A direct proof of Davis' non-immersion theorem and a generalization

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Abstract

We show how to deduce a generalized version of Davis' elegant and strong non-immersion theorem ([3]) applicable to any 2-torsion lens space. The methods rely on an conjectured formula which, in the case of real projective spaces, is verified in an amazingly simple way (yielding Davis' theorem). For higher torsion lens spaces the formula is verified within a range of dimensions leading to (restricted) non-immersion results.

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1 Introduction

Let $L(2^k)^{2n+1}$ denote the $(2n+1)$-dimensional 2^k-torsion lens space, that is, the oriented manifold arising as the orbit space of the $\mathbb{Z}/2^k$ (diagonal) action on the unitary vectors in $\mathbb{C}^{n+1}$. The present work is motivated by the following problem: Find the smallest possible Euclidean space where $L(2^k)^{2n+1}$ can be immersed. For $k = 1$ this is the elusive immersion problem for (odd dimensional) real projective spaces. Among the many results available in this particular case the following one combines a theoretically simple approach with a remarkably strong statement. Here $\alpha(d)$ stands for the number of ones in the dyadic expansion of d.

Theorem 1.1 (Davis [3]) *The real projective space $L(2)^{2(d+\alpha(d)-1)}$ cannot be immersed in a Euclidean space of dimension $4d - 2\alpha(d)$.*

More generally, the next conjecture, which appeared first in [7], aims to describe the analogous situation for lens spaces of any 2-torsion (note that the only even dimensional lens spaces which admit a manifold structure are the real projective spaces).

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Conjecture 1.2 *For $1 \leq k \leq \alpha(d)$ the lens space $L(2^k)^{2(d+\alpha(d)-k)+1}$ cannot be immersed in a Euclidean space of dimension $4d - 2\alpha(d)$.*

The case $k = \alpha(d)$ can be proved by using K-theoretic techniques (see [5] or Remark 3.2 below). By a geometrical argument, this also describes in a rather close way the situation for larger torsion lens spaces ($k > \alpha(d)$) and, in the limit, for complex projective spaces (see Section 4 in [4] for a discussion). Although K-theory can also be used to prove a few of the lower torsion cases, one needs better suited techniques. Davis uses Brown-Peterson cohomology —$BP\langle 2\rangle$ is enough— in the case $k = 1$ (Theorem 1.1). His proof relies on a complete description (in a certain range) of the Abelian groups $BP\langle 2\rangle^*(P \times P')$, where P and P' are projective spaces. Although elegant, Davis' methods are mainly computational and, in particular, they give no indication on a possible generalization in the direction of Conjecture 1.2. Our approach is also computationally motivated, but it does apply in the general case. Besides, we end up with a much easier and a bit more conceptual proof of Theorem 1.1. This is discussed in Section 4.

Our whole program for proving Conjecture 1.2 depends on the validness of certain general relation (formula (3)) in the BP-cohomology of a (suitable) product of a projective space and a larger torsion lens space. The author has verified the formula in a certain range with the help of a computer. In Section 3 we show how those calculations allow us to deduce the next result (compare with [2] and [13]).

Theorem 1.3 *Conjecture 1.2 holds if, in addition, $\alpha(d) - k \leq 6$.*

As in [3], proving formula (3) in the case $k = 1$ requires only a general knowledge of the 2-divisibility properties of the coefficients in the 2-series for Brown-Peterson homology —the fact that half of them are even. All the rest is elementary. For general k, the corresponding information on the $[2^k]$-series is already available (Theorem 2.1 below), but it seems to be necessary a deeper analysis of the interaction of the 2-series with the 2^k-series. This represents work in progress. At any rate, the reader can get a more concise idea of the complexities one should deal with for $k > 1$ and which do not show up when $k = 1$.

