So there is the possibility to utilize the oscillatios of the members in this sum. The result is

Proposition 8.6. Let $cd \geq 2$ and $q << Q^{1/4}$ then in the case $c \leq \frac{\sqrt{M}}{Q}$ for any fixed $R > 1$ we have

$$(8.53) \qquad \sum_{\substack{N=mq \\ (m,q)=1}} e^{i\hat{y}(\xi)} S(\nu,N;c)\,\omega\left(\frac{N}{M}\right)g(N) << M^{-R} \max_{N \asymp M} |g(N)$$

and in the case $\frac{\sqrt{M}}{Q} \leq c \leq \sqrt{M}$ this sum is estimated by the quantity

$$(8.54) \qquad << q^2(\nu d^2)^{1/4} M^{3/4} \ .$$

Firstly we replace N by mq and further instead of the condition $(m,q) = 1$ we write $m \equiv m_0 (\mathrm{mod}\ q)$ where m_0 runs the reduced system $(\mathrm{mod}\ q)$. So our sum equals to

$$(8.55)\ \frac{1}{q}\sum_{r=1}^{q}\ \sum_{(m_0,q)=1} e\left(-\frac{m_0 r}{q}\right) \sum_{m} e\left(\frac{mr}{q}\right) S(\nu,mq;c) e^{i\hat{y}(\xi)} \omega\left(\frac{mq}{M}\right) g(Mq), \quad \xi = \frac{4\pi\sqrt{\nu qm}}{c}$$

Now by the usual way we replace m by n+mqc with $1 \le n \le qc$, $m = 0,1,2,\ldots$, , and use the Fourier expansion. It gives for the sum (8.55) the expression

(8.56) $$\frac{1}{q}\sum_{r=1}^{q}\sum_{(m_0,q)=1} e\left(-\frac{m_0 r}{q}\right)\sum_{n=1}^{qc} e\left(\frac{nr}{q}\right)S(\nu,nq;c)\sum_{m=-\infty}^{\infty} e\left(\frac{mn}{qc}\right)B_m ,$$

(8.57) $$B_m = \int_{-\infty}^{\infty} \omega\left(\frac{cq^2x}{M}\right)g(cq^2x)\exp\left(-2\pi imx + i\hat{y}\left(\frac{4\pi q\sqrt{x\nu}}{\sqrt{c}}\right)\right)dx$$

$$= \frac{2M}{cq^2}\int \omega(x^2)g(Mx^2)\exp\left(-2\pi im\frac{M}{cq^2}x^2 + i\hat{y}\left(\frac{4\pi\sqrt{\nu M}}{c}x\right)\right) dx$$

Here the integration is doing over the interval $\sqrt{b_1} \le x \le \sqrt{b_2}$ for some fixed positive b_1, b_2 . Remember with $a = (1 + 16\pi zy\sqrt{\nu M}\,x\,c^{-1}(y - 4\pi\sqrt{\nu M}\,c^{-1}x)^{-2})^{-1/2}$

(8.58) $$\hat{y}\left(\frac{4\pi\sqrt{\nu M}}{c}x\right) = -4\pi\sqrt{\nu d^2 M}\cdot x + \frac{4\pi(1+u)\sqrt{\nu M}\,x}{c\,a} + \frac{y(1+u)}{a} ;$$

so the derivative over x is near to $-4\pi\sqrt{\nu d^2 M}\,(1 - \frac{1}{cd})$.

If $m=0$ we can integrate by parts any times and it immediately gives the estimate $B_0 << M^{-R}$ for any fixed R . If $m \neq 0$ and $m > 0$ then there are no teroes for the derivative

$$\frac{\partial}{\partial x}\left(-2\pi m\frac{M}{cq^2}x^2 + \hat{y}\right)$$

and for this reason we come to the same estimate $B_m << (m\sqrt{M})^{-R}$ again. If $m < 0$ then zero of this derivative is near to

(8.59)
$$x_0 = \frac{cq^2\sqrt{\nu d^2}}{\sqrt{M}}\left(1 - \frac{1}{cd}\right) \cdot \frac{1}{|m|}$$

Here $\nu d^2 \leq Q^\alpha$ with α which is near to 1; so for $c \leq \frac{\sqrt{M}}{Q}$ for all $m \leq -1$ the point x_0 is on the left of the interval $(\sqrt{b_1}, \sqrt{b_2})$ where the integrand is not zero. It means the etsimate $B_m << (|m|\sqrt{M})^{-R}$ hold in this case.

Finally, if

(8.60)
$$\frac{\sqrt{M}}{q^2\sqrt{\nu d^2}} << c \leq \sqrt{M}$$

then there are $O(q^2\sqrt{\nu d^2})$ values of m for which $x_0 \in [\sqrt{b_1}, \sqrt{b_2}]$. For each such m we have

(8.61)
$$B_m << \left[\frac{M}{cq^2}\right]^{1/2} \cdot \frac{1}{|m|^{1/2}} << \frac{M^{3/4}}{cq^2(\nu d^2)^{1/4}} .$$

At the same time

(8.62)
$$\sum_{n=1}^{qc} e\left(\frac{nr}{q}\right)S(\nu,nq;c)e\left(\frac{mn}{qc}\right) = \sum_{(a,c)=1} e\left(\frac{a'\nu}{c}\right) \sum_{n=1}^{qc} e\left(\frac{n}{qc}(m+aq^2+rc)\right)$$

and for any given m the inner sum on the right side equals qc at most for one value of a for $c > q$.

So the sum on the left side does not exceed qc. Two summations over m_0 and r's give the additional factor q^2 and (8.54) follows.

For the full sum $\sum^{(0)}$ the obtained estimate is not sufficient because the mian terms have the order

$$(8.63)\qquad \sum_{\frac{\sqrt{M}}{Q}\le c\le\sqrt{M}} \frac{1}{c} \sum_{d^2\nu\le Q^\alpha} |t(\nu)t(\nu d^2)| \cdot M^{3/4}L^{5/4}q^2Q^{1/2} << \frac{(ML)^{3/4}}{\epsilon}\cdot Q^\alpha q^2 \log Q$$

here $\alpha > 1$ so this term is not $O(\epsilon^{-1}(ML)^{3/4})$. At the same time this estimate is more than sufficient to reject all terms in the expression for $\tilde{\psi}_v$ except the main ones and reject all members with $c \le Q^{-1}\sqrt{M}$. It means that really we have in this expression

$$(8.64)\qquad \xi << Q^{1+\alpha/2}\ ,\quad \alpha \text{ is near to 1 and } \alpha > 1\ .$$

Furthermore, with the acceptable accuracy we have

$$(8.65)\qquad J_{k-1}(y)J_{k-1}(\hat{y}) \cong \frac{1}{\pi\sqrt{y\hat{y}}}(\sin(\hat{y}+y) + \cos(\hat{y}-y))$$

and after the integrations by parts over y the terms with $y+\hat{y}$ give a negligible remainder term. So it is sufficient to remain only the terms with $\cos(\hat{y}-y)$ where

$$(8.66)\qquad \hat{y}-y = (1-2z)\xi + uy - (2z-1)u\xi - \frac{2z\xi^2(1+u)(3y-\xi)}{(y-\xi)^2} +$$

$$+ (1+u)(y+\xi)\sum_{m\ge 2}\binom{-1/2}{m}\left[\frac{4z\xi y}{(y-\xi)^2}\right]^m - 4\pi\sqrt{\nu d^2 N}$$

and now it is obvious that we can remain only two terms $(1-2z)\xi + uy$ from the expansion under the signe of sine.

By the same reason we can replace $\log \frac{1}{1+u}$ by $-u$ in the argument of φ_ϵ ; using the identities (8.40) and (8.41) and introducing the cutting function $(1-\eta(\xi/Q\sqrt{\nu}))$ (it is 0 for $\xi >> Q\sqrt{\nu}$ what corresponds to $c \lesssim Q^{-1}\sqrt{M}$) we obtain the following representation for the sum $\sum^{(0)}$.

Proposition 8.7. Under our assumptions for the parameters and with the additional condition $Q >> q^4$ we have

(8.67)
$$\sum^{(0)} =$$

$$= L^{5/4} \sum_{\nu d^2 \leq Q^\alpha} \frac{t(\nu)t(\nu d^2)}{(\nu d^2)^{1/4}} \sum_{\substack{N \\ N=mq,(m,q=1}} \omega(\tfrac{N}{M}) \cdot \sqrt{2} \cos(4\pi\sqrt{\nu d^2 N} + \tfrac{\pi}{4}) \sum_{c \geq 1} \frac{S(\nu,N;c)}{c} \cdot$$

$$\cdot\ \psi_\nu(\frac{4\pi\sqrt{\nu N}}{c}) + O(\frac{(ML)^{3/4}}{\epsilon} Q^{-2})$$

where with the notation

$$\Phi(u) = \int_{-1}^{1} \varphi_1(x)\cos(ux)dx$$

for the Fourier transform of the function φ_1 (it is φ_ϵ for $\epsilon=1$) we have

(8.68)
$$\psi_\nu(\xi) = J_{k-1}(\xi)(1-\eta(\frac{\xi}{Q\sqrt{\nu}})) \int_0^\infty y\ \omega_{a_2,b_2}(y^2)(1-\eta(\frac{\xi_0^2}{\xi^2}))\Phi(2\pi y\sqrt{\frac{\nu d^2}{Q}})dy \ ,$$

where $\xi_0^2 = 4\pi\nu d^2 y^2$ and really $a_2 \le y^2 \le b_2$.

Later we shall see (in the beginning § 10) the function $\Phi(u)$ is etsimated for u large as $O(\exp(-\sqrt{u}))$. So the condition $\nu d^2 \le Q^\alpha$ we can replace by the more strong one $\nu d^2 << Q \log^2 Q$ (or reject any conditions since the series with Φ is convergent).

8.8. *The coefficients in the spectral representation.*

On this step of the obtaining the asymptotic fomrula for $\sum^{(0)}$ we use the idetity (2.15) to transform the inner sum in (8.67). To have the integral transforms (2.18) and (2.19) be calculated for the function (8.68) we write this function in the form

(8.69) $$\psi_\nu(\xi) = J_{k-1}(\xi) \cdot \frac{1}{i\pi} \int_{(\sigma)} \xi^{-2\rho} \alpha_\nu(\rho) d\rho \ .$$

Here $\int_{(\sigma)}$ denotes the integral over the line $\operatorname{Re} \rho = \sigma$; α_ν is the Mellin transform of the multiplier in (8.68); it is the entire function in ρ and it is obvious for $|\rho|$ large with $\sigma = \operatorname{Re} \rho$ be fixed, $\sigma > 0$, we have for any $R > 1$

(8.70) $$|\alpha_\nu(\rho) << \frac{(Q\sqrt{\nu})^{2\sigma}}{|\rho|^R} \ .$$

We denote by $h_\nu(r)$ and $h_{\nu,e}$ accordingly the integrals (2.18) and (2.15) with $\varphi = \psi_\nu$.

