ON THE GEOMETRY OF THE ORBITS OF HERMANN ACTIONS

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Abstract. We investigate the submanifold geometry of the orbits of Hermann actions on Riemannian symmetric spaces. After proving that the curvature and shape operators of these orbits commute, we calculate the eigenvalues of the shape operators in terms of the restricted roots. As applications, we get a formula for the volumes of the orbits and a new proof of a Weyl-type integration formula for Hermann actions.

Contents

1. Introduction and Results

An isometric action of a compact Lie group on a Riemannian manifold M is called *polar* if it admits a *section*, i.e. a connected submanifold Σ of M that meets all orbits perpendicularly at each point of intersection. If the section is flat, the action is called *hyperpolar*.

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In this paper, $M = G/K$ will denote a Riemannian symmetric space of compact type. As the classification of hyperpolar actions on irreducible symmetric spaces of compact type [10] shows, all examples of such actions of cohomogeneity at least two are orbit equivalent to the so-called *Hermann actions*, i.e. actions of symmetric subgroups of G. Recall that a subgroup $H \subset G$ is called *symmetric* if there exists an involutive automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ with fixed point algebra $\mathfrak{h}$.

In Section 3, we prove the following theorem.

Theorem. *Let $H \subset G$ be a symmetric subgroup, $p \in M$ regular and $v, w \in \nu_p Hp$. Then the tangent space $T_p Hp$ is an invariant subspace of the curvature operator $R_v(x) = R(x, v)v$ and the restriction of R_v to $T_p Hp$ commutes with the shape operator A_w of Hp.*

Therefore the curvature and shape operators of Hp can be simultaneously diagonalized. The eigenspaces of the curvature operators are given by the root spaces of M; more precisely, a coarser version of the root space decomposition obtained by regarding only the restrictions of the roots to the tangent space of the section is relevant here – see Section 4. As a corollary of the above theorem, we obtain that for singular orbits, the restricted curvature operator R_v commutes with the shape operator A_w if v and w lie in the same section.

In Section 5, we restrict ourselves to the case where H can be conjugated in such a way that the involutions corresponding to H and K commute[1], which is possible except in a few cases [4]. We can completely determine the eigenspaces of the shape operators in terms of the restrictions of the roots (Theorem 5.3), thereby generalizing [15] where the case $H = K$ is treated.

In the general case, which is treated in Section 6, we can show how the eigenvalues of A_v change if the normal direction v is varied (Proposition 6.1).

Using the methods of [5], where the case $H = K$ is treated, we calculate in Section 7 the volumes of the principal orbits; furthermore we reprove a Weyl-type integration formula for actions of Hermann type ([6], which is a generalization of Theorem I.5.10 of [8]) using our calculations of the shape operators.

We would like to remark that, with slight modifications, our results are also true in the noncompact case, the only difference being some sign changes and some replacements of trigonometric functions by hyperbolic ones. Nevertheless, for better readability, we will present the proofs only for the compact case. Note that in the noncompact case, H

[1]Note that in this case the triple (G, H, K) is called a *symmetric triad* in [3].

can always be conjugated in such a way that the two involutions commute, see [1], Lemma 10.2. The shape operators in the noncompact case are also calculated in [11], but in a completely different way.

2. Preliminaries

Let $M = G/K$ be a Riemannian symmetric space of compact type and set $p = eK$. Then G is a semisimple compact Lie group, and we assume the metric on M to be induced by the Killing form of G. The Lie algebra $\mathfrak{g}$ can be identified with the Lie algebra of Killing vector fields on M, with the bracket being the negative of the bracket of the Killing vector fields. Considering the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

and using the identification of $\mathfrak{g}$ with the Killing vector fields, we have

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid X(p) = 0\} \quad \text{and} \quad \mathfrak{m} = \{X \in \mathfrak{g} \mid (\nabla X)_p = 0\}, \tag{1}$$

see [14], Lemma 6.8. The Killing vector fields in $\mathfrak{m}$ are those induced by transvections along geodesics through p.

If $X, Y, Z \in \mathfrak{m}$, we can express the curvature of M at the point p by

$$R(X(p), Y(p))Z(p) = -[[X, Y], Z](p). \tag{2}$$

Note that this equality remains valid if we assume only two of the Killing vector fields to be induced by transvections – if e.g. $X \in \mathfrak{g}$ is arbitrary, this follows from $[[\mathfrak{k}, \mathfrak{m}], \mathfrak{m}] \subset [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$.

Let now $\mathfrak{a} \subset \mathfrak{m}$ be a maximal abelian subalgebra, denote the set of restricted roots by Δ and a choice of positive roots by Δ^+. Then the corresponding root space decomposition of M is

$$\mathfrak{k} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \oplus \sum_{\alpha \in \Delta^+} \mathfrak{k}_\alpha \quad \text{and} \quad \mathfrak{m} = \mathfrak{a} \oplus \sum_{\alpha \in \Delta^+} \mathfrak{m}_\alpha, \tag{3}$$

where

$$\mathfrak{k}_\alpha = \{X \in \mathfrak{k} \mid \mathrm{ad}_W^2(X) = -\alpha(W)^2 X \text{ for all } W \in \mathfrak{a}\} \tag{4}$$

and

$$\mathfrak{m}_\alpha = \{X \in \mathfrak{m} \mid \mathrm{ad}_W^2(X) = -\alpha(W)^2 X \text{ for all } W \in \mathfrak{a}\}. \tag{5}$$

We call $X \in \mathfrak{k}_\alpha$ and $\mathfrak{m}_\alpha$ *related* if $[W, X] = -\alpha(W)Y$ and $[W, Y] = \alpha(W)X$ for all $W \in \mathfrak{a}$ (see [12], p. 61). For any $X \in \mathfrak{k}_\alpha$ there exists a related vector $Y \in \mathfrak{m}_\alpha$, and vice versa; in particular, the vector spaces $\mathfrak{k}_\alpha$ and $\mathfrak{m}_\alpha$ are isomorphic.

For $v \in T_pM$, the *curvature operator* R_v is defined to be the endomorphism of T_pM given by $R_v(u) = R(u, v)v$.

The *shape operator* $A_\xi : T_pN \to T_pN$ of a submanifold $N \subset M$ in the normal direction $\xi \in \nu_pN$ is defined as $A_\xi x = -(\nabla_x \xi)^T$; with this choice of sign, a Jacobi field J along the normal geodesic γ in direction ξ is an N-Jacobi field if and only if $J(0) \in T_pN$ and $J'(0) + A_\xi J(0) \in \nu_pN$.

Let $H \subset G$ act on $M = G/K$. If $p \in M$ is regular, the fact that the slice representation at p is trivial implies that we can extend normal vectors to well-defined H-equivariant normal vector fields on Hp. If the H-action is polar, these are automatically parallel with respect to the normal connection, see [13], Theorem 5.6.7. For any such vector field ξ, we thus get

$$A_{\xi(p)} x = -\nabla_x \xi. \tag{6}$$

The exponential map of G will be denoted by exp and the one of M by Exp. For parallel translation along a curve γ, we will write $\gamma||_{t_0}^{t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M$.

3. Proof of the Main Theorem

In this section, we will first prove the following theorem about the principal orbits of Hermann actions; the singular orbits will be dealt with in Corollary 3.3.

Theorem 3.1. *Let $H \subset G$ be a symmetric subgroup, $p \in M$ regular and $v, w \in \nu_pHp$. Then the tangent space T_pHp is an invariant subspace of the curvature operator $R_v(x) = R(x, v)v$ and the restriction of R_v to T_pHp commutes with the shape operator A_w of Hp.*

First we need a lemma.

Lemma 3.2. *Let $X \in \mathfrak{h}$ and ξ be an H-equivariant normal vector field on Hp. Then $[X, \xi] = 0$.*

Proof. Let $\gamma(t) = \mathrm{Exp}(t\xi(p))$. Then we have

$$\begin{aligned}\nabla_{X(p)}\xi &= \frac{\nabla}{ds}\bigg|_{s=0} \xi(\exp(sX)\cdot p) = \frac{\nabla}{ds}\bigg|_{s=0} \frac{d}{dt}\bigg|_{t=0} \exp(sX)\cdot\gamma(t) \\ &= \frac{\nabla}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} \exp(sX)\cdot\gamma(t) = \frac{\nabla}{dt}\bigg|_{t=0} X(\gamma(t)) = \nabla_{\xi(p)}X.\end{aligned}$$

□

Proof of Theorem 3.1. The invariance of T_pHp under R_v follows from the fact that for any $x \in T_pHp$ and any $u \in \nu_pHp$, we have

$$\langle R(x, v)v, u\rangle = -\langle R(v, u)v, x\rangle = 0$$

because the action is polar.