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2 Preliminaries

Although all our results work just as well for the spectra $BP\langle i\rangle$ ($i \geq 2$) introduced in [14], we will stick calculations with the spectrum BP representing Brown-Peterson homology at the prime 2. Only at the end of this section we will describe a few calculation with $BP\langle 2\rangle$. A general reference for the material reviewed in this section is [12]. Recall that the coefficients $BP_* = BP^{-*}$ form a polynomial ring over the 2-local integers on homotopy classes $v_i \in BP_{2^{i+1}-2}$, $i \geq 1$. Let $\mu \in BP^*[[x,y]]$ stand for the corresponding formal sum, that is, the BP-Euler class of the tensor product of the two Hopf bundles over the axes in $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ (x, y and μ lie in degree 2). For any integer k, the Brown-Peterson 2^k-series is defined inductively by $[2^{k+1}](x) = \mu([2^k](x),[2^k](x))$, with $[1](x) = x$. Topologically, this is the Brown-Peterson Euler class for $\iota^*(H)$, where $\iota : L(2^k)^\infty \to \mathbb{C}P^\infty$ is the fiber inclusion of the 2^k-power map $\mathbb{C}P^\infty \to \mathbb{C}P^\infty$ and H is the Hopf bundle. Set $[2^k](x) = \sum_{s\geq 0} a_{k,s}x^{s+1}$, so that $a_{k,s} \in BP_{2s}$.

Theorem 2.1 (Johnson-Nakos ($k = 1$) [9], González [6]) *For $s \geq 0$ the highest power of 2 dividing $a_{k,s}$ is given by $\nu(a_{k,s}) = ke_0 + (k-1)e_1 + \cdots + e_{k-1}$, where $s+1 = e_0 + 2e_1 + 4e_2 + \cdots$ is the dyadic expansion of $s+1$.*

The relevance of these objects to the immersion problem of 2-torsion lens spaces is established with the help of a standard geometrical argument based on Hirsch's characterization of the Euclidean dimension in an optimal immersion of a given manifold ([8]). For lens spaces this leads to the identification of an algebraic obstruction to the the type of (non-)immersions in Conjecture 1.2. For the details of the next result see either one of [1, 7, 11]. For integers n and m consider the ring

$$R_{k,n,m} = BP^*[x,y] \Big/ (x^{n+1}, y^{m+1}, [2^k](x), [2](y)) \,.$$

Proposition 2.2 *If the lens space $L(2^k)^{2(d+\alpha(d)-k)+1}$ can be immersed in the Euclidean space $\mathbb{R}^{4d-2\alpha(d)}$, then $(x-y)^e$ vanishes in $R_{k,n,m}$, where $e = 2^L - d - \alpha(d) + k - 1$, $n = d + \alpha(d) - k$, $m = 2^L - 2d + \alpha(d) - 1$ and $L \gg 0$.*

In view of the relations $x^{n+1} = y^{m+1} = 0$, the obstruction above can be rewritten in the form

$$\sum_{t=0}^{T} \pm c_t x^{n-t} y^{m-T+t} \tag{1}$$

where $T = 3\alpha(d) - 2k$, $c_t = \binom{2^L-n-1}{n-t}$ and where the "$\pm$" mark means that signs alternate. We will set up a program for reducing (suitable multiples of) expressions of this sort to the form $v_1^{k-1}v_2^a x^n y^m$ with $a \geq 0$. The next result is therefore the conclusion of that program. For a proof see [7] or [10].

Theorem 2.3 *The element $v_1^{k-1}x^n y^m \in R_{k,n,m}$ is v_2-torsion free.*

We close the section by recalling some auxiliary formuæ taken from [7].

Lemma 2.4 *The following relations hold in* $R_{k,n,m}$*:*

(i) $2^a y^b = 0$ *provided* $a + b \geq m + 1$ *and* $b > 0$.

(ii) $2^a x^b = 0$ *provided* $a + b \geq m + k$ *and* $b > 0$.

(iii) $2^a y^{m-a} = v_1^a y^m$ *provided* $0 \leq a < m$.

(iv) $2^a x^b y^c = 0$ *provided* $3a + b + c \geq n + m + 2k - 1$, $b > 0$ *and* $b + c > n + 1$.

3 The weight filtration

Once a binomial coefficient in (1) is divisible by 2, the relation imposed by the 2-series modifies the corresponding summand. As shown by the next result, this is not a terrible problem if at least one such binomial coefficient is not divisible by 2^k. Indeed, in such a case connective K-theory (that is, $BP\langle 1\rangle$) suffices to detect the obstruction. As in Theorem 2.1, the highest power of 2 dividing an integer c is denoted by $\nu(c)$ ($\nu(0) = \infty$).