Using the special case of the Weber–Schafheitlin integral we come to the following representations for this coefficients:

$$(8.71)\ h_\nu(r) = -\frac{i^k}{2\pi i}\int_{(\sigma)} 2^{-2\rho}\sin(\pi\rho)(2\rho+1)\cdot\frac{\Gamma(\frac{k-1}{2}-\rho+ir)\Gamma(\frac{k-1}{2}-p+ir)}{\Gamma(\frac{k+1}{2}+\rho+ir)\Gamma(\frac{k+1}{2}+\rho-ir}\,\alpha_\nu(\rho)d\rho$$

$$(8.72)\ h_{\nu,\ell} = -\frac{i^k}{2\pi i}\int_{(\sigma)} 2^{-2\rho}\sin(\pi\rho)\Gamma(2\rho+1)\cdot\frac{\Gamma(\frac{\ell+k}{2}-1-\rho)\Gamma(\frac{\ell-k}{2}-\rho)}{\Gamma(\frac{\ell+k}{2}+\rho)\Gamma(\frac{\ell-k}{2}+1+\rho)}\,\alpha_\nu(\rho)d\rho\ .$$

It follows immediately from these representations:

Proposition 8.8. For all $r >> 1$ we have

$$(8.73)\qquad |h_\nu(r)| << \frac{1}{r^2},\quad |h_{\nu,\ell}| << \frac{1}{\ell^2}$$

and at the same time for $r^2 >> Q\sqrt{\nu}$, $\ell^2 >> Q\sqrt{\nu}$ for any $\sigma > 1$ we have

$$(8.74)\qquad |h_\nu(r)| << \frac{1}{r^2}\left[\frac{Q^2\nu}{r^4}\right]^\sigma,\quad |h_{\nu,\ell}| << \frac{1}{\ell^2}\left[\frac{Q^2\nu}{\ell^4}\right]^\sigma\ .$$

Of course it is a consequence of the Stirling expansion for gamma–function; we integrate over the line $\mathrm{Re}\,\rho = 0$ to obtain (8.73) and move the path of the integration to the right; it gives (8.74).

Since $\nu << Q\log^2 Q$ in the sums on the right side (2.15) for the case $\varphi = \psi_\nu$ we can reject all members with $\kappa_j^4 >> Q^{3\alpha}$, $\ell^4 >> Q^{3\alpha}$ for any $\alpha > 1$.

It allows us to give the explicit form for the sum over N's ; but we begin the new section to avoid four–digit numeration for the next formulae.

§ 9. The sums $\sum^{(0)}$ and $\sum^{(1)}$ (the end).

9.1. *Two additional sum formulae.*

Lemma 9.1. Let $f \in C^{\infty}(0,\infty)$ be a function with a bounded support. Then for any integer $c \geq 1$ and a which is coprime with c we have for $j \geq 1$

$$(9.1) \qquad \frac{4\pi}{c} \sum_{m=1}^{\infty} t_j(m) e(\tfrac{ma}{c}) f(\tfrac{4\pi\sqrt{m}}{c}) =$$

$$= \sum_{m=1}^{\infty} t_j(m) \int_0^{\infty} (e(-\tfrac{md}{c}) k_0(x\sqrt{m}, \tfrac{1}{2}+i\kappa_j) + \epsilon_j\, k_1(x\sqrt{m}, 1/2+i\kappa_j) e(\tfrac{md}{c})) f(x) x\, dx$$

where ϵ_j is the eigenvalue of the reflection operator ($\epsilon_j = +1$ for even eigenfuntions and $\epsilon_j = -1$ for odd ones), d is defined by the congruence $ad \equiv 1 (\text{mod } c)$ and the kernels k_0, k_1 are expressed in terms of the Bessel functions by the equalities

$$(9.2) \qquad k_0(x,\nu) = \frac{1}{2\cos(\pi\nu)} (J_{2\nu-1}(x) - J_{1-2\nu}(x)) \ ,$$

$$(9.3) \qquad k_1(x,\nu) = \frac{2}{\pi} \sin(\pi\nu) K_{2\nu-1}(x) \ .$$

This analogue of (4.22) is an easy consequence of the functional equation for the corresponding Hecke series; the details are in my doctoral dissertation (LOMI, 1981).

The following similar sum formula corresponds to the continuous spectrum of the Hecke operators.

Lemma 9.2. Under the same assumptions as in Lemma 9.1 we have for any $\nu \in C$

$$(9.4) \qquad \frac{4\pi}{c} \sum_{m=1}^{\infty} \tau_\nu(m) e\left(\frac{ma}{c}\right) f\left(\frac{4\pi\sqrt{m}}{c}\right) =$$

$$= 2 \int_0^{\infty} \left(\zeta(2\nu) \left(\frac{x}{4\pi}\right)^{2\nu} + \zeta(2-2\nu) \left(\frac{x}{4\pi}\right)^{2-2\nu} \right) f(x) dx +$$

$$+ \sum_{m=1}^{\infty} \tau_\nu(m) \int_0^{\infty} \left(e\left(-\frac{md}{c}\right) k_0(x\sqrt{m},\nu) + e\left(\frac{md}{c}\right) k_1(x\sqrt{m},\nu) \right) f(x) x \, dx \ .$$

9.2. *The second summation over* N's .

The very special case of the identities (4.22), (9.1) and (9.4) will be used here. After the replacement of the inner sum in (8.67) by the corresponding bilinear form of the eigenvalues of the Hecke operators we come to the following sums

$$(9.5) \qquad T_q(g) = \tilde{t}(q) \sum_{(m,q)=1} \tilde{t}(m) \omega\left(\frac{mq}{M}\right) \sqrt{2} \cos(4\pi\sqrt{mg} + \pi/4) \ ,$$

where ω is written instead of ω_{b_1,b_2}, g is an integer and

$$\tilde{t}(m) = t_j(m), \ t_{j,\ell}(m) \ \text{or} \ \tau_{1/2+ir}(m), \ r \in \mathbb{R} \ .$$

In our special case we have $g = \nu q d^2$ and we can assume that for every fixed (small) $\alpha > 0$

$$g,\ \kappa_j,\ \ell,\ |r| << M^{\alpha}\ . \tag{9.6}$$

It means the usual asymptotic expansions can be used for the Bessel functions; in particular the integrals with the kernel $k_1(x,\nu)$ ($\nu = 1/2+ir$ or $\nu = 1/2+i\kappa_j$) contribute $O(\exp(-\sqrt{M}))$.

For these conditions we have the following asymptotic formula for the sum (9.5).

Proposition 9.1. Let for $c\,|\,q$

$$A_{q,c} = \frac{1}{cq^2}\Big(\sum_{(a,c)=1} 1\Big) \sum_{a\,|\,q/c} a\,\mu(\tfrac{q}{a}) \tag{9.7}$$

where μ is the Möbius function. Then

$$T_q(g) = 2M^{3/4}(q\,|\,g)^{1/4}\Big(\sum_{c\,|\,q} A_{q,c}\tilde{t}(q)\tilde{t}(c^2 g)\Big)\Big(\int_0^{\infty} \sqrt{x}\,\omega(x^2)dx + O\big((\tfrac{g^2 M}{q})^{-1/4}\big)\Big)\ . \tag{9.8}$$

First of all, for any quantities $z(m)$ we have

$$\sum_{(m,q)=1} z(m) = \sum_{\substack{(m_0\,,\,q)=1\\ 1\le m_0<q}} \ \sum_{m\equiv m_0(\mathrm{mod}\ q)} z(m) \tag{9.9}$$

$$= \sum_{(m_0\,,q)=1} \frac{1}{q} \sum_{c\,|\,q} \ \sum_{(a,c)=1} \ \sum_{m} e\Big[\frac{(m-m_0)a}{c}\Big]\, z(m)$$

and in the inner sum m's run all integers without any conditions. For this inner sum we can use (9.1), (9.4) or (4.22). If $\tilde{\tau}(m) = t_j(m)$ or $\tau_{1/2+ir}(m)$ we have

$$(9.10)\quad k_0(x,1/2+ir) = \sqrt{\frac{2}{\pi x}}\left\{\cos(x+\pi/4)\cdot(1+O(\tfrac{1}{x^2})) + \frac{\gamma(r)}{x}\sin(x+\pi/4) + O(\tfrac{1}{x^3})\right\}$$

and for an even integer ℓ we have the same main term

$$(9.11)\qquad J_{\ell-1}(x) = i^{\ell}\left(\frac{2}{\pi x}\right)^{1/2}\cos(x+\tfrac{\pi}{4}) + \ldots$$

Now all integrals

$$(9.12)\qquad \int_0^\infty k_0(x\sqrt{m},1/2+ir)\cos(x\sqrt{c^2g}+\pi/4)\,\omega\left[\frac{qc^2x^2}{(4\pi)^2M}\right]\sqrt{x}\,dx$$

with $m \neq c^2g$ can be rejected; the same is true for the Mellin integral

$$(9.13)\qquad \int x^{1\pm 2ir}\cos(x+\pi/4)\,\omega\left[\frac{x^2}{(4\pi g)^2M}\right]dx$$

since $|r|$ is small in the comparison with $\sqrt{M}$.

So one term with $m = c^2g$ rests; in this term $\ell(\frac{ma}{c}) = 1$ and for the Ramanujan sum we have the explicit form,

$$(9.14)\quad \sum_{(m_0,q)=1} e\left(-\frac{m_0a}{c}\right) = \sum_{(m_0,q)=1} e\left(-\frac{m_0}{q}\cdot q/c\right) = \sum_{a|(q,q|c)} a\mu\left(\frac{q}{a}\right) = \sum_{a|q/c} a\mu(q/a)\,,$$

and the assertion (9.8) follows.

9.3. *The asymptotic representation for the sum* $\sum^{(0)}$.

