On the Trace Density in Deformation Quantization

Boris Fedosov*

Abstract

We derive a variation formula for the trace functional resulting in a differential equation for the trace density. It enables one in principle to calculate completely the trace density. We illustrate this process computing several few terms.

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1 Introduction

In this paper we consider a standard deformation quantization on a symplectic manifold M, ω and a canonical trace on the algebra of quantum observables, see e.g. [1, 2]. Our result consists in a variation formula for a trace density c which enables one to compute it in principle. We illustrate this process by computing several few terms.

In the simplest scalar-valued case we obtain

$$\begin{aligned} c(x,h) = 1 &+ \frac{h^2}{48}\partial_{i_1}\partial_{i_2}R_{j_1j_2i_3j_3}\omega^{i_1j_1}\omega^{i_2j_2}\omega^{i_3j_3} \\ &- \frac{h^2}{64}R_{i_1i_2i_3j_3}R_{j_1j_2i_4j_4}\omega^{i_1j_1}\omega^{i_2j_2}\omega^{[i_3j_3}\omega^{i_4j_4]} + O(h^3). \end{aligned} \tag{1.1}$$

Here R_{ijkl} is a curvature tensor of a symplectic connection $\partial = \partial^M$ on M, ω^{ij} is inverse to ω_{ij}, square brackets mean alternation (alternative averaging

*Mathématique Physique, Université de Bourgogne, BP 47870, F-21078, Dijon Cedex, France. e-mail: fedosov@u-bourgogne.fr

over the symmetric group S_4). Of course, we use the Einstein summation convention, moreover, in the sequel we will omit the tensor factors $\omega^{i_k j_k}$ meaning a contraction by $\omega^{i_k j_k}$ with respect to a pair of indices i_k, j_k with the same subscript k. Thus, (1.1) may be rewritten in a simplified form

$$c(x,h) = 1 + \frac{h^2}{48}\partial_{i_1}\partial_{i_2}R_{j_1 j_2 i_3 j_3} - \frac{h^2}{64}R_{i_1 i_2 [i_3 j_3}R_{|j_1 j_2| i_4 j_4]} + \dots,$$

the vertical dashes mean that the indices $j_1 j_2$ between them are excluded from alternation.

Our quantization gives a star-product for which the identity function 1 is a unit with respect to $*$, that is

$$1 * a = a * 1 = a. \tag{1.2}$$

It is always possible to pass to an equivalent star-product $*'$ (the so-called closed star-product introduced in [3] such that the integral

$$\mathrm{Tr}\,'a = \frac{(2\pi h)^{-n}}{n!}\int_M a(x,h)\omega^n$$

possesses a trace property

$$\mathrm{Tr}\,'a *' b = \mathrm{Tr}\,'b *' a.$$

Of course, in general the normalization (1.2) is violated for the closed product $*'$, but may be easily reconstructed. Indeed, let

$$c(x,h) = 1 + hc_1 + h^2 c_2 + \dots \tag{1.3}$$

be a unit with respect to $*'$, then

$$a * b = ((ca) *' (cb))\frac{1}{c}$$

defines a star-product $*$ satisfying (1.2), but the trace functional for $*$ looks more complicated

$$\mathrm{Tr}\,a = \mathrm{Tr}\,'ca = \frac{(2\pi h)^{-n}}{n!}\int_M ca\frac{\omega^n}{n!}. \tag{1.4}$$

Thus, the unit in the $*'$-algebra gives a trace density in the $*$- algebra and vice versa. We will use the canonical deformation quantization for which (1.2) is fulfilled automatically and the trace functional is given by (1.4).

If M is compact, we obtain by the index theorem for deformation quantization

$$\operatorname{Tr} 1 = \frac{(2\pi h)^{-n}}{n!} \int_M c\omega^n = \int_M \exp \frac{\omega}{2\pi h} \det{}^{-1/2} \left(\frac{\sinh \kappa/2}{\kappa/2} \right)$$

where

$$\kappa = \kappa_M = \frac{1}{4\pi i} \omega^{im} R_{mjkl} dx^k \wedge dx^l$$

is the curvature 2-form of the connection ∂^M. In the last term of (1.1) one can easily recognize the Pontrjagin polynomial, being multiplied by $\omega \wedge \omega$ it is proportional to the first Pontrjagin form. The second term in (1.1) is a complete covariant derivative, so it vanishes under integration.

The computation of the higher-oder terms in the trace density is a challenging problem. We mean, of course, precise computation not only up to complete covariant derivatives leading to the index formula. Our direct computation of the lower-order terms may be of interest since it illustrates in detail the process of "miraculous cancelation" resulting in the elegant formula (1.1).

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2 Preliminaries

To fix notation, let us recall necessary notions and constructions from [1, 2]. We will consider a canonical quantization with nontrivial coefficients, so we need a vector bundle E with a connection $\partial = \partial^E$. It defines a connection on the homomorphism bundle $K = \operatorname{Hom}(E, E)$ called a coefficient bundle. Further, W means a bundle over M whose fiber W_x is the Weyl algebra defined on the tangent space $T_x M$ with respect to the constant symplectic form $1/2\omega_{ij}(x)dy^i \wedge dy^j$ defined on $T_x M$ by the form ω on M. The symplectic connection $\partial = \partial^M$ on M gives a connection on the bundle W which in local Darboux coordinates has the form

$$\partial a = d_x a(x, y, h) + \left[\frac{i}{2h}\Gamma_{ijk} y^i y^j dx^k, a\right].$$

Here $a(x, y, h) \in C^\infty(M, W)$ is a section of W, Γ_{ijk} the connection coefficients of ∂^M. Next we consider the bundle $W \otimes K$ of Weyl algebras with coefficients in K with a connection $\partial = \partial^M \otimes 1 + 1 \otimes \partial^E$ on it. Finally, construct a connection

$$D = \delta + \partial + [s, \cdot\,]$$

where

$$\delta a = dx^i \frac{\partial a}{\partial y^i} = \frac{i}{h}[\omega_{ij} y^i dx^j, a],$$

and s subject to the following conditions:

(1) $s = ir/h$ where $r \in C^\infty(M, W \otimes K \otimes \Lambda^1)$ with deg $r \geq 3$ (recall that $\deg y^i = 1, \quad \deg h = 2$),

(2) $\delta^{-1} s = 0$, where δ^{-1} is defined as follows

$$\delta^{-1} a_{pq} = \frac{1}{p+q} y^k i \left(\frac{\partial}{\partial x^k}\right) a_{pq}$$

for a homogeneous form a_{pq} of degree p in y^i and q in dx^j if $p + q > 0$ and $\delta^{-1} a_{00} = 0$,

(3) $D^2 \equiv 0$ (the so-called Abelian property). More precisely, we impose a sufficient condition

$$\delta s = R + \partial s + s^2 \tag{2.1}$$

where

$$R = \frac{1}{2} R^E_{i_1 i_2} dx^{i_1} \wedge dx^{i_2} + \frac{i}{4h} R_{i_1 i_2 i_3 i_4} y^{i_1} y^{i_2} dx^{i_3} \wedge dx^{i_4}$$

is the curvature form of the connection ∂ on the bundle $W \otimes K$.

These conditions define D uniquely. It may be shown that they are equivalent to the following equation

$$s = s_1 + \delta^{-1}\left(\partial s + s^2\right) \tag{2.2}$$

where

$$s_1 = \frac{1}{2} R^E_{i_1 i_2} y^{i_1} dx^{i_2} + \frac{i}{8h} R_{i_1 i_2 i_3 i_4} y^{i_1} y^{i_2} y^{i_3} dx^{i_4} \tag{2.3}$$

which may be solved by iterations.

We define W_D to be the algebra of flat sections of $W \otimes K$ with respect to D, that is

$$W_D = \{a \in C^\infty(M, W \otimes K) : Da \equiv 0\}.$$

We also consider a complex of differential forms with values in $W \otimes K$

$$0 \longrightarrow W \otimes K \xrightarrow{D} W \otimes K \otimes \Lambda^1 \xrightarrow{D} \dots .$$

This complex is exact except the zero-order term where $\operatorname{Ker} D = W_D$ (see [2]). In particular, $Da = 0$ has a unique solution such that $a|_{y=0} = a_0 \in C^\infty(M, K)$. It may be found by iterations of the equation

$$a = a_0 + \delta^{-1}\left(\partial a + [s, a]\right). \tag{2.4}$$

The map

$$Q : a_0 \mapsto a \in W_D$$

will be called a canonical quantization map or simply quantization. From (2.4) we can obtain an "explicit" form of the operator Q

$$\begin{aligned} Qa &= \sum_{k=0}^{\infty} (\delta^{-1}(\partial + \operatorname{ad} s))^k a_0 \\ &= (1 - \delta^{-1}(\partial + \operatorname{ad} s))^{-1} a_0. \end{aligned} \tag{2.5}$$

In such a form Q may be applied to any section of the bundle $W \otimes K$, the result will be not necessarily flat in general.

Further, the equation

$$Da = b$$

with a one-form b satisfying $Db = 0$ has a unique solution a such that $a|_{y=0} = 0$. We denote it $D^{-1}b$. It can be found from the equation similar to (2.4)

$$a = -\delta^{-1}b + \delta^{-1}\left(\partial a + [s, a]\right) \tag{2.6}$$

which is equivalent to to a system

$$Da = b, \quad a|_{y=0} = 0$$

with a solvability condition $Db = 0$. Using the operator Q the result may be written in the form

$$D^{-1}b = -Q(\delta^{-1}b). \tag{2.7}$$

Next we recall the definition of the canonical trace on W_D. Locally, using the Darboux charts we can trivialize the manifold M and the bundle E and construct a connection D_0 on the restriction of the bundle $W \otimes K$ to the coordinate neighborhood $O \subset M$ starting with the de Rham differential d_x as an initial connection ∂.

It turns out that the two algebras $W_D|_O$ and $(W|_O)|_{D_0}$ defined over O are isomorphic, moreover, this isomorphism may be taken in the form

$$\begin{aligned} A : W_D|_O &\longrightarrow (W|_O)_{D_0} \\ Aa &= U \circ a \circ U^{-1}. \end{aligned} \tag{2.8}$$

Here $a \in W_D, U$ is a section of $W|_O$ extended by negative powers of h so that the total degree of each term with respect to y^i and h is positive (see [1, 2]), the product $\circ$ is a fiberwise Weyl product in the bundle W. Applying D_0 to (2.8), we come to the following equation for U

$$\left[U^{-1} \circ D_0 U - \Delta\gamma, a\right] = 0$$

with

$$\Delta\gamma = \frac{i}{2h}\Gamma_{ijk}y^i y^j dx^k + \Gamma_i^E dx^i + s.$$

We set

$$U^{-1} \circ D_0 U = \Delta\gamma. \tag{2.9}$$

This equation has a unique solution satisfying the normalization condition $U|_{y=0} = 1$. In fact the equation (2.9) together with $U|_{y=0} = 1$ is equivalent to the following equation

$$U = 1 + \delta^{-1}\left(dU + U \circ \Delta\gamma\right) \tag{2.10}$$

which may be solved by iterations.

Now, for a section $a \in W_D$ with support in a coordinate chart O we define

$$\operatorname{Tr} a = \frac{(2\pi h)^{-n}}{n!} \int_{\mathbb{R}^{2n}} \operatorname{tr}\ (Aa)|_{y=0}\, \omega^n \tag{2.11}$$

where tr means coefficient trace in $K = \operatorname{Hom}(E, E)$. In general, we define $\operatorname{Tr} a$ by linearity using coordinate Darboux covering $\{O_i\}$, a partition of unity $\{\rho_i\}$ and local trivializations A_i defined by (2.8), (2.9). It turns out that this is a correct definition, in particular, (2.11) does not depend on the choice of local coordinates and trivialization isomorphisms. The trace (2.11) is unique up to a normalization factor, the choice of this factor as in (2.11) gives us the so-called canonical trace.