Proposition 3.1 *An expression of the form (1) is nontrivial in the ring* $R_{k,n,m}$ *if* $\nu(c_t) \leq \min\{T - t, k - 1\}$ *for some* t *with* $0 \leq t \leq T$.

Note. Here we are not assuming T nor the coefficients c_t have any particular form. We do require though $T < \min\{n, m\}$ which, except for trivial well-known cases, holds in Proposition 2.2.

Proof. For $0 \leq t \leq T$ set $\nu_t = \nu(c_t)$ and choose t_0 such that $\nu_{t_0} = \min\{\nu_t\}$ where the minimum is taken over those t satisfying $\nu_t \leq \min\{T - t, k - 1\}$. By Lemma 2.4 *(iii)*, the $x^{t_0} y^{T - t_0 - \nu_{t_0}}$-multiple of the t_0-th summand in (1) is up to a unit $2^{\nu_{t_0}} x^n y^{m - \nu_{t_0}} = v_1^{\nu_{t_0}} x^n y^m$, which is nontrivial in view of Theorem 2.3. We shown that the corresponding multiple of any other summand is trivial, that is, $2^{\nu_t} x^{n - t + t_0} y^{m + t - t_0 - \nu_{t_0}} = 0$, for $t \neq t_0$. This is obvious if $t < t_0$ (for $x^{n+1} = 0$) or if $\nu_t - \nu_{t_0} + t - t_0 > 0$, by Lemma 2.4 *(i)*. In any other case we would have $t > t_0$, $\nu_t - \nu_{t_0} + t - t_0 \leq 0$ and in fact $\nu_t < \nu_{t_0}$. But from $T - t \geq \min\{T - t_0, k - 1\} + t_0 - t \geq \nu_{t_0} + t_0 - t \geq \nu_t > \min\{T - t, k - 1\}$ (the last inequality is implied from the choice of t_0) it would follow $T - t > k - 1$. However all these inequalities are incompatible with $t > t_0$. ■

Remark 3.2 When $k = \alpha(d)$, the hypothesis in Proposition 3.1 holds for the obstruction of Proposition 2.2 (take $t = 1$ and recall $\nu\binom{2^L - d - 1}{d - 1} = \alpha(d) - 1$). Therefore we will assume from now on that (1) reduces to

$$\sum_{t=1}^{R} \pm c_t x^{n-t} y^{m-k-R+t} \tag{2}$$

with $R = 3(\alpha(d) - k) > 0$ and where each $c_t = \binom{2^L - n - 1}{n - t}$ is divisible by 2^k. This type of expressions will be managed by introducing a suitable filtration in $R_{k,n,m}$ and identifying the (nontrivial) term carrying the lowest filtration.

Definition 3.3 Consider the multiplicative decreasing BP^*-modulo *weight filtration* (WF for short) defined in $BP^*[x, y]$ and in its quotient $R_{k,n,m}$ by setting $W(2) = 3$, $W(v_1) = 2$, $W(x) = W(y) = 1$ and $W(v_i) = 0$ for $i \geq 2$.

The next result sets a certain "weight vanishing line" in $R_{k,n,m}$. Note that the element in Theorem 2.3 lies in the border.

Lemma 3.4 *Let $b > 0$, $b + c - a_2 > n + 1$ and $3a_1 + 2a_2 + b + c \geq n + m + 2k - 1$. Then the relation $2^{a_1} v_1^{a_2} x^b y^c = 0$ holds in $R_{k,n,m}$.*

Proof. Induction on a_2, the case $a_2 = 0$ being Lemma 2.4 *(iv)*. The first few terms of the relation imposed by the 2-series yield $v_1 y^2 \in (2y, y^4)$. Therefore, for $a_2 \geq 1$, the element $2^{a_1} v_1^{a_2} x^b y^c$ lies in the ideal generated by $2^{a_1+1} v_1^{a_2-1} x^b y^{c-1}$ and $2^{a_1} v_1^{a_2-1} x^b y^{c+2}$. But these two elements are trivial by induction. ■

In order to motivate the type of relations needed, we indicate some easy calculations with this filtration.