Let us write the definition (8.68) in the form

$$\psi_\nu(\xi) = \psi_\nu^{(0)}(\xi) - \psi_\nu^{(1)}(\xi) \tag{9.15}$$

where

$$\psi_\nu^{(0)}(\xi) = J_{k-1}(\xi)\int_0^\infty y\omega_2(y^2)(1-\eta(\frac{\xi_0^2}{\xi^2}))\Phi(2\pi y\sqrt{\frac{\nu d^2}{Q}})dy, \quad \xi_0^2 = 4\pi\nu d^2 y^2 , \tag{9.16}$$

and $\psi_\nu^{(1)}(\xi) = \eta(\xi/Q\sqrt{\nu})\psi^{(0)}(\xi)$. Accordingly to this subdividing the coefficients $h_\nu(r)$, $h_{\nu,\ell}(r)$ are the quantities

$$h_\nu(r) = h_\nu^{(0)}(r) - h_\nu^{(1)}(r)\,, \quad h_{\nu,\ell}(r) = h_{\nu,\ell}^{(0)} - h_{\nu,\ell}^{(1)} , \tag{9.17}$$

so that

$$h_\nu^{(0)}(r) = \frac{i\pi}{2sh(\pi r)}\int_0^\infty (J_{2ir}(\xi) - J_{-2ir}(\xi))\psi_\nu^{(0)}(\xi)\frac{d\xi}{\xi} , \tag{9.18}$$

$$h_\nu^{(1)}(r) = b(\nu)\frac{i\pi}{2sh(\pi r)}\int_0^\infty (J_{2ir}(\xi) - J_{-2ir}(\xi))I_{k-1}(\xi)\eta(\frac{\xi}{Q\sqrt{\nu}})\frac{d\xi}{\xi} ; \tag{9.19}$$

in the last equality

$$b(\nu) = \int_0^\infty y\ \omega_2(y^2)\Phi(2\pi y\sqrt{\frac{\nu d^2}{Q}})dy\ .$$

This possibility to write $b(\nu)$ outside of the sign of the integration over ξ is the consequence of the definition of η : we have $\eta(\frac{4\pi\nu d^2 y^2}{\xi^2}) \equiv 0$ for $y \asymp 1$, $\nu d^2 << Q^\alpha$ and $\xi >> Q\sqrt{\nu}$. The similar representations we have for the coefficients $h^{(0)}_{\nu,\ell}$ and $h^{(1)}_{\nu,\ell}$.

Now, for brevity, let us denote (for the given function h and for an integer $n,m \geq 1$) by $Z^{dis}(n,m;h)$ $Z^{con}(n,m;h)$ and $Z^{cusp}(n,m;\{h_\ell\})$ three sum on the right side (2.15). For example,

$$(9.20) \qquad Z^{dis}(n,m;h) = \sum_{j\geq 1} \alpha_j t_j(n) t_j(m) h(\kappa_j)$$

and by the similar manner the quantities Z^{con} and Z^{cusp} are defined.

With these notations we have the following asymptotic expansion.

Proposition 9.2. With the same remainder term as in (8.67) we have

$$(9.21) \quad \sum^{(0)} = 2i^k a_0 M^{3/4} L^{5/4} \sum_{c|q} A_{q,c} \sum_{\nu d^2 \leq Q \log^2 Q} \sum_{m|(\nu,q)} \frac{t(\nu)t(\nu d^2)}{\sqrt{\nu d^2}} \cdot$$

$$\cdot\left\{Z^{dis}(\frac{\nu q}{m^2},\nu q c^2 d^2;h^{(0)}_\nu)+Z^{con}(\frac{\nu q}{m^2},\nu q c^2 d^2;h^{(0)}_\nu)+Z^{cusp}(\frac{\nu q}{m^2},\nu q c^2 d^2;\left\{h^{(0)}_{\nu,\ell}\right\})\right\} +$$

$$+ O(\frac{(ML)^{3/4}}{\epsilon} \cdot Q^{-2}) ,$$

where

$$a_0 = \pi \int_0^\infty \sqrt{x}\, \omega_{a_1,b_1}(x^2)dx \ . \tag{9.22}$$

This assertion is the same that we can reject the sum

$$Z^{dis}(...;h_\nu^{(1)}) + Z^{con}(...;h_\nu^{(1)}) + Z^{cusp}(...;\{h_{\nu,\ell}^{(1)}\}) \tag{9.23}$$

because all the rest follows from (8.67) and (9.8); remember that

$$t_j(q)t_j(\nu) = \sum_{m\,|\,(\nu,q)} t_j(\frac{\nu q}{m^2})$$

and the same relations we have for $t_{j,\ell}$ and τ_ν.

The possibility of this rejection follows from the following fact.

Proposition 9.3. Let $Q > q^{4+\delta}$ for some positive δ; then the sum (9.23) is zero.

Let us read the identity (2.15) from the right to the left; then we see that the sum (9.23) is the sum of the Kloosterman sums

$$\sum_{c\geq 1} \frac{1}{c} S(\frac{\nu q}{m^2}, \nu q c^2 d^2; c) V\left[\frac{4\pi \nu q c_1 d}{mc}\right] \tag{9.24}$$

where $c_1 \,|\, q$, $m \,|\, (\nu,q)$; the taste function V is equal to

$$V(x) = J_{k-1}(x)\eta(\frac{x}{Q\sqrt{\nu}}) \ . \tag{9.25}$$

In the sum (9.24) we have

$$\nu d^2 << Q \log^2 Q\ ,\ \ c_1 \leq q,\ \ m \geq 1\ ;$$

for this reason

$$(9.26) \qquad \frac{1}{Q\sqrt{\nu}} \cdot \frac{\nu q c_1 d}{mc} << \frac{q^2 \log Q}{\sqrt{Q}} \cdot \frac{1}{c}\ .$$

Since $\eta(x) \equiv 0$ for $x \leq x_1$ for some positive fixed x_1 we have $V\left[\frac{4\pi\nu q c_1 d}{mc}\right] = 0$ for all $c \geq 1$ if the quantity $\frac{q^2 \log Q}{\sqrt{Q}}$ is small enough.

9.4. *The sum* $\sum^{(1)}$.

To finish the consideration of the sum $\sum_1$ it rests to estimate the sum $\sum^{(1)}$. It will be proved here that this sum is small enough in the comparison with the main terms of the sum $\sum_1$.

<u>Proposition 9.4</u>.

$$(9.27) \qquad \sum^{(1)} << L^{5/4} M^{1/4} \log L\ .$$

As it was early we use the identity (4.22) for the inner sum over ν's in (8.17). This allow us to replace our sum by the short sum of the integrals. The number of these integrals is $O(Q^{1+\alpha})$ for every fixed $\alpha > 0$.

Let us write

$$x(u) = (1+u)(\sqrt{n}+\sqrt{N})\left(1 + \frac{4z\sqrt{nN}}{(\sqrt{n}-\sqrt{N})^2}\right)^{-1/2} \tag{9.28}$$

and with this notation

$$\Omega(n,N;z,u) = \left(1-\eta\left(\frac{n}{LM}\right)\right)\omega_1\left(\frac{N}{M}\right)\omega_2\left(\frac{x^2(u)}{L}\right)(nN)^{-1/4} \ . \tag{9.29}$$

We subdivide the half–axis $z \geq 0$ on to three interval $(0,1-\delta)$, $(1-\delta,1+\delta)$, $(1+\delta,+\infty)$ and define three function A_1, A_2, A_3 :

$$A_1 = \Omega \cdot (1-\beta)\sqrt{x(u)-\sqrt{N}}\, \frac{\partial x}{\partial u} \cdot (1-\tilde{\varphi}_\delta(z-1)) \ , \tag{9.30}$$

$$A_2 = \Omega \cdot (1-\beta) \cdot \sqrt{x(u)-\sqrt{N}}\, \frac{\partial x}{\partial u} \frac{\partial x}{\partial z} \tilde{\varphi}_\delta(z-1) \ , \tag{9.31}$$

$$A_3 = -\Omega \cdot \frac{1}{\omega_2\left(\frac{x^2(u)}{L}\right)} \cdot \frac{\partial}{\partial z} \left\{\omega_2\left(\frac{x^2(u)}{L}\right)\sqrt{x(u)-\sqrt{N}} \cdot \frac{\partial x}{\partial u} \cdot (1-\beta) \cdot \tilde{\varphi}_\delta(z-1)\right\} \tag{9.32}$$

Then as the result of the first summation we have the following representation:

(9.33)
$$\sum_{\nu} t(\nu)\tilde{v}_{\epsilon}(n,\nu,N) =$$

$$= 4\pi i^k \sum_{\nu\leq Q^{1+\alpha}} t(\nu) \int_{-\epsilon}^{\epsilon} \varphi_\epsilon(2\log\tfrac{1}{1+u})\Big\{\int_0^\infty W'(z)J_{k-1}(4\pi\sqrt{\nu}(x(u)-\sqrt{N}))\ A_1 dz +$$

$$+\int_0^\infty W(z)(4\pi\sqrt{\nu}\ J_k(4\pi\sqrt{\nu}(x(u)-\sqrt{N})) - \frac{k-1}{x(u)\sqrt{N}} J_{k-1}(4\pi\sqrt{\nu}(x(u)-\sqrt{N})))\ A_2 dz +$$

$$+\int_0^\infty W(z)J_{k-1}(4\pi\sqrt{\nu}(x(u)-\sqrt{N}))\ A_3 dz\Big\} + O((ML)^{-1}) .$$

Now we have for $0 \leq z \leq \sqrt{\frac{n}{N}}$

(9.34)
$$-\frac{\partial x}{\partial z} >> \sqrt{N}$$

and for $\sqrt{\frac{n}{N}} \leq z$

(9.35)
$$-\frac{\partial x}{\partial z} >> z^{-3/2}N^{-1/4}n^{3/4} .$$

It means we can repeate the considerations what were done in the proof of Proposition 8.3 and integrate by parts any times.

If for some positive α we have in (9.33)

(9.36)
$$\delta = Q^{\alpha}(\nu M)^{-1/2}$$

then only the neighbourhood of the point $z=0$ gives a noticeable contribution to the asymptotic of the first integral (with A_1). Since

$x\big|_{z=0} = (\sqrt{n}+\sqrt{N})(1+u) = \sqrt{n} + \sqrt{N} + O(\epsilon\sqrt{n})$ and

$$\frac{\partial x}{\partial z}\Big|_{z=0} = -2(1+u)\sqrt{N}\,\frac{n+\sqrt{nN}}{(\sqrt{n}-\sqrt{N})^2} \tag{9.38}$$

one can easily check that our integral with A_1 is equal to

$$\frac{1}{8\pi\sqrt{\nu nN}}\cdot\frac{1}{1+u}\cdot\frac{(\sqrt{n}-\sqrt{N})^2}{\sqrt{n}+\sqrt{N}}\cdot W'(z)A_1\Big|_{z=0}(J_{k-1}(4\pi\sqrt{\nu n}+4\pi u\sqrt{\nu}(\sqrt{n}+\sqrt{N}))+ \tag{9.39}$$

$$+ O(\frac{1}{(\nu n)^{1/4}\sqrt{\nu N}}))$$

At the same time the argument of the Bessel functions in the integrals with A_2 and A_3 is near to

$$4\pi\sqrt{\nu}(\sqrt{n}+\sqrt{N})(1-\frac{2z\sqrt{N}}{\sqrt{n}}+\ldots) - 4\pi\sqrt{\nu N} = 4\pi\sqrt{\nu n} - 8\pi z\sqrt{\nu N} + \ldots \tag{9.40}$$

where z is near to 1 and $\nu << Q^{1+\alpha}$ for any $\alpha > 0$.