Without loss of generality, we may assume that $p = eK$. Note that then $\mathfrak{t} := \mathfrak{h}^\perp \cap \mathfrak{m} \cong \nu_p Hp$ is an abelian subalgebra of $\mathfrak{g}$.

Let $\vartheta \in \mathfrak{t}$ be the Killing vector field on M with $\vartheta(p) = v$ and $(\nabla\vartheta)(p) = 0$ and let ξ be the H-equivariant normal vector field on Hp with $\xi(p) = w$. Furthermore, let $\xi' \in \mathfrak{t}$ be the Killing vector field with $\xi'(p) = w$ and $(\nabla\xi')(p) = 0$ and set $g(t) = \exp(t\xi(p))$. For $X \in \mathfrak{h}$, we have

$$
\begin{aligned}
R_v(A_w(X(p))) &\overset{(6)}{=} -R(\nabla_{X(p)}\xi, v)v \overset{3.2}{=} -R(\nabla_{\xi(p)}X, v)v \\
&= -\nabla_{\xi(p)}(R(X,\vartheta)\vartheta) + R(X(p), \underbrace{\nabla_{\xi(p)}\vartheta}_{=0})v + R(X(p), v)\underbrace{\nabla_{\xi(p)}\vartheta}_{=0}
\end{aligned}
$$

($\nabla R = 0$ on a symmetric space)

$$
\begin{aligned}
&= -\left.\frac{\nabla}{dt}\right|_{t=0} R(X(g(t)p), \vartheta(g(t)p))\vartheta(g(t)p) \\
&= -\left.\frac{\nabla}{dt}\right|_{t=0} d(g(t))(R(\mathrm{Ad}_{g(t)^{-1}}X, \mathrm{Ad}_{g(t)^{-1}}\vartheta)\mathrm{Ad}_{g(t)^{-1}}\vartheta) \\
&= -\left.\frac{\nabla}{dt}\right|_{t=0} d(g(t))R(\mathrm{Ad}_{g(t)^{-1}}X, \vartheta(p))\vartheta(p)
\end{aligned}
$$

($\mathfrak{t}$ is abelian)

$$
\begin{aligned}
&= -\left.\frac{\nabla}{dt}\right|_{t=0} {}_\gamma||_0^t R(\mathrm{Ad}_{g(t)^{-1}}X, \vartheta(p))\vartheta(p) \\
&= -\left.\frac{d}{dt}\right|_{t=0} R(\mathrm{Ad}_{g(t)^{-1}}X, \vartheta(p))\vartheta(p) \\
&= \left.\frac{d}{dt}\right|_{t=0} [[\mathrm{Ad}_{g(t)^{-1}}X, \vartheta], \vartheta](p)
\end{aligned}
$$

(ϑ is induced by transvections; see (2))

$$
\begin{aligned}
&= [[[X, \xi'], \vartheta], \vartheta](p) \\
&= [[[X, \vartheta], \xi'], \vartheta](p) + [[X, \underbrace{[\xi', \vartheta]}_{=0}], \vartheta](p) \\
&= [[[X, \vartheta], \vartheta], \xi'](p) \\
&= -\underbrace{\nabla_{[[X,\vartheta],\vartheta](p)}\xi'}_{=0\,(\xi'\in\mathfrak{p})} + \nabla_{\xi(p)}[[X, \vartheta], \vartheta]
\end{aligned}
$$

(note the sign of the bracket in $\mathfrak{g}$)

$$
\begin{aligned}
&= -\nabla_{R(X(p),v)v}\xi + [[[X, \vartheta], \vartheta], \xi](p) \\
&= A_w(R_v(X(p))) + [[[X, \vartheta], \vartheta], \xi](p).
\end{aligned}
$$

Since X and ϑ are Killing vector fields on M, we have $[[X,\vartheta],\vartheta] \in [[\mathfrak{h},\mathfrak{h}^\perp],\mathfrak{h}^\perp] \subset [\mathfrak{h}^\perp,\mathfrak{h}^\perp] \subset \mathfrak{h}$, where for the last inclusion we used that H is a symmetric subgroup of G. Hence, Lemma 3.2 implies that $[[[X,\vartheta],\vartheta],\xi](p) = 0$. □

For the singular orbits, we have the following corollary.

Corollary 3.3. *Let $H \subset G$ be a symmetric subgroup and $p \in M$ arbitrary. Then for all $v \in \nu_p Hp$, the tangent space T_pHp is invariant under the curvature operator R_v. If Σ is a section of the H-action passing through p and $v, w \in T_p\Sigma$, then the restriction of R_v to T_pHp commutes with the shape operator A_w of Hp.*

Proof. Conjugate H such that p is the origin. Let $\vartheta \in \mathfrak{h}^\perp \cap \mathfrak{m}$ be the Killing vector field induced by transvections with $\vartheta(p) = v$. Then for any $X \in \mathfrak{h}$, equation (2) yields

$$R(X(p),\vartheta(p))\vartheta(p) = -[[X,\vartheta],\vartheta](p) \in T_pHp$$

because $\mathfrak{h}$ is a symmetric subgroup.

Then the operators commute because the regular points in Σ are dense in Σ. □

Corollary 3.4. *Let $H \subset G$ be a symmetric subgroup. Then $\{A_v, R_v \mid v \in T_p\Sigma\}$ is a commuting family of endomorphisms of T_pHp, where Σ is a section passing through p.*

Proof. We first assume that p is regular. Then the R_v commute because ν_pHp is abelian as one sees by combining (2) with the Jacobi identity. The Ricci equation implies

$$\begin{aligned}\langle [A_v, A_w]x, y\rangle &= \langle R(x,y)v, w\rangle - \left\langle R^\perp(x,y)v, w\right\rangle \\ &= \langle R(v,w)x, y\rangle = 0\end{aligned}$$

for all $x, y \in T_pHp$ and all $v, w \in \nu_pHp$ since the normal bundle of Hp is flat. Therefore, the shape operators of Hp commute.

If p is not regular, the claim then follows from the fact that the regular points in Σ are dense in Σ. □

Remark. A submanifold N of a Riemannian manifold M is called *curvature-adapted* if T_pN is invariant under the curvature operator R_u and if the restriction of R_u to T_pN commutes with the shape operator A_u of N for any $p \in N$ and all $u \in \nu_pM$. Theorem 3.1 and Corollary 3.3 immediately imply that all the orbits of the H-action are curvature-adapted submanifolds of M.

4. Adapted Root Space Decomposition

The root space decomposition adapted to the H-action has already been described in [6]; we include it for the convenience of the reader.

Let $H \subset G$ act on M in a hyperpolar fashion; later we will assume that it is a symmetric subgroup such that the corresponding involution commutes with the one of $\mathfrak{k}$. Consider the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

and choose a maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{h}^\perp \cap \mathfrak{m}$. Then $\Sigma = \operatorname{Exp}(\mathfrak{t}) \subset M$ is a section for the H-action on M. Note that Σ is a torus since we assume the metric to be induced by the Killing form on $\mathfrak{g}$; see [9], Theorem 2.3. Note that

$$\mathfrak{m} = \operatorname{pr}_{\mathfrak{m}}\mathfrak{h} \oplus (\mathfrak{h}^\perp \cap \mathfrak{m}) \cong T_pHp \oplus \nu_pHp.$$

Let further $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{m}$ containing $\mathfrak{t}$. The set of restricted roots of M with respect to $\mathfrak{a}$ shall be denoted by Δ and a choice of positive roots by Δ^+.

Consider the corresponding root space decomposition

$$\mathfrak{k} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \oplus \sum_{\alpha\in\Delta^+} \mathfrak{k}_\alpha \quad \text{and} \quad \mathfrak{m} = \mathfrak{a} \oplus \sum_{\alpha\in\Delta^+} \mathfrak{m}_\alpha, \tag{7}$$

where the root spaces $\mathfrak{k}_\alpha$ and $\mathfrak{m}_\alpha$ are given by (4) and (5).