Example 3.5 By analyzing the first few coefficients one can see that $W([2](y)) = 4$ with $[2]y \equiv 2y + v_1 y^2 + v_2 y^4$ modulo larger WF. More generally, using the inductive relation $[2^{k+1}](x) = [2]([2^k](x))$ one gets $W([2^k](x)) = 3k + 1$ with $[2^k](x) \equiv 2^k x + 2^{k-1} v_1 x^2 + 2^{k-1} v_2 x^4$ modulo larger WF. In the ring $R_{k,n,m}$ these two formulæ yield

$$\begin{aligned} 2^k x^i y^j &\equiv 2^{k-1} v_1 x^{i+1} y^j + 2^{k-1} v_2 x^{i+3} y^j \\ &\equiv 2^k x^{i+1} y^{j-1} + 2^{k-1} v_2 x^{i+1} y^{j+2} + 2^{k-1} v_2 x^{i+3} y^j \end{aligned}$$

modulo WF $> 3k+i+j$ (so that all terms are of order 2). Since we are ultimately interested in cases where $m \gg 0$ we can safely rewrite this as $2^k \equiv 2^k x y^{-1} + 2^{k-1} v_2 x (x^2 + y^2)$ modulo WF $> 3k$. We now iterate so that the first summand in the right hand side vanishes ($x^{n+1} = 0$ and $m \gg n$) and all others, except for two, cancel out. There results the relation $2^k \equiv 2^{k-1} v_2 xy(x + y) = 2^{k-1} v_2 xy(x - y)$ modulo WF $> 3k$.

Conjecture 3.6 *For $m \gg n > R$ the following formula in $R_{k,n,m}$ applies to the summands in an expression of the form (2):*

$$2^k = 2^{k-1} v_2 xy(x - y) + \sum_{b=1}^{k} 2^{k-b} x^b y^b (x - y)^b Q_b. \tag{3}$$

Here the Q_b are elements having positive weight filtration.

Proposition 3.7 *Conjecture 3.6 implies Conjecture 1.2.*

Proof. All coefficients in (2) are assumed to be divisible by 2^k, thus (3) yields

$$\sum_{t=1}^{R} \pm c_t x^{n-t} y^{m-k-R+t} = \sum_{t=1}^{R} \pm \tfrac{1}{2^k} c_t \left(2^{k-1} v_2 x^{n-t+1} y^{m-k-R+t+1} (x-y) + \right.$$
$$\left. + \sum_{b=1}^{k} 2^{k-b} x^{n+b-t} y^{m-k-R+b+t} (x-y)^b Q_b \right).$$

In any summation of the form $\sum_t \pm c_t (x-y) A_t$ we have $x A_t = y A_{t-1}$ and $c_R y A_R = c_1 x A_1 = 0$ (Lemma 2.4). Therefore, by regrouping, the above expression is

$$\sum_{t=2}^{R} \pm c'_t v_2 x^{n-t+2} y^{m-k-R+t+1} + \sum_{b=1}^{k} \tfrac{1}{2^{b-1}} \sum_{t=2}^{R} \pm c'_t x^{n+b+1-t} y^{m-k-R+b+t} (x-y)^{b-1} Q_b \qquad (4)$$

where $c'_t = (c_t + c_{t-1})/2$. Note that $\nu(c'_t) \geq k-1$ for $t = 2, \ldots, R$. If equality ever holds we are done: by Proposition 3.1 and its proof, a certain multiple of the first summation in (4) is nontrivial, whereas the corresponding multiple of all other summations will vanish in view of Lemma 3.4 (recall $m \gg 0$). On the other hand, if $\nu(c'_t) \geq k$ for $t = 2, \ldots, R$, then terms with $t = 2$ and $t = R$ are trivial by Lemma 2.4 and, after re-indexing, (4) takes the form

$$v_2 \sum_{t=1}^{R'} \pm c'_{t+2} x^{n-t} y^{m-k-R'+t} + \sum_{b=1}^{k} \tfrac{1}{2^{b-1}} \sum_{t=1}^{R'} \pm c'_{t+2} x^{n+b-1-t} y^{m-k-R'+b-1+t} (x-y)^{b-1} Q_b$$

