Kleinian orbifolds uniformized by $\mathcal{RP}$ groups with an elliptic and a hyperbolic generators

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Abstract

We consider non-elementary Kleinian groups Γ without invariant plane generated by an elliptic and a hyperbolic elements with their axes lying in one plane. We find presentations and a complete list of orbifolds uniformized by such Γ.

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Key words: discrete group, Kleinian group, hyperbolic geometry, hyperbolic orbifold.

1 Introduction

This work is a part of the program on description of all 2-generator Kleinian groups with real parameters. We study $\mathcal{RP}$ groups, that is, 2-generator subgroups of $\mathrm{PSL}(2,\mathbb{C})$ with real parameters β, β', and γ (see Section 2 for exact definitions). Since discreteness questions were answered for elementary groups and groups with invariant hyperbolic plane (in particular, all Fuchsian groups were described), we concentrate on the non-elementary $\mathcal{RP}$ groups without invariant plane, which we call *truly spatial* $\mathcal{RP}$ groups.

This paper deals with the most complicated and filled with discrete groups case of $\mathcal{RP}$ groups with one generator elliptic and the other one hyperbolic. 'Truly spatial' for this class means that the elliptic generator is not a half-turn and the axes of the generators either (1) are disjoint lines lying in a hyperbolic plane or (2) intersect non-orthogonally at a point of $\mathbb{H}^3$. In terms of parameters, we have here $\beta \in (-4,0)$, $\beta' \in (0,\infty)$, and γ for (1) and (2) belongs to the intervals $(-\infty,0)$ and $(0,-\beta\beta'/4)$, respectively [9, Theorem 1 and Table 2]. In the previous papers [7, 9, 10] necessary and sufficient conditions for discreteness of all such groups were found constructively. Here we use the construction (we reproduce it in Section 3) to determine fundamental sets, presentations, and orbifolds for all truly spatial discrete $\mathcal{RP}$ groups with an elliptic and a hyperbolic generators (Section 4).

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Note that throughout this paper f is elliptic and g is hyperbolic. The other cases of f and g with real traces that generate a truly spatial $\mathcal{RP}$ group and the question when the group is discrete were investigated in earlier papers [6, 8, 11]. The final results including the results of the present paper are collected in [12] (mostly without proofs), where parameters, presentations and orbifolds for *all* truly spatial discrete $\mathcal{RP}$ groups *with real traces of the generators* are given.

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2 Preliminaries

Definitions and notation

We identify $\mathrm{PSL}(2,\mathbb{C})$ with the full group of orientation preserving isometries of hyperbolic 3-space $\mathbb{H}^3$.

A two-generator subgroup $\Gamma = \langle f, g\rangle$ of $\mathrm{PSL}(2,\mathbb{C})$ is determined up to conjugacy by its parameters $\beta = \beta(f) = \mathrm{tr}^2 f - 4$, $\beta' = \beta(g) = \mathrm{tr}^2 g - 4$, and $\gamma = \gamma(f,g) = \mathrm{tr}[f,g] - 2$ whenever $\gamma(f,g) \neq 0$ [3].

The class of $\mathcal{RP}$ *groups* (two-generator groups with real parameters) is defined as follows:

$$\mathcal{RP} = \{\Gamma \,|\, \Gamma = \langle f, g\rangle \text{ for some } f, g \in \mathrm{PSL}(2,\mathbb{C}) \text{ with } \beta, \beta', \gamma \in \mathbb{R}\}.$$

Note that the requirement of discreteness is not included in the definition of $\mathcal{RP}$ groups.

We recall that an element $f \in \mathrm{PSL}(2,\mathbb{C})$ with real $\beta = \beta(f)$ is *elliptic*, *parabolic*, *hyperbolic*, or π-*loxodromic* according to whether $\beta \in [-4, 0)$, $\beta = 0$, $\beta \in (0, +\infty)$, or $\beta \in (-\infty, -4)$. If $\beta \notin [-4, \infty)$, i.e. $\mathrm{tr} f$ is not real, then f is called *strictly loxodromic*.

An elliptic element f of order n is said to be *primitive* if it is a rotation through $2\pi/n$ (with $\beta = -4\sin^2(\pi/n)$); otherwise, it is called *non-primitive* (and then $\beta = -4\sin^2(\pi q/n)$, where q and n are coprime and $1 < q < n/2$).

A plane divides $\mathbb{H}^3$ into two components; we will call the closure of either of them a *half-space* in $\mathbb{H}^3$. A connected subset P of $\mathbb{H}^3$ with non-empty interior is said to be a *(convex) polyhedron* if it is the intersection of a family $\mathcal{H}$ of half-spaces with the property that each point of P has a neighbourhood meeting at most a finite number of boundaries of elements of $\mathcal{H}$. A closed polyhedron with finite number of faces bounded by planes $\alpha_1, \ldots, \alpha_k$ is denoted by $\mathcal{P}(\alpha_1, \ldots, \alpha_k)$.

We define a *tetrahedron* T to be a polyhedron which in the projective ball model is the intersection of the hyperbolic space $\mathbb{H}^3$ with a Euclidean tetrahedron T_E (possibly with vertices on the sphere $\partial\mathbb{H}^3$ at infinity or beyond it) so that the intersection of each edge of T_E with $\mathbb{H}^3$ is non-empty.

A tetrahedron T (possibly of infinite volume) in $\mathbb{H}^3$ is uniquely determined up to isometry by its dihedral angles. Let T have dihedral angles π/p_1, π/p_2, π/p_3 at some face and let π/q_1, π/q_2, π/q_3 be dihedral angles of T that are opposite to π/p_1, π/p_2, π/p_3, respectively. Then a standard notation for such a T is $T[p_1, p_2, p_3; q_1, q_2, q_3]$.

We denote the reflection in a plane κ by R_κ. The axis of an element $h \in \mathrm{PSL}(2,\mathbb{C})$ with two distinct fixed points in $\partial\mathbb{H}^3$ is denoted by the same h if this does not lead to any confusion.

We use symbols ∞ and $\overline{\infty}$ with the following convention. We assume that $\overline{\infty} > \infty > x$ and $x/\infty = x/\overline{\infty} = 0$ for every real x; $\infty/x = \infty$ and $\overline{\infty}/x = \overline{\infty}$ for every positive real x; in particular, $(\infty, k) = (\overline{\infty}, k) = k$ for every positive integer k. If we denote the dihedral angle between two planes by π/p ($1 < p \leq \overline{\infty}$), then the planes intersect when p is finite, they are parallel when $p = \infty$ and disjoint when $p = \overline{\infty}$.

Convention on pictures

Since the methods we use here are essentially geometrical, the paper contains many pictures of hyperbolic polyhedra. In those pictures, shaded polygons are not faces of polyhedra, but are drawn to underline the combinatorial structure of the corresponding polyhedron. They are just intersections of the polyhedron with appropriate planes.

The edges of polyhedra are marked with the values of corresponding dihedral angles; we omit labels $\pi/2$. Some other lines on such pictures are labeled with positive integers $k \geq 2$, which means that such a line is the axis of an elliptic element of order k that belongs to Γ^* (see below).

3 Fundamental polyhedra and parameters

From here on f is a primitive elliptic element and g is hyperbolic. The main tool in the study of discreteness of $\Gamma = \langle f, g\rangle$ in [7, 9, 10] was a construction of a 'convenient' finite index extension Γ^* of Γ together with a fundamental polyhedron for each discrete Γ^*. In this section, we reproduce the construction of Γ^* and describe the fundamental polyhedra for all discrete Γ^*. This is a preliminary part for Section 4, where we will work with the groups Γ themselves to list the corresponding orbifolds.

3.1 Geometric description of discrete groups for the case of disjoint axes

Theorem 3.1 below gives necessary and sufficient conditions for discreteness of Γ for the case of *disjoint axes* of the generators f and g; a complete proof can be found in [7]. We also repeat the geometric construction from [7] and recall fundamental polyhedra for the series of discrete groups Γ^* corresponding to Items 2(i)–2(iii) of Theorem 3.1.

Theorem 3.1 ([7]). *Let $f \in \mathrm{PSL}(2,\mathbb{C})$ be a primitive elliptic element of order $n \geq 3$, $g \in \mathrm{PSL}(2,\mathbb{C})$ be a hyperbolic element, and let their axes be disjoint lines lying in a hyperbolic plane. Then*

(1) *There exists* $h \in \mathrm{PSL}(2,\mathbb{C})$ *such that* $h^2 = fgf^{-1}g^{-1}$ *and* $(hg)^2 = 1$.

(2) $\Gamma = \langle f, g\rangle$ *is discrete if and only if one of the following holds:*

   (*i*) h *is a hyperbolic, parabolic, or primitive elliptic element of order* $p \geq 3$*;*

   (*ii*) $n \geq 5$ *is odd,* $h = x^2$*, where* x *is a primitive elliptic element of order* n*, and* $y = hgfx^{-1}f$ *isa hyperbolic, parabolic, or primitive elliptic element of order* $q \geq 4$*;*

   (*iii*) $n = 3$*,* $h = x^2$*, where* x *is a primitive elliptic element of order* 5*, and* $z = hgf(x^{-1}f)^3$ *is a hyperbolic, parabolic, or primitive elliptic element of order* $r \geq 3$.

Let f and g be as in Theorem 3.1 and let ω be the plane in which the (disjoint) axes of f and g lie.

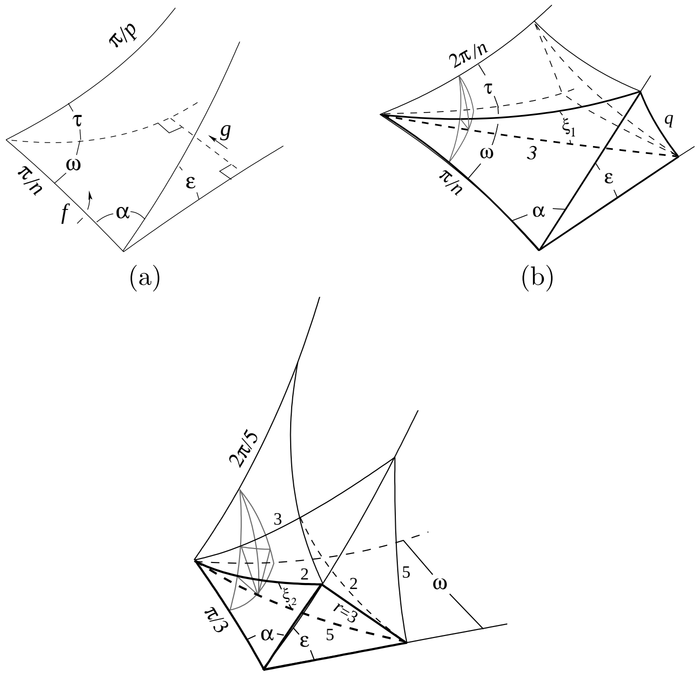


Figure 1: Fundamental polyhedra for Γ^* ($\gamma < 0$)

Denote by ε the plane that passes through the common perpendicular to the axes of f and g and is orthogonal to ω. Let α and τ be the planes such that $f = R_\alpha R_\omega$ and $g = R_\tau R_\varepsilon$, and let $\mathcal{P} = \mathcal{P}(\omega, \varepsilon, \alpha, \tau)$. The planes ω and α make a dihedral angle of π/n; the planes ε and τ are disjoint so that the axis of g is their common perpendicular. Moreover, α is orthogonal to ε, and τ is orthogonal to ω. The planes α and τ either intersect non-orthogonally or are parallel or disjoint. We denote the dihedral angle of $\mathcal{P}$ between these planes by π/p, $p > 2$, where $p = \infty$ if α and τ are parallel and $p = \overline{\infty}$ if they are disjoint.