One can observe, carefully following the formulas (2.2), (2.4), (2.10), that the integrand in (2.11) is a linear combination of covariant derivatives of the original section $a_0 \in C^\infty(W, K)$, that is

$$\operatorname{tr}\ (Aa)|_{y=0} = \sum_I \operatorname{tr} c^I \partial_I a_0$$

where $I = (i_1, i_2, \ldots, i_k)$ means a set of indices. Integration by parts in (2.11) gives

$$\operatorname{Tr} a = \frac{(2\pi h)^{-n}}{n!} \int_{\mathbb{R}^{2n}} \operatorname{tr} \sum_I (-1)^{|I|} \partial_{I'} c^I a_0 \tag{2.12}$$

where $I' = (i_k i_{k-1}, \ldots, i_1)$. The function

$$c(x, h) = \sum_I (-1)^{|I|} \partial_{I'} c^I \in C^\infty(M, K) \tag{2.13}$$

is a correctly defined section of $C^\infty(M, K)$, in particular, it does not depend upon the choice of local coordinates and trivializations of the bundle E. Thus, we have come to the notion of the trace density which may be expressed by the formula

$$\operatorname{Tr} a = \frac{(2\pi h)^{-n}}{n!} \int_M \operatorname{tr} c a_0 \omega^n.$$

In principle, we are even able to compute it using (2.2)–(2.13), but very soon it becomes evident that the direct approach is rather hopeless. Nevertheless, these formulas lead to the following important conclusions:

1. the trace density depends only on connection coefficients Γ_{ijk} and Γ_i^E in local coordinates and local frames and their derivatives,

2. to compute $c(x, h)$ at a point $x \in M$, we can use arbitrary local coordinates and frames in a small neighborhood of x, the result does not depend on the choices,

3. using normal Darbooux coordinates (see [2]) centered at $x \in M$, we see that $c(x, h)$ depends on jets of connection coefficients at the origin. They, in turn, may be expressed through curvature tensors and their covariant derivatives.

3 The Variation of the Trace

Let us fix a classical observable $a_0(x) \in C^\infty(M, K)$, quantize it obtaining $a = Q(a_0) \in W_D$ and take its canonical trace $\operatorname{Tr} a$. We are interested how this quantity depends on the connection ∂.

Proposition 3.1. *The following variation formula holds*

$$\frac{d}{dt} \operatorname{Tr} a = \operatorname{Tr} \left([D^{-1}\dot{\gamma}, a] - D^{-1}[\dot{\gamma}, a] \right) \tag{3.1}$$

where

$$\dot{\gamma} = \Delta\dot{\gamma} = \frac{i}{2h} \dot{\Gamma}_{ijk} y^i y^j dx^k + \dot{\Gamma}_i^E dx^i + \dot{s}.$$

Proof. Considering variations of the connection ∂ in a coordinate neighborhood O, we can use the local formula (2.11) implying

$$\frac{d}{dt} \operatorname{Tr} a = \frac{(2\pi h)^{-n}}{n!} \int_M \operatorname{tr} \left. \frac{d}{dt}(Aa) \right|_{y=0} \omega^n.$$

Further, (2.8) yields

$$\frac{d}{dt}(Aa) = U \circ \{[U^{-1} \circ \dot{U}, a] + \dot{a}\} \circ U^{-1}. \tag{3.2}$$

Differentiating (2.9) and using that D_0 is a constant connection and thus independent of t, we obtain

$$U^{-1} \circ D_0 \dot{U} - U^{-1} \circ \dot{U} \circ U^{-1} \circ D_0 U = \Delta\dot{\gamma}.$$

The first term may be transformed further

$$U^{-1} \circ D_0 \dot{U} = D_0(U^{-1} \circ \dot{U}) + U^{-1} \circ D_0 U \circ U^{-1} \circ \dot{U},$$

so the left-hand side of the previous equation may be written as

$$\begin{gathered} D_0(U^{-1} \circ \dot{U}) + [U^{-1} \circ D_0 U, U^{-1} \circ \dot{U}] \\ = D_0(U^{-1} \circ \dot{U}) + [\Delta\gamma, U^{-1} \circ \dot{U}] = D(U^{-1} \circ \dot{U}). \end{gathered}$$

Here we have made use that $U^{-1} \circ D_0 U$ is equal to $\Delta\gamma$ in virtue of (2.9) and further that

$$D_0 + [\Delta\gamma, \cdot\,] \equiv D.$$

Finally, we arrive to the equation for $U^{-1} \circ \dot{U}$

$$D(U^{-1} \circ \dot{U}) = \Delta\dot{\gamma}. \tag{3.3}$$

The solvability condition $D(\Delta\dot{\gamma}) = 0$ follows from the construction of D. In terms of the connection D_0 and the form $\Delta\gamma$ we can rewrite the equation (2.1) in the form

$$D_0(\Delta\gamma) + (\Delta\gamma)^2 = 0.$$

The differentiation yields

$$D_0(\Delta\dot{\gamma}) + [\Delta\gamma, \Delta\dot{\gamma}] = D(\dot{\Delta\gamma}) = 0.$$

Thus, from (3.3) we obtain

$$U^{-1} \circ \dot{U} = (D^{-1}(\Delta\dot{\gamma}) + b)$$

where b is an arbitrary flat section, that is $b \in W_D|_O$.

Now differentiating $Da = 0$, we come to the following equation for $\dot{a}$

$$D\dot{a} + [\Delta\dot{\gamma}, a] = 0.$$

The solvability condition is fulfilled since

$$D[\Delta\dot{\gamma}, a] = [D(\Delta\dot{\gamma}), a] + [\Delta\dot{\gamma}, Da] = 0.$$

We also have

$$\dot{a}\big|_{y=0} = \dot{a}_0 = 0,$$

implying

$$\dot{a} = -D^{-1}[\Delta\dot{\gamma}, a].$$

Thus, the right-hand side of (3.2) becomes

$$U \circ \big\{\big(\big[D^{-1}\dot{\gamma} + b, a\big] - D^{-1}\left[\dot{\gamma}, a\right]\big)\big\} \circ U^{-1},$$

where we have replaced $\Delta\dot{\gamma}$ by a shorter notation $\dot{\gamma}$.