where $R' = R - 3$. Hence the process repeats with a new application of (3) on the first summation and on that with $b = 1$ in the second (double) summation above. For summations with $b > 1$ we already have the required $(x-y)$ factor so that (3) is not needed (and may not be applicable!). Letting $a = b - 1$ such summations evolve as

$$\tfrac{1}{2^a} \sum_{t=1}^{R'} \pm c'_t x^{n+a-t} y^{m-k-R'+a+t} (x-y)^a Q_{a+1} =$$
$$= \tfrac{1}{2^{a-1}} \sum_{t=2}^{R'} \pm \frac{c'_t + c'_{t-1}}{2} x^{n+a+1-t} y^{m-k-R'+a+t} (x-y)^{a-1} Q_{a+1}$$

which are terms similar to those in the second (double) summation of (4) —the next expected output. In order for this process to be successful it should stop with an application of Proposition 3.1 and before reaching a point where the summations involved were empty (they reduce length by 3 each time the process applies, except for the first time, where the reduction is by 1). This is guaranteed from the original values of R and the c_t in (2). Indeed, after the i-th application of the process the summations run for $t = 2i, \cdots, R-i+1$ (without any re-indexing) and the corresponding coefficients are $\frac{1}{2^i}\binom{2^L-n+i-1}{n-t+i}$. Therefore when $i = \alpha(d) - k$ (if the process gets this far) the resulting summations have two terms and one of the coefficients (that for $t = 2\alpha(d) - 2k + 1$) is just $\frac{1}{2^{\alpha(d)-k}}\binom{2^L-d-1}{d-1}$ which, as indicated in Remark 3.2, is not divisible by 2^k. ∎

Remark 3.8 It is possible to use Proposition 3.7 together with partial information on (3) to deduce explicit non-immersion results for 2-torsion lens spaces. Indeed, from the proof above, we observe that the original obstruction (2), which in principle has WF at least $n+m+2k-R$, is detected in WF at most $n+m+2k-3$ —an increment in filtration of at most $e = 3(\alpha(d) - k - 1)$. Since all filtrations will be brought up to the border of the "weight vanishing line" (with an application of Proposition 3.1), the arguments in the proof of Proposition 3.7 will only require a version of (3) *modulo WF larger than* $3k + e$. Therefore Theorem 1.3 follows from the next $BP\langle 2\rangle$-result.

Proposition 3.9 *Consider Araki generators for* BP_* *and set* $v_i = 0$ *for* $i \geq 3$. *Then Conjecture 3.6 holds modulo weight filtration larger than* $3k + 15$.

Just as in Example 3.5 (which proves Conjecture 3.6 modulo WF $> 3k$) this is verified in a straightforward way: using Davis' reduced 2-series (computed modulo WF > 19) one substitutes the value of v_1 in the 2^k-series (computed modulo WF $> 3k + 16$) and applies the trick in Example 3.5 to obtain the desired relations. These manipulations can be handled with the help of a computer for $k \leq 6$. In other cases one can take advantage of the fact that, for $k \geq 7$ and modulo WF $> 3k + 16$, the 2^k-series equals twice the 2^{k-1}-series (recall the recursive relation $[2^k](x) = [2]([2^{k-1}](x))$). To give the flavor of the type of calculations, we close the section by reporting the computer output and showing the way it leads to (3) when $k = 2$.

Modulo WF > 22, the 4-series is

$$\begin{aligned}[4](x) \equiv\ & 4x + 6v_1x^2 + 24v_1^2x^3 + (9v_1^3 + 18v_2)x^4 + (4v_1^4 + 8v_1v_2)x^5 \\ & + (6v_1^5 + 12v_1^2v_2)x^6 + (3v_1^4v_2 + v_1v_2^2)x^8 + 4v_1^2v_2^2x^9 + 6v_1^3v_2^2x^{10} \\ & + v_1^5v_2^2x^{12} + 2v_1^2v_2^3x^{12} + v_2^5x^{16}.\end{aligned}$$