For the group $\Gamma = \langle f, g\rangle$, we consider two finite index extensions of it: $\widetilde{\Gamma} = \langle f, g, e\rangle$, where $e = R_\varepsilon R_\omega$, and $\Gamma^* = \langle f, g, e, R_\omega\rangle$. $\widetilde{\Gamma}$ is the orientation preserving subgroup of index 2 in Γ^*, and $\widetilde{\Gamma}$ contains Γ as a subgroup of index at most 2. In Section 4, we shall see when $\Gamma = \widetilde{\Gamma}$ and when $\Gamma \neq \widetilde{\Gamma}$.

It was shown in [7] that $h = R_\alpha R_\tau$ is the only element that satisfies both $h^2 = [f, g]$ and $(hg)^2 = 1$. There are three series of discrete groups Γ depending on how $\mathcal{P}$ is decomposed into fundamental polyhedra for Γ^*. The series correspond to the conditions $2(i)$, $2(ii)$, and $2(iii)$ of Theorem 3.1.

1. h is hyperbolic, parabolic, or a primitive elliptic element of order $p \geq 3$ (that is $2(i)$ holds) if and only if the dihedral angle of $\mathcal{P}$ between α and τ is of the form π/p with $p = \overline{\infty}$, $p = \infty$, or $p \in \mathbb{Z}$, $p \geq 3$, respectively. This is the first series of the discrete groups. In this case the polyhedron $\mathcal{P}$ is a fundamental polyhedron for Γ^*. In Figure 1(a) $\mathcal{P}$ is drawn under assumption that $1/n + 1/p > 1/2$.

The other discrete groups appear only if h is the square of a primitive elliptic element $x = R_\kappa R_\tau$, where κ is the bisector of the dihedral angle of $\mathcal{P}$ made by α and τ. Fundamental polyhedra for Γ^* corresponding to these two series are obtained by decomposing $\mathcal{P}$ into smaller polyhedra as follows (see [7] for the proof).

2. Let Γ be determined by the condition $2(ii)$. In this case, $n \geq 5$ is odd, the dihedral angle of $\mathcal{P}$ between α and τ is $2\pi/n$, and κ and ω make a dihedral angle of $\pi/3$. Hence, ξ_1, where $\xi_1 = R_\kappa(\omega)$, and ω also make a dihedral angle of $\pi/3$, and ξ_1 and α are orthogonal. The planes ξ_1 and ε either intersect at an angle of π/q, where $q \in \mathbb{Z}$, $q \geq 4$, or are parallel or disjoint ($q = 3$ is not included, because then ε and τ must intersect). One can show that if $y = R_\varepsilon R_{\xi_1}$, then $y = hgfx^{-1}f$. The polyhedron $\mathcal{P}(\omega, \varepsilon, \alpha, \xi_1)$ is a fundamental polyhedron for Γ^*. For $q = 4$ or 5 and $n = 5$, $\mathcal{P}(\omega, \varepsilon, \alpha, \xi_1)$ is compact and we show it in Figure 1(b) by bold lines.

3. Let Γ be determined by the condition $2(iii)$. In this case $n = 3$ and the dihedral angle of $\mathcal{P}$ between α and τ is $2\pi/5$. Denote $\xi_2 = R_\kappa(\omega)$. The planes κ and ω make a dihedral angle of $2\pi/5$ and, hence, ξ_2 and ω make an angle of $\pi/5$. It can be shown that ξ_2 and α are orthogonal. The planes ε and ξ_2 either intersect at an angle of π/r, where $r \in \mathbb{Z}$, $r \geq 3$, or are parallel or disjoint. In this case $z = R_\varepsilon R_{\xi_2} = hgf(x^{-1}f)^3$. The polyhedron $\mathcal{P}(\omega, \varepsilon, \alpha, \xi_2)$ is a fundamental polyhedron for Γ^* (see Figure 1(c), where $\mathcal{P}(\omega, \varepsilon, \alpha, \xi_2)$ is drawn for $r = 3$).

3.2 Parameters for discrete groups in the case of disjoint axes

Let $\mathcal{U} = \{u : u = i\pi/p, p \in \mathbb{Z}, p \geq 2\} \cup [0, +\infty)$. Define a function $t : \mathcal{U} \to \{2, 3, \dots\} \cup \{\infty, \overline{\infty}\}$ as follows:

$$t(u) = \begin{cases} p & \text{if} \quad u = i\pi/p, \\ \infty & \text{if} \quad u = 0, \\ \overline{\infty} & \text{if} \quad u > 0. \end{cases}$$

The purpose of introducing the function $t(u)$ is to shorten statements that involve parameters (β, β', γ). We use it in Theorems 3.2, 3.3, 4.1, 4.2, and 4.5.

Now we give a parameter version of Theorem 3.1 with a proof. Theorem 3.2 is new and did not appear before, however, we did use a similar technique in earlier papers.

Theorem 3.2. *Let $f, g \in \mathrm{PSL}(2, \mathbb{C})$, $\beta = -4\sin^2(\pi/n)$, $n \geq 3$, $\beta' \in (0, +\infty)$, and $\gamma \in (-\infty, 0)$. Then the group $\Gamma = \langle f, g \rangle$ is discrete if and only if one of the following holds:*

1. *$\gamma = -4\cosh^2 u$, where $u \in \mathcal{U}$ and $t(u) \geq 3$;*
2. *$n \geq 5$, $(n, 2) = 1$, $\gamma = -(\beta + 2)^2$, and $\beta' = 4(\beta + 4)\cosh^2 u - 4$, where $u \in \mathcal{U}$ and $t(u) \geq 4$;*
3. *$\beta = -3$, $\gamma = (\sqrt{5} - 3)/2$, and $\beta' = 2(7 + 3\sqrt{5})\cosh^2 u - 4$, where $u \in \mathcal{U}$ and $t(u) \geq 3$.*

Proof. $\beta = -4\sin^2(\pi/n)$, $n \in \mathbb{Z}$, if and only if f is a primitive elliptic element of order n, and $\beta' \in (0, +\infty)$ if and only if g is hyperbolic. Since $n \geq 3$ and $\gamma \in (-\infty, 0)$, Γ is non-elementary and the axes of f and g are disjoint by [9, Theorem 1]. So, the hypotheses of Theorem 3.2 and Theorem 3.1 are equivalent.

Let us find explicit values of β' and γ for each of the discrete groups from part (2) of Theorem 3.1. The idea is to use the fundamental polyhedra described in Section 3.1. Since $\gamma = \mathrm{tr}[f, g] - 2$ and h is a square root of $[f, g]$, it is not difficult to get conditions on γ.

The element $h = R_\alpha R_\tau$ is hyperbolic if and only if the planes α and τ (see Figure 1(a)) are disjoint. Therefore, $\mathrm{tr}[f, g] = \mathrm{tr}h^2 = -2\cosh(2d)$, where d is the hyperbolic distance between α and τ. Here $\mathrm{tr}[f, g]$ must be negative, because γ is negative for all values of β and β' that satisfy the hypotheses of the theorem.

The element h is parabolic if and only if $[f, g]$ is parabolic and if and only if $\mathrm{tr}[f, g] = -2$ which is equivalent to $\gamma = -4$ ($\mathrm{tr}[f, g] = 2$ would give $\gamma = 0$).

Thus, h is hyperbolic or parabolic if and only if

$$\gamma = \mathrm{tr}[f, g] - 2 = -2\cosh(2d) - 2 = -4\cosh^2 d, \quad d \geq 0. \tag{3.1}$$

Now suppose that h is an elliptic element with rotation angle ϕ, where $\phi/2 = \pi/p < \pi/2$ is the dihedral angle of $\mathcal{P}(\omega, \varepsilon, \alpha, \tau)$ made by α and τ.

Then $[f,g]=h^2$ is also elliptic with rotation angle 2ϕ. Since $\mathrm{tr}[f,g]$ is well-defined, we conclude that $\mathrm{tr}[f,g]=-2\cos\phi$ by continuity. On the other hand, if $\mathrm{tr}[f,g]\in(-2,2)$ is given, we can use the formula $\mathrm{tr}[f,g]=-2\cos\phi$, $\phi<\pi$, to determine the rotation angle ϕ of the element h from Theorem 3.1.

Hence, h is a primitive elliptic element of order p ($p\geq 3$), that is, $\phi=2\pi/p$, if and only if

$$\gamma=\mathrm{tr}[f,g]-2=-2\cos(2\pi/p)-2=-4\cos^2(\pi/p),\ p\in\mathbb{Z},\ p\geq 3. \tag{3.2}$$

Now we can combine the formulas (3.1) and (3.2) for γ and write them as

$$\gamma=-4\cosh^2 u,\ \text{where } u\in\mathcal{U} \text{ and } t(u)\geq 3.$$

It is clear that for the groups from Item $2(i)$ of Theorem 3.1, we have no further restrictions on n and β'. So, part 1 of Theorem 3.2 is equivalent to $2(i)$ of Theorem 3.1.

Further, in $2(ii)$, $n\geq 5$ is odd and h is the square of a primitive elliptic element of order n (that is, $\phi=4\pi/n$) if and only if $n\geq 5$, $(n,2)=1$, and $\gamma=-4\cos^2(2\pi/n)=-(\beta+2)^2$.

So it remains to specify β' for $2(ii)$. Now β' depends on the order of element y defined in Theorem 3.1. Since $g=R_\tau R_\varepsilon$, $\beta'=\mathrm{tr}^2 g-4=4\sinh^2 T$, where T is the distance between the planes ε and τ.

Elementary calculations in the plane ω show that for the groups from Item $2(ii)$,

$$\beta'=\begin{cases} 4(\beta+4)\cos^2(\pi/q)-4 & \text{if} \quad 4\leq q<\infty,\\ 4(\beta+4)-4 & \text{if} \quad q=\infty,\\ 4(\beta+4)\cosh^2 d_1-4 & \text{if} \quad q=\overline{\infty},\end{cases}$$

where d_1 is the distance between ε and ξ_1 if they are disjoint, and π/q is the angle between ε and ξ_1 if they intersect. Hence, β' can be written in general form as follows:

$$\beta'=4(\beta+4)\cosh^2 u-4,\ \text{where } u\in\mathcal{U} \text{ and } t(u)\geq 4.$$

Analogously, for the groups from Item $2(iii)$, we have $n=3$ and $\phi=4\pi/5$, and therefore

$$\beta=-3, \quad \text{and } \gamma=-4\cos^2(2\pi/5)=(\sqrt{5}-3)/2.$$

Moreover,

$$\beta'=\begin{cases} 2(7+3\sqrt{5})\cos^2(\pi/r)-4 & \text{if} \quad 3\leq r<\infty,\\ 2(7+3\sqrt{5})-4 & \text{if} \quad r=\infty,\\ 2(7+3\sqrt{5})\cosh^2 d_2-4 & \text{if} \quad r=\overline{\infty},\end{cases}$$

where d_2 is the distance between ε and ξ_2 if they are disjoint, and π/r is the angle between ε and ξ_2 if they intersect, and hence,

$$\beta'=2(7+3\sqrt{5})\cosh^2 u-4,\ \text{where } u\in\mathcal{U} \text{ and } t(u)\geq 3.$$

$\square$

3.3 Geometric description of discrete groups for the case of intersecting axes

Now we consider $\Gamma = \langle f, g \rangle$ with f primitive elliptic of order $n > 2$ and g hyperbolic with *non-orthogonally intersecting axes.* In [9] and [10], criteria for discreteness of such groups were found for n even and odd, respectively. In this section we recall the criteria in terms of parameters and remind the construction of a fundamental polyhedron for each discrete group Γ^*.