In the following, the restrictions of the roots to $\mathfrak{t}$ will be of greater importance than the roots themselves. We define

$$\Delta_{\mathfrak{t}} = \{\alpha|_{\mathfrak{t}} \mid \alpha \in \Delta\} \setminus \{0\}.$$

Note that for two roots $\alpha, \alpha' \in \Delta^+$, it is possible that $\alpha|_{\mathfrak{t}} = -\alpha'|_{\mathfrak{t}}$ (see the example in Section 8). Therefore, we let $\Delta_{\mathfrak{t}}^+ \subset \Delta_{\mathfrak{t}}$ be the set of nonzero restrictions of elements in Δ^+ to $\mathfrak{t}$, but if this occurs, we include only one of $\alpha|_{\mathfrak{t}}$ and $\alpha'|_{\mathfrak{t}}$ in $\Delta_{\mathfrak{t}}^+$. For any $\beta \in \Delta_{\mathfrak{t}}^+$, we set

$$\mathfrak{k}_\beta^{\mathfrak{t}} = \{X \in \mathfrak{k} \mid \operatorname{ad}_w^2(X) = -\beta(w)^2 X \text{ for all } w \in \mathfrak{t}\}$$

and

$$\mathfrak{m}_\beta^{\mathfrak{t}} = \{X \in \mathfrak{m} \mid \operatorname{ad}_w^2(X) = -\beta(w)^2 X \text{ for all } w \in \mathfrak{t}\}.$$

Lemma 4.1. *For any $\beta \in \Delta_{\mathfrak{t}}^+$, we have*

$$\mathfrak{k}_\beta^{\mathfrak{t}} = \sum_{\alpha\in\Delta^+ :\, \alpha|_{\mathfrak{t}} = \pm\beta} \mathfrak{k}_\alpha \quad \textit{and} \quad \mathfrak{m}_\beta^{\mathfrak{t}} = \sum_{\alpha\in\Delta^+ :\, \alpha|_{\mathfrak{t}} = \pm\beta} \mathfrak{m}_\alpha.$$

Furthermore,

$$\mathfrak{z}_{\mathfrak{m}}(\mathfrak{t}) = \mathfrak{a} \oplus \sum_{\alpha\in\Delta^+ :\, \alpha|_{\mathfrak{t}} = 0} \mathfrak{m}_\alpha \quad \textit{and} \quad \mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}) = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \oplus \sum_{\alpha\in\Delta^+ :\, \alpha|_{\mathfrak{t}} = 0} \mathfrak{k}_\alpha.$$

Proof. We only prove the first equality. For all $\beta \in \Delta_{\mathfrak{t}}^+$ and all $\alpha \in \Delta^+$ such that $\alpha|_{\mathfrak{t}} = \pm\beta$, we have $\mathfrak{k}_\alpha \subset \mathfrak{k}_\beta^{\mathfrak{t}}$ by definition. The desired equality then follows from (7). □

Consequently we have the decompositions

$$\mathfrak{k} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}) \oplus \sum_{\beta\in\Delta_{\mathfrak{t}}^+} \mathfrak{k}_\beta^{\mathfrak{t}} \quad \text{and} \quad \mathfrak{m} = \mathfrak{z}_{\mathfrak{m}}(\mathfrak{t}) \oplus \sum_{\beta\in\Delta_{\mathfrak{t}}^+} \mathfrak{m}_\beta^{\mathfrak{t}}. \tag{8}$$

From now on until the end of Section 5, we assume that H and K are symmetric subgroups of G corresponding to commuting involutions. Then the two Cartan decompositions

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} = \mathfrak{h} \oplus \mathfrak{p} \tag{9}$$

are compatible in the sense that

$$\mathfrak{g} = \mathfrak{k} \cap \mathfrak{p} \oplus \mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{m} \cap \mathfrak{h} \oplus \mathfrak{m} \cap \mathfrak{p}. \tag{10}$$

Lemma 4.2. *For all $\beta \in \Delta_{\mathfrak{t}}^+$, we have*

$$\mathfrak{k}_\beta^{\mathfrak{t}} = \mathfrak{k}_\beta^{\mathfrak{t}} \cap \mathfrak{p} \oplus \mathfrak{k}_\beta^{\mathfrak{t}} \cap \mathfrak{h} \tag{11}$$

$$\mathfrak{m}_\beta^{\mathfrak{t}} = \mathfrak{m}_\beta^{\mathfrak{t}} \cap \mathfrak{p} \oplus \mathfrak{m}_\beta^{\mathfrak{t}} \cap \mathfrak{h}. \tag{12}$$

Furthermore $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}) = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}) \cap \mathfrak{p} \oplus \mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}) \cap \mathfrak{h}$ *and* $\mathfrak{z}_{\mathfrak{m}}(\mathfrak{t}) = \mathfrak{z}_{\mathfrak{m}}(\mathfrak{t}) \cap \mathfrak{p} \oplus \mathfrak{z}_{\mathfrak{m}}(\mathfrak{t}) \cap \mathfrak{h}$.

Proof. Let $X \in \mathfrak{k}_\beta^{\mathfrak{t}}$. According to (10), we can decompose X as $X = X_{\mathfrak{p}} + X_{\mathfrak{h}}$, where $X_{\mathfrak{p}} \in \mathfrak{k} \cap \mathfrak{p}$ and $X_{\mathfrak{h}} \in \mathfrak{k} \cap \mathfrak{h}$. First of all we have

$$-\beta(w)^2 X_{\mathfrak{p}} - \beta(w)^2 X_{\mathfrak{h}} = \mathrm{ad}_w^2(X_{\mathfrak{p}}) + \mathrm{ad}_w^2(X_{\mathfrak{h}}).$$

for all $w \in \mathfrak{t}$. Since

$$\mathrm{ad}_w^2 X_{\mathfrak{p}} \in [\mathfrak{t}, [\mathfrak{t}, \mathfrak{k} \cap \mathfrak{p}]] \subset [\mathfrak{m} \cap \mathfrak{p}, [\mathfrak{m} \cap \mathfrak{p}, \mathfrak{k} \cap \mathfrak{p}]] \subset [\mathfrak{m} \cap \mathfrak{p}, \mathfrak{m} \cap \mathfrak{h}] \subset \mathfrak{k} \cap \mathfrak{p}$$

and

$$\mathrm{ad}_w^2 X_{\mathfrak{h}} \in [\mathfrak{t}, [\mathfrak{t}, \mathfrak{k} \cap \mathfrak{h}]] \subset [\mathfrak{m} \cap \mathfrak{p}, [\mathfrak{m} \cap \mathfrak{p}, \mathfrak{k} \cap \mathfrak{h}]] \subset [\mathfrak{m} \cap \mathfrak{p}, \mathfrak{m} \cap \mathfrak{p}] \subset \mathfrak{k} \cap \mathfrak{h},$$

we conclude $\mathrm{ad}_w^2(X_{\mathfrak{p}}) = -\beta(w)^2 X_{\mathfrak{p}}$ and $\mathrm{ad}_w^2(X_{\mathfrak{h}}) = -\beta(w)^2 X_{\mathfrak{h}}$. Thus we have proven (11). The proof of the rest of the lemma is similar. □

Remark. Note that the equations (11) and (12) do not have analogues for the root spaces $\mathfrak{k}_\alpha$ and $\mathfrak{m}_\alpha$ since they do not necessarily respect the decomposition (10).

We can now refine decomposition (8) as follows:

$$\mathfrak{k} = (\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}) \cap \mathfrak{h} \oplus \sum_{\beta\in\Delta_{\mathfrak{t}}^+} \mathfrak{k}_\beta^{\mathfrak{t}} \cap \mathfrak{h}) \oplus (\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}) \cap \mathfrak{p} \oplus \sum_{\beta\in\Delta_{\mathfrak{t}}^+} \mathfrak{k}_\beta^{\mathfrak{t}} \cap \mathfrak{p}) \tag{13}$$

and

$$\mathfrak{m} = (\mathfrak{z}_{\mathfrak{m}}(\mathfrak{t}) \cap \mathfrak{h} \oplus \sum_{\beta \in \Delta_{\mathfrak{t}}^+} \mathfrak{m}_\beta^{\mathfrak{t}} \cap \mathfrak{h}) \oplus (\mathfrak{t} \oplus \sum_{\beta \in \Delta_{\mathfrak{t}}^+} \mathfrak{m}_\beta^{\mathfrak{t}} \cap \mathfrak{p}). \tag{14}$$

5. Eigenvalues of the Shape Operator: Commuting Involutions

In this section, $H \subset G$ is a symmetric subgroup corresponding to an involution commuting with the one of K. Recall the refined Cartan decomposition (10)

$$\mathfrak{g} = \mathfrak{k} \cap \mathfrak{p} \oplus \mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{m} \cap \mathfrak{h} \oplus \mathfrak{m} \cap \mathfrak{p}$$

and that $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{m}$ containing a maximal abelian subspace $\mathfrak{t}$ of $\mathfrak{m} \cap \mathfrak{p}$. Then $\mathfrak{a}$ can be written as $\mathfrak{a} = \mathfrak{t} \oplus \mathfrak{t}'$, where $\mathfrak{t}' \subset \mathfrak{m} \cap \mathfrak{h}$.

Let $w \in \mathfrak{t}$ and set $p = \operatorname{Exp}(w)$ (which we do not assume to be regular) and $\gamma(t) = \operatorname{Exp}(tw)$. Our first goal is to express the tangent space T_pHp in terms of the restricted roots. Note that the case $H = K$ in the following proposition is the content of Proposition 3 of [15].