For this reason we can repeat the same considerations what we have had in Proposition 8.6; it gives the same integrals as in (8.57) (with $c=1$ in our case).

All these integrals are small enough and after the summation over N's we can reject the members with A_2 and A_3 without any loss for the remainder term.

Now the sum of quantities (9.39) over n's may be reduced to one integral (we use the summation formula (4.22) again); as an immediate consequence we come to the estimate (9.27).

§ 10. The integral with the sum $\sum_2$.

Our next problem is the consideration of the integral with the sum $\sum_2$ which is defined by the equality (6.12).

It seems the members of the series for this sum are very different nature than in the considered sum $\sum_1$. But it will be seen later the integral with $\sum_2$ may be expressed in the same terms as in (9.21) with one natural distinction. Namely, in the integrals which define the coefficients of this representation the cutting function $(1-\eta(\frac{\xi_0^2}{\xi^2}))$ will be replaced by $\eta(\xi_0^2/\xi^2)$. So the coefficients in the representation for the sum $\sum_1 + \int \sum_2(s)d\chi(s)$ are defined by the full integrals

$$(10.1) \quad \frac{i\pi}{sh(\pi r)} \int_0^\infty (J_{2ir}(\xi)-J_{-2ir}(\xi))J_{k-1}(\xi)\,\frac{d\xi}{\xi}, \quad 2(\ell-1)\int_0^\infty J_{\ell-1}(\xi)J_{k-1}(\xi)\,\frac{d\xi}{\xi} .$$

But the first integral is 0 and the second one differs from 0 only for $\ell = k$. Since each term in the final representation contains $t_{j,k}(q)$ our main Theorem 1 will be an immediate consequence of this result.

For the beginning we consider the function $\tilde{\varphi}_\epsilon$ in the integral (6.12).

10.1. *The asymptotic formula for* $\hat{\varphi}_\epsilon$.

Proposition 10.1. Let $x_0 = \log \omega$ with a positive $\omega > 1$ and $s = 1/2+it$, $t \in \mathbb{R}$. Then for the integral transform (6.13) we have

$$\hat{\varphi}_\epsilon(s,x_0) \equiv \hat{\varphi}_\epsilon(1/2+it,\log \omega) \tag{10.2}$$

$$= (b(s)\omega^{s-1/2} + b(1-s)\omega^{1/2-s})\Phi(\epsilon t) + O(|s|^{1/2-k}\omega^{-2}\exp(-\sqrt{\epsilon|s|}))$$

where $\phi(\cdot)$ is the Fourier cosine–transform of φ_1,

$$\Phi(x) = \int_{-1}^{1} \cos(x\eta)\varphi_1(\eta)d\eta \ , \tag{10.3}$$

and

$$b(s) = \frac{\Gamma(k)\Gamma(s-1/2)}{2\sqrt{\pi}\ \Gamma(k-1+s)} \ . \tag{10.4}$$

In the addition we have for $x \longrightarrow \pm\infty$

$$|\Phi(x)| << |x|^{-3/4}\exp(-|x|^{1/2}) \ . \tag{10.5}$$

Firstly we can rewrite the definition (6.13) in the form

$$\hat{\varphi}_\epsilon(s,x_0) = \int_{-\epsilon}^{\epsilon} \psi(\log v,s)\varphi_\epsilon(\eta)d\eta \tag{10.6}$$

where for $x_0 = \log \omega$ the variable v is equal to $e^\eta\omega$. Here it is convenient to use the Goursat square transformation for the hypergeometric function in (3.9) (see [8], equality (36) in the section 2.11):

(10.7) $$F(k-1+s,s;k;z) =$$

$$= (1+\sqrt{z})^{-2k+2-2s} F(k-1+s,k-1/2;2k-1;\frac{4\sqrt{z}}{(1+\sqrt{z})^2}) .$$

For the case $\xi = \log v$ it gives

(10.8) $$\psi(\log v,s) = 2^{1-2k} v^{1/2-s} (1-\frac{1}{v^2})^{k-1/2} F(k-1+s,k-1/2;2k-1;1-\frac{1}{v^2}) .$$

Finally, the Kummer relations between the hypergeometric functions of argument z and $1-z$ ([8], subsection 2.10) give the representation

(10.9) $$\psi(\log v,s) = (b(s) v^{s-1/2} F(k-s,k-1/2;3/2-s;\frac{1}{v^2}) +$$

$$+ b(1-s) v^{1/2-s} F(k-1+s,k-1/2;1/2+s;\frac{1}{v^2}))(1-\frac{1}{v^2})^{k-1/2}$$

with $b(s)$ from (10.4). Hence, replacing the hypergeometric functions by their power series expansions and substituting the power series developments of $(1-\frac{1}{v^2})^{k-1/2}$, we find

(10.10) $$\psi(\log v,s) = b(s) \sum_{m=0}^{\infty} \beta_{m,k}(s) v^{s-1/2-2m} + \{\text{the same with } s \longmapsto 1-s\}$$

where the coefficients $\beta_{m,k}(s)$ are bounded by the quantities $(m+1)^B$ with a fixed B uniformly over s on the line $\mathrm{Re}\, s = 1/2$.

The term with $m=0$ gives the term of (10.2) since $\beta_{0,k} = 1$. All other terms give the remainder term of (10.2); it remains to estimate the integrals

$$\phi_m = \int_{-\epsilon}^{\epsilon} e^{(it-2m)\eta}\varphi_\epsilon(\eta)d\eta \tag{10.11}$$

$$= C\int_{-1}^{1} \exp(\epsilon(it-2m)\eta - \frac{1}{1-\eta^2})d\eta \ .$$

For any real t we have $|\phi_m| \leq e^{2m\epsilon}$, so for $|\epsilon t| \leq 1$ we have

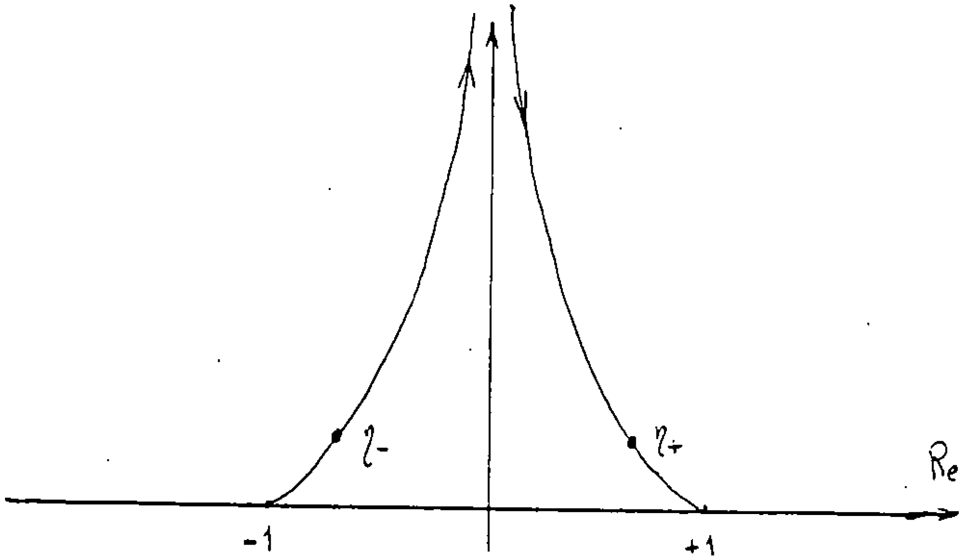
$$\sum_{m\geq 1} |\beta_{m,k}(s)\omega^{-2m}\phi_m| << \omega^{-2} \ . \tag{10.12}$$

If $|t|$ is large in that measure that $\epsilon|t|$ is large also then we can use the same trivial estimate for $m \geq m_0 = A\sqrt{\epsilon|t|}/\log \omega$ with a sufficiently large constant A. So it remains to estimate the integrals ϕ_m for the case $\epsilon t \longrightarrow +\infty$, $m \leq m_0$. Now we shall write the integral ϕ_m in the form

$$\phi_m = \int_{\ell-} e^{i\epsilon t\eta - \frac{1}{2(1+\eta)}} g_-(\eta)d\eta + \int_{\ell+} e^{i\epsilon t\eta - \frac{1}{2(1-\eta)}} g_+(\eta)d\eta \tag{10.13}$$

where $\ell_\pm$ are paths from ± 1 to $+i\infty$ shown in the picture and

$$g_\pm(\eta) = c\exp(-2\epsilon m\eta - \tfrac{1}{2}(1\pm\eta)^{-1}) \ . \tag{10.14}$$



The paths of the integration.

The main contribution to $\int_{\ell\pm}$ comes from the neighbourhood of the saddle points

$$\eta_\pm = \pm\,(1 - \tfrac{1}{2}\sqrt{\tfrac{2}{\epsilon t}}\,e^{\mp\, i\pi/4})$$

where the derivative of the expression

$$i\epsilon t\eta - \tfrac{1}{2}(1 \mp \eta)^{-1}$$

vanishes.

Now by applying the standard formulae of the saddle–point method we see that the sum of the integrals in (10.14) does not exceed the quantity $O((\epsilon t)^{-3/4}\exp(-\sqrt{\epsilon t} + 2\tau m))$. Hence for ϵt large

$$(10.15) \qquad \Big|\sum_{m\geq 1} \beta_{m,k}(s)\omega^{-2m}\psi_m\Big| << (\epsilon t)^{-3/4}\exp(-\sqrt{\epsilon t})\,\omega^{-2}$$

and the proposition is proved since $|b(1/2 \pm it)| << t^{1/2-k}$.

10.2. *The coefficients in* (4.35); r *and* ℓ *are large.*

As it follows from the inequality

$$|\hat{\varphi}_\epsilon(s,x_0)| << |s|^{1/2-k}\exp(-\sqrt{\epsilon|s|}) , \quad \mathrm{Re}\, s = 1/2 ,$$

the essential part for all integrals with this function in (5.24) is defined by the interval $|s| << \epsilon^{-1}\log^2 \frac{1}{\epsilon}$.

There is a similar sharp bound for the sums over κ_j and ℓ in the representation (4.35). Firstly the multiple integration by parts will be done and on this way we come to the following estimates for $s = 1/2+it$ with a positive large t .

Proposition 10.2. For any fixed positive integer $m \geq 1$ we have

$$|h_T(r,s)| = |h_L(r,s)| << t^{1-k} \left[\frac{\xi_0^2+1}{r^4}\right]^m , \tag{10.16}$$

$$|h_{T,\ell}(s)| = |h_{L,\ell}(s)| << t^{1-k} \left[\frac{\xi_0^2+1}{\ell^4}\right]^m , \tag{10.17}$$

where $\xi_0^2 = T^{-1}t^2 = L^{-1}t^2$.