Now we can return to global expressions. To this end observe that

$$[D^{-1}\dot{\gamma}, a] - D^{-1}[\dot{\gamma}, a]$$

is a flat section. Indeed, applying D, we get

$$D[D^{-1}\dot{\gamma}, a] = [\dot{\gamma}, a]$$

since $DD^{-1}\dot{\gamma} = \dot{\gamma}$ and $Da = 0$. On the other hand,

$$D(D^{-1}[\dot{\gamma}, a]) = [\dot{\gamma}, a],$$

implying the assertion.

Now for a flat section $s \in W_D$ with support in a neighborhood O we have by definition

$$\frac{(2\pi h)^{-n}}{n!} \int_{\mathbb{R}^{2n}} \operatorname{tr} U \circ s \circ U^{-1} \omega^n = \operatorname{Tr} s.$$

It means that the integral of the right-hand side of (3.2) is nothing but

$$\operatorname{Tr}\left([D^{-1}\dot{\gamma}, a] - D^{-1}[\dot{\gamma}, a] + [b, a]\right).$$

The last summand may be dropped since the trace of the commutator vanishes. $\square$

To calculate the section $D^{-1}\dot{\gamma} \in C^\infty(M, W \otimes K)$ (which is not flat), let us write equation (2.6) for it. Denoting $u = D^{-1}\dot{\gamma}$, we have

$$u = -\delta^{-1}\dot{\gamma} + \delta^{-1}\left(\partial u + [s, u]\right).$$

Taking into account that $\delta^{-1}\dot{s} \equiv 0$, so that

$$\delta^{-1}\dot{\gamma} = \delta\dot{\Gamma} = \frac{i}{6h}\dot{\Gamma}_{ijk}y^i y^j y^k + \dot{\Gamma}^E_i y^i,$$

we obtain an equation

$$u = -\frac{i}{6h}\dot{\Gamma}_{ijk}y^i y^j y^k + \dot{\Gamma}^E_i y^i + \delta^{-1}\left(\partial u + [s, u]\right) \tag{3.4}$$

which gives a computational algorithm by the iteration method.The result may be represented as

$$D^{-1}\dot{\gamma} = -Q\dot{\Gamma} = -Q(\dot{\Gamma}^E_i y^i + \frac{i}{6h}\dot{\Gamma}_{ijk}y^i y^j y^k) \tag{3.5}$$

with Q given by (2.5) The formula (3.1) can be immediately translated into the language of trace density. Indeed, by the definition of the trace density the left-hand side of (3.1) is equal to (up to a normalizing factor)

$$\int_M \operatorname{tr} \dot{c} a_0 \omega^n.$$

Similarly, the right-hand side is

$$\begin{aligned} &\textstyle\int_M \operatorname{tr} c\left([D^{-1}\dot{\gamma}, a] - D^{-1}[\dot{\gamma}, a]\right)\big|_{y=0} \omega^n \\ &= \textstyle\int_M \operatorname{tr} c\left[D^{-1}\dot{\gamma}, a\right]\big|_{y=0} \omega^n, \end{aligned}$$

since $D^{-1}\left[\dot{\gamma}, a\right]\big|_{y=0} = 0$. Thus, the variation formula may be rewritten as

$$\int_M \operatorname{tr} \dot{c} a_0 \omega^n = \int_M \operatorname{tr} c\left[D^{-1}\dot{\gamma}, Q(a_0)\right]\big|_{y=0} \omega^n. \tag{3.6}$$

Finally, we can get rid of a_0 since it is an arbitrary function and obtain an equation for c. To this end introduce a pairing

$$\langle a, b\rangle = \int_M \operatorname{tr} a \circ b|_{y=0}\, \omega^n$$

on the space of all sections $a, b \in C^\infty(M, W \otimes K)$

Lemma 3.2. *The pairing $\langle\cdot,\cdot\rangle$ has the following properties:*

(1) *it is non-degenerate, that is $\langle a,b\rangle = 0$ for any $b \in C^\infty(M, W\otimes K)$ if and only if $a = 0$,*

(2) *for homogeneous sections*

$$a = a_p = \frac{1}{p!} a_{i_1 i_2 \dots i_p} y^{i_1} y^{i_2} \dots y^{i_p}, \quad b = b_q = \frac{1}{q!} b_{j_1 j_2 \dots j_q} y^{j_1} y^{j_2} \dots y^{j_q}$$

$\langle a,b\rangle = 0$ if $p \neq q$ and

$$\langle a_p, b_p\rangle = (-1)^p \langle b_p, a_p\rangle, \tag{3.7}$$

(3)

$$\langle a, \delta^{-1}\partial b\rangle = -\frac{ih}{2}\langle i(\omega^{-1})\partial\delta e a, b\rangle \tag{3.8}$$

where $ea_p = a_p/p$ for $p > 0$, $ea_0 = 0$ and $i(\omega^{-1})$ means a contraction with the tensor ω^{ij}.