Theorem 3.3 ([9] and [10]). *Let $f, g \in \mathrm{PSL}(2, \mathbb{C})$, $\beta = -4\sin^2(\pi/n)$, $n \geq 3$, $\beta' > 0$, and $0 < \gamma < -\beta\beta'/4$. Then $\Gamma = \langle f, g \rangle$ is discrete if and only if (β, β', γ) is one of the triples listed in Table 1.*

Remark 3.4. Note that if a formula in Table 1 involves $u \in \mathcal{U}$ such that $(t(u), 2) = 1$, then $t(u)$ is finite and odd, while for $u \in \mathcal{U}$ with $(t(u), 2) = 2$, $t(u)$ can be not only finite (and even), but ∞ or $\overline{\infty}$, which implies that the formula is applicable also to $u \geq 0$.

Table 1: All parameters for discrete $\mathcal{RP}$ groups generated by a primitive elliptic element f of order $n \geq 3$ and a hyperbolic element g whose axes intersect non-orthogonally.

	$\beta = \beta(f)$	$\gamma = \gamma(f, g)$	$\beta' = \beta(g)$
	$n \geq 4$, $(n, 2) = 2$, $u, v \in \mathcal{U}$, $1/n + 1/t(u) < 1/2$		
P_1	$-4\sin^2 \frac{\pi}{n}$, $n \geq 4$	$4\cosh^2 u + \beta$, $(t(u), 2) = 2$	$\dfrac{4}{\gamma}\cosh^2 v - \dfrac{4\gamma}{\beta}$, $t(v) \geq 3$
P_2	$-4\sin^2 \frac{\pi}{n}$, $n \geq 4$	$4\cosh^2 u + \beta$, $(t(u), 2) = 1$	$\dfrac{4(\gamma - \beta)}{\gamma}\cosh^2 v - \dfrac{4\gamma}{\beta}$, $t(v) \geq 3$
P_3	-2	$2\cos(2\pi/m)$, $m \geq 5$, $(m, 2) = 1$	$\gamma^2 + 4\gamma$
	$n \geq 3$, $(n, 2) = 1$, $u, v \in \mathcal{U}$, $1/n + 1/t(u) < 1/2$; $S = -2\dfrac{(\gamma-\beta)^2\cos\frac{\pi}{n} + \gamma(\gamma+\beta)}{\gamma\beta}$, $T = -2\dfrac{(\beta+2)^2\cos\frac{\pi}{n}}{\beta+1} - 2\dfrac{(\beta^2+6\beta+4)}{\beta}$		
P_4	$-4\sin^2 \frac{\pi}{n}$, $n \geq 3$	$4\cosh^2 u + \beta$, $(t(u), 2) = 2$	$\dfrac{2}{\gamma}(\cosh v - \cos\dfrac{\pi}{n}) + S$, $t(v) \geq 2$
P_5	$-4\sin^2 \frac{\pi}{n}$, $n \geq 3$	$4\cosh^2 u + \beta$, $(t(u), 2) = 1$	$\dfrac{2(\gamma-\beta)}{\gamma}\cosh v + S$, $t(v) \geq 2$
P_6	$-4\sin^2 \frac{\pi}{n}$, $n \geq 7$	$(\beta+4)(\beta+1)$	$\dfrac{2(\beta+2)^2}{\beta+1}\cosh v + T$, $t(v) \geq 2$
P_7	$-4\sin^2 \dfrac{\pi}{n}$, $n \geq 5$, $(n, 3) = 1$	$\beta + 3$	$\dfrac{2}{\beta}\left((\beta-3)\cos\dfrac{\pi}{n} - 2\beta - 3\right)$
P_8	$-4\sin^2 \dfrac{\pi}{n}$,	$2(\beta+3)$	$-\dfrac{6}{\beta}\left(2\cos\dfrac{\pi}{n} + \beta + 2\right)$

Table 1: (continued)

	$\beta=\beta(f)$	$\gamma=\gamma(f,g)$	$\beta'=\beta(g)$
	$n\geq 5$, $(n,3)=1$		
P_9	-3	$2\cos(2\pi/m)-1$, $m\geq 7$, $(m,2)=1$	$\frac{2}{\gamma}\left(\gamma^2+2\gamma+2\right)$
P_{10}	-3	$2\cos(\pi/m)-1$, $m\geq 4$, $(m,3)=1$	$\gamma^2+4\gamma$
P_{11}	-3	$2\cos(2\pi/m)$, $m\geq 7$, $(m,4)\leq 2$	2γ
P_{12}	-3	$(\sqrt{5}+1)/2$	$\sqrt{5}$
P_{13}	-3	$(\sqrt{5}-1)/2$	$\sqrt{5}$
P_{14}	-3	$(\sqrt{5}-1)/2$	$\sqrt{5}-1$
P_{15}	$(\sqrt{5}-5)/2$	$(\sqrt{5}-1)/2$	$\sqrt{5}$
P_{16}	$(\sqrt{5}-5)/2$	$(\sqrt{5}-1)/2$	$(3\sqrt{5}-1)/2$
P_{17}	$(\sqrt{5}-5)/2$	$(\sqrt{5}-1)/2$	$3(\sqrt{5}+1)/2$
P_{18}	$(\sqrt{5}-5)/2$	$(\sqrt{5}+1)/2$	$3(\sqrt{5}+1)/2$
P_{19}	$(\sqrt{5}-5)/2$	$\sqrt{5}+2$	$(5\sqrt{5}+9)/2$

Let f and g be as in Theorem 3.3, that is, let f be a primitive elliptic element of order $n\geq 3$, g be a hyperbolic element, and let their axes intersect non-orthogonally. Let ω be a plane containing f and g, and let e be a half-turn whose axis is orthogonal to ω and passes through the point of intersection of f and g.

Again, we define two finite index extensions of $\Gamma=\langle f,g\rangle$ as follows: $\widetilde{\Gamma}=\langle f,g,e\rangle$ and $\Gamma^*=\langle f,g,e,R_\omega\rangle$.

Let e_f and e_g be half-turns such that $f=e_fe$ and $g=e_ge$. The lines e_f and e lie in a plane, denote it by ε, and intersect at an angle of π/n; ε and ω are mutually orthogonal; e_g is orthogonal to ω and intersects g.

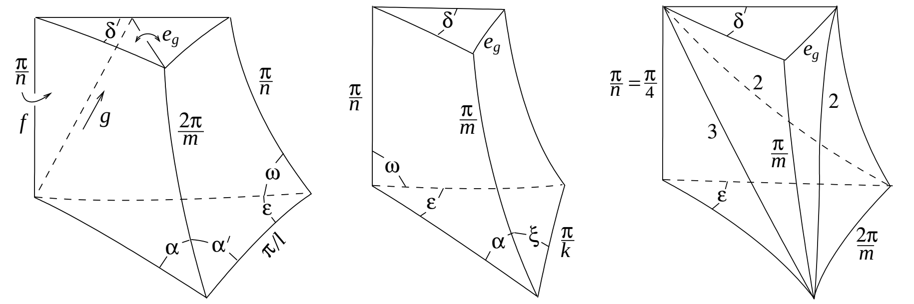
Let α be a hyperbolic plane such that $f=R_\omega R_\alpha$ and let $\alpha'=e_g(\alpha)$. There exists a plane δ which is orthogonal to the planes α, ω, and α'. The plane δ passes through the common perpendicular to f and $e_g(f)$ orthogonally to ω. It is clear that $e_g\subset\delta$.

From here on, we discribe the cases of even n and odd n separately (n is the order of the elliptic generator f).

$n\geq 4$ is even

Let $\mathcal{P}=\mathcal{P}(\alpha,\omega,\alpha',\delta,\varepsilon)$. The polyhedron $\mathcal{P}$ can be compact or non-compact; in Figure 2(a), $\mathcal{P}$ is drawn as compact.

The polyhedron $\mathcal{P}$ has five right dihedral angles; the dihedral angles formed by ω with α and α' equal π/n. The planes α and α' can either intersect, or be parallel or disjoint; the same is true for ε and α'. Denote the angle between ε



(a): P_1 (b): P_2 (c): P_3

Figure 2: Fundamental polyhedra for Γ^* in the case of even n $(0 < \gamma < -\beta\beta'/4)$

and α' by π/ℓ, where $\ell \in (2,\infty) \cup \{\infty, \overline{\infty}\}$ and denote the angle between α and α' by $2\pi/m$, where $m \in (2,\infty) \cup \{\infty, \overline{\infty}\}$, $1/n + 1/m < 1/2$.

For each triple of parameters with n even in Table 1, we know a fundamental polyhedron for Γ^* [9] and we describe all such polyhedra below.

$\mathbf{P}_1$. $\mathcal{P}$ is a fundamental polyhedron for Γ^* if and only if $m \in \mathbb{Z} \cup \{\infty, \overline{\infty}\}$, m is even $(1/m + 1/n < 1/2)$, and $\ell \in \mathbb{Z} \cup \{\infty, \overline{\infty}\}$ $(\ell \geq 3)$. In terms of the function t, $m = t(u)$ and $\ell = t(v)$ (cf. Table 1).

$\mathbf{P}_2$. Note that in this case $m = t(u)$ is finite and odd. Let ξ be the bisector of the dihedral angle of $\mathcal{P}$ at the edge $\alpha \cap \alpha'$. It is clear that ξ passes through e_g and is orthogonal to ω. The polyhedron $\mathcal{P}(\alpha, \delta, \xi, \varepsilon, \omega)$ is bounded by reflection planes of Γ^* (see Figure 2(b)) and, therefore, it is a fundamental polyhedron for Γ^* if and only if ξ and ε intersect at an angle of π/k, where $k \geq 3$, or are parallel or disjoint ($k = \infty$ or $k = \overline{\infty}$, respectively). In Table 1, $k = t(v)$ for the parameters P_2.

$\mathbf{P}_3$. In this case $n = 4$ and the dihedral angle of $\mathcal{P}(\alpha, \delta, \xi, \varepsilon, \omega)$ at the edge $\xi \cap \varepsilon$ is $2\pi/m$, where $m = t(u)$ is odd, $5 \leq m < \infty$. The polyhedron $\mathcal{P}(\alpha, \delta, \xi, \varepsilon, \omega)$ is decomposed by reflection planes of Γ^* into three (possibly infinite volume) tetrahedra $[2, 2, 4; 2, 3, m]$, each of which is a fundamental polyhedron for Γ^* (see Figure 2(c)).