It follows from these inequalities it is sufficient to take in our sums only the terms with

$$\kappa_j << Q^{1/4+\alpha} , \quad \ell << Q^{1/4+\alpha}$$

for any fixed $\alpha > 0$ since we can assume $\xi_0^2 << Q \log^2 Q$.

To give (10.16) and (10.17) we use the following simple fact which may be checked by the direct differentiation.

Lemma 10.1. Let Y_1, Y_2 are any solutions of the differential equations

$$Y_1'' = P_1 Y_1\,,\ Y_2'' = P_2 Y_2\,,\ P_1, P_2 \in C^2\,; \tag{10.18}$$

then the product $Y = Y_1 Y_2$ is a solution of the equation

$$(P_1 - P_2)Y = -Y^{(4)} + ((P_1+P_2)Y)'' + (P_1+P_2)Y\,. \tag{10.19}$$

To apply this assertion to our case we use the representation (4.30) in the following form (the change of the variable $y \longmapsto \mathrm{cth}^2\,\xi/2$ is done so that $\psi(\log \frac{\sqrt{y+1}}{\sqrt{y-1}}, s) = \psi(\xi, s))$:

$$h_L(r,s) = \int_0^\infty \psi(\xi,s)(v(\xi;s,r) + v(\xi;s,-r))\eta((L\,\mathrm{th}^2\xi/2)^{-1})\frac{d\xi}{\mathrm{sh}^2\xi/2}\,, \tag{10.20}$$

where

$$v(\xi;s,r) = \tag{10.21}$$

$$= \gamma(t,r)(\mathrm{th}\,\xi/2)^{1/2+2ir}(\mathrm{ch}\,\xi/2)^{1-2s}F(\tfrac{k-1}{2}+s+ir, \tfrac{1-k}{2}+s+ir; 1+2ir; \mathrm{th}^2\,\tfrac{\xi}{2})$$

(10.22) $$\gamma(t,r) = \frac{\pi}{4} i^{k+1}(2\pi)^{1-2s} \cdot \frac{\Gamma(\frac{k-1}{2}+s+ir)}{sh(\pi r)\Gamma(1+2ir)\Gamma(\frac{k+1}{2}-s-ir)}$$

The function v is the solution of the differential equation

(10.23) $$\frac{d^2v}{d\xi^2} + (t^2 + \frac{r^2}{sh^2\xi/2} + \frac{1}{4\ sh^2\xi} + \frac{(k-1)^2}{4\ ch^2\xi/2})\ v = 0$$

and the similar equation we have for $\psi(\xi,s)$.

Applying (10.19) we have the differential equation for the product $\psi \cdot v$ with the difference of the potentials

$$P_1 - P_2 = (sh\ \xi/2)^{-2}(r^2 + \frac{(k-1)^2}{4}) \ .$$

Now let D be the differential operator which acts as

(10.24) $$Df = -((2\ sh\ \xi/2)^4 f)^{(4)} + (P_1+P_2)((2sh\ \xi/2)^4 f)'' +$$

$$+ (((2sh\ \xi/2)^4(P_1+P_2)f)'' \ .$$

Here

(10.25) $$P_1 = -t^2 + \frac{(k-1)^2}{4sh^2\xi/2} - \frac{1}{4sh^2\xi} - \frac{(k-1)^2}{4ch^2\xi/2} \ ,$$

$$P_2 = -t^2 - \frac{r^2}{sh^2\xi/2} - \frac{1}{4sh^2\xi} - \frac{(k-1)^2}{4ch^2\xi/2} \ .$$

As a consequence (10.19) we have for any $m \geq 1$

$$h_L(s,r) = (4r^2+(k-1)^2)^{-2m} \int_0^{\infty} \psi(\xi,s)(v(\xi;s,r) + v(\xi;s,-r))D^m f\, d\xi \tag{10.26}$$

with

$$f = (\mathrm{sh}\ \xi/2)^{-2}\eta((L\ \mathrm{th}^2\xi/2)^{-1}) \ . \tag{10.27}$$

After the change of the variable $\xi \longmapsto L^{-1/2}x$ in (10.26) we have $x << 1$ and with

$$p(x,L) = (2\sqrt{L}\ \mathrm{sh}\ \frac{x}{2\sqrt{L}})^4 = x^4 + \frac{1}{12L}x^6 + \dots$$

our differential operator D has the form

$$D= -\left[\frac{d}{dx}\right]^4 p(x,L)-\left[\frac{2t^2}{L} + \frac{4r^2}{x^2} + \dots\right]\left[\frac{d}{dx}\right]^2 p(x,L)- \frac{d^2}{dx^2}(p(x,L)\left[\frac{2t^2}{L} + \frac{4r^2}{x^2} + \dots\right]) \tag{10.28}$$

Now one can easily see that the result of the differentiation in (10.26) may be estimated as

$$D^m f << L\, x^{-2}(\frac{t^2}{L} + r^2)^m \ . \tag{10.29}$$

It rests to use the simple estimates

$$|\psi(\xi,s)| << \min(\sqrt{\xi},t^{1/2-k}) \ , \tag{10.30}$$

(10.31) $$|v(\xi;s,r)| << \min(\sqrt{\xi/r}, t^{-1/2})$$

for t large; then it follows from (10.26)

(10.32) $$|h_L(s,r)| << t^{1-k_r-4m}(\frac{t^2}{L}+r^2)^m << t^{1-k_r-4m}(Q\log^2Q+r^2)^m \ .$$

The same considerations give the second inequality (10.17).

10.3. *The coefficients in* (4.35); r *and* ℓ *are small.*

The equation (10.23) is near to

(10.33) $$Y'' + (t^2 + \frac{4r^2+1/4}{\xi^2})Y = 0 \ ,$$

so any solution of this equation must be near to a combination of the function $\sqrt{\xi}\, J_{2ir}(t\xi)$ and $\sqrt{\xi}\, J_{-2ir}(t\xi)$.

Let

(10.34) $$Y = \sqrt{\xi/2} \cdot (\mathrm{sh}(\pi r))^{-1}(J_{2ir}(t\xi) - J_{-2ir}(t\xi))$$

and the coefficients A_n, B_n are defined by the recurrent relations

(10.35) $$A_0 = 1,\ A_n' = -\tfrac{1}{2}B_n'' + \tfrac{1}{2}f\,B_n\ ,\ A_n(0) = 0 \ \text{for}\ n \geq 1 \ ,$$

with

$$f = r^2\left(\frac{4}{\xi^2} - (\mathrm{sh}\ \xi/2)^{-2}\right) + \frac{1}{4}\left(\frac{1}{\xi^2} - \frac{1}{\mathrm{sh}^2\xi}\right) - \frac{(k-1)^2}{4\mathrm{ch}^2\xi/2}$$

and for $n \geq 0$

(10.36) $$B_n = \frac{1}{2}(A_n'' - f\,A_n) + \frac{16r^2+1}{4\xi}\left[\frac{B_n}{\xi}\right]', \quad B_n(0) = 0\ .$$

Proposition 10.3. For every integer $m \geq 1$ there is the solution v_0 of the differential equation (10.23) such that for all $\xi \geq 0$

(10.37) $$v_0 = Y \sum_{n=0}^{m} t^{-2n}A_n + Y' \sum_{n=1}^{m} t^{-2n}B_n + O\left(t^{-2m-2}\min\left(\sqrt{\xi}, \frac{1}{r}, \frac{1}{\sqrt{t}}\right)\cdot(r^2+1)^m\right)$$

It is the well known fact for the equations of this type.

Proposition 10.4. For t large and $r^4 << t$ we have

(10.38) $$v(\xi;s,r) + v(\xi;s,-r) = \gamma_0(t,r)v_0(\xi;t,r)$$

where

(10.39) $$\gamma_0(t,r) = \pi e^{i\phi(t)}\left(1 + O\left(\frac{r^2+1}{t}\right)\right), \quad \phi(t) = 2t\log\frac{t}{2\pi} - 2t\ .$$

It follows from the comparison both solutions at $\xi = 0$ and the Stirling expansion for the gamma–function.

By the same way we obtain

Proposition 10.5. Let $y_\ell(\xi)$ be the solution of the differential equation

(10.40) $$\frac{d^2y_\ell}{d\xi^2} + \left(t^2 - \frac{(\ell-1)^2}{4sh^2\xi/2} + \frac{1}{4sh^2\xi} + \frac{(k-1)^2}{4ch^2\xi/2}\right)y_\ell = 0$$

which has the asymptotic expansion

(10.41) $$y_\ell = \sqrt{\xi/2}\, J_{\ell-1}(t\xi) \sum_{n\geq 0} \frac{\tilde{A}_n}{t^{2n}} + (\sqrt{\xi/2}\, J_{\ell-1}(t\xi))' \sum_{n\geq 1} \frac{\tilde{B}_n}{t^{2n}}$$

where $\tilde{A}_n$, $\tilde{B}_n$ are defined by (10.35) and (10.36) with $r = \frac{i(\ell-1)}{2}$.

Then for all $\xi \geq 0$ we have

(10.42) $$\frac{1}{2}\sin(\pi s)(2\pi)^{1-2s}\frac{\Gamma(\frac{\ell-k}{2}-1+s)\Gamma(\frac{\ell-k}{2}+s)}{\Gamma(\ell)} v_\ell(cth^2\xi/2,s) =$$

$$= \frac{\pi}{2} i^{\ell-1} e^{i\phi(t)}(1 + O(\tfrac{\ell^2}{t}))y_\ell(\xi)\,, \quad \phi(t) = 2t\log\frac{t}{2\pi} - 2t\,.$$

As the special case of the same expansions we have (for $\ell = k$)

(10.43) $$\phi(\xi,s) = \Gamma(k)t^{1-k}\Big\{\sqrt{\xi/2}\, J_{k-1}(t\xi)\sum_{n\geq 0}\frac{\tilde{A}_n}{t^{2n}} +$$

$$+ (\sqrt{\xi/2}\, J_{k-1}(t\xi))' \sum_{n\geq 1}\frac{\tilde{B}_n}{t^{2n}}\Big\}\,.$$

10.4. *The shortened functional equations.*

The Rankin series (more precisely, the products $\zeta(2s)R_j(s)$ and $\zeta(2s)R_{j,\ell}(s)$) have been integrated over s at the concluding step. For this purpose it is necessary to represent our Rankin series by finite part of the corresponding Dirichlet series.