Proof. The first two properties are obvious, so we prove only (3.8). Let a_p, b_{p-1} be two homogeneous sections of degree p and $p-1$ respectively. Then

$$\delta^{-1}\partial b = \frac{1}{p!}\partial_j b_{j_1 \dots j_{p-1}} y^{j_1} \dots y^{j_{q-1}} y^j,$$

so, the left-hand side is equal to

$$\int_M \operatorname{tr}\left(-\frac{ih}{2}\right)^p \frac{1}{p!} a_{i_1 \dots i_p} \partial_j b_{j_1 \dots j_{p-1}} \omega^{i_1 j_1} \dots \omega^{i_{p-1} j_{p-1}} \omega^{i_p j} \omega^n.$$

On the right-hand side we have

$$\delta e a = \frac{1}{p!} a_{i_1 \dots i_{p-1} i_p} y^{i_1} \dots y^{p-1} dx^{i_p},$$

$$\begin{aligned} i(\omega^{-1})\partial\delta e a &= i(\omega^{-1})\tfrac{1}{p!}\partial_j a_{i_1 \dots i_{p-1} i_p} y^{i_1} \dots y^{i_{p-1}} dx^j \wedge dx^{i_p} \\ &= \tfrac{1}{p!}\partial_j a_{i_1 \dots i_{p-1} i_p} y^{i_1} \dots y^{i_{p-1}} \omega^{j i_p}, \end{aligned}$$

so that

$$-\frac{ih}{2}\mathrm{tr}\,(i(\omega^{-1})\partial\delta e a)\circ b|_{y=0}$$
$$=\left(-\frac{ih}{2}\right)^p \frac{1}{p!}\mathrm{tr}\,\partial_j a_{i_1\dots i_p} b_{j_1\dots j_{p-1}}\omega^{j i_p}\omega^{i_1 j_1}\dots\omega^{i_{p-1}j_{p-1}}.$$

After integration by parts we obtain the same expression as on the left-hand side. □

Non-degeneracy allows one to define a transposition for operators acting on sections, so that

$$\langle A' a, b\rangle = \langle a, Ab\rangle.$$

For example, if $A = \mathrm{ad}\, c_p$ where c_p is homogeneous in y of degree p then, using (3.7), we obtain

$$\langle a, (\mathrm{ad}\, c)b\rangle = \langle a, c\circ b - b\circ c\rangle = \langle a\circ c, b\rangle - (-1)^p\langle c\circ a, b\rangle,$$

so that

$$(\mathrm{ad}\, c_p)' a = a\circ c - (-1)^p c\circ a.$$

Now, we can remove the operator Q from a_0 using this transposition. We also use the fact that $a_0 \in C^\infty(M, K)$, that is does not depend on y. The latter observation implies

$$\delta^{-1}(\mathrm{ad}\, s)a_0 = (\mathrm{ad}\,\delta^{-1}s)a_0 = 0$$

because of the condition $\delta^{-1}s = 0$. Thus,

$$Qa_0 = a_0 + Q\delta^{-1}(\partial + \mathrm{ad}\, s)a_0 = a_0 + Q\delta^{-1}\partial a_0, \tag{3.9}$$

and we come to the following theorem.

Theorem 3.3. *The trace density satisfies the following differential equation*

$$\dot{c} = \frac{ih}{2}\omega^{ij}\partial_i\frac{\partial}{\partial y^j}Q'(\mathrm{ad}\, Q(\delta^{-1}\dot{\Gamma}))' c|_{y=0} \tag{3.10}$$

where

$$Q' = \left(1 + \frac{ih}{2} i(\omega^{-1})(\partial + (\mathrm{ad}\, s)'\delta e)\right)^{-1}.$$

Proof. The equation easily follows from (3.6), (3.5) and (3.9) by Lemma 3.2. ☐

This theorem gives an algorithm to compute the trace density. In a local chart introduce a family of connections with

$$\Gamma(t) = t(\Gamma_i^E dx^i + \frac{i}{2h}\Gamma_{ijk}y^i y^j dx^k), \ 0 \le t \le 1. \tag{3.11}$$

It is clear that the coefficients of the trace density are polynomials in t, so the Taylor series gives a complete answer:

$$c = c(1) = \sum_{k=0}^{\infty} \frac{1}{k!}c^{(k)}(0).$$

On the other hand, the derivatives $c^{(k)}(0)$ may be computed successively from (3.10). This yields an expression for c in terms of connection coefficients and their derivatives. But we know that this expression is independent of the choice of local Darboux coordinates and local frames. So, using normal Darboux coordinates, we can rewrite this expression in terms of curvature tensors and their covariant derivatives.

4 Calculations

In this section we calculate the coefficients c_1 and c_2 of the expansion (1.3) using the differential equation (3.10) for the trace density. So, we introduce a family of connections in local coordinates and frames

$$\Gamma(t) = \Gamma_i^E(t)dx^i + \frac{i}{2h}\Gamma_{i_1i_2i_3}(t)y^{i_1}y^{i_2}dx^{i_3},$$

so that $\Gamma(0) = 0$ and $\Gamma(1)$ coincides with the given connection form Γ. For example, we may take a linear family (3.11).

Let

$$s = s_1 + s_2 + \dots$$

be the solution of (2.2), (2.3) where s_k denotes the term of degree k in y, h. We calculate the right-hand side of (3.10) up to degree 4. So,

$$Q(\delta^{-1}\dot{\Gamma}(t)) = u_1 + u_2 + u_3 + \dots$$

with

$$
\begin{gathered}
u_1 = \delta\dot{\Gamma}(t) = \dot{\Gamma}_i^E y^i + \tfrac{i}{6h}\dot{\Gamma}_{i_1 i_2 i_3}(t) y^{i_1} y^{i_2} y^{i_3} \\
u_2 = \delta^{-1}\partial u_1 \\
u_3 = \delta^{-1}[s_1, u_1] + \delta^{-1}\partial\delta^{-1}\partial u_1.
\end{gathered} \tag{4.1}
$$

Further, define $v = v_1 + v_2 + v_3 + \ldots$ to be

$$
\begin{gathered}
v = (\mathrm{ad}\, Q(\delta^{-1}\dot{\Gamma}))' c \\
= (1 + hc_1 + \ldots)(u_1 + u_2 + u_3 + \ldots) + (u_1 - u_2 + u_3 \ldots)(1 + hc_1 + \ldots) \\
= 2u_1 + 2u_3 + h(u_1 c_1 + c_1 u_1) + \ldots,
\end{gathered}
$$

so that $v_2 \equiv 0$. Thus, up to degree 4 the right-hand side of (3.10) may be written in the form

$$
\begin{gathered}
\tfrac{ih}{2}\omega^{ij}\partial_i \tfrac{\partial}{\partial y^j} Q' v\Big|_{y=0} \\
= \tfrac{ih}{2}\omega^{ij}\partial_i \tfrac{\partial}{\partial y^j} \Big\{ v_1 + v_3 - \tfrac{ih}{2} i(\omega^{-1})(\partial + (\mathrm{ad}\, s)')\delta e v_1 \\
+ \left(\tfrac{ih}{2} i(\omega^{-1})\partial\delta e\right)^2 v_1 \Big\}\Big|_{y=0}.
\end{gathered} \tag{4.2}
$$