Proposition 5.1. *The tangent space T_pHp coincides with the parallel displacement of $(\mathfrak{z}_{\mathfrak{m}}(\mathfrak{t}) \cap \mathfrak{h}) \oplus V_1 \oplus V_2 \subset \mathfrak{m}$ along γ, where*

$$V_1 = \sum_{\beta \in \Delta_{\mathfrak{t}}^+,\, \beta(w) \notin \frac{\pi}{2} + \pi\mathbf{Z}} \mathfrak{m}_\beta^{\mathfrak{t}} \cap \mathfrak{h} \quad \text{and} \quad V_2 = \sum_{\beta \in \Delta_{\mathfrak{t}}^+,\, \beta(w) \notin \pi\mathbf{Z}} \mathfrak{m}_\beta^{\mathfrak{t}} \cap \mathfrak{p}.$$

Proof. Regarding the elements of $\mathfrak{h}$ as Killing fields on M, the equations in (1) yield

$$\mathfrak{m} \cap \mathfrak{h} = \{X \in \mathfrak{h} \mid (\nabla X)_{eK} = 0\}, \quad \mathfrak{k} \cap \mathfrak{h} = \{X \in \mathfrak{h} \mid X(eK) = 0\}. \tag{15}$$

Of course $T_pHp = U_1 + U_2$, where $U_1 = \{X(p) \mid X \in \mathfrak{h} \cap \mathfrak{m}\}$ and $U_2 = \{X(p) \mid X \in \mathfrak{h} \cap \mathfrak{k}\}$.

For any $\alpha \in \Delta^+$, let $\{X_i^\alpha\}_{i \in I_\alpha}$ be an orthonormal basis of $\mathfrak{m}_\alpha$; furthermore let $\{X_i^0\}_{i \in I_0}$ be an orthonormal basis of $\mathfrak{t}'$. Let E_i^α and E_i^0 be the parallel fields along γ with $E_i^\alpha(0) = X_i^\alpha(eK)$ and $E_i^0(0) = X_i^0(eK)$, respectively.

For $X \in \mathfrak{h}$ let $Y = X|_\gamma$ be the Jacobi field along γ obtained by restricting X to γ. Since $Y(t)$ is tangent to the orbit through $\gamma(t)$ for all t, it follows from the description of Jacobi fields on symmetric spaces that

$$Y(t) = \sum_{i\in I_0}(a_i + b_i t)E_i^0(t) + \sum_{\alpha\in\Delta^+}\sum_{i\in I_\alpha}(c_i \sin(\alpha(w)t) + d_i\cos(\alpha(w)t))E_i^\alpha(t) \tag{16}$$

for some constants a_i, b_i, c_i, d_i.

Consider first the case $X \in \mathfrak{m}_\beta^{\mathfrak{t}} \cap \mathfrak{h}$ for some $\beta \in \Delta_{\mathfrak{t}}^+$ and let $v = X(eK)$. According to Lemma 4.1, we can write

$$X = \sum_{\alpha\in\Delta^+ \colon \alpha|_{\mathfrak{t}} = \pm\beta}\sum_{i\in I_\alpha}\lambda_{\alpha,i}X_i^\alpha$$

for some constants $\lambda_{\alpha,i} \in \mathbf{R}$. Since $X \in \mathfrak{m}$, we have $Y'(0) = 0$ because of (15), so we get $b_i = c_i = 0$. It follows that

$$\begin{aligned} Y(t) &= \sum_{\alpha\in\Delta^+ \colon \alpha|_{\mathfrak{t}} = \pm\beta}\sum_{i\in I_\alpha}\lambda_{\alpha,i}\cos(\alpha(w)t)E_i^\alpha \\ &= \cos(\beta(w)t)\sum_{\alpha\in\Delta^+ \colon \alpha|_{\mathfrak{t}} = \pm\beta}\sum_{i\in I_\alpha}\lambda_{\alpha,i}E_i^\alpha(t) \\ &= \cos(\beta(w)t)_\gamma||_0^t v. \end{aligned}$$

We thus have shown that if $\beta(w) \notin \frac{\pi}{2} + \pi\mathbf{Z}$, then the parallel transport of $\mathfrak{m}_\beta^{\mathfrak{t}} \cap \mathfrak{h}$ along γ is contained in U_1.

Let now $X \in \mathfrak{z}_{\mathfrak{m}}(\mathfrak{t}) \cap \mathfrak{h}$ and $v = X(eK)$. Lemma 4.1 yields

$$X = \sum_{i\in I_0}\mu_i X_i^0 + \sum_{\alpha\in\Delta^+ \colon \alpha|_{\mathfrak{t}} = 0}\lambda_{\alpha,i}X_i^\alpha$$

for some constants μ_i and $\lambda_{\alpha,i}$. We obtain

$$Y(t) = \sum_{i\in I_0}\mu_i E_i^0(t) + \sum_{\alpha\in\Delta^+ \colon \alpha|_{\mathfrak{t}} = 0}\sum_{i\in I_\alpha}\lambda_{\alpha,i}\cos(\alpha(w)t)E_i^\alpha = {}_\gamma||_0^t v,$$

so the parallel transport of $\mathfrak{z}_{\mathfrak{m}}(\mathfrak{t})\cap\mathfrak{h}$ along γ is contained in U_1. It is now clear that U_1 is the direct sum of the parallel transport of $\mathfrak{z}_{\mathfrak{m}}(\mathfrak{t})\cap\mathfrak{h}\oplus V_1$.

It remains to describe U_2. For any $\alpha \in \Delta^+$, let $\{Z_i^\alpha\}_{i\in I_\alpha}$ be the orthonormal basis of $\mathfrak{k}_\alpha$ which is related to $\{X_i^\alpha\}$ by $[Z_i^\alpha, u] = \alpha(u)X_i^\alpha$ for all $u \in \mathfrak{a}$. For any $\alpha \in \Delta^+$, we have

$$[\mathfrak{k}_\alpha, u] = \begin{cases} \mathfrak{m}_\alpha & \text{if } \alpha(u) \neq 0 \\ 0 & \text{if } \alpha(u) = 0. \end{cases}$$

Lemma 4.1 now yields that an analogous relation is true for the root spaces with respect to $\mathfrak{t}$: for all $\beta \in \Delta_{\mathfrak{t}}^+$, we have

$$[\mathfrak{k}^{\mathfrak{t}}_\beta, u] = \begin{cases} \mathfrak{m}^{\mathfrak{t}}_\beta & \text{if } \beta(u) \neq 0 \\ 0 & \text{if } \beta(u) = 0. \end{cases}$$

For $X \in \mathfrak{k}$ we have that

$$\begin{aligned} Y(t) = X(\operatorname{Exp}(tw)) &= \frac{d}{ds}\Big|_{s=0} \exp(sX)\exp(tw)K \\ &= \frac{d}{ds}\Big|_{s=0} \exp(\operatorname{Ad}_{\exp sX} tw)K = (d\operatorname{Exp})_{tw}(t[X,w]). \end{aligned}$$

Hence, those $X \in \mathfrak{h}$ which lie in $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}) \cap \mathfrak{h}$ do not contribute to U_2. We thus have

$$U_2 = (d\operatorname{Exp})_w \sum_{\beta\in\Delta^+_{\mathfrak{t}}} [\mathfrak{k}^{\mathfrak{t}}_\beta \cap \mathfrak{h}, w] = (d\operatorname{Exp})_w \sum_{\beta\in\Delta^+_{\mathfrak{t}},\, \beta(w)\neq 0} \mathfrak{m}^{\mathfrak{t}}_\beta \cap \mathfrak{p}.$$

Let $v \in \mathfrak{m}^{\mathfrak{t}}_\beta \cap \mathfrak{p}$, where $\beta(w) \neq 0$, and write

$$v = \sum_{\alpha\in\Delta^+ :\, \alpha|_{\mathfrak{t}} = \pm\beta} \sum_{i\in I_\alpha} \lambda_{\alpha,i} X^\alpha_i.$$

for some constants $\lambda_{\alpha,i}$. Define $X \in \mathfrak{k}^{\mathfrak{t}}_\beta \cap \mathfrak{h}$ to be

$$X = \sum_{\alpha\in\Delta^+ :\, \alpha|_{\mathfrak{t}} = \pm\beta} \sum_{i\in I_\alpha} \pm\lambda_{\alpha,i} Z^\alpha_i.$$

By definition, we have $[X, w] = \beta(w)v$.