We introduce the new "cutting" function α ; this function from $C^\infty(0,\infty)$ is 0 on the halfline $x \geq b$ for some $b > 1$ and $\alpha(x) \equiv 1$ for small x . It is convenient to assume

$$\alpha(x) + \alpha(\tfrac{1}{x}) \equiv 1 , \quad x > 0 .$$

For the definiteness we assume $\alpha(x) \equiv 0$ for $x \leq 3/4$ (and hence $\alpha(x) \equiv 0$ for $x > 4/3$) and we suppose α be monotonic. The circumstance of no small importance is the fact that for our aims it is sufficient to consider only the case $\kappa_j << L^{\delta_1}$, $\ell << L^{\delta_1}$ for some fixed (and small) δ_1 and $t >> \sqrt{L}$. There are infinitely many forms of the shortened functional equation for the Hecke series (many years ago the similar thought was expressed by Gelfond for the case of the Riemann zeta–function). It seems, the more suitable are those where the "cutting" function will be an infinitely smooth function with a bounded support. To give an acceptable form we shall introduce some additional notations.

Let B_n be the n–th Bernoulli polynomial ($B_0 \equiv 1$, $B_1(x) = x-1/2$, $B_2(x) = x^2-x+1/6$,...) and γ be a positive number with $\gamma < 1$. We define the polynomial $b_n(u,\gamma)$ in u (and γ will be a parameter) by the recurrent relation

$$(10.45) \qquad b_{n+1}(u,\gamma) = \sum_{m=0}^{n} \left(1 - \frac{m}{n+1}\right)\tilde{B}_{n-m+2}(u,\gamma) b_m(u,\gamma) , \quad b_0 \equiv 1 ,$$

where for $n \geq 2$

$$(10.46)\ \tilde{B}_n(u,\gamma) = \frac{(-n)^n}{n(n-1)}\left((1-\gamma)^{-n+1}+(1+\gamma)^{-n+1}\right)\left(i^{n-1}B_n(\tfrac{k}{2}+u)-i^{-n+1}B_n(\tfrac{k}{2}-u)\right)$$

Finally, let for $z > 0$

$$(10.47) \qquad \alpha_n(z,\gamma) = b_n(-z\frac{\partial}{\partial z},\gamma)\alpha(z) \ .$$

Lemma 10.2. Let $\mu_j(n)$ be the n–th coefficient of the Dirichlet series for $\zeta(2s)R_j(s)$,

$$(10.48) \qquad \mu_j(n) = \sum_{d^2|n} t(\tfrac{n}{d^2})\rho_j(\tfrac{n}{d^2}) \ .$$

Let x, y be the positive numbers with the conditions $x,y >> 1$ and

$$(10.49) \qquad xy = \frac{(t^2-\kappa_j^2)^2}{(2\pi)^4} \ .$$

Then for t large and $\kappa_j = \overset{=}{0}(t)$ we have for any fixed $m \geq 2$

$$(10.50) \qquad \zeta(2s)R_j(s) = \sum_{n\geq 1} n^{-s}\mu_j(n)\alpha(\tfrac{n}{x}) -$$

$$- e^{-i\chi(t,\kappa_j)}\sum_{n\geq 1} n^{-1+s}\mu_j(n)\beta_m(\tfrac{n}{y};\tfrac{\kappa_j}{t},t) + O(y^{1/2}t^{-m+1}|\rho_j(1)|)$$

where $s = 1/2+it$, for $0 < r < t$

(10.51) $$\chi(t,r) = 2t \log \frac{t^2-r^2}{4\pi^2} + 2r \log \frac{t+r}{t-r} - 4t \quad ,$$

and with α_ℓ from (10.47) for $x > 0$, $0 < \gamma < 1$

(10.52) $$\beta_m(x,\gamma,t) = \sum_{\ell=0}^{m} t^{-\ell} \alpha_\ell(x,\gamma) \quad .$$

Let s be in the half–plane $\operatorname{Re} s > 1$. Then for any $x > 0$ we have (since $\alpha(x) + \alpha(\frac{1}{x}) \equiv 1$)

(10.53) $$\zeta(2s)R_j(s) = \sum_{n \geq 1} n^{-s} \mu_j(n) \alpha(\tfrac{n}{x}) + \sum_{n \geq 1} n^{-s} \mu_j(n) \alpha(\tfrac{x}{n})$$

The first sum contains $<< x$ terms. The second sum may be written in the form

(10.54) $$\frac{1}{2\pi i} \int_{(\delta)} \zeta(2s-2u) R_j(s-u) \hat{\alpha}(u) x^{-u} du \; , \quad \delta > 0$$

where $\hat{\alpha}(u)$ is the Mellin transform of α ,

(10.55) $$\hat{\alpha}(u) = \int_0^{\infty} \alpha(x) x^{u-1} dx \quad ,$$

and δ is taken so small that $\operatorname{Re} s - \delta > 1$.

It is clear from the definition of α that $\hat{\alpha}(u)$ is the regular function in the half–plane $\operatorname{Re} u > 0$. Further,

$$\hat{\alpha}(u) = -\frac{1}{u}\int_0^\infty \alpha'(x)x^u dx \ ,$$

so $u\hat{\alpha}(u)$ is the entire function. If $u \longrightarrow \infty$ with a fixed value of the real part of u , then for any fixed $B > 1$ we have

$$(10.56) \qquad |\hat{\alpha}(u)| << |u|^{-B} \ , \ \text{Re}\, u = \text{const} \ , \ u \longrightarrow \infty \ .$$

So we can integrate in (10.54) on any line $\text{Re}\, u = \sigma$ and this integral represents the second sum on the right side (10.53) for any s .

In the case $\text{Re}\, s = 1/2$ we integrate on the line $\text{Re}\, u = 1/2+\delta$ with some (small) $\delta > 0$. Using the functional equation (4.10) we come to the representation for this integral

$$(10.57) \quad \frac{(2\pi)^{4s-2}}{2\pi i} \int_{(1/2+\delta)} \frac{\Gamma(z)\Gamma(z')}{\Gamma(k-z)\Gamma(k-z')} \zeta(2-2s+2u)R_j(1+u-s)\left[\frac{(2\pi)^4}{x}\right]^u \hat{\alpha}(u)du$$

where $z = \frac{k+1}{2} - s+u+i\kappa_j$, $z' = \frac{k+1}{2} - s+u-i\kappa_j$. Now for $s = 1/2 + it$ with a positive t we have on the line $\text{Re}\, u = 1/2+\delta$, $\delta > 0$,

$$\text{Re}(1+u-s) > 1 \ .$$

For this reason we can replace the function $\zeta(2-2s+2u)R_j(1-s+u)$ by the absolutely convergent Dirichlet series, after this it will be sufficient to calculate the arising integrals. The following form of the Stirling expansion will be convenient here:

$$(10.58) \qquad \log \Gamma(z+v) = (z+v-1/2)\log z - z + \frac{1}{2}\log(2\pi) +$$

$$+\frac{B_2(v)}{1\cdot 2}z^{-1}+\ldots+\frac{(-1)^{n+1}B_{n+1}(v)}{n(n+1)}z^{-n}+O(|\tfrac{v}{z}|^{n+1})$$

where B_n is the n–th Bernoulli polynomial. Uisng this asymptotic series with $z = \pm i(t \pm r)$ and $v = \frac{k}{2} \pm u$ we obtain the equality

$$\frac{\Gamma(\frac{k}{2}-i(t+r)+u)\Gamma(\frac{k}{2}-i(t-r)+u)}{\Gamma(\frac{k}{2}+i(t+r)-u)\Gamma(\frac{k}{2}+i(t-r)-u)}= \tag{10.59}$$

$$=\exp(-i\chi(t,r)-i(k-1)\pi+\sum_{n\geq 1}\frac{(-1)^{n+1}}{n(n+1)}t^{-n}(\frac{1}{(1+\gamma)^n}+\frac{1}{(1+\gamma)^n})(i^nB_{n+1}(\tfrac{k}{2}+u)$$

$$-i^{-n}B_{n+1}(\tfrac{k}{2}-u)))\cdot(t^2-r^2)^{+2u}$$

Because k is an even integer we have $\exp(-i(k-1)\pi) = -1$ here; so it rests to write

$$\exp(\sum_{n\geq 1} t^{-n}b_{n+1}(n,\gamma))$$

with a given polynomial b_n in the asymptotic form

$$\sum_{n\geq 0} t^{-n}\,\tilde{b}_n(n,\gamma)$$

where $\tilde{b}_n$ are some polynomials in u . It is easy to do this by the comparison of the logarithmic derivatives for both sides. Finally, for any polynomial $P(u)$ we have for $x > 0$

$$\frac{1}{2\pi i}\int P(u)\,\hat{\alpha}(u)x^{-u}du = P(-x\frac{\partial}{\partial x})\,\alpha(x) \tag{10.60}$$

and the union of these formulae gives the assertion (10.50).

We shall use the shortened functional equation (10.50) later in the situation when κ_j is very small in the comparison with t. In this case

$$(10.61) \qquad \chi(t,r) = 4t \log \frac{t}{2\pi} - 4t + \frac{2r^2}{t} + O\left(\frac{r^4+1}{t^2}\right)$$

and, taking in (10.50) $m=1$, $x = y = \frac{t^2}{4\pi^2}$, we obtain the representation

$$(10.62) \qquad \zeta(1+2it)R_j(1/2+it) =$$

$$= 2ie^{-2it \log \frac{t}{2\pi}+2it-\frac{r^2}{t}} \sum_{n\le t^2/4\pi^2} \frac{\mu_j(n)}{\sqrt{n}}\, \alpha\left[\frac{4\pi^2 n}{t^2}\right] \sin\left[2t \log \frac{t}{2\pi\sqrt{n}} - 2t + \frac{r^2}{t}\right] -$$

$$- \frac{i}{t}\, e^{-4it \frac{t}{2\pi} + 4it} \sum_{n\le t^2/4\pi^2} \frac{\mu_j(n)}{n^{1/2-it}}\, \tilde{\alpha}\left[\frac{4\pi^2 n}{t^2}\right] + O\left[\frac{r^4+1}{t}\right]$$

where r is written on the right side instead of κ_j and

$$(10.63) \qquad \tilde{\alpha}(x) = ((k-1)^2 - \tfrac{1}{3})\alpha(x) + 4x\alpha'(x) + 4x^2\alpha''(x) \ .$$

Of course, the shortened functional equation of the same kind holds for the Rankin series $R_{j,\ell}(s)$. By the same way as in the preceding case one can write the following representation.