The first observation is that only linear terms with respect to y in curly brackets do contribute to (4.2). Hence, the second term in (4.1) may be dropped since it is at least cubic in y. Retaining only linear terms in the first summand of (4.1), we obtain the following expression for v_3 in (4.2)

$$
v_3 = 2\delta^{-1}\left([s_1, u_1]\big|_{y=0} \right) + h(c_1 u_1 + u_1 c_1). \tag{4.3}
$$

Next, taking into account that v_1 (which is equal to $2u_1$) contains linear and cubic terms in y, we see that $\partial\delta e v_1$ contains only even-degree terms in y, so, this term does not contribute to (4.2).

Thus, retaining only essential terms, we rewrite (3.10) in the form

$$
\begin{aligned}
& h\dot{c}_1 + h^2\dot{c}_2 + \dots \\
&= ih\omega^{ij}\partial_i \tfrac{\partial}{\partial y^j} u_1\Big|_{y=0} \\
&+ \tfrac{ih^2}{2}\omega^{ij}\partial_i \tfrac{\partial}{\partial y^j}(c_1 u_1 + u_1 c_1)\Big|_{y=0} \\
&+ ih\omega^{ij}\partial_i \tfrac{\partial}{\partial y_j}\delta^{-1}\left([s_1, u_1]|_{y=0}\right) \\
&+ \tfrac{h^2}{2}\omega^{ij}\partial_i \tfrac{\partial_i}{\partial y^j} i(\omega^{-1})(\operatorname{ad} s_1)'\delta e u_1\Big|_{y=0} \\
&- \tfrac{ih^3}{4}\omega^{ij}\partial_i \tfrac{\partial}{\partial y^j}(i(\omega^{-1})\partial\delta e)^2 u_1\Big|_{y=0} + \dots
\end{aligned}
\tag{4.4}
$$

We have replaced s by its leading term s_1, this does not affect the terms of degree ≤ 4.

Comparing linear terms in h, we get

$$
\begin{aligned}
\dot{c}_1 &= i\omega^{ij}\partial_i \tfrac{\partial}{\partial y^j} u_1\Big|_{y=0} = i\omega^{ij}\partial_i \dot{\Gamma}^E_j \\
&= \tfrac{i}{2}\omega^{ij}\left(\partial_i\dot{\Gamma}^E_j - \partial_j\dot{\Gamma}^E_i\right) = \tfrac{i}{2}\dot{R}^E_{ij}\omega^{ij},
\end{aligned}
$$

so that

$$
c_1 = \frac{i}{2} R^E_{ij}
$$

(we again drop ω^{ij} in the final expression). Substitute this result into (4.4) and compare the coefficients at h^2. The second term in (4.4) yields

$$
-\frac{1}{4}\partial_i\left(R^E_{i_1 j_1}\dot{\Gamma}^E_j + \dot{\Gamma}^E_j R^E_{i_1 j_1}\right) \tag{4.5}
$$

(contraction in i, j and i_1, j_1 is understood). Passing to the third term, we first compute

$$
\begin{aligned}
[s_1, u_1]|_{y=0} &= \left[\tfrac{1}{2}R^E_{i_1 i_2} y^{i_1} dx^{i_2}, \dot{\Gamma}^E_i y^i\right]\Big|_{y=0} \\
&+ \left[\tfrac{i}{8h}R_{i_1 i_2 i_3 i_4} y^{i_1} y^{i_2} y^{i_3} dx^{i_4}, \tfrac{i}{6h}\dot{\Gamma}_{i_5 i_6 i_7} y^{i_5} y^{i_6} y^{i_7}\right]\Big|_{y=0} \\
&= -\tfrac{ih}{4}\left(R^E_{i_1 i_2}\dot{\Gamma}^E_{j_1} + \dot{\Gamma}^E_{j_1} R^E_{i_1 i_2}\right) dx^{i_2} \\
&\quad -\tfrac{ih}{32} R_{i_1 i_2 i_3 i_4}\dot{\Gamma}_{j_1 j_2 j_3} dx^{i_4}.
\end{aligned}
$$

Thus, the third term yields

$$\tfrac{1}{4}\partial_{i_2}\left(R^E_{i_1j_2}\dot{\Gamma}^E_{j_1}+\dot{\Gamma}^E_{j_1}R^E_{i_1j_2}\right)$$
$$+\tfrac{1}{32}\partial_{i_4}\left(R_{i_1i_2i_3j_4}\dot{\Gamma}_{j_1j_2j_3}\right). \tag{4.6}$$

The fourth term requires more work. First of all we have

$$(\mathrm{ad}\, s_1)'\delta eu_1=\delta eu_1\circ s_1-s_1\circ\delta eu_1,$$

the sign $-$ before the second summand appears since we have to change the sign twice: once because s_1 contains the terms of odd degrees in y, and once more because s_1 and δeu_1 are one-forms. The terms linear in y are

$$\tfrac{1}{2}\left(\dot{\Gamma}^E_{i_1}R^E_{i_2i_3}+R^E_{i_2i_3}\dot{\Gamma}^E_{i_1}\right)y^{i_2}dx^{i_1}\wedge dx^{i_3}$$
$$+\tfrac{1}{16}\dot{\Gamma}_{i_1i_2i_3}R_{(j_1j_2i_4)i_5}y^{i_4}dx^{i_3}\wedge dx^{i_5},$$

where parentheses mean symmetrization. Thus, the fourth summand becomes

$$\tfrac{1}{4}\partial_{i_1}\left(\dot{\Gamma}^E_{i_2}R^E_{j_1j_2}+R^E_{j_1j_2}\dot{\Gamma}^E_{i_2}\right)$$
$$+\tfrac{1}{32}\partial_{i_4}\left(\dot{\Gamma}_{i_1i_2i_3}R_{(j_1j_2j_4)j_3}\right).$$