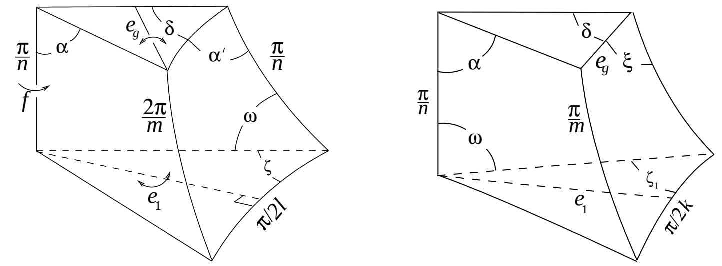
$n \geq 3$ is odd

Denote $e_1 = f^{(n-1)/2}e$. Note that e_1 makes angles of $\pi/(2n)$ with α and ω.

We can forget about the plane ε, because now we need another plane, denote it by ζ, for the construction of a fundamental polyhedron for Γ^*. To construct ζ we use an auxiliary plane κ that passes through e_1 orthogonally to α'. The plane ζ then passes through e_1 orthogonally to κ. (Note that ζ is not orthogonal to each of α and ω if $m \neq 2n$.) In fact, the planes ζ and α' can either intersect, or be parallel, or disjoint. Note that if $\zeta \cap \alpha' \neq \emptyset$ then e_1 is orthogonal to $\zeta \cap \alpha'$. Let $\mathcal{P} = \mathcal{P}(\alpha, \omega, \alpha', \delta, \zeta)$. In Figure 3(a), $\mathcal{P}$ is drawn for the compact case.

Consider the dihedral angles of $\mathcal{P}$. The angles between δ and ω, δ and α, δ and α' are all of $\pi/2$; the angles formed by ω with α and α' equal π/n; since ζ passes through e_1, which is orthogonal to f, the sum of the angles formed by ζ with ω and α equals π. The planes α and α' can either intersect or be parallel or disjoint. The same is true for ζ and α'. Denote the angle between α and α' by $2\pi/m$ and the angle between α' and ζ by $\pi/(2\ell)$.

Fundamental polyhedra for groups Γ^* for all triples of parameters with odd n from Table 1 were constructed in [10]. Now we describe them.



(a): P_4 (b): P_5

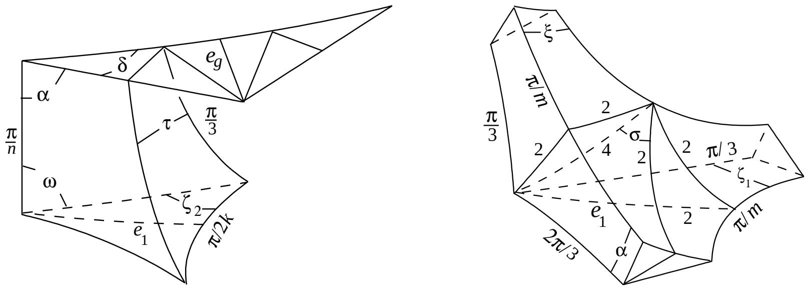
Figure 3:

$\mathbf{P}_4$. $\mathcal{P}$ itself is a fundamental polyhedron for Γ^* if and only if m is even $(1/m + 1/n < 1/2)$, $m = \infty$, or $m = \overline{\infty}$, and $\ell \in \mathbb{Z} \cup \{\infty, \overline{\infty}\}$, $\ell \geq 2$. In terms of the function t, $m = t(u)$ and $\ell = t(v)$ (cf. Table 1).

$\mathbf{P}_5$. Let ξ be the bisector of the dihedral angle of $\mathcal{P}$ at the edge $\alpha \cap \alpha'$. Clearly, ξ passes through e_g orthogonally to ω and δ. Construct a plane ζ_1 in a similar way as ζ above (now ξ plays the role of α'). The polyhedron $\mathcal{Q} = \mathcal{P}(\alpha, \delta, \xi, \zeta_1, \omega)$ is a fundamental polyhedron for Γ^* (see Figure 3(b)) if and only if m is odd and ξ and ζ_1 make an angle of $\pi/(2k)$, where $k \geq 2$ is an integer, ∞ or $\overline{\infty}$.

$\mathbf{P}_6$. In this case, the dihedral angle of $\mathcal{Q}$ at the edge $\alpha \cap \xi$ equals $2\pi/n$ (i.e. $m = n/2$), $n \geq 7$ is odd. Let ρ be the bisector of this dihedral angle and let $\tau = R_\rho(\omega)$. The bisector ρ makes an angle of $\pi/3$ with ω and, therefore, so does τ. It is clear that then τ is orthogonal to α (we have one of Knapp's triangles in δ). Construct a plane ζ_2 similarly to the planes ζ and ζ_1 above (but using τ). The polyhedron $\mathcal{P}(\alpha, \delta, \omega, \tau, \zeta_2)$ (see Figure 4(a), where we show also a part of the plane δ) is a fundamental polyhedron for Γ^* if and only if ζ_2 and τ intersect at an angle of $\pi/(2k)$, where $k \geq 2$ is an integer, or are parallel or disjoint ($k = \infty$ or $k = \overline{\infty}$, respectively). In Table 1, $t(v)$ corresponds to k.

$\mathbf{P}_9$. The dihedral angles of $\mathcal{Q}$ at the edges $\alpha \cap \alpha'$ and $\zeta_1 \cap \alpha'$ equal π/m, m is odd. The plane ζ_1 makes dihedral angles of $2\pi/3$ and $\pi/3$ with α and ω, respectively. Let σ be the bisector of the dihedral angle of $\mathcal{Q}$ at $\alpha \cap \zeta_1$. It is clear that σ is orthogonal to ω. $\mathcal{P}(\alpha, \omega, \xi, \delta, \mu)$, where μ is the plane that passes through $\sigma \cap \omega$ orthogonally to α, is a fundamental polyhedron for Γ^* (see Figure 4(b)). The dihedral angle of $\mathcal{P}(\alpha, \omega, \xi, \delta, \mu)$ at $\mu \cap \omega$ equals $\pi/4$.



(a): P_6 (b): P_9

Figure 4:

Fundamental polyhedra for remaining discrete groups Γ^* are obtained after decomposition of $\mathcal{P}$ (which is shown in Figuree 3(a)) by the planes of reflections from Γ^*, that is, $(m, 2) = 2$, $m \geq 4$, and ℓ is fractional. We first consider the cases where $R_\zeta \in \Gamma^*$.

A compact convex polyhedron in $\mathbb{H}^3$ whose skeleton is a trivalent graph is uniquely determined by its dihedral angles up to isometry of $\mathbb{H}^3$ [5]. Given n, m, and ℓ, all the dihedral angles of the polyhedron $\mathcal{P} \cup e_1(\mathcal{P})$ are defined. Therefore, the dihedral angle ϕ of $\mathcal{P}$ at $\alpha \cap \zeta$ can be obtained. So to determine a compact $\mathcal{P}$ it is sufficient to indicate only n, m, and ℓ, but we shall also give the value of ϕ for convenience. If $\mathcal{P}$ has infinite volume, but $\ell < \infty$ and $m < \infty$ (then $2/m + 1/n + 1/\ell < 1$), $\mathcal{P}$ is also determined by the values of n, m, and ℓ, since we can cut off a compact polyhedron from $\mathcal{P} \cup e_1(\mathcal{P})$ by a plane orthogonal to ζ, α, α' and a plane orthogonal to ζ, ω, α'.

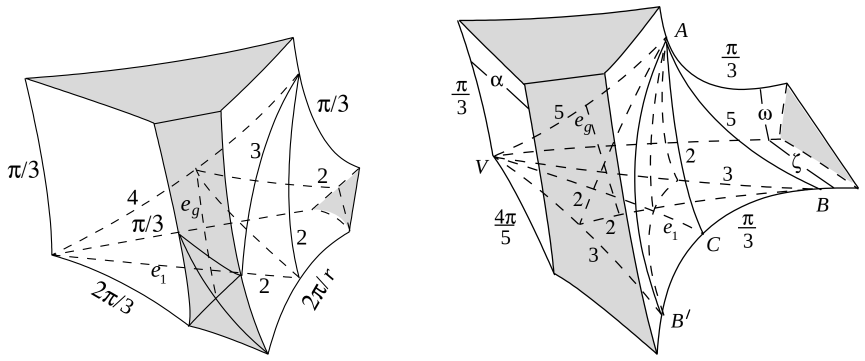
There are no discrete groups for which $m = \infty$ or $\ell \geq \infty$ except for those with parameters of type P_4. When $m = \overline{\infty}$ we also indicate the distance d between α and α' to determine $\mathcal{P}$. In fact, given d, one can find ϕ, but we shall give ϕ explicitely for convenience.

In all of these cases $R_\delta \notin \Gamma^*$, so we do not show δ (but indicate e_g) in figures in order to simplify the picture. By the same reason we draw only those parts of the decomposition (including ω) that are important for the reconstruction of the action of Γ^* and help to determine positions of e_1 and e_g.

$\mathbf{P}_{11}$. $n = 3$, $m = \overline{\infty}$, $\ell = r/4$, $(r, 4) \leq 2$, $r \geq 7$, $\phi = 2\pi/3$, and $\cosh d = 2\cos^2(\pi/r) - 1/2$. $\mathcal{P}(\alpha, \alpha', \omega, \zeta)$ is decomposed into tetrahedra $\mathcal{T} = T[2, 3, r; 2, 2, 4]$ each of which is a fundamental polyhedron for Γ^* (Figure 5(a)).

$\mathbf{P}_{12}$. $n = 3$, $m = \overline{\infty}$, $\ell = 3/2$, $\phi = 4\pi/5$, and $\cosh d = (3 + \sqrt{5})/4$. $\mathcal{P}(\alpha, \alpha', \omega, \zeta)$ is decomposed into tetrahedra $\mathcal{T} = T[2, 2, 3; 2, 5, 3]$. A half of $\mathcal{T}$ is a fundamental polyhedron for Γ^* (Figure 5(b)).

$\mathbf{P}_{13}$. $n = 3$, $m = 10$, $\ell = 3/2$, $\phi = 3\pi/5$. $\mathcal{P}(\alpha, \alpha', \omega, \zeta)$ is decomposed into tetrahedra $\mathcal{T} = T[2, 3, 5; 2, 3, 2]$. A half of $\mathcal{T}$ is a fundamental polyhedron for Γ^* (Figure 6(a)).



(a): P_{11} (b): P_{12}

Figure 5:

$\mathbf{P}_{14}$. $n = 3$, $m = 10$, $\ell = 5/4$, $\phi = 2\pi/3$. $\mathcal{P}(\alpha, \alpha', \omega, \zeta)$ is decomposed into tetrahedra $\mathcal{T} = T[2, 3, 5; 2, 2, 4]$. A half of $\mathcal{T}$ is a fundamental polyhedron for Γ^* (Figure 6(b)).

$\mathbf{P}_{15}$. $n = 5$, $m = 4$, $\ell = 3/2$, $\phi = \pi/5$. $\mathcal{P}(\alpha, \alpha', \omega, \zeta)$ is decomposed into tetrahedra $\mathcal{T} = T[2, 3, 5; 2, 3, 2]$. $\mathcal{T}$ is a fundamental polyhedron for Γ^* (Figure 6(c)).

$\mathbf{P}_{17}$. $n = 5$, $m = 4$, $\ell = 5/2$, $\phi = \pi/3$. $\mathcal{P}(\alpha, \alpha', \omega, \zeta)$ is decomposed into tetrahedra $\mathcal{T} = T[2, 3, 5; 2, 2, 5]$. A half of $\mathcal{T}$ is a fundamental polyhedron for Γ^* (Figure 6(d)).