Since Y is the unique Jacobi field along γ with $Y(0) = 0$ and $Y'(0) = [X, w] = \beta(w)v$, we get

$$Y(t) = \sum_{\alpha\in\Delta^+ :\, \alpha|_{\mathfrak{t}} = \pm\beta} \sum_{i\in I_\alpha} \pm\lambda_{\alpha,i} \sin(\alpha(w)t) E^\alpha_i(t) = \sin(\beta(w)t)_\gamma||^t_0 v.$$

It follows that $Y(1)$ vanishes if and only if $\beta(w) \notin \pi\mathbf{Z}$. We have thus proven that U_2 is the parallel displacement of V_2 along γ. □

Corollary 5.2. *The point $p = \operatorname{Exp}(w) \in \Sigma$ is a regular point of the H-action if and only if*

1. $\beta(w) \notin \frac{\pi}{2} + \pi\mathbf{Z}$ *for all* $\beta \in \Delta^+_{\mathfrak{t}}$ *with* $\mathfrak{m}^{\mathfrak{t}}_\beta \cap \mathfrak{h} \neq \{0\}$ *and*
2. $\beta(w) \notin \pi\mathbf{Z}$ *for all* $\beta \in \Delta^+_{\mathfrak{t}}$ *with* $\mathfrak{m}_\beta \cap \mathfrak{p} \neq \{0\}$.

Choose a vector $u \in \mathfrak{t}$, let $c(t) = \operatorname{Exp}(w + tu)$ and $u(p) = \dot{c}(0)$. By Corollary 3.4, the shape and curvature operators can be simultaneously diagonalized. A concrete such diagonalization is given in the following theorem.

Theorem 5.3. *The decomposition of T_pHp into parallel displacements of the root spaces described in Proposition 5.1 is compatible with the decomposition into the eigenspaces of the shape operator $A_{u(p)}$ of Hp. More precisely,*

(1) *For $v \in \mathfrak{m}_\beta^\mathfrak{t} \cap \mathfrak{h}$ with $\beta(w) \notin \frac{\pi}{2} + \pi\mathbf{Z}$, we have*
$$A_{u(p)}({}_\gamma||_0^1 v) = \beta(u)\tan(\beta(w))_\gamma||_0^1 v. \tag{17}$$
(2) *For $v \in \mathfrak{m}_\beta^\mathfrak{t} \cap \mathfrak{p}$ with $\beta(w) \notin \pi\mathbf{Z}$, we have*
$$A_{u(p)}({}_\gamma||_0^1 v) = -\beta(u)\cot(\beta(w))_\gamma||_0^1 v. \tag{18}$$
(3) *For $v \in \mathfrak{z}_\mathfrak{m}(\mathfrak{t}) \cap \mathfrak{h}$, we have*
$$A_{u(p)}({}_\gamma||_0^1 v) = 0. \tag{19}$$

Proof. For any $s \in [0,1]$, let $\gamma_s(t) := \mathrm{Exp}(t(w+su))$. Note that $\gamma_0 = \gamma$.

First of all, let $X \in \mathfrak{m}_\beta^\mathfrak{t} \cap \mathfrak{h}$, where $\beta(w) \notin \frac{\pi}{2} + \pi\mathbf{Z}$, and set $v = X(eK)$. Let Y_s be the Jacobi field obtained by restriction of the Killing field X to γ_s. The initial values of Y_s are $Y_s(0) = v$ and $Y_s'(0) = 0$; as in the proof of Proposition 5.1, we get
$$Y_s(t) = \cos(\beta(w+su)t)_{\gamma_s}||_0^1 v, \tag{20}$$
since v is contained in a sum of root spaces corresponding to roots whose restrictions to $\mathfrak{t}$ coincide. We are interested in the Hp-Jacobi field $Y(t) = Y_t(1)$ along c. Its initial values are $Y(0) = \cos(\beta(w))_\gamma||_0^1 v$ and
$$\begin{aligned} Y'(0) &= \left.\frac{d}{dt}\right|_{t=0} \cos(\beta(w+tu))_{\gamma_t}||_0^1 v \\ &= -\beta(u)\sin(\beta(w))_\gamma||_0^1 v, \end{aligned} \tag{21}$$
since $\left.\frac{\nabla}{dt}\right|_{t=0} {}_{\gamma_t}||_0^1 v = 0$ (use Lemma 8.3.2 of [2], together with the fact that the γ_s lie in the flat section Σ). The fact that Y is an Hp-Jacobi field along c now implies
$$\begin{aligned} &Y'(0) + A_{u(p)}Y(0) \\ &\qquad = -\beta(u)\sin(\beta(w))_\gamma||_0^1 v + \cos(\beta(w))A_{u(p)}({}_\gamma||_0^1 v) \in \nu_p Hp, \end{aligned}$$
so we get
$$A_{u(p)}({}_\gamma||_0^1 v) = \beta(u)\tan(\beta(w))_\gamma||_0^1 v, \tag{22}$$
which is Equation (17).

In order to prove Equation (18), choose some vector $v \in \mathfrak{m}_\beta^\mathfrak{t} \cap \mathfrak{p}$ with $\beta(w) \notin \pi\mathbf{Z}$. Let $X \in \mathfrak{k}_\beta^\mathfrak{t}$ be such that $[H, X] = -\beta(H)v$ for all $H \in \mathfrak{t}$. We have $X \in \mathfrak{h}$ since $\beta(w) \neq 0$. Now continue exactly as above: let Y_s be the Jacobi field obtained by restriction of the Killing field X to γ_s.

Its initial values are $Y_s(0) = 0$ and $Y_s'(0) = [X, w + su] = \beta(w + su)v$, so we get

$$Y_s(t) = \sin(\beta(w + su)t)_{\gamma_s}||_0^t v. \tag{23}$$

The Hp-Jacobi field $Y(t) = Y_t(1)$ has initial values $Y(0) = \sin(\beta(w))_\gamma||_0^1 v$ and

$$\begin{aligned} Y'(0) &= \left.\frac{d}{dt}\right|_{t=0} \sin(\beta(w + tu))_{\gamma_t}||_0^1 v \\ &= \alpha(u)\cos(\beta(w))_\gamma||_0^1 v, \end{aligned} \tag{24}$$

so we obtain

$$\begin{aligned} Y'(0)+&A_{u(p)}Y(0) \\ &= \beta(u)\cos(\beta(w))_\gamma||_0^1 v + \sin(\beta(w))A_{u(p)}({}_\gamma||_0^1 v) \in \nu_p Hp; \end{aligned}$$

thus, Equation (18) follows.

Finally, let $X \in \mathfrak{z}_\mathfrak{m}(\mathfrak{t}) \cap \mathfrak{h}'$ and set $v = X(eK)$. Let again Y_s be the restriction of X along γ_s and $Y(t) = Y_t(1)$. Then we see that $Y_s(t) = {}_{\gamma_s}||_0^1 v$ and hence Y satisfies the initial conditions $Y(0) = {}_\gamma||_0^1 v$ and $Y'(0) = 0$. Equation (19) follows immediately.

□

6. Eigenvalues of the Shape Operator: General Case

In this section we will determine, as far as possible, the eigenvalues of the shape operators in the general case of an arbitrary Hermann action. This is independent of the calculations in section 5.

The following proposition shows the dependence of the normal direction. Let the origin be a regular point, denoted by p. Note that $T_pHp \cong \mathrm{pr}_\mathfrak{m}\mathfrak{h} = \sum_\beta \mathfrak{m}_\beta^\mathfrak{t} \oplus (\mathfrak{z}_\mathfrak{m}(\mathfrak{t}) \cap \mathrm{pr}_\mathfrak{m}\mathfrak{h})$.

Proposition 6.1. (1) *There exists a refinement* $\mathfrak{m}_\beta^\mathfrak{t} = \sum_i V_{\beta,i}$ *of the root spaces, together with constants* $c_{\beta,i}$*, such that for* $v \in \nu_p Hp$ *and all* $x \in V_{\beta,i}$*, we have*

$$A_v x = c_{\beta,i}\beta(v)x.$$

(2) *For* $x \in \mathfrak{z}_\mathfrak{m}(\mathfrak{t}) \cap \mathrm{pr}_\mathfrak{m}\mathfrak{h}$*, we have*

$$A_v x = 0.$$

Proof. Let $x \in T_pHp$ be a common eigenvector of all curvature and shape operators in normal directions.

Let the linear form $f : \nu_p Hp \to \mathbf{R}$, depending on x, be defined by $A_v x = f(v)x$. Choose $X \in \mathfrak{h}$ such that $X(p) = x$. We write $X = X_\mathfrak{k} + X_\mathfrak{m}$ with $X_\mathfrak{k} \in \mathfrak{k}$ and $X_\mathfrak{m} \in \mathfrak{m}$.