The completely symmetric component $R_{((j_1j_2j_4)j_3)}$ of the tensor $R_{(j_1j_2j_4)j_3}$ vanishes, hence

$$R_{(j_1j_2j_4)j_3}+R_{(j_1j_2j_3)j_4}+R_{(j_1j_3j_4)j_2}+R_{(j_3j_2j_4)j_1}=0.$$

After contraction with the completely symmetric tensor $\dot{\Gamma}_{i_1i_2i_3}$ this identity yields

$$3\dot{\Gamma}_{i_1i_2i_3}R_{(j_1j_2j_4)j_3}=-\dot{\Gamma}_{i_1i_2i_3}R_{j_1j_2j_3j_4},$$

so that the fourth term in (4.4) becomes

$$\tfrac{1}{4}\partial_{i_1}\left(\dot{\Gamma}^E_{i_2}R^E_{j_1j_2}+R^E_{j_1j_2}\dot{\Gamma}^E_{i_2}\right)$$
$$-\tfrac{1}{96}\partial_{i_4}\left(\dot{\Gamma}_{i_1i_2i_3}R_{j_1j_2j_3j_4}\right). \tag{4.7}$$

Finally, the last term in (4.4) gives

$$\frac{1}{24}\partial_{i_1}\partial_{i_2}\partial_{i_3}\dot{\Gamma}_{j_1j_2j_3}. \tag{4.8}$$

Summing up (4.5) - (4.8), we obtain

$$\begin{aligned}\dot{c}_2 = -\tfrac{1}{4}\partial_{i_1}\left(\dot{\Gamma}^E_{j_1}R^E_{i_2j_2} + R^E_{i_2j_2}\dot{\Gamma}^E_{j_1}\right) \\ +\tfrac{1}{2}\partial_{i_1}\left(\dot{\Gamma}^E_{i_2}R^E_{j_1j_2} + R^E_{j_1j_2}\dot{\Gamma}^E_{i_2}\right) \\ +\tfrac{1}{24}\left(\partial_{i_1}\partial_{i_2}\partial_{i_3}\dot{\Gamma}_{j_1j_2j_3} - \partial_{i_4}(\dot{\Gamma}_{i_1i_2i_3}R_{j_1j_2j_3j_4})\right).\end{aligned} \tag{4.9}$$

Now we can watch an amazing interplay of the symmetry properties of connection coefficients and curvature tensors resulting in the fact that the right-hand side is a complete derivative in t. Clearly, the terms corresponding to ∂^E and ∂ may be treated separately. First consider the terms corresponding to ∂^E dropping temporarily the superscript E:

$$\begin{aligned}&-\tfrac{1}{4}(R_{i_1j_1}\partial_{i_2}\dot{\Gamma}_{j_2} + \partial_{i_2}\dot{\Gamma}_{j_2}R_{i_1j_1}) \\ &-\tfrac{1}{4}(\partial_{i_2}R_{i_1j_1}\dot{\Gamma}_{j_2} + \dot{\Gamma}_{j_2}\partial_{i_2}R_{i_1j_1}) \\ &+\tfrac{1}{2}(\partial_{i_1}\dot{\Gamma}_{i_2}R_{j_1j_2} + R_{j_1j_2}\partial_{i_1}\dot{\Gamma}_{i_2}) \\ &+\tfrac{1}{2}(\dot{\Gamma}_{i_2}\partial_{i_1}R_{j_1j_2} + \partial_{i_1}R_{j_1j_2}\dot{\Gamma}_{i_2}).\end{aligned}$$

Note, that the second and the fourth terms cancel out. Indeed, changing summation indices, rewrite them as

$$\begin{aligned}&-\tfrac{1}{4}(\partial_{i_2}R_{i_1j_1} + 2\partial_{i_1}R_{j_1i_2})\dot{\Gamma}_{j_2} \\ &-\tfrac{1}{4}\dot{\Gamma}_{j_2}(\partial_{i_2}R_{i_1j_1} + 2\partial_{i_1}R_{j_1i_2}).\end{aligned}$$

Expressions in parentheses vanish by Bianchi's identity:

$$\begin{aligned}\partial_{i_2}R_{i_1j_1} + 2\partial_{i_1}R_{j_1j_2} &= \partial_{i_2}R_{i_1j_1} + \partial_{i_1}R_{j_1j_2} - \partial_{j_1}R_{i_1j_2} \\ &= \partial_{i_2}R_{i_1j_1} + \partial_{i_1}R_{j_1j_2} + \partial_{j_1}R_{j_2i_1} = 0.\end{aligned}$$

In the remaining summands everything may be expressed in terms of the curvature tensor. Since $R_{j_1j_2}$ is antisymmetric and because j_1, j_2 are contracted with i_1, i_2, we may antisymmetrize with respect to i_1, i_2, that is

$$\begin{aligned}\partial_{i_1}\dot{\Gamma}_{i_2}R_{j_1j_2} &= \tfrac{1}{2}(\partial_{i_1}\dot{\Gamma}_{i_2} - \partial_{i_2}\dot{\Gamma}_{i_1})R_{j_1j_2} \\ &= \tfrac{1}{2}\dot{R}_{i_1i_2}R_{j_1j_2}.\end{aligned}$$

Thus, the remaining terms give

$$\begin{aligned}
&-\tfrac{1}{8}(R_{i_1j_1}\dot R_{i_2j_2}+R_{i_2j_2}R_{i_1j_1})\\
&+\tfrac{1}{4}(\dot R_{i_1i_2}R_{j_1j_2}+R_{j_1j_2}\dot R_{i_1i_2})\\
&=-\tfrac{1}{8}\tfrac{d}{dt}(R_{i_1j_1}R_{i_2j_2}-2R_{i_1i_2}R_{j_1j_2})\\
&=-\tfrac{1}{8}\tfrac{d}{dt}(R_{i_1j_1}R_{i_2j_2}+R_{i_1i_2}R_{j_2j_1}+R_{i_1j_2}R_{j_1j_2})\\
&=-\tfrac{3}{8}\tfrac{d}{dt}R_{[i_1j_1}R_{i_2j_2]}.
\end{aligned}$$