$\mathbf{P}_{18}$. $n = 5$, $m = 6$, $\ell = 5/4$, $\phi = \pi/3$. $\mathcal{P}(\alpha, \alpha', \omega, \zeta)$ is decomposed into tetrahedra $\mathcal{T} = T[2, 3, 5; 2, 2, 5]$. A half of $\mathcal{T}$ is a fundamental polyhedron for Γ^* (Figure 6(e)).

Now consider discrete groups for which $R_\zeta \notin \Gamma^*$. In all these cases $\ell = p/3$, where $(p, 3) = 1$. Let η be the plane through $\alpha' \cap \zeta$ that makes a dihedral angle of $2\pi/p$ with α' and let $\overline{\mathcal{P}} = \mathcal{P}(\alpha, \alpha', \omega, \delta, \eta)$. Denote by θ_1 and θ_2 dihedral angles of $\overline{\mathcal{P}}$ at $\eta \cap \alpha$ and $\eta \cap \omega$, respectively.

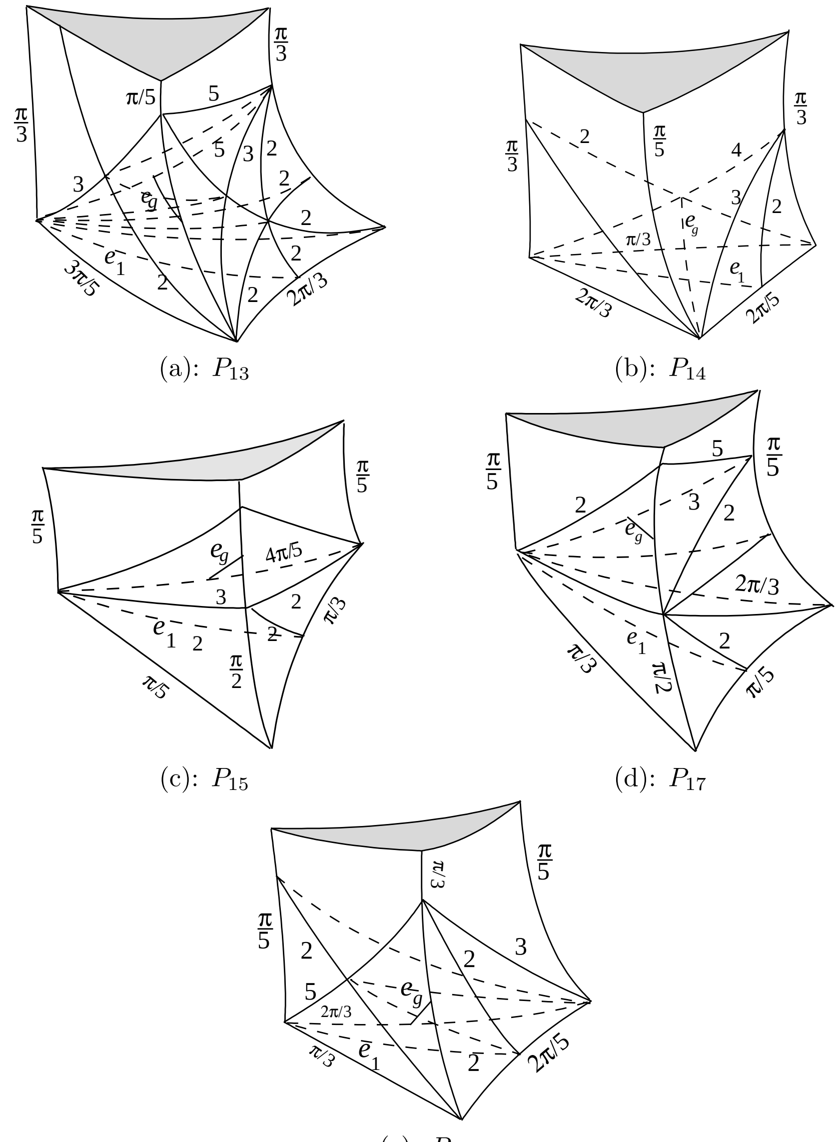
If $\mathcal{P}$ is compact or non-compact with $m < \infty$, $\overline{\mathcal{P}}$ is determined by values n, m, ℓ, θ_1, and θ_2. For $m = \overline{\infty}$, we give the distance d between α and α'.

$\mathbf{P}_7$. $n \geq 5$, $(n, 3) = 1$, $m = 6$, $\ell = n/3$, $\theta_1 = \pi/3$, $\theta_2 = \pi/2$. $\mathcal{P}(\alpha, \alpha', \omega, \eta)$ is decomposed into tetrahedra $\mathcal{T} = T[2, 3, n; 2, 3, n]$. A quarter of $\mathcal{T}$ is a fundamental polyhedron for Γ^* (Figure 7(a)).

$\mathbf{P}_8$. $n \geq 5$, $(n, 3) = 1$, $m = \overline{\infty}$, $\ell = n/3$, $\theta_1 = \pi/2$, $\theta_2 = \pi/n$, and $\cosh d = 2\cos^2(\pi/n)$. $\mathcal{P}(\alpha, \alpha', \omega, \eta)$ is decomposed into tetrahedra $\mathcal{T} = T[2, 2, 4; 2, n, 4]$. A half of $\mathcal{T}$ is a fundamental polyhedron for Γ^* (Figure 7(b)).

$\mathbf{P}_{10}$. $n = 3$, $m \geq 8$ is even, $(m, 3) = 1$, $\ell = m/6$, $\theta_1 = \pi/2$, $\theta_2 = \pi/3$. $\mathcal{P}(\alpha, \alpha', \omega, \eta)$ is decomposed into tetrahedra $\mathcal{T} = T[2, 3, m/2; 2, 3, 3]$. A half of $\mathcal{T}$ is a fundamental polyhedron for Γ^* (Figure 7(c)).

$\mathbf{P}_{16}$. $n = 5$, $m = 4$, $\ell = 5/3$, $\theta_1 = \pi/5$, $\theta_2 = 2\pi/3$. $\mathcal{P}(\alpha, \alpha', \omega, \eta)$ is decomposed into tetrahedra $\mathcal{T} = T[2, 3, 5; 2, 3, 2]$. A half of $\mathcal{T}$ is a fundamental polyhedron

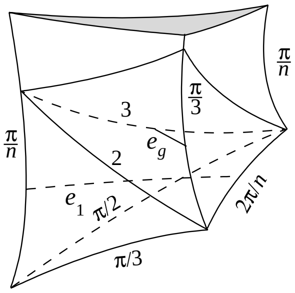


(a): P_{13} (b): P_{14}

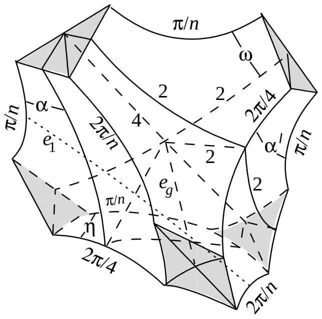
(c): P_{15} (d): P_{17}

(e): P_{18}

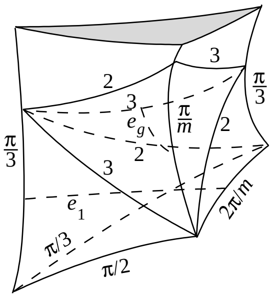
Figure 6:



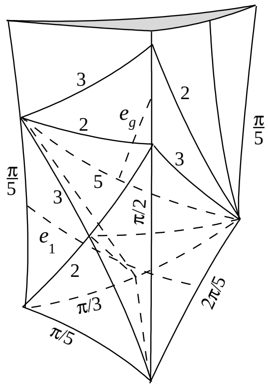
(a): P_7



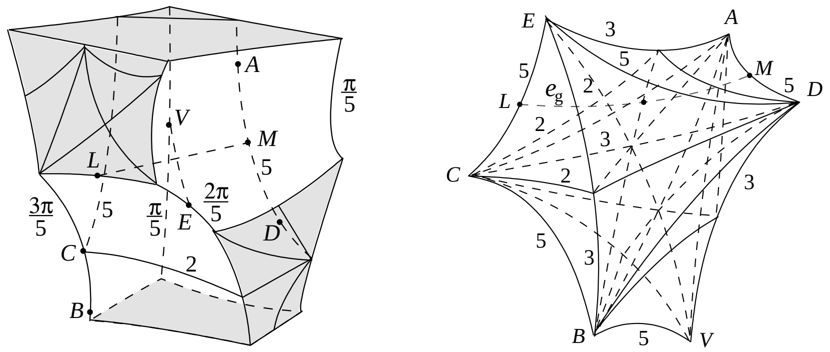
(b): P_8



(c): P_{10}



(d): P_{16}



(e): P_{19}

Figure 7:

for Γ^* (Figure 7(d)).

$\mathbf{P}_{19}$. $n = 5$, $m = \overline{\infty}$, $\ell = 5/3$, $\theta_1 = 3\pi/5$, and $\cosh d = (5 + \sqrt{5})/4$. The planes η and ω are disjoint. $\mathcal{P}(\alpha, \alpha', \omega, \eta)$ is decomposed into tetrahedra $\mathcal{T} = T[2, 2, 3; 2, 5, 3]$. A half of $\mathcal{T}$ is a fundamental polyhedron for Γ^*, see Figure 7(e), where $LM = e_g$ and $VE = e_1$.

4 Kleinian orbifolds and their fundamental groups

Let Γ be a non-elementary Kleinian group, and let $\Omega(\Gamma)$ be the discontinuity set of Γ. Following [1], we say that the *Kleinian orbifold* $Q(\Gamma) = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ is an orientable 3-orbifold with a complete hyperbolic structure on its interior $\mathbb{H}^3/\Gamma$ and a conformal structure on its boundary $\Omega(\Gamma)/\Gamma$.

In this section we shall describe the Kleinian orbifold $Q(\Gamma)$ and a presentation for each truly spatial discrete $\mathcal{RP}$ group Γ generated by an elliptic and a hyperbolic elements. Since a fundamental polyhedron for Γ^* (a finite index extension of Γ) was shown, it remains to construct a fundamental polyhedron for Γ itself and identify the equivalent points on the boundary of the new polyhedron to get the corresponding orbifold.

In figures, we schematically draw singular sets and boundary components of the orbifolds using fat vertices and fat edges. A fat vertex is either a singular point of $Q(\Gamma)$, or corresponds to a puncture, or is deleted together with its neighbourhood (which means that the orbifold has a non-empty boundary) depending on the indices of the incident edges. A fat edge can be labelled by an integer, ∞, or $\overline{\infty}$. If the index at a fat edge is ∞, then the egde corresponds to a cusp, and if the index is $\overline{\infty}$, the edge must be removed together with its regular neighbourhood.

More details on how to 'decode' an orbifold with fat edges and vertices are given in [12]. We do not discuss them here since fundamental polyhedra for all Γ will be found, so it is not difficult to reconstruct the orbifolds.