On the other hand, with the notation suggested in Example 3.5, Davis' reduced 2-series (on y) is

$$-v_1 \equiv 10y^{-1} + 9v_2y^2 + 4v_2^2y^5 + 5v_2^3y^8 + 2v_2^4y^{11} + v_2^5y^{14}$$

modulo WF > 17. Substitution yields the following formula modulo WF > 21 and modulo terms of the form $4A$, with A of positive WF

$$\begin{aligned} 4 \equiv\ & 4xy^{-1} - 2v_2x(x-y)^2 - 2v_2^2x^3y^{-1}(x-y)^4 + v_2^3x^3y^2(x-y)^4 \\ & - 2v_2^3xy^4(x-y)^4 - v_2^5x^3(x^3-y^3)^4 - 2v_2^5(x^7y^4+xy^{10})(x-y)^4 \\ & - 2v_2^6x^7y^7(x-y)^4 + v_2^7x^7y^{10}(x-y)^4. \end{aligned}$$

We only need the simplified expression $4 \equiv 4xy^{-1} - 2v_2x(x-y)^2 - x^2y(x-y)^3Q + 4A$ modulo WF > 21, where both Q and A have positive WF. Iterating this formula on the resulting terms $4x^iy^{-i}$ we get $4 \equiv 2v_2xy(x-y) + x^2y^2(x-y)^2Q + 4A_1$ for A_1 of positive WF. Then (3) follows by iterating (at most 15 times) the last formula on the resulting (higher filtration) terms $4A_i$. ∎

4 Projective spaces

In this final section we show that our approach yields a rather elementary proof of Theorem 1.1. Indeed, for even dimensional real projective spaces the analogue of Proposition 2.2 claims that an obstruction for immersing $L(2)^{2(d+\alpha(d)-1)}$ in a Euclidean space of dimension $4d - 2\alpha(d)$ is given by the element $(x-y)^e \in R_{1,n,m}$, where $e = 2^L - d - \alpha(d) + 1$, $n = d + \alpha(d) - 1$, $m = 2^L - 2d + \alpha(d) - 1$ and $L \gg 0$. This element can be handled just as before: it can be rewritten as

$$\sum_{t=0}^{T} \pm \binom{2^L - n}{n - t} x^{n-t} y^{m-T+t}$$

where $T = 3(\alpha(d) - 1)$. If any of these binomial coefficients is odd we are done in view of Proposition 3.1 (for instance if $\alpha(d) = 1$). Otherwise we apply our method. The proof of Proposition 3.7 shows that, after the $\alpha(d) - 1$st time the substitution in (3) is applied (if we have got this far), the lowest filtration part of the obstruction is given by $cv_2^{\alpha(d)-1}x^ny^m$, where $c = \frac{1}{2^{\alpha(d)-1}}\binom{2^L-d}{d}$. The process stops here since c is odd (compare with Remark 3.2). We now validate our approach.

Proposition 4.1 *Conjecture 3.6 is true for $k = 1$.*

Proof. For $s \geq 0$ let a_s denote the sth coefficient in the 2-series. The first few terms are $a_0 = 2$, $a_1 = v_1$ and $a_3 \equiv v_2$ modulo 2; moreover $a_{2s} \equiv 0$ modulo 2 in view of Theorem 2.1. Thus, as in Example 3.5, the relations $[2](x) = [2](y) = 0$ in $R_{1,n,m}$ yield

$$2 \ = \ -v_1x - \sum_{s\geq 2} a_sx^s \ = \ x\left(2y^{-1} + \sum_{s\geq 2} a_sy^{s-1}\right) - \sum_{s\geq 2} a_sx^s$$

$$
\begin{aligned}
&= 2xy^{-1} - \sum_{s\geq 2} a_s x \left(x^{s-1} - y^{s-1}\right) \\
&= 2xy^{-1} + 2A - v_2 x(x-y)^2 - \sum_{s\geq 2} a_{2s+1} x \left(x^s - y^s\right)^2
\end{aligned}
$$

where A has positive WF (recall $(a-b)^2 \equiv a^2 - b^2$ modulo 2). In particular we get an expression of the form $2 = 2xy^{-1} + 2A - v_2x(x-y)^2 - x(x-y)^2Q$, where Q also has positive WF. From this point the proof goes as that indicated at the end of the previous section. ∎

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