For $v \in \nu_p Hp$, we denote by $\xi' \in \mathfrak{m}$ the Killing vector field induced by transvections with $\xi'(p) = v$ and by ξ the H-equivariant parallel normal vector field with $\xi(p) = v$. Then we have

$$\begin{aligned}[][\xi', X_\mathfrak{k}](p) &= -\nabla_v X_\mathfrak{k} + \nabla_{\underbrace{X_\mathfrak{k}(p)}_{=0}} \xi' = -\nabla_v X = -\nabla_x \xi - \underbrace{[\xi, X]}_{=0} \\ &= A_v x = f(v)x;\end{aligned}$$

since $[\xi', X_\mathfrak{k}] \in [\mathfrak{m}, \mathfrak{k}] \subset \mathfrak{m}$, it follows that

$$[\xi', X_\mathfrak{k}] = f(v) X_\mathfrak{m}. \tag{25}$$

Let us now first regard the case of $x \in \mathfrak{m}_\beta^\mathfrak{t}$ for some root β. Since (25) is valid for all $\xi' \in \mathfrak{m}$, we can write $X_\mathfrak{k} = X_{\mathfrak{k},0} + X_{\mathfrak{k},1}$ with $X_{\mathfrak{k},0} \in \mathfrak{z}_\mathfrak{k}(\mathfrak{m})$ and $X_{\mathfrak{k},1} \in \mathfrak{k}_\beta^\mathfrak{t}$; the vector $Z \in \mathfrak{m}_\beta^\mathfrak{t}$ related to $X_{\mathfrak{k},1}$ is a multiple of $X_\mathfrak{m}$, i.e. there exists some constant c, independent of v, with $f(v) = c \cdot \alpha(v)$.

If $x \in \mathfrak{m}$ is such that $[X_\mathfrak{m}, \mathfrak{t}] = 0$, it follows from (25) that

$$f(v) \langle X_\mathfrak{m}, X_\mathfrak{m} \rangle = \langle [\xi', X_\mathfrak{k}], X_\mathfrak{m} \rangle = - \langle X_\mathfrak{k}, [\xi', X_\mathfrak{m}] \rangle = 0;$$

hence $A_v x = 0$. □

Remark. The explicit description of the eigenvalues in the case of commuting involutions was possible because there existed a point $p \in M$ such that every H-Killing vector field could be written as the sum of one vanishing at p and one with derivative vanishing at p, leading to eigenvalues containing either a cotangent or a tangent.

In the general case such a point does not exist, but for each Killing vector field we can choose a point where either it vanishes or its derivative. For the following calculation we choose to express the $c_{\beta,i}$ in terms of zeros of the Killing vector fields themselves; hence, only the cotangent occurs.

Now we can investigate how the $c_{\beta,i}$ and the eigenvalues of the shape operators change when varying the orbit. In the notation above, let $Y_w := X \circ \gamma_w$, where $\gamma_w(t) = \mathrm{Exp}(tw)$ is the geodesic in direction w. For $x = X(p) \in V_{\beta,i} \subset \mathfrak{m}_\beta^\mathfrak{t}$, we have

$$Y_w(t) = (-c_{\beta,i} \sin(\beta(w)t) + \cos(\beta(w)t))_{\gamma_w} ||_0^t x.$$

If $t_{\beta,i}$ is a fixed zero of some Y_{w_0} with $\beta(w_0) = 1$, we can write $c_{\beta,i} = \cot(t_{\beta,i})$. Regarding the $H\gamma_w(1)$-Jacobi field $Y(t) := Y_{w+tv}(1)$, we can determine the shape operator of the orbit $H\gamma_w(1)$:

$$\begin{aligned} 0 &= Y'(0) + A_v Y(0) \\ &= (-c_{\beta,i}\beta(v)\cos(\beta(w)) - \beta(v)\sin(\beta(w)))_{\gamma_w} ||_0^1 x + A_v Y_w(1). \end{aligned}$$

Hence

$$\begin{aligned}A_v({}_{\gamma_w}||_0^1 x) &= \beta(v)\frac{c_{\beta,i}\cos(\beta(w))+\sin(\beta(w))}{-c_{\beta,i}\sin(\beta(w))+\cos(\beta(w))}{}_{\gamma_w}||_0^1 x\\ &= \beta(v)\frac{\cos(t_{\beta,i})\cos(\beta(w))+\sin(t_{\beta,i})\sin(\beta(w))}{-\cos(t_{\beta,i})\sin(\beta(w))+\sin(t_{\beta,i})\cos(\beta(w))}{}_{\gamma_w}||_0^1 x\\ &= \beta(v)\cot(t_{\beta,i}-\beta(w))_{\gamma_w}||_0^1 x.\end{aligned}$$

7. Applications

Let $H \subset G$ be a symmetric subgroup such that the corresponding involutions commute. Let $p \in M$ be regular and $\Sigma = \mathrm{Exp}(\mathfrak{t})$ be the section through p. Denote the generalized Weyl group of the action by W and define a function $\vartheta : \mathfrak{t} \to \mathbf{R}$ by

$$\vartheta(w) = \prod_{\beta\in\Delta_{\mathfrak{t}}^+} |\sin(\beta(w))|^{p_\beta}|\cos(\beta(w))|^{h_\beta}, \tag{26}$$

where the exponents p_β and h_β are the relative root multiplicities defined by $p_\beta = \dim \mathfrak{m}_\beta^{\mathfrak{t}} \cap \mathfrak{p}$ and $h_\beta = \dim \mathfrak{m}_\beta^{\mathfrak{t}} \cap \mathfrak{h}$. Since ϑ is invariant under the reflections in the singular hyperplanes in $\mathfrak{t}$, it can be regarded as a function on Σ. We will reprove the following theorem from [6] using our calculation of the shape operators. The proof is similar to the arguments in [5], where the case $H = K$ is treated.

Theorem 7.1. *For any integrable function f on M, we have*

$$\int_M f(x)\,dx = \frac{1}{|W|\cdot\vartheta(p)}\int_\Sigma\left(\int_{H/H_p} f(hq)\,d(hH_p)\right)\vartheta(q)\,dq, \tag{27}$$

where the Riemannian measure on H/H_p is chosen to be the one induced by $H\cdot p \subset M$.

If f is additionally H-invariant, we have

$$\int_M f(x)\,dx = \frac{\mathrm{Vol}(M)}{\int_\Sigma \vartheta(q)\,dq}\int_\Sigma f(q)\vartheta(q)\,dq. \tag{28}$$

Remark. If H is an arbitrary symmetric subgroup of G, the theorem is true with ϑ defined as follows: Conjugate H such that the regular point p is the origin $eK \in M$ and consider the decomposition

$$T_pHp = (\mathfrak{z}_{\mathfrak{m}}(\mathfrak{h}) \cap \mathrm{pr}_{\mathfrak{m}}\mathfrak{h}) \oplus \sum_{\beta\in\Delta_{\mathfrak{t}}^+}\sum_i V_{\beta,i}$$

of Proposition 6.1. Choose nonzero $x_{\beta,i} \in V_{\beta,i}$ and $X_{\beta,i} \in \mathfrak{h}$ with $X_{\beta,i}(p) = x_{\beta,i}$. Then choose $w_{\beta,i} \in \mathfrak{t}$ with $\beta(w_{\beta,i}) = 1$ and let $t_{\beta,i}$ be a zero of the Jacobi field $X_{\beta,i} \circ \gamma_{\beta,i}$, where $\gamma_{\beta,i}$ is the geodesic in direction $w_{\beta,i}$. Then define $\vartheta : \mathfrak{t} \to \mathbf{R}$ as

$$\vartheta(w) = \prod_{\beta \in \Delta_{\mathfrak{t}}^+} \prod_i |\sin(\beta(w) - t_{\beta,i})|^{\dim V_{\beta,i}}. \tag{29}$$

Since the proof is completely analogous using the calculations in Section 6 instead of Theorem 5.3, we will prove the theorem only in the case of commuting involutions.

We also remark that in the noncompact case, the theorem remains true if ϑ is defined using hyperbolic functions.

Lemma 7.2. (Generalized Cavalieri Principle) *Let M be a Riemannian manifold such that a subset $U \subset M$ of full measure can be written as $L \times N$, equipped with a Riemannian metric of the form $g(q,r) = \begin{pmatrix} h(q,r) & 0 \\ 0 & k(r) \end{pmatrix}$, where $(q,r) \in L \times N$. Then for any integrable function f on M, we have*

$$\int_M f(p)\, dM = \int_{L\times N} f(q,r)\, d(L\times N) = \int_N \left(\int_{L_r} f(q,r)\, dL_r \right) dN.$$

Proof. Applying Fubini's theorem in coordinates $\phi : V \times W \xrightarrow{\sim} V' \times W' \subset U$ yields

$$\begin{aligned} \int_{V'\times W'} f(q,r)\, dM &= \int_{V\times W} f\circ\phi(x,y) \sqrt{\det(g_{ij}(x,y))}\, d(x,y) \\ &= \int_W \left(\int_V f\circ\phi(x,y)\sqrt{\det(h_{ij}(x,y))}\, dx \right) \sqrt{\det(k_{ij}(y))}\, dy. \\ &= \int_{W'} \left(\int_{V_r'} f(q,r)\, dV_r' \right) dW'. \end{aligned}$$

□

Applying Lemma 7.2 to the regular set in M, we obtain that for any integrable function f on M,

$$\begin{aligned} \int_M f(x)\, dx &= \int_{\bar{Q}} \left(\int_{H\cdot \mathrm{Exp}(w)} f(x)\, dx \right) dw \\ &= \frac{1}{|W|} \int_\Sigma \left(\int_{Hq} f(x)\, dx \right) dq, \end{aligned} \tag{30}$$

where $\bar{Q} \subset \mathfrak{t}$ is a generalized Weyl chamber, and W is the generalized Weyl group of the H-action on M.