Denote

1. $GT[n, m; q] = \langle f, g \mid f^n, g^m, [f, g]^q \rangle$
2. $PH[n, m, q] = \langle x, y, z \mid x^n, y^2, z^2, (xz)^2, [x, y]^m, (yxyz)^q \rangle$
3. $H[p; n, m; q] = \langle x, y, s \mid s^2, x^n, y^m, (xy^{-1})^p, (sxsy^{-1})^q, (sx^{-1}y)^2 \rangle$
4. $P[n, m, q] = \langle w, x, y, z \mid w^n, x^2, y^2, z^2, (wx)^2, (wy)^2, (yz)^2, (zx)^q, (zw)^m \rangle$
5. $Tet[p_1, p_2, p_3; q_1, q_2, q_3] = \langle x, y, z \mid x^{p_1}, y^{p_2}, z^{p_3}, (yz^{-1})^{q_1}, (zx^{-1})^{q_2}, (xy^{-1})^{q_3} \rangle$. The group $Tet[2, 2, n; 2, q, m]$ is denoted by $Tet[n, m; q]$ for simplicity.
6. $GTet_1[n, m, q] = \langle x, y, z \mid x^n, y^2, (xy)^m, [y, z]^q, [x, z] \rangle$
7. $GTet_2[n, m, q] = \langle x, y, z \mid x^n, y^2, (xy)^m, (xz^{-1}y^{-1}zy)^q, [x, z] \rangle$

8. $\mathcal{S}_2[n,m,q] = \langle x, L \,|\, x^n, (xLxL^{-1})^m, (xL^2x^{-1}L^{-2})^q \rangle$

9. $\mathcal{S}_3[n,m,q] = \langle x, L \,|\, x^n, (xLxL^{-1})^m, (xLxLxL^{-2})^q \rangle$

10. $R[n,m;q] = \langle u, v \,|\, (uv)^n, (uv^{-1})^m, [u,v]^q \rangle$

In the presentations 1–10, the exponents $n, m, q, \ldots$ may be integers (greater than 1), ∞, or $\overline{\infty}$. We employ the symbols ∞ and $\overline{\infty}$ in the following way. If we have relations of the form $w^n = 1$, where $n = \overline{\infty}$, we remove them from the presentation (in fact, this means that the element w is hyperbolic in the Kleinian group). Further, if we keep the relations $w^\infty = 1$, we get a Kleinian group presentation where parabolics are indicated. To get an abstract group presentation, we need to remove such relations as well.

The reader can find the orbifolds that correspond to the above presentations in Figures 9, 10, and 12.

We start with description of presentations and orbifolds for all truly spatial discrete groups generated by a primitive elliptic and a hyperbolic elements with *disjoint axes* (Theorem 4.1). As usual, we can also use the theorem when the elliptic generator is non-primitive, using recalculation formulas for parameters (see [4] or [9]).

Theorem 4.1. *Let $\Gamma = \langle f, g \rangle$ be a discrete $\mathcal{RP}$ group with $\beta = -4\sin^2(\pi/n)$, $n \geq 3$, $\beta' \in (0, +\infty)$, and $\gamma \in (-\infty, 0)$. The one of the following occurs:*

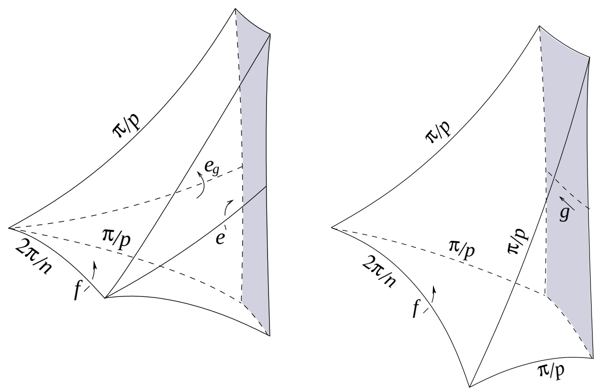
1. *$\gamma = -4\cosh^2 u$, where $u \in \mathcal{U}$, $(t(u), 2) = 2$, and $t(u) \geq 4$; Γ is isomorphic to $GT[n, \overline{\infty}; t(u)/2]$.*
2. *$\gamma = -4\cosh^2 u$, where $u \in \mathcal{U}$, $(t(u), 2) = 1$, and $t(u) \geq 3$; Γ is isomorphic to $Tet[n, \overline{\infty}; t(u)]$.*
3. *$n \geq 5$, $(n, 2) = 1$, $\gamma = -(\beta + 2)^2$, and $\beta' = 4(\beta + 4)\cosh^2 u - 4$, where $u \in \mathcal{U}$ and $t(u) \geq 4$; Γ is isomorphic to $Tet[n, t(u); 3]$.*
4. *$\beta = -3$, $\gamma = (\sqrt{5} - 3)/2$, and $\beta' = 2(7 + 3\sqrt{5})\cosh^2 u - 4$, where $u \in \mathcal{U}$ and $t(u) \geq 3$; Γ is isomorphic to $Tet[3, t(u); 5]$.*

Proof. All parameters for discrete groups in the statement of Theorem 4.1 are described in Theorem 3.2. We shall obtain a presentation for each discrete group by using the Poincaré polyhedron theorem.

Let Γ have parameters as in part 1 of Theorem 3.2. In Section 3.1, a fundamental polyhedron for the group Γ^* was described. Since $\widetilde{\Gamma}$ is the orientation preserving index 2 subgroup of Γ^*, we can take $\widetilde{\mathcal{P}} = \mathcal{P}(\varepsilon, \alpha, \tau, R_\omega(\alpha))$ as a fundamental polyhedron for $\widetilde{\Gamma}$ (see Figure 8(a)). In our notation $p = t(u)$.

Let $e_g = R_\tau R_\omega$. It is clear that $e_g = ge$. By applying the Poincaré polyhedron theorem to $\widetilde{\mathcal{P}}$ and face pairing transformations e, e_g, and f, we get

$$\widetilde{\Gamma} = \langle e, e_g, f \,|\, e^2, e_g^2, f^n, (fe)^2, (fe_g)^p \rangle,$$



(a) (b)

Figure 8: Fundamental polyhedra for $\widetilde{\Gamma}$ and Γ in case of disjoint axes

where p is an integer, ∞, or $\overline{\infty}$. Since $g = e_g e$,

$$\widetilde{\Gamma} = \langle f, g, e \,|\, f^n, e^2, (fe)^2, (ge)^2, (fge)^p \rangle.$$

If p is odd, then from the relations for $\widetilde{\Gamma}$ it follows that $e = (fgf^{-1}g^{-1})^{(p-1)/2} fg$. Hence, in this case $\widetilde{\Gamma} = \Gamma$ and $\Gamma \cong Tet[n, \overline{\infty}; p]$. Identifying faces of $\widetilde{\mathcal{P}}$, we get the orbifold $Q(\Gamma)$ shown in Figure 9(d).

If p is even, ∞, or $\overline{\infty}$, then Γ is a subgroup of index 2 in $\widetilde{\Gamma}$. To see this we apply the Poincaré theorem to the polyhedron $\mathcal{P}(\alpha, \tau, R_\omega(\alpha), R_\varepsilon(\tau))$ (see Figure 8(b)). Then

$$\Gamma = \langle f, g \,|\, f^n, (fgf^{-1}g^{-1})^{p/2} \rangle \cong GT[n, \overline{\infty}; p/2].$$

The orbifold $Q(\Gamma)$ is shown in Figure 9(a).

Now consider the groups with parameters from part 2 of Theorem 3.2. In this case $t(u) = q$ from Theorem 3.1. By applying the Poincaré theorem to the polyhedron $\mathcal{P}(\varepsilon, \alpha, \xi_1, R_\omega(\alpha), R_\omega(\xi_1))$ and the group generated by f, e, and s, where $s = R_\omega R_{\xi_1}$, we get the following presentation for the group $\langle f, e, s \rangle$:

$$\langle f, e, s \,|\, f^n, e^2, s^3, (fe)^2, (fs)^2, (se)^q \rangle.$$

Since $x = R_\kappa R_\tau$, we have $x^2 = h$ and $x = fs^{-1}$. Therefore, $g = e_g e = f^{-1}he = f^{-1}x^2 e = f^{-1}(fs^{-1})^2 e = s^{-1}fs^{-1}e$, and hence $\Gamma \subseteq \langle f, e, s \rangle$.

Since $h^n = 1$, n is odd and $h^2 = [f, g]$, we have that $h = [f, g]^{-(n-1)/2} \in \Gamma$. Further, $e_g = f^{-1}h$, and so $e = e_g g = f^{-1}hg \in \Gamma$. From $x^n = 1$ we have that $x = h^{-(n-1)/2} \in \Gamma$ and, therefore, $s = xe_g = xge \in \Gamma$. Thus, $\langle f, e, s \rangle \subseteq \Gamma$.

We see that $\langle f, e, s \rangle = \Gamma$ and Γ is isomorphic to the group $Tet[n, q; 3]$, where $q \geq 4$ is an integer, ∞, or $\overline{\infty}$.

The orbifold $Q(\Gamma)$ is shown in Figure 9(d).

Similarly, one can show that the groups with parameters from part 3 of Theorem 3.2 are isomorphic to $Tet[3, t(u); 5]$, where $t(u) \geq 3$ is an integer, ∞, or $\overline{\infty}$. $\square$

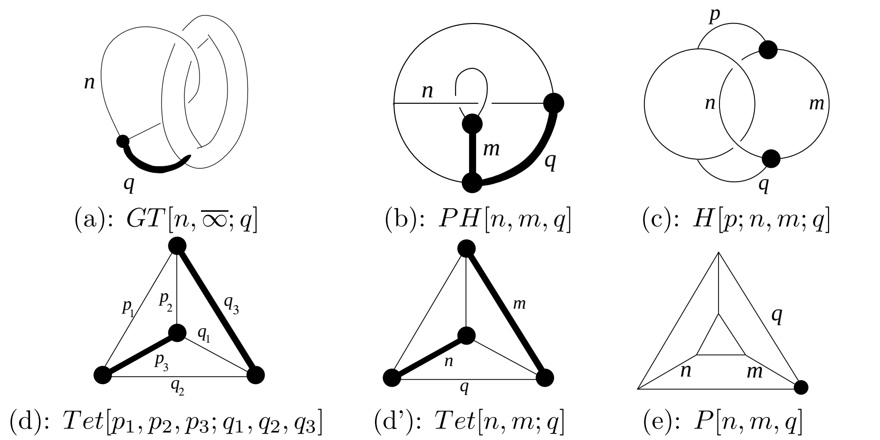


Figure 9: Orbifolds embedded in $\mathbb{S}^3$

Let $T(p)$, $p \in \mathbb{Z}$, be a Seifert fibred solid torus obtained from a trivial fibred solid torus $D^2 \times \mathbb{S}^1$ by cutting it along $D^2 \times \{x\}$ for some $x \in \mathbb{S}^1$, rotating one of the discs through $2\pi/p$ and glueing back together.

Denote by $\mathcal{S}(p)$ a space obtained by glueing $T(p)$ with its mirror symmetric copy along their boundaries fibre to fibre. Clearly, $\mathcal{S}(p)$ is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ and is p-fold covered by trivially fibred $\mathbb{S}^2 \times \mathbb{S}^1$. There are two critical fibres in $\mathcal{S}(p)$ whose length is p times shorter than the length of a regular fibre.