Lemma 10.3. Let the real positive t be large, the even integer ℓ be small in the sense that $\ell^4 = O(t)$ and k be a fixed even integer. Then for $s = 1/2+it$

$$\zeta(2s)R_{j,\ell}(s) = 2i \exp\left(-i\psi(t) + \frac{i(\ell-1)^2}{4t}\right) \cdot \tag{10.64}$$

$$\cdot \sum_{n \le t^2/4\pi^2} \frac{\mu_{j,\ell}(n)}{\sqrt{n}} \alpha\left[\frac{4\pi^2 n}{t^2}\right] \sin\left[2t \log \frac{t}{2\pi\sqrt{n}} - 2t - \frac{(\ell-1)^2}{4t}\right] -$$

$$- \frac{i}{t} \exp(-2i\psi(t)) \sum_{n \le \frac{t^2}{4\pi^2}} \frac{\mu_{j,\ell}(n)}{\sqrt{n}} \cdot n^{it} \tilde{\alpha}\left[\frac{4\pi^2 n}{t^2}\right] + O\left[\frac{\ell^4+1}{t^2}\right] ,$$

where $\tilde{\alpha}$ is the same what was occured in (10.63) and in the accordance with (4.2)

$$\mu_{j,\ell}(n) = \sum_{d^2|n} t\left(\frac{n}{d^2}\right) t_{j,\ell}\left(\frac{n}{d^2}\right) . \tag{10.65}$$

10.5. *The first summation.*

Here we consider the sum

$$z(N,t) = \sum_{\nu} t(\nu)\left(\frac{L}{\nu}\right)^{1/4} \omega\left[\frac{(\sqrt{\nu}+\sqrt{N})^2}{L}\right] \hat{\varphi}_\epsilon(s, 2\log(\sqrt{\nu}+\sqrt{N})) , \quad \omega = \omega_2 , \tag{10.66}$$

$$s = 1/2+it .$$

As it was earlier we replace this sum (using the identity (4.22)) by the short sum of the integrals

$$(10.67)\ 4\pi i^k \sum_{\nu \le Q \log^2 Q} t(\nu) \int_0^\infty J_{k-1}(4\pi x\sqrt{\nu})\hat{\varphi}_\epsilon(s, 2\log(x+\sqrt{N}))\,\omega\!\left(\frac{(x+\sqrt{N})^2}{L}\right)\sqrt{x}\,dx\ ,$$

$$Q = \frac{1}{\epsilon^2 L}\ .$$

The integrals in this sum may be calculated with ease by the standard method of the stationary phase (see [13]) and after this we obtain the same sums over m's what were considered in § 9. The certain time is needed to remain the more terms in the asymptotic expansions than it is required for the final result. But after the integration over t all terms in these expansions, excepting the main ones, may be rejected without any loss for the remainder term. The main terms for $\hat{\varphi}_\epsilon$ are equal to

$$(10.68)\qquad \frac{i^k\Gamma(k)}{\sqrt{\pi}}\, t^{1/2-k}\Phi(\epsilon t)\cos(2t\log(x+\sqrt{N}) + \tfrac{\pi}{4})$$

(we use the Stirling expansion) and the integrals in (10.67) may be written as

$$(10.69)\quad \frac{i^k\Gamma(k)}{\sqrt{\pi}}\, t^{1/2-k}\Phi(\epsilon t)\mathrm{Re}\Big\{\int_0^\infty J_{k-1}(4\pi x\sqrt{\nu})e^{2it\log(x+\sqrt{N})+i\pi/4}\,\mathcal{E}(t,x)\ \cdot$$

$$\cdot\ \omega\left(\frac{(x+\sqrt{N})^2}{L}\right) L^{1/4}\sqrt{x}\,dx\Big\}$$

where here and later $\mathcal{E}$ denotes an asymptotic series in t^{-1} (not necessary the same for other cases) with the main term 1. The main property what is sufficient for our purposes is the possibility to integrate by parts any times (really four times is sufficient); it is ensured by the inequalities

(10.70) $$(\frac{\partial}{\partial t})^R \gamma << t^{-R}, \ (\frac{\partial}{\partial x})^R \gamma << x^{-R}$$

which hold for every fixed integer $R \geq 1$.

Now the asymptotic calculation of the integrals in (10.69) which is reduced to the substitution of the special values of the variables in the standard formulae gives the following representation for the sum (10.67).

Proposition 10.6.

(10.71) $$z(N,t)=i^k\sqrt{2}\,\frac{\Gamma(k)}{t^k}\cdot\Phi(\epsilon t)\cdot \mathrm{Re}\Big\{\sum_{\nu} t(\nu)\cdot\Big[\frac{t}{2\pi\sqrt{\nu}}\Big]\cdot\exp(i\phi_\nu(t)+4\pi i\sqrt{\nu N}-\frac{i\pi}{4})$$

$$\omega_2\Big[\frac{t^2}{4\pi^2\nu L}\Big](\frac{L}{\nu})^{1/4}\gamma(t,x_\nu)\Big\}$$

where

(10.72) $$\phi_\nu(t) = 2t\log\frac{t}{2\pi\sqrt{\nu}} - 2t\ , \ \ x_\nu = \frac{2\pi\sqrt{\nu N}}{t}$$

and γ is the asymptotic series in t^{-1} with the main term 1; we have for any fixed $R \geq 1$

(10.73) $$(\frac{\partial}{\partial t})^R \gamma << t^{-R}, \ (\frac{\partial}{\partial x})^R \gamma\,(t,x) << x^{-R} .$$

10.6. *The second summation.*

Now we have the sum over N's which is almost identical with the sums what we considered in 9.2. The members with $\cos(4\pi\sqrt{\nu N}+\pi/4)$ are the same nature and the sums with sine are smaller any power of M^{-1}. So all details may be omitted and we have

Proposition 10.7. Let $\tilde{t}(m)=t_j(m)$, $\tau_{1/2+ir}(m)$ or $t_{j,\ell}(m)$; then

(10.74)
$$\sum_{(m,q)=1}\tilde{t}(mq)z(mq,t)\omega_1(\tfrac{mq}{M})=$$

$$=\frac{2i^k\Gamma(k)}{t^k}\Phi(\epsilon t)M^{3/4}\sum_{c|q}A_{q,c}\tilde{t}(q)\sum_{\nu}\frac{t(\nu)\tilde{t}(c^2q\nu)}{\sqrt{\nu}}\left[\frac{t}{2\pi\sqrt{\nu}}\right]\operatorname{Im}(e^{i\psi}\nu\tilde{\omega}(t,\nu)\,\gamma(t,\nu))$$

where $A_{q,c}$ are the same as in (9.7),

(10.75)
$$\tilde{\omega}(t,\nu)=L^{1/4}\omega_2\left(\frac{t^2}{4\pi^2\nu L}\right)\int\omega_1(x^2)\sqrt{x}\,dx$$

and γ is the asymptotic series in t^{-1} with the main term 1.

10.7. *The integration over* t.

This integration is an easy and pleasurable work. First of all the remainder term from the shortened functional equations gives $O((ML)^{3/4}\sqrt{Q})$ which is more smaller than our main term.

Further, the oscillating multiplier $\exp(-i\psi(t))$, $\psi(t)=2t\log\frac{t}{2\pi}-2t$, in (10.62) and (10.64) is compensated by the multiplier $\exp(+i\psi(t))$ which results in the coefficients $h_L(r,s)$, $h_{L,\ell}(s)$ from the asymptotic formulae of the functions $v(\xi;s,r)$ and $v_\ell(\xi,s)$.

So we have the integrals

(10.76) $$\int \sin(\phi_n(t))\,{}^{\sin}_{\cos}(\phi_\nu(t))\,t^{3/2}\omega\left[\frac{t^2}{4\pi^2\nu L}\right] \gamma(t)dt$$

where $\phi_\nu(t) = 2t \log \frac{t}{2\pi\sqrt{\nu}} - 2t$, γ is a smooth function with the small derivatives, $\gamma^{(m)} << t^{-m}$. Here L is large and for any $\alpha > 0$ we have $\nu << L^\alpha$. So it is obvious that after integration over t only the terms with $n = \nu$ will be survived; all terms with $n \neq \nu$ give the small contribution to the remainder term and may be rejected.

Furthermore, since ν is very small in the coparison with t , $t \asymp \sqrt{\nu L}$, the cutting function $\alpha(\frac{4\pi^2\nu}{t^2})$ in (10.62) and (10.64) is equal to 1 in the remained integrals.

The explicit form for these integrals comes by the union of (10.74), (10.62) and (10.64), the definitions $h_L(s,r)$, $h_{L,\ell}(s)$ and the corresponding asymptotic formulae (10.38), (10.42) and (10.43); finally, it follows from (5.7) for t large

(10.77) $$d\chi(1/2+it) = \frac{t^{2k-1}}{2\Gamma^2(k)}(1+O(\frac{1}{t^2}))dt$$

Firstly we consider the sum over discrete spectrum. To facilitate the control for the coefficients we write the integrand in the form $\{\ \}_{(n_1)} \cdot \{\ \}_{(n_2)} \cdots$ where $(n_1), (n_2),\ldots$ is the number of the corresonding formula. Now we have under the sign of the summations over $c|q$, ν and κ_j (for the brevity we write r instead of κ_j):

$$(10.78)\ \frac{1}{2}\cdot\int\left\{2i\,\nu^{-1/2}\mu_j(\nu)\right\}_{(10.62)}\cdot\left\{\frac{2i^k\Gamma(k)}{t^k}\,\Phi(\epsilon t)M^{3/4}A_{q,c}t_j(q)t(\nu)t_j(c^2q\nu)\cdot\frac{1}{\sqrt{\nu}}\cdot\right.$$

$$\left.\cdot\left[\frac{t}{2\pi\sqrt{\nu}}\right]\cdot\omega_2\left[\frac{t^2}{4\pi^2\nu L}\right]\cdot L^{1/4}\right\}_{(10.74)}\cdot\left\{\frac{t^{2k-1}}{2\Gamma^2(k)}\right\}_{(10.77)}\cdot$$

$$\cdot\int_0^\infty\left\{\frac{\Gamma(k)}{t^{k-1}}J_{k-1}(\xi)\right\}_{(10.20),(10.43)}\left\{\frac{\pi}{4sh(\pi r)}(J_{2ir}(\xi)-J_{-2ir}(\xi))\right\}_{(10.20),(10,38),(10.34)}\cdot$$

$$\cdot\left\{2\eta\left[\frac{4t^2}{L\xi^2}\right]\frac{d\xi}{\xi}\right\}_{(10.20)}dt$$

Here the coefficient $(1/2)$ before the integral is the mean value of $\sin^2\psi_\nu(t)$ and in the integral (10.20) the change of the variable $\xi\longmapsto t^{-1}\xi$ is done and the function $\eta((L\,th^2\xi/2t)^{-1})$ is replaced by $\eta\left[\frac{4t^2}{L\xi^2}\right]$; further, the variable x comes from (10.38) and we have $x\asymp 1$, $t^{-1}\sqrt{\nu M}<<(M/L)^{1/2}<<Q^{-2}$ (see (8.14)).

