GLOBAL PROPERTIES OF THE MODULI OF CALABI–YAU MANIFOLDS II

(Teichmüller Theory)

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MPI/1990–90

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#O. INTRODUCTION.

This is the second part of the article "The Weil-Petersson Geometry of the moduli space of SU(n>2)(Calabi-Yau manifolds I." (See [T].) In this article we will study the global properties of the Teichmüler space of Calabi-Yau manifold. The definition of the Teichmüller space of any complex manifold is the following one:

Definition.

Let I(M) be the set of all integrable complex structures on M, then by definition the Teichmüller space of M is $\mathfrak{T}(M):=I(M)/Diff_o(M)$, where $Diff_o(M)$ is the group of diffeomorphisms of M isotopic to the identity.

Remark. Up to now we have defined the Teichmüller space set theoretically.

The content of this article is the following one:

In Chapter 1 we make a review of the results of [Ti] and [T].

In Chapter 2 we prove that the Teichmüller space exists in the category of complex analytic spaces and even more using the results of [Ti] and [T] we prove that the Teichmüller space is a non-singular complex analytic manifold of dimension equal to $\dim_{\mathbb{C}} H^1(M,\Omega^{n-1})$. This is exactly Theorem 2.2.2.

In Chapter 3 we prove the following Theorem:

THEOREM 3.

Ler M_o be a Calabi-Yau manifold with a Kähler-Einstein metric $(g_{\alpha\overline{\beta}})$. Let

$$\lambda(t)=\sum_{n=1}^{\infty}\lambda_n t^n \in C^{\infty}(M_o,\Omega_o^{0,1}\otimes\Theta_o)$$

be such that a) $\lambda(t)=\lambda_1 t+\frac{1}{2}\overline{\partial}^*G[\lambda(t),\lambda(t)]$, b) $\lambda_1\in\mathbf{H}^1(M_o,\Theta_o)$, i.e. λ_1 is a harmonic Dalbealt form with respect to the Kähler-Einstein metric $(g_{\alpha\overline{\beta}})$, $\overline{\partial}^*$is the conjugate to $\overline{\partial}$ with respect to $(g_{\alpha\overline{\beta}})$ and G is the Green operator.

Then for $|t|<\epsilon$, $\lambda(t)$ defines in $\mathfrak{T}(M_o)$ a totally geodesic real two dimensional submanifold with respect to the Weil-Petersson metric.

In #4. we prove that $\lambda(t)$ defined as in THEOREM 3 is defined for all $t\in\mathbb{C}$. Using that fact we define the extended Teichmüller space $\tilde{\mathfrak{T}}(M_o)$ and then we prove:

THEOREM 4.

The extended Teichmüller space $\tilde{\mathfrak{T}}(M_o)$ of a Calabi-Yau manifold M_o is complete with respect to the Weil-Petersson metric.

Let me remind you how we define the Weil-Petersson metric on the Teichmüller space of a Calabi-Yau manifold. From [Ti] and [T] we know that the tangent space at a point $t\in\mathfrak{T}(M)$ can be identified with $H^1(M_t,\Omega_t^{n-1})$. From Kodaira-Spencer theory it follows that the tangent space $T_{t,\mathfrak{T}(M)}=H^1(M_t,\Theta_t)$. On each M_t for $t\in\mathfrak{T}(M)$ we fix $\omega_t(n,0)$ (holomorphic n-form on the Calabi-Yau manifold M_t) such that:

$$\int_M \omega_t(n,0)\wedge\omega_t(0,n)=1,\ \omega_t(0,n)=\overline{\omega_t(,n,0)}$$

then the map

$$\phi\in H^1(M_t,\Theta_t)\rightarrow \phi\perp\omega_t(2,0)\in H^1(M_t,\Omega_t^{n-1})$$

gives the desired identification. Then the Weil-Petersson metric is defined as follows:

$$<\phi_1,\phi_2>_{W.P.}=(-1)^{\frac{n(n+1)}{2}}(i)^{n-2}\int_M(\phi_1\perp\omega_t(2,0))\wedge\overline{\phi_2\perp\omega_t(2,0))}$$

In Chapter 5 we study the Torelli problem for Calabi-Yau manifolds. Namely let

$$p:\mathfrak{T}(M)\rightarrow Gr$$

be the period map, where Gr is the Griffith's domain, i.e. the space that classifies all Hodge structures of weight n on the primitive part of $H^n(M,\mathbf{Q})$ that have the same data as the Hodge structure on M. The map

$$p:\mathfrak{T}(M)\rightarrow Gr$$

is defined as follows:

$$p(t):=\text{The Hodge structure on } H^n(M_t,\mathbf{Q})_o$$

In #5 the following Theorem is proved:

THEOREM 5.4. The map $p:\mathfrak{T}(M)\rightarrow Gr$ is an embedding.

This is the famous Torelli problem for Calabi-Yau manifolds. See [D].

In Chapter 6 we prove the following Theorem:

THEOREM 6.