Next two theorems describe presentations and orbifolds for all truly spatial discrete groups $\Gamma = \langle f, g \rangle$ whose generators have *intersecting axes*, g is hyperbolic and f is primitive elliptic of even order (Theorem 4.2) or odd order (Theorem 4.5). In both theorems there are series of orbifolds embedded into $\mathbb{S}^2 \times \mathbb{S}^1$, and in case when f has odd order some orbifolds are embedded into $\mathbb{R}P^3$.

Theorem 4.2. *Let $\Gamma = \langle f, g \rangle$ be a discrete $\mathcal{RP}$ group so that $\beta = -4\sin^2(\pi/n)$, $n \geq 4$, $(n, 2) = 2$, $\beta' \in (0, +\infty)$, and $\gamma \in (0, -\beta\beta'/4)$. Then $\gamma = 4\cosh^2 u + \beta$, where $u \in \mathcal{U}$, $1/n + 1/t(u) < 1/2$, and one of the following occurs:*

1. *$(t(u), 2) = 2$ and $\beta' = 4\cosh^2 v/\gamma - 4\gamma/\beta$, where $v \in \mathcal{U}$, $t(v) \geq 3$, and $(t(v), 2) = 1$; Γ is isomorphic to $PH[n, t(u)/2; t(v)]$.*

2. *$(t(u), 2) = 2$ and $\beta' = 4\cosh^2 v/\gamma - 4\gamma/\beta$, where $v \in \mathcal{U}$, $t(v) \geq 4$, and $(t(v), 2) = 2$; Γ is isomorphic to $\mathcal{S}_2[n, t(u)/2; t(v)/2]$.*

3. *$(t(u), 2) = 1$ and $\beta' = 4(\gamma - \beta)\cosh^2 v/\gamma - 4\gamma/\beta$, where $v \in \mathcal{U}$, $t(v) \geq 3$, and $(t(v), 2) = 1$; Γ is isomorphic to $P[n, t(u), t(v)]$.*

4. $(t(u),2)=1$ *and* $\beta' = 4(\gamma-\beta)\cosh^2 v/\gamma - 4\gamma/\beta$*, where* $v \in \mathcal{U}$*,* $t(v) \geq 4$*, and* $(t(v),2)=2$*;* Γ *is isomorphic to* $GTet_1[n, t(u), t(v)/2]$.

5. $\beta = -2$*,* $(t(u),2)=1$*,* $t(u) \geq 5$*, and* $\beta' = \gamma^2 + 4\gamma$*;* Γ *is isomorphic to* $Tet[4, t(u); 3]$.

Proof. The idea of the proof is the same as for Theorem 4.1. We refer now to the part of Section 3.3 where n is even.

1. Let Γ have parameters as in row P_1 of Table 1. A fundamental polyhedron $\mathcal{P}(\alpha,\alpha',\delta,\varepsilon,\omega)$ for Γ^* is shown in Figure 2(a). A fundamental polyhedron for $\widetilde{\Gamma}$ is $\mathcal{P}(\alpha,\alpha',R_\omega(\alpha),R_\omega(\alpha'),\delta,\varepsilon)$, whose faces are identified by face pairing transformations f, $f' = R_\omega R_{\alpha'}$, $e_2 = f^{n/2}e$, and e_g. By the Poincaré polyhedron theorem, we get that

$$\widetilde{\Gamma} = \langle f, f', e_g, e_2 \,|\, f^n, (f')^n, e_g^2, e_2^2, (fe_2)^2, e_g f^{-1} e_g f', (f^{-1}f')^{m/2}, (e_2 f')^\ell \rangle.$$

Since $e_g = ge$ and $e_2 = f^{n/2}e$, we have that

$$\widetilde{\Gamma} = \langle f, g, e \,|\, f^n, e^2, (fe)^2, (ge)^2, (gfg^{-1}f)^{m/2}, (f^{n/2}g^{-1}fge)^\ell \rangle.$$

If ℓ is odd, $e \in \langle f, g\rangle$. Therefore, in this case $\Gamma = \widetilde{\Gamma} \cong PH[n, m/2; \ell]$, where $m/2$ is an integer, ∞, or $\overline{\infty}$, and ℓ is odd; the orbifold $Q(\Gamma)$ is shown in Figure 9(b).

Suppose now that ℓ is even. Then $|\widetilde{\Gamma} : \Gamma| = 2$ and a fundamental polyhedron for Γ is $\mathcal{P}' = \mathcal{P}(\alpha,\alpha',\varepsilon,R_\omega(\alpha),R_\omega(\alpha'),e_g(\varepsilon))$. Indeed, let L be a π-loxodromic element such that $L = e_g e_2$. Then L identifies the faces of $\mathcal{P}'$ lying in ε and $e_g(\varepsilon)$ and the group generated by f, f', and L has the following presentation:

$$\langle f, f', L \,|\, f^n, (f')^n, (f^{-1}f')^{m/2}, L^{-1}f'Lf, (L^{-1}fLf')^{\ell/2} \rangle.$$

Further, since $L = e_g e_2 = gf^{n/2}$, $\langle f, f', L\rangle = \Gamma$ and $\Gamma \cong \mathcal{S}_2[n, m/2; \ell/2]$, where $m/2$ and $\ell/2$ are integers, ∞, or $\overline{\infty}$; the orbifold $Q(\Gamma)$ is shown in Figure 10(a), see also remarks after the proof.

2. Now let Γ have parameters as in row P_2 of Table 1. A fundamental polyhedron for $\widetilde{\Gamma}$ is $\mathcal{P}(\alpha,\delta,\varepsilon,\xi,R_\omega(\alpha))$, whose faces are identified by f, e_2, $x = R_\delta R_\omega$, and $y = R_\omega R_\xi$. (We doubled the fundamental polyhedron for Γ^* shown in Figure 2(b).) Then

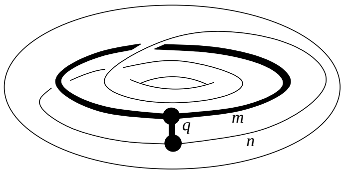
$$\langle f, e_2, x, y \,|\, f^n, e_2^2, x^2, y^2, (xy)^2, (xf)^2, (fe_2)^2, (ye_2)^k, (fy)^m \rangle.$$

Using the facts that $e_g = xy = ge$ and $xfx = f^{-1}$, we get $yfy = (yx)(xfx)(xy) = gef^{-1}ge = gfg^{-1}$. Therefore, since m is odd, $y = y(f,g)$. Furthermore, since $e_2 = f^{n/2}e$, $\langle f, e_2, x, y\rangle = \widetilde{\Gamma}$. Similarly to part 1 above, if k is odd, since in this case $e = e(f,g) \in \Gamma$, we have that $\widetilde{\Gamma} = \Gamma \cong P[n, m, k]$, where $m < \infty$ is also odd. The orbifold $Q(\Gamma)$ is shown in Figure 9(e).

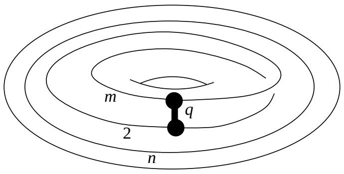
If k is even, Γ is an index 2 subgroup in $\widetilde{\Gamma}$. The polyhedron $\mathcal{P}(\alpha,\varepsilon,\xi,R_\omega(\alpha),R_\delta(\varepsilon))$, whose faces are identified by f, y, and $z = xe_2 =$

$ygf^{n/2} \in \Gamma$, is a fundamental polyhedron for Γ. Then Γ is isomorphic to $GTet_1[n, m, k/2]$, where $m < \infty$ is odd and $k/2$ is an integer, ∞, or $\overline{\infty}$. The orbifold $Q(\Gamma)$ is shown in Figure 10(d).

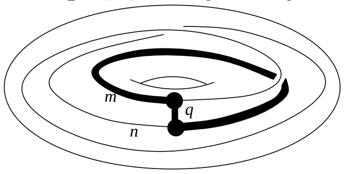
3. If Γ has parameters as in row P_3 of Table 1, it is easy to show that $\Gamma = \widetilde{\Gamma}$ and Γ is isomorphic to a tetrahedron group $Tet[4, m; 3]$, where $5 \leq m < \infty$ is odd. $\square$



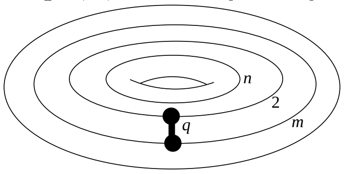
(a) Orbifolds embedded in $\mathcal{S}(2)$; $\pi_1^{orb}(Q) \cong \mathcal{S}_2[n, m, q]$



(b) Orbifolds embedded in $\mathcal{S}(2)$; $\pi_1^{orb}(Q) \cong GTet_2[n, m, q]$



(c) Orbifolds embedded in $\mathcal{S}(3)$; $\pi_1^{orb}(Q) \cong \mathcal{S}_3[n, m, q]$



(d) Orbifolds embedded in $\mathbb{S}^2 \times \mathbb{S}^1$; $\pi_1^{orb}(Q) \cong GTet_1[n, m, q]$

Figure 10: Orbifolds embedded in Seifert fibre spaces; only the torus that contains cone points or boundary components is shown.

Remark 4.3. Note that when $Q = Q(\mathcal{S}_2[n, m/2, \ell/2])$, due to the action of the face pairing transformation of the fundamental polyhedron, Q is embedded in a Seifert fibre space $\mathcal{S}(2)$ and the singular set is placed in $\mathcal{S}(2)$ in such a way that the axis of order m (if $m < \infty$) lies on a critical fibre of $\mathcal{S}(2)$ and the axis of order n lies on a regular one. In Figure 10(a) we draw only the solid torus that contains singular points (or boundary components). The other fibred torus is meant to be attached and is not shown.

Remark 4.4. Consider the case when parameters of Γ are as in row P_1 and $t(u) = \ell$ is even. Denote $Q = Q(\Gamma)$ and $\widetilde{Q} = Q(\widetilde{\Gamma})$, where $\Gamma \cong \mathcal{S}_2[n, m/2, \ell/2]$ and $\widetilde{\Gamma} \cong PH[n, m/2, \ell]$. Let us show the structure of the orbifold covering $\pi : Q \to \widetilde{Q}$. Assume for simplicity that $m, \ell < \infty$. Draw the orbifold Q (same as in Figure 10(a), but with the change of indices $q \to \ell/2$, $m \to m/2$) in the spherical shell $\mathbb{S}^2 \times I$ as shown in Figure 11; keep in mind that the inner and outer spheres are identified. Let σ be a circle in the plane (x, y) such that the inversion in the sphere for which σ is a big circle identifies the inner and the outer spheres. Let s be the order 2 automorphism of Q with the axis σ. Then s determines $\pi : Q \to \widetilde{Q}$ and $\langle \pi_1^{orb}(Q), s \rangle = \pi_1^{orb}(\widetilde{Q})$. The underlying space of $\widetilde{Q}$ is $\mathbb{S}^3$.