Now after the change of the variable $t=2\pi\sqrt{\nu L}\cdot y$ we obtain in (10.78)

$$(10.79)\ 2\pi i^k\int\omega_2(y^2)y\cdot\Phi\left(2\pi y\sqrt{\frac{\nu}{Q}}\right)\left\{\frac{i\pi}{2sh(\pi r)}\int_0^\infty(J_{2ir}(\xi)-J_{-2ir}(\xi))J_{k-1}(\xi)\eta\left(\frac{\xi_0^2}{\xi^2}\right)\frac{d\xi}{\xi}\right\}dy$$

$$\cdot M^{3/4}L^{5/4}A_{q,c}\cdot\frac{t_j(q)t(\nu)t_j(c^2q\nu)\mu_j(\nu)}{\sqrt{\nu}}$$

where $\xi_0^2=4\pi^2\nu y^2$.

Finally,

(10.80) $$\mu_j(\nu) = \sum_{d^2|\nu} t(\frac{\nu}{d^2}) t_j(\frac{\nu}{d^2})$$

and after the replacement ν by νd^2 we obtain the same expression as in (9.21) with the coefficients in the spectral representation

(10.81) $$\frac{i\pi}{2sh(\pi r)} \int_0^\infty (J_{2ir}(\xi) - J_{-2ir}(\xi)) J_{k-1}(\xi) \eta(\frac{\xi_0^2}{\xi^2}) \frac{d\xi}{\xi} .$$

In (9.21) we have the similar integral with $(1-\eta(\frac{\xi_0^2}{\xi^2}))$; so the sum of two representations (for the sum $\sum^{(0)}$ and for the integral with $\sum_2(s)$) is reduced to the sum $Z^{dis} + Z^{con} + Z^{cusp}$ where the coefficients are defined by the integral transformations (2.18) and (2.19) for the function $\varphi(x) = J_{k-1}(x)$ with even $k \geq 12$. The integral (2.18) equals to zero in this case and the integrals (2.19) are zeroes for all $\ell \neq k$; if $\ell = k$ we have

(10.82) $$h_k = \frac{i^k}{2(k-1)} .$$

It gives the following asymptotic formula for the average (6.4) of the left side (5.24).

<u>Proposition 10.8</u>. Let $\epsilon \longrightarrow 0$, M and L are taken in the form (6.6) and $Q = (\epsilon^2 L)^{-1} >> q^{4+\delta}$ for some positive δ . Then we have

(10.83) $$\sum_{(m,q)=1} \omega_{a_1,b_1}\left(\frac{mq}{M}\right)\sum_{\nu} \omega_{a_2,b_2}\left(\frac{\nu}{L}\right)t(\nu)\Big\{\text{the left side of (5.24) with } N = mq\,,$$

$$T = L \text{ and } \varphi = \varphi_{\epsilon}(x-2\log(\sqrt{\nu}+\sqrt{mq}))\Big\} =$$

$$= C_{k,q}\frac{(LM)^{3/4}}{\epsilon} + \Theta + O\left(\frac{(LM)^{3/4}}{\epsilon}Q^{-2}\right)$$

with

(10.84) $$\Theta = \frac{a_0}{k-1} M^{3/4}L^{5/4}\sum_{j=1}^{\nu_k} \alpha_{j,k}\sum_{c|q} A_{q,c}\sum_{\nu,d}\frac{b(\nu d^2)}{\sqrt{\nu d^2}}$$

$$t(\nu)t(\nu d^2)t_{j,k}(q)t_{j,k}(\nu)t_{j,k}(\nu q c^2 d^2)$$

here the quantities $b(\nu)$, a_0, $A_{q,c}$ and $C_{k,q}$ are defined by the equalities (9.19), (9.22), (9.7) and (7.11) accordingly.

10.8. *The right side of* (5.24).

The right side of (5.24) and Θ in (10.83) are the linear combination of the quantities $t_{j,k}(q)$. So the case $t_{j,k}(q) = 0$ for all j , $1 \leq j \leq \nu_k$, is impossible and our Theorem 1 follows from the asymptotic representation (10.83).

Now we consider the same average of the right side (5.24); it will be expressed in the same explicit form. Really all is ready for the calulation of this average. For the sums over ν's and over m's , $(m,q) = 1$, we have the representation (10.74) (with $\tilde{t}(m) = t_{j,k}(m)$) , the Rankin series is given by (10.64) with $\ell = k$ and instead of $h_{L,k}(s)$ we have the multiplier

(10.85) $$\frac{\Gamma(s)(2\pi)^{-2s}}{\Gamma(k-s)} = \frac{(it)^{1-k}}{2\pi} e^{i\phi(t)}(1+O(\tfrac{1}{t})), \quad \phi(t) = 2t \log \frac{t}{2\pi} - 2t$$

The simple calculation of the coefficients gives 4Θ for this integral over t. So it rests to estimate the double average for the first term from the right side (5.24). Using the representations (10.71) for the sum over ν's and estimating by modulus the result of the summation over m's we obtain

Proposition 10.9.

(10.86) $$\sum_{(m,q)=1} \omega_1(\tfrac{mq}{M})t(mq) \sum_{\nu} \omega_2(\tfrac{(\sqrt{\nu}+\sqrt{N})^2}{L})t(\nu)$$

$$\mathrm{Re}\Big\{ \int_{(1/2)} \frac{\Gamma(s)\,\zeta(2s)}{(2\pi\sqrt{N})^{2s}\Gamma(k-s)} \hat{\varphi}_\epsilon(s,x_0)d\chi(s)\Big\} =$$

$$= O(M^{1/2}(LQ)^{5/4}) \frac{|t(q)|}{q^{3/2}}$$

Here

$$M^{1/2}(LQ)^{5/4} = \frac{(ML)^{3/4}}{\epsilon} \cdot Q^{3/4} \cdot M^{-1/4}$$

and this term is smaller than the remainder term in (10.83).

We conclude this subsection by the last

Proposition 10.10. Let Θ be defined by (10.84); then

(10.87) $$\Theta = \frac{1}{3} C_{k,q} \frac{(ML)^{3/4}}{\epsilon} (1 + O(Q^{-\alpha})) \ .$$

§ 11. The cases k = 12,16,18,20,22 and 26 ; the estimate from below.

For all enumerated cases we have $\nu_k = 1$ in (10.84) so we can omit the indexes j, k in the notations for the eigenvalues of the Hecke operators. Of course now we can forget a lot of the parameters and remain the large parameter Q and two arbitrary function φ and ω.

The first function is an even from $C^{\infty}(-\infty,+\infty)$ for which the support is the symmetrical interval $(-a,a)$ for some $a > 0$. Our identity (10.87) is proved only for the realization (6.3); nevertheless it is obviously enough that we can take other models of the similar function.

Let φ be such a function; we define

(11.1) $$\Phi(x) = \int_{-\infty}^{\infty} \varphi(y)\cos(xy)dy$$

and note that

(11.2) $$\varphi(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(x)dx \ .$$

The second function ω has the support which is strongly separated from zero; of course we assume $\omega \in C^{\infty}(0,\infty)$; we note again that (10.87) is proved for the specialization of the kind (4.31) but we can take any function with the similar properties.

It is known for any $f \in \mathcal{M}_k$ the explicit form for the residue of the corresponding Rankin series:

$$\frac{\Gamma(k)}{(4\pi)^k} \, |a(1)|^2 \mathrm{Res}\Big[\sum_{n=1}^{\infty} \frac{t^2(n)}{n^s} \Big]_{s=1} = \frac{3}{\pi} \, \|f\|^2 \; . \tag{11.3}$$

Finally, assume q be a prime. This q was almost fixed in our considerations; nevertheless if our estimates will be examined then one can see that is is allowable to take the growing q under the condition $q << Q^{1/4+\epsilon}$ for an arbitarary small fixed $\epsilon > 0$.

Now we can reformulate the assertion (10.87) in the following form.

Lemma 11.1. Let $\varphi \in C^{\infty}(-\infty,\infty)$ be an even function with a bounded support $(-a,a)$ and Φ is the Fourier cosine–transformation of φ . Let for $X > 0$ and for q prime

$$\mathcal{G}_q(X;\Phi) = t(q) \sum_{\nu,d} \frac{t^2(\nu)\, t(\nu d^2)}{\sqrt{\nu d^2}} \left(t(\nu q d^2) - \frac{1}{q}\, t(\nu q^3 d^2)\right) \Phi\Big(2\pi \sqrt{\frac{\nu d^2}{X}}\Big) \tag{11.4}$$

where $t(n)$, $n = 1,2,\ldots$, are the eigenvalues of the Hecke operators in $\mathcal{M}_k$ and is assumed $\dim \mathcal{M}_k = 1$ (i.e. $k = 12,16,18,20,22$ or 26). Now let $\omega \in C^{\infty}(0,\infty)$ has a bounded support which is strongly separated from zero. Then for $Q \longrightarrow +\infty$ and for $q << Q^{1/4+\epsilon}$ with a sufficiently small fixed $\epsilon > 0$ we have

$$\int_0^{\infty} \omega(y) \Big\{ \frac{y}{\sqrt{Q}} \, \mathcal{G}_q\Big(\frac{Q}{y^2};\Phi\Big) \Big\} dy = \frac{1}{q} \, \gamma_k \int_0^{\infty} \omega(y) dy \cdot \int_{-\infty}^{\infty} \Phi(x) dx + O(Q^{-1/2}) \tag{11.5}$$

where for k with the condition $\dim \mathcal{M}_k = 1$

$$\gamma_k = \frac{8(4\pi)^{2k-3}}{\Gamma(k)\Gamma(k-1)} \|f\|^2 , \quad f = \sum_{n=1}^{\infty} n^{(k-1)/2} t(n) e(nz) \quad . \tag{11.6}$$

The equality (11.5) is a reformulation of (10.87) for the case dim $\mathcal{M}_k = 1$ (we replaced $\omega_{a_2,b_2}(y^2)$ by $\omega(y)$); it means that in the average

$$\frac{1}{\sqrt{X}} \mathcal{G}_q(X;\phi) \sim \frac{1}{q} \gamma_k \int_{-\infty}^{\infty} \Phi(x)dx \tag{11.7}$$

if X is large enough.

As a consequence (11.5) we obtain the estimate from below for the eigenvalues of the Hecke operators.

Theorem 11.1. Let q be a prime, $q \longrightarrow \infty$; let $q^{(k-1)/2}t(q)$ be the q–th Fourier coefficient for the cusp form in $\mathcal{M}_k$ and dim $\mathcal{M}_k = 1$. Then for any positive $\epsilon > 0$ we have

$$|t(q)| >> q^{-1-\epsilon} \tag{11.8}$$

Of course it is the immediate consequence of the asymptotic formula (11.7). Since for every $\delta > 0$ we have $|t(\nu)| << \nu^{\delta}$ (the Deligne estimate in the rough form) we can estimate the quantity $\frac{1}{\sqrt{X}} \mathcal{G}_q(X;\Phi)$ by $O(|t(q)| q^{\delta} X^{4\delta})$. At the same time this quantity is $>> \frac{1}{q}$. Since we can take $X = q^{4+\delta_1}$ for some small $\delta_1 > 0$ the inequality (11.8) follows.

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