Lemma 7.3. *The Riemannian densities μ and μ' of the H-orbits through two regular points $p = \mathrm{Exp}(w)$ and $q = \mathrm{Exp}(w + tu)$ are related by the* *formula*

$$\mu' \circ \varphi = F_p(q) \cdot \mu,$$

where

$$F_p(q) = \prod_{\beta \in \Delta_{\mathfrak{t}}^+} (\cos(\beta(tu)) + \cot(\beta(w)) \sin(\beta(tu)))^{p_\beta} \times (\cos(\beta(tu)) - \tan(\beta(w)) \sin(\beta(tu)))^{h_\beta}. \tag{31}$$

Here the exponents are the relative root multiplicities $h_\beta = \dim \mathfrak{m}_\beta^{\mathfrak{t}} \cap \mathfrak{h}$ *and* $p_\beta = \dim \mathfrak{m}_\beta^{\mathfrak{t}} \cap \mathfrak{p}$.

Proof. Let ξ be the H-equivariant vector field on Hp determined by $\xi(p) = u(p)$. The polarity of the H-action implies that ξ is a parallel normal vector field ([13], Theorem 5.7.1) and the map $\varphi := \mathrm{Exp}_{Hp} \circ (t\xi)$ is a diffeomorphism between Hp and Hq. Since ξ is H-equivariant, so is φ, and consequently, for $X \in \mathfrak{h}$, we have $d\varphi(X \cdot p) = X \cdot q$. In other words, applying $d\varphi$ to $X \cdot p$ is the same as evaluating the Jacobi field obtained by restriction of the Killing vector field X to the geodesic $c(t) = \mathrm{Exp}_{Hp}(tu(p))$ at time t.

In order to compare the Riemannian densities of Hp and Hq, we need to calculate the determinant of $d\varphi_p$; the argumentation above shows that this amounts to explicitly calculate these Jacobi fields.

Let $X \in \mathfrak{m}_\beta^{\mathfrak{t}} \cap \mathfrak{h}$, $v = X(eK)$ and $v' = {}_\gamma||_0^1 v \in T_p Hp$. The initial values of the Jacobi field $Y_v(t) = X \cdot c(t)$ along c have been calculated in the proof of Theorem 5.3. Therefore

$$\frac{1}{\cos(\beta(w))} Y_v(t) = (\cos(\beta(tu)) - \tan(\beta(w)) \sin(\beta(tu)))_c||_0^t v'. \tag{32}$$

If $X \in \mathfrak{z}_{\mathfrak{m}}(\mathfrak{t}) \cap \mathfrak{h}$, $v = X(eK)$ and $v' = {}_\gamma||_0^1 v$, then

$$Y_v(t) = {}_c||_0^t v' \tag{33}$$

is the restriction of X to c.

If $v \in \mathfrak{m}_\beta^{\mathfrak{t}} \cap \mathfrak{p}$, $v' = {}_\gamma||_0^1 v$ and $X \in \mathfrak{k}_\beta^{\mathfrak{t}} \cap \mathfrak{h}$ is such that $[H, X] = -\beta(H)v$ for all $H \in \mathfrak{t}$, we analogously see that the Jacobi field Y_v, defined by $Y_v(t) = X \cdot c(t)$, is given by

$$\frac{1}{\sin(\beta(w))} Y_v(t) = (\cos(\beta(tu)) + \cot(\beta(w)) \sin(\beta(tu)))_c||_0^t v'. \tag{34}$$

If μ and μ' are the Riemannian densities of Hp and Hq, respectively, we conclude from (32), (33) and (34) that

$$\mu'(\varphi(p)) = \mu(p)\cdot \prod_{\beta\in\Delta_{\mathfrak{t}}^+} (\cos(\beta(tu)) + \cot(\beta(w))\sin(\beta(tu)))^{p_\beta} \times (\cos(\beta(tu)) - \tan(\beta(w))\sin(\beta(tu)))^{h_\beta},$$

since the Jacobi fields in question are multiples of parallel vector fields and therefore stay orthogonal to each other. □

Corollary 7.4. *With the same notation as in Lemma 7.3, the volumes of Hp and Hq are related by*

$$\mathrm{Vol}(Hq) = \mathrm{Vol}(Hp)\cdot F_p(q).$$

Lemma 7.3 now enables us to replace the inner integral in Equation (30) by an integral over a fixed regular orbit Hp, where $p\in\Sigma$. If we denote the Riemannian density of the orbit Hq by μ_q, and $\varphi_q : Hp \to Hq$ is the equivariant diffeomorphism introduced in Lemma 7.3, we have

$$\begin{aligned}\int_M f(x)\,dx &= \frac{1}{|W|}\int_\Sigma \left(\int_{Hq} f(x)\,dx\right) dq\\ &= \frac{1}{|W|}\int_\Sigma \left(\int_{Hp} f(\varphi_q(x))\frac{\mu_q(\varphi_q(x))}{\mu_p(x)}\,dx\right) dq\\ &= \frac{1}{|W|}\int_\Sigma \left(\int_{H/H_p} f(hq)\,dh\right) F_p(q)\,dq. \qquad (35)\end{aligned}$$

A short calculation using the addition theorems for the trigonometric functions shows that

$$\frac{\mu_q\circ\varphi_q}{\mu_p} = F_p(q) = \frac{\vartheta(q)}{\vartheta(p)}. \qquad (36)$$

Now equality (27) follows from (35) and (36).

If we additionally assume f to be H-invariant, we can argue as in Theorem 3.5 of [5]: on the one hand, we clearly have

$$\int_M f(x)\,dx = \frac{1}{|W|}\int_\Sigma \mathrm{Vol}(Hq) f(q)\,dq.$$

On the other hand, Corollary 7.4 and Equation (36) show that

$$\mathrm{Vol}(Hq) = \frac{\mathrm{Vol}(Hp)}{\vartheta(p)}\cdot\vartheta(q),$$

so the quotient $\mathrm{Vol}(Hp)/\vartheta(p)$ does not depend on p, and we may denote it by $V(M)$. Then we have

$$\mathrm{Vol}(M) = \frac{1}{|W|}\int_\Sigma \mathrm{Vol}(Hq)\,dq = \frac{V(M)}{|W|}\cdot\int_\Sigma \vartheta(q)\,dq.$$

Combining these three equalities, we have shown (28). On the way, we have also proven

Proposition 7.5. *For any regular point p,*

$$\mathrm{Vol}(Hp) = \mathrm{Vol}(M)|W|\frac{\vartheta(p)}{\int_\Sigma \vartheta(q)dq}. \tag{37}$$

8. Example: $\mathrm{U}(p+q)$ Acting on $\mathrm{SO}(2p+2q)/\mathrm{S}(\mathrm{O}(2p)\times\mathrm{O}(2q))$

In this section, we determine the function ϑ introduced in Section 7 for the $H = \mathrm{U}(p+q)$-action on the symmetric space

$$M = \mathrm{SO}(2p+2q)/\mathrm{S}(\mathrm{O}(2p)\times\mathrm{O}(2q)).$$

Let us assume that $p \le q$.

On the Lie algebra $\mathfrak{g} = \mathfrak{so}(2p+2q)$, we consider the two involutions $\sigma_1(X) = I_{2p,2q}XI_{2p,2q}$ and $\sigma_2(X) = J_{p+q}XJ_{p+q}^{-1}$, where

$$I_{2p,2q} = \begin{pmatrix} I_p & 0 & 0 & 0\\ 0 & I_p & 0 & 0\\ 0 & 0 & -I_q & 0\\ 0 & 0 & 0 & -I_q\end{pmatrix} \quad \text{and } J_{p+q} = \begin{pmatrix} 0 & I_p & 0 & 0\\ -I_p & 0 & 0 & 0\\ 0 & 0 & 0 & I_q\\ 0 & 0 & -I_q & 0\end{pmatrix}.$$

Clearly, σ_1 and σ_2 commute. The corresponding fixed point algebras are $\mathfrak{k} = \mathfrak{g}^{\sigma_1} = \mathfrak{so}(2p)\oplus\mathfrak{so}(2q)$ and

$$\mathfrak{h} = \mathfrak{g}^{\sigma_2} = \left\{ \left(\begin{array}{cc|cc} a & -b & c & -d\\ b & a & d & c\\ \hline -c^t & -d^t & e & -f\\ \underbrace{d^t}_{p} & \underbrace{-c^t}_{p} & \underbrace{f}_{q} & \underbrace{e}_{q}\end{array}\right) \middle| \begin{array}{cc} a^t = -a & e^t = -e\\ b^t = b & f^t = f\end{array} \right\} \cong \mathfrak{u}(p+q).$$