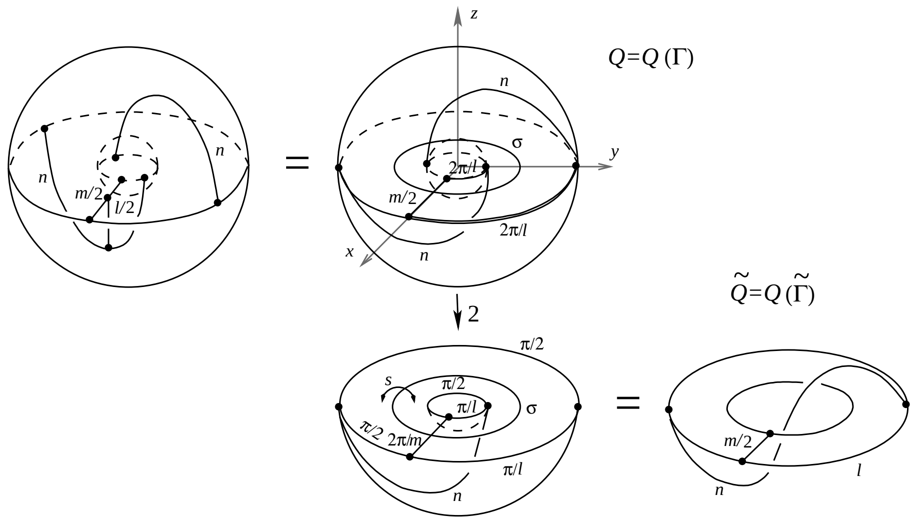


Figure 11: Branched covering $\pi : Q \to \widetilde{Q}$

Theorem 4.5. *Let $\Gamma = \langle f, g\rangle$ be a discrete $\mathcal{RP}$ group so that $\beta = -4\sin^2(\pi/n)$, $n \geq 3$, $(n,2) = 1$, $\beta' \in (0, +\infty)$, and $\gamma \in (0, -\beta\beta'/4)$. Then one of the following occurs:*

1. *$\gamma = 4\cosh^2 u + \beta$, where $u \in \mathcal{U}$, $(t(u), 2) = 2$, $1/n + 1/t(u) < 1/2$, and $\beta' = \frac{2}{\gamma}(\cosh v - \cos(\pi/n)) - \frac{2}{\gamma\beta}\left((\gamma-\beta)^2\cos(\pi/n) + \gamma(\gamma+\beta)\right)$, where $v \in \mathcal{U}$; Γ is isomorphic to $\mathcal{S}_3[n, t(u)/2, t(v)]$.*

2. *$\gamma = 4\cosh^2 u + \beta$, where $u \in \mathcal{U}$, $(t(u), 2) = 1$, $1/n + 1/t(u) < 1/2$, and $\beta' = \frac{2(\gamma-\beta)}{\gamma}\cosh v - \frac{2}{\gamma\beta}\left((\gamma-\beta)^2\cos(\pi/n) + \gamma(\gamma+\beta)\right)$, where $v \in \mathcal{U}$; Γ is isomorphic to $GTet_2[n, t(u), t(v)]$.*

3. *$n \geq 7$, $\gamma = (\beta+4)(\beta+1)$ and $\beta' = 2(\beta+2)^2(\cosh v - \cos(\pi/n))/(\beta+1) - 2\left(\beta^2 + 6\beta + 4\right)/\beta$, where $v \in \mathcal{U}$; Γ is isomorphic $GTet_2[n, 3, t(v)]$.*

4. *$\beta = -3$, $\gamma = 2\cos(2\pi/m) - 1$, where $m \geq 7$, $(m, 2) = 1$, and $\beta' = 2(\gamma^2 + 2\gamma + 2)/\gamma$; Γ is isomorphic to $GTet_1[m, 3, 2]$.*

5. *$n \geq 5$, $(n, 3) = 1$, $\gamma = \beta + 3$, and $\beta' = 2\left((\beta - 3)\cos(\pi/n) - 2\beta - 3\right)/\beta$; Γ is isomorphic to $H[2; 3, n; 2]$.*

6. *$(\beta, \gamma, \beta') = ((\sqrt{5} - 5)/2, (\sqrt{5} \pm 1)/2, 3(\sqrt{5} + 1)/2)$; Γ is isomorphic to $H[2; 2, 5; 3]$.*

7. *$(\beta, \gamma, \beta') = (-3, (\sqrt{5} \pm 1)/2, \sqrt{5})$ or $(\beta, \gamma, \beta') = ((\sqrt{5} - 5)/2, (\sqrt{5} - 1)/2, \sqrt{5})$, or $(\beta, \gamma, \beta') = ((\sqrt{5} - 5)/2, \sqrt{5} + 2, (5\sqrt{5} + 9)/2)$; in all cases Γ is isomorphic to $H[2; 2, 3; 5]$.*

8. $(\beta, \gamma, \beta') = ((\sqrt{5}-5)/2, (\sqrt{5}-1)/2, (3\sqrt{5}-1)/2)$; Γ *is isomorphic to* $Tet[3,3;5]$.

9. $\beta = -3$, $\gamma = 2\cos(2\pi/m)$, *where* $m \geq 7$, $(m,4) = 1$, *and* $\beta' = 2\gamma$; Γ *is isomorphic to* $Tet[3,4;m]$.

10. $n \geq 5$, $(n,3) = 1$, $\gamma = 2(\beta+3)$, *and* $\beta' = -6(2\cos(\pi/n)+\beta+2)/\beta$; Γ *is isomorphic to* $R[n,2;2]$.

11. $\beta = -3$, $\gamma = 2\cos(2\pi/m)$, *where* $m \geq 8$, $(m,4) = 2$, *and* $\beta' = 2\gamma$, *then* Γ *is isomorphic to* $H[m;3,3;2]$.

12. $\beta = -3$, $\gamma = 2\cos(2\pi/m) - 1$, *where* $m \geq 4$, $(m,3) = 1$, *and* $\beta' = \gamma^2 + 4\gamma$; Γ *is isomorphic to* $T[2,3,3;2,3,m]$.

Sketch of proof. Now we shall use fundamental polyhedra for Γ^* described in Section 3.3 for n odd and the Poincaré theorem to find a presentation for Γ.

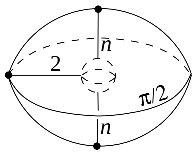
1. Let Γ have parameters as in row P_4 of Table 1. Consider the polyhedron bounded by α, α', $R_\omega(\alpha)$, $R_\omega(\alpha')$, ζ, $R_\omega(\zeta)$, $e_g(\zeta)$, and $R_\omega(e_g(\zeta))$, which is the union of four copies of $\mathcal{P}$. Its faces are identified by f, $f' = e_g f e_g$, and two loxodromic elements $L = e_g e_1$ and $L' = e_g e_1 f^{-1} = Lf^{-1}$. Using the Poincaré theorem, one can show that

$$\langle f, f', L, L' \rangle = \langle f, L \mid f^n, (LfL^{-1}f)^{m/2}, (LfL^{-1}fL^{-1}fL)^{\ell} \rangle.$$

Further, since $L = e_g e_1 = gef^{(n-1)/2}e = gf^{-(n-1)/2}$, the group $\langle f, L \rangle = \Gamma$. Hence, Γ is isomorphic to $\mathcal{S}_3[n, m/2, \ell]$, where m is even $(1/n + 1/m \leq 1/2)$, or $m = \infty$ or $\overline{\infty}$, and $\ell \geq 2$ is an integer. It is clear that $\Gamma \neq \widetilde{\Gamma}$ for all admissible values of m and ℓ. The orbifold $Q(\Gamma)$ is shown in Figure 10(c).

2. For groups Γ with parameters as in rows P_5 or P_6, one can show that Γ is isomorphic to $GTet_2[n,m,\ell]$ and $GTet_2[n,3,\ell]$, respectively, where m is odd $(1/n + 1/m < 1/2)$ and $\ell \geq 2$ is an integer, ∞, or $\overline{\infty}$. In this case Γ is always an index 2 subgroup of $\widetilde{\Gamma}$. The orbifold is shown in Figure 10(b).

3. If Γ has parameters as in row P_9, one can show that a fundamental polyhedron of Γ (compare with Figure 4(b)) is bounded by ω, ξ, $R_\alpha(\omega)$, $R_\alpha(\xi)$, μ, and $R_\delta(\mu)$, and its faces are identified by f, $h = R_\xi R_\alpha$, and $L = R_\delta R_\mu$. Then $\Gamma \cong GTet_1[m,3,2]$ and the orbifold $Q(\Gamma)$ is shown in Figure 10(d).



$\pi_1^{orb}(Q) \cong R[n,2;2]$

Figure 12: Underlying space is $\mathbb{R}P^3 \backslash \mathbb{B}^3$

4. Analogously, one can produce orbifolds and presentations of their fundamental groups for the rest of the parameters. The corresponding orbifolds are embedded in $\mathbb{S}^3$ for all of these groups except $R[n, 2; 2]$, for which the underlying space is $\mathbb{R}P^3 \backslash \mathbb{B}^3$. This group has parameters P_8. $\square$

Remark 4.6. When $Q = Q(\mathcal{S}_3[n, m; q])$, the singular set of Q is placed into $\mathcal{S}(3)$ in such a way that the curve consisting of cone points of indices n and m lies on a regular fibre. When $Q = Q(GTet_2[n, m; q])$, the curve consisting of cone points of indices m and 2 lies on a regular fibre, and the singular component of index n lies on a critical fibre.

References

[1] M. Boileau and J. Porti, *Geometrization of* 3*-orbifolds of cyclic type. With an appendix: Limit of hyperbolicity for spherical* 3*-orbifolds by Michael Heusener and Joan Porti*, Astérisque. 272. Paris: Sosiété Mathématique de France. 2001.

[2] D. B. A. Epstein and C. Petronio, *An exposition of Poincaré's polyhedron theorem*, L'Enseignement Mathématique **40** (1994), 113–170.

[3] F. W. Gehring and G. J. Martin, *Stability and extremality in Jørgensen's inequality*, Complex Variables Theory Appl. **12** (1989), no. 1-4,

[4] F. W. Gehring and G. J. Martin, *Chebyshev polynomials and discrete groups*, Proc. of the Conf. on Complex Analysis (Tianjin, 1992), 114–125, Conf. Proc. Lecture Notes Anal., I, Internat. Press, Cambridge, MA, 1994.277–282.

[5] C. D. Hodgson and I. Rivin, *A characterization of compact convex polyhedra in hyperbolic* 3*-space*, Invent. Math. **111** (1993), no. 1, 77–111.

[6] E. Klimenko, *Discrete groups in three-dimensional Lobachevsky space generated by two rotations*, Siberian Math. J. **30** (1989), no. 1, 95–100.

[7] E. Klimenko, *Some remarks on subgroups of* $\mathrm{PSL}(2, \mathbb{C})$, Q & A in General Topology **8** (1990), no. 2, 371–381.

[8] E. Klimenko, *Some examples of discrete groups and hyperbolic orbifolds of infinite volume*, J. Lie Theory **11** (2001), 491–503.

[9] E. Klimenko and N. Kopteva, *Discreteness criteria for $\mathcal{RP}$ groups*, Israel J. Math. **128** (2002), 247–265.

[10] E. Klimenko and N. Kopteva, *All discrete $\mathcal{RP}$ groups whose generators have real traces*, Int. J. Algebra Comput. **15** (2005), no. 3, 577–618.

[11] E. Klimenko and N. Kopteva, *Discrete $\mathcal{RP}$ groups with a parabolic generator*, Sib. Math. J. **46** (2005), no. 6, 1069–1076.

[12] E. Klimenko and N. Kopteva, *Two-generator Kleinian orbifolds*, 2006, preprint, available at `http://www.arxiv.org/math.GT/0606066`.

[13] B. Maskit, *Kleinian groups*, Grundlehren der Math. Wiss. 287, Springer-Verlag, 1988.

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