The terms corresponding to ∂ are proportional to

$$\partial_{i_1}\partial_{i_2}\partial_{i_3}\dot\Gamma_{j_1j_2j_3}-\partial_{i_4}(\dot\Gamma_{i_1i_2i_3}R_{j_1j_2j_3j_4}). \tag{4.10}$$

We transform the first term using the relation

$$\begin{aligned}
\partial_{i_3}\dot\Gamma_{j_1j_2j_3}\omega^{i_3j_3}&=\tfrac{1}{2}(\partial_{i_3}\dot\Gamma_{j_1j_2j_3}-\partial_{j_3}\Gamma_{j_1j_2j_3})\omega^{i_3j_3}\\
&=\tfrac{1}{2}\dot R_{j_1j_2i_3j_3}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\partial_{i_1}\partial_{i_2}\partial_{i_3}\dot\Gamma_{j_1j_2j_3}&=\tfrac{1}{2}\tfrac{d}{dt}(\partial_{i_1}\partial_{i_2}R_{j_1j_2i_3j_3})\\
&-\tfrac{1}{2}\partial_{i_1}\dot\partial_{i_2}R_{j_1j_2i_3j_3}-\tfrac{1}{2}\dot\partial_{i_1}\partial_{i_2}R_{j_1j_2i_3j_3}.
\end{aligned}$$

The last summand is equal to zero. Indeed, the only free index in $\partial_{i_2}R_{j_1j_2i_3j_3}$ is j_1, so

$$\dot\partial_{i_1}\partial_{i_2}R_{j_1j_2i_3j_3}=\dot\Gamma_{i_1j_1i_4}\partial_{i_2}R_{j_4j_2i_3j_3}=0$$

since Γ_{pqr} is symmetric in all the indices, so that $\dot\Gamma_{i_1j_1i_4}\omega^{i_1j_1}\equiv 0$. Now,

$$\dot\partial_{i_2}R_{j_1j_2i_3j_3}=\dot\Gamma_{i_2j_1i_4}R_{j_4j_2i_3j_3}+\dot\Gamma_{i_2j_2i_4}R_{j_1j_4i_3j_3},$$

and again the last summand is equal to 0, so (4.10) becomes

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}(\partial_{i_1}\partial_{i_2}R_{j_1j_2i_3j_3})-\tfrac{1}{2}\partial_{i_4}(\dot\Gamma_{i_1i_2j_4}R_{j_1j_2i_3j_3})\\
&-\partial_{i_4}(\dot\Gamma_{i_1i_2i_3}R_{j_1j_2j_3j_4}).
\end{aligned}$$

The second and the third terms considered as tensors in i_3,j_3,i_4,j_4 (the indices i_1,j_1 and i_2,j_2 in both terms are contracted and we consider them as dumb ones) are treated similarly to the ∂^E-case. Namely,

$$\begin{aligned}
&-\tfrac{1}{2}\partial_{i_4}\dot\Gamma_{i_1i_2j_4}R_{j_1j_2i_3j_3}-\partial_{i_4}\dot\Gamma_{i_1i_2i_3}R_{j_1j_2j_3j_4}\\
&=-\tfrac{1}{4}\dot R_{i_1i_2i_4j_4}R_{j_1j_2i_3j_3}-\tfrac{1}{2}\dot R_{i_1i_2i_4i_3}R_{j_1j_2j_3j_4}\\
&=-\tfrac{1}{8}\tfrac{d}{dt}(R_{i_1i_2i_4j_4}R_{j_1j_2i_3j_3}+R_{i_1i_2i_4i_3}R_{j_1j_2j_3j_4}+R_{i_1i_2i_4j_3}R_{j_1j_2j_4i_3})\\
&=-\tfrac{3}{8}\tfrac{d}{dt}R_{i_1i_2[i_3j_3}R_{|j_1j_2|i_4j_4]},
\end{aligned}$$

while the remaining terms

$$-\frac{1}{2}\Gamma_{i_1 i_2 j_4}\partial_{i_4} R_{j_1 j_2 i_3 j_3} - \dot{\Gamma}_{i_1 i_2 i_3}\partial_{i_4} R_{j_1 j_2 j_3 j_4}$$

vanish in virtue of the Bianchi identity.

Let us formulate the final result as a proposition.

Proposition 4.1. *The trace density up to degree* 4 *is given by the expression*

$$\begin{aligned} c = 1 + \tfrac{ih}{2} R_{ij} - \tfrac{3}{8} h^2 R_{[i_1 j_1} R_{i_2 j_2]} \\ -\tfrac{h^2}{64} R_{i_1 i_2 [i_3 j_3} R_{|j_1 j_2| i_4 j_4]} + \tfrac{h^2}{48} \partial_{i_1} \partial_{i_2} R_{j_1 j_2 i_3 j_3}. \end{aligned}$$

After multiplication by the volume form $\omega^n/(2\pi h)^n n!$ one can easily recognize characteristic forms of connections ∂^E and ∂^M. So, the first three terms give

$$1 + \left(-\frac{1}{2\pi i}\frac{1}{2} R^E_{ij} dx^i \wedge dx^j\right) + \frac{1}{2}\left(-\frac{1}{2\pi i}\frac{1}{2} R^E_{ij} dx^i \wedge dx^j\right)^2$$

times $\exp(\omega/2\pi h)$, the forth term yields

$$\frac{1}{12}\mathrm{tr}\left(\frac{1}{2\pi i}\frac{1}{4} R_{ij} dx^i \wedge dx^j\right)^2 \frac{\omega^{n-2}}{(2\pi h)^{n-2}(n-2)!}.$$

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