Let the respective Cartan decompositions of $\mathfrak{g}$ be $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}$ and $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{p}$. We have

$$\mathfrak{m} = \left\{ \left(\begin{array}{c|c} 0 & a\\ \hline -a^t & 0\end{array}\right) \in \mathfrak{g} \right\}$$

and

$$\mathfrak{p} = \left\{ \left(\begin{array}{cc|cc} u & v & w & x \\ v & -u & x & -w \\ \hline -w^t & -x^t & y & z \\ -x^t & w^t & z & -y \end{array} \right) \middle| \begin{array}{ll} u^t = -u & v^t = -v \\ y^t = -y & z^t = -z \end{array} \right\}.$$

The intersections $\mathfrak{m} \cap \mathfrak{p}$ and $\mathfrak{m} \cap \mathfrak{h}$ are given by

$$\mathfrak{m} \cap \mathfrak{p} = \left\{ \left(\begin{array}{cc|cc} 0 & 0 & w & x \\ 0 & 0 & x & -w \\ \hline -w^t & -x^t & 0 & 0 \\ -x^t & w^t & 0 & 0 \end{array} \right) \in \mathfrak{g} \right\}$$

and

$$\mathfrak{m} \cap \mathfrak{h} = \left\{ \left(\begin{array}{cc|cc} 0 & 0 & c & -d \\ 0 & 0 & d & c \\ \hline -c^t & -d^t & 0 & 0 \\ d^t & -c^t & 0 & 0 \end{array} \right) \in \mathfrak{g} \right\}.$$

For $1 \leq i \leq p$ and $1 \leq j \leq q$, let $E_{i,j}$ be the $p \times q$-matrix having 1 in the (i,j)-th entry and zeros elsewhere, and define

$$Q_{i,j} = \left(\begin{array}{cc|cc} 0 & 0 & E_{i,j} & 0 \\ 0 & 0 & 0 & -E_{i,j} \\ \hline -E_{i,j}^t & 0 & 0 & 0 \\ 0 & E_{i,j}^t & 0 & 0 \end{array} \right) \in \mathfrak{m} \cap \mathfrak{p},$$

$$R_{i,j} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & E_{i,j} \\ 0 & 0 & E_{i,j} & 0 \\ \hline 0 & -E_{i,j}^t & 0 & 0 \\ -E_{i,j}^t & 0 & 0 & 0 \end{array} \right) \in \mathfrak{m} \cap \mathfrak{p},$$

$$F_{i,j} = \left(\begin{array}{cc|cc} 0 & 0 & E_{i,j} & 0 \\ 0 & 0 & 0 & E_{i,j} \\ \hline -E_{i,j}^t & 0 & 0 & 0 \\ 0 & -E_{i,j}^t & 0 & 0 \end{array} \right) \in \mathfrak{m} \cap \mathfrak{h},$$

$$G_{i,j} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & -E_{i,j} \\ 0 & 0 & E_{i,j} & 0 \\ \hline 0 & -E_{i,j}^t & 0 & 0 \\ E_{i,j}^t & 0 & 0 & 0 \end{array} \right) \in \mathfrak{m} \cap \mathfrak{h}.$$

Then we define abelian subalgebras by

$$\mathfrak{t} = \sum_{i=1}^{p} \mathbf{R} Q_{i,i} \subset \mathfrak{m} \cap \mathfrak{p}, \quad \mathfrak{t}' = \sum_{i=1}^{p} \mathbf{R} F_{i,i} \subset \mathfrak{m} \cap \mathfrak{h}, \quad \mathfrak{a} = \mathfrak{t} \oplus \mathfrak{t}' \subset \mathfrak{m}$$

and consider the root space decomposition of $(\mathfrak{g}, \mathfrak{k})$ with respect to $\mathfrak{a}$: For $1 \leq i \leq 2p$, let linear forms $\lambda_i : \mathfrak{a} \to \mathbf{R}$ be defined by

$$\lambda_i(Q_{j,j}) = \begin{cases} \delta_{i,j} & \text{if } i \leq p \\ -\delta_{i-p,j} & \text{if } i > p \end{cases}, \quad \lambda_i(F_{j,j}) = \begin{cases} \delta_{i,j} & \text{if } i \leq p \\ \delta_{i-p,j} & \text{if } i > p \end{cases}.$$

The restricted roots are $\pm\lambda_i$ for $1 \leq i \leq 2p$ and $\pm(\lambda_i \pm \lambda_j)$ for $1 \leq i < j \leq 2p$; we choose as positive roots the λ_i and the $\lambda_i \pm \lambda_j$ for $i < j$. Denoting the restriction of λ_i to $\mathfrak{t}$ by $\lambda_i^{\mathfrak{t}}$, we obtain the adapted root space decomposition with respect to $\mathfrak{t}$

$$\mathfrak{m} = \mathfrak{z}_{\mathfrak{m}}(\mathfrak{t}) \oplus \sum_{1 \leq i \leq p} \mathfrak{m}^{\mathfrak{t}}_{\lambda_i^{\mathfrak{t}}} \oplus \sum_{\substack{1 \leq i \leq p \\ i < j \leq 2p}} \mathfrak{m}^{\mathfrak{t}}_{\lambda_i^{\mathfrak{t}} \pm \lambda_j^{\mathfrak{t}}}$$

where

$$\mathfrak{m}^{\mathfrak{t}}_{\lambda_i^{\mathfrak{t}}} = \underbrace{\sum_{k=p+1}^{q} \mathbf{R}F_{i,k} \oplus \mathbf{R}G_{i,k}}_{\subset \mathfrak{m} \cap \mathfrak{h}} \oplus \underbrace{\sum_{k=p+1}^{q} \mathbf{R}Q_{i,k} \oplus \mathbf{R}R_{i,k}}_{\subset \mathfrak{m} \cap \mathfrak{p}}$$

for $1 \leq i \leq p$,

$$\mathfrak{m}^{\mathfrak{t}}_{\lambda_i^{\mathfrak{t}} \pm \lambda_j^{\mathfrak{t}}} = \underbrace{\mathbf{R}(F_{i,j} \pm F_{j,i})}_{\subset \mathfrak{m} \cap \mathfrak{h}} \oplus \underbrace{\mathbf{R}(Q_{i,j} \pm Q_{j,i})}_{\subset \mathfrak{m} \cap \mathfrak{p}}$$

for $1 \leq i < j \leq p$ and

$$\mathfrak{m}^{\mathfrak{t}}_{\lambda_i^{\mathfrak{t}} \pm \lambda_j} = \underbrace{\mathbf{R}(G_{i,j-p} \pm G_{j-p,i})}_{\subset \mathfrak{m} \cap \mathfrak{h}} \oplus \underbrace{\mathbf{R}(R_{i,j-p} \pm R_{j-p,i})}_{\subset \mathfrak{m} \cap \mathfrak{p}}$$

for $1 \leq i \leq p$ and $p < j \leq 2p$. Therefore,

$$\begin{aligned} \vartheta &= \prod_{i=1}^{p} \prod_{j=1}^{2p} |\sin(\lambda_i^{\mathfrak{t}} + \lambda_j^{\mathfrak{t}})||\sin(\lambda_i^{\mathfrak{t}} - \lambda_j^{\mathfrak{t}})||\cos(\lambda_i^{\mathfrak{t}} + \lambda_j^{\mathfrak{t}})||\cos(\lambda_i^{\mathfrak{t}} - \lambda_j^{\mathfrak{t}})| \\ &\quad \times \prod_{i=1}^{p} |\sin(\lambda_i^{\mathfrak{t}})|^{2(q-p)}|\cos(\lambda_i^{\mathfrak{t}})|^{2(q-p)} \\ &= \prod_{i,j=1}^{p} |\sin(\lambda_i^{\mathfrak{t}} + \lambda_j^{\mathfrak{t}})|^2|\sin(\lambda_i^{\mathfrak{t}} - \lambda_j^{\mathfrak{t}})|^2|\cos(\lambda_i^{\mathfrak{t}} + \lambda_j^{\mathfrak{t}})|^2|\cos(\lambda_i^{\mathfrak{t}} - \lambda_j^{\mathfrak{t}})|^2 \\ &\quad \times \prod_{i=1}^{p} |\sin(\lambda_i^{\mathfrak{t}})|^{2(q-p)}|\cos(\lambda_i^{\mathfrak{t}})|^{2(q-p)}, \end{aligned}$$

where we have used that $\lambda_{i+p}^{\mathfrak{t}} = -\lambda_i^{\mathfrak{t}}$ for $1 \leq i \leq p$.

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