Let $\pi^*:\mathcal{M}^*\rightarrow D^*$ a family of CALABI-YAU manifolds over the punctured disk $D^*:=\left\{t\in\mathbf{C}|0<|t|<1\right\}$ such that the monodtromy operator T which acts on the middlie homology group, i.e. on $H_n(M_t,\mathbf{Z})$ is trivial, i.e. T=id, then there exists a family over $D:=\left\{t\in\mathbf{C}|\ |t|<1\right\}$ $\pi:\mathcal{M}\rightarrow D$ of nonsingular complex manifolds such that on $M_o:=\pi^{-1}(0)$ there exists a holomorphic n-form $\omega_o(n,0)$ without zeroes.

In Chapter 7 we prove the following Theorem:

THEOREM 7.1.

a) The Teichmüller space $\mathfrak{T}(M)$ of a Calabi-Yau manifold M is C^{∞} diffeomorphic to $\mathbf{R}^{2N}$, where $N=\dim_{\mathbf{C}} H^1(M,\Omega^{n-1})$.

b) $\mathfrak{T}(M)$ is a Stein manifold.

The proof of THEOREM 7.1. follows the lines of the proof of the analogous theorems in the classical Teichmüller theory given by Fisher and Tromba in case of a) and Tromba in case of b) .(See [F,Tr] and [Tr1].) The main idea is to use the potential of the Weil-Petersson metric as a Morse function on The Teichmüller space. This idea in the context of the Teichmüller theory was introduced by A.Tromba.

There is an important difference between the method of Tromba and ours, namely Tromba uses the energy functional of the harmonic maps between Riemann surfaces, while in the case of Calabi-Yau manifolds we do not have a theory of harmonic maps. So we use the potential of the Weil-Petersson metric as a substitute for the energy functional of the harmonic maps. This potential is defined via deformation theory and uses the existence of a Calabi-Yau metric on M. One can prove more general result about the potential of Weil-Petersson metric on moduli of manifolds for which c_1 is positive. Namely each point of the moduli space define a $\overline{\partial}$ operator. So we can define the determinant line bundle of these operators and this determinant line bundle has the so called Quillen metric. See [Q]. The curvature of this metric is just the Weil-Petersson metric. See [T2].

In #8 we prove the analogue of the Nielson realization problem for Calabi-Yau manifolds. See [Wo], [K] & [Tr2].

Aknowledgements.

This articles was finished during my visist in UC Santa Cruz. I want to thank the Department of Mathematics of UCSC for excellent conditions for work. I want to thank A. Tromba and A. Fisher for their interest in this work.

#1. REVIEW OF SOME DEFINITIONS AND RESULTS FROM [T].

Definition 1.1.

Let M be a compact Kähler manifold such that:

a) $H^{o}(M,\Omega^{i})=0$ for $0<i<\dim_{\mathbf{C}}M=n\geq 3$.

b) $H^{o}(M,\Omega^{n})\simeq \mathbf{C}\omega_{M}(n,0)$, where $\omega_{M}(n,0)$ is a holomorphic n-form on M without zeroes then M will be called a Calabi-Yau manifold.

Remark 1.2.

Condition 1.1.b. is equivelent to the fact that $K_{M}:=\Lambda^{n}\Omega_{M}^{1,0}$ is a trivial bundle and so $c_1(M)=0$.

Definition 1.3.

A pair (M,L) will be called a polarized Calabi-Yau manifold if M is a Calabi-Yau manifold where $L\in H^{1,1}(M,\mathbf{R})$, $L=[g'_{\alpha,\overline{\beta}}]$ and $(g'_{\alpha,\overline{\beta}})$ is a Kähler metric on M.

Definition 1.4.

A Kähler metric $(g_{\alpha,\overline{\beta}})$ will be called a Calabi-Yau metric if

$$\text{Ricci}(g_{\alpha,\overline{\beta}})=\partial\overline{\partial}\ (\log\det(g_{\alpha,\overline{\beta}}))=0$$

THEOREM 1.5.(YAU).

Suppose that M is a compact manifold with $c_1(M)=0$ and $(g'_{a,\overline{\beta}})$ is a Kähler metric on M, then there exists a unique Calabi-Yau metric $(g_{a,\overline{\beta}})$ such that

$$[\text{Im}(g'_{a,\overline{\beta}})]=[\text{Im}(g_{a,\overline{\beta}})]$$

For the proof of this Theorem see [Y].

1.6. BOCNER's principle.

Bochner proved that for every holomorphic tensor ϕ on a compact complex manifold we have

(1.6.1.) $\nabla\phi=0$

where ∇ is the Levi-Chevita connection of the Calabi-Yau metric $(g_{\alpha,\overline{\beta}})$ on M. So from this principle it follows that

(1.6.2.) $\nabla\omega_{M}(n,0)=0$

(For the proof of Bocner principle see [B].)

1.7. Review of local deformation theory of CALABI-YAU manifolds.See [T].

Definition 1.7.1.a.

Let M be a compact real even dimensional manifold. An almost complex structure on M is by definition an element $I\in\Gamma(M,Hom(T^*,T^*)\simeq\Gamma(M,T^*\otimes T)$ such that $IoI=-id$. T is the tangent bundle of M and T^* is the cotangent bundle.

1.7.1.a. is equivelent to the following definition.

Definition 1.7.1.b.

An almost complex structure on M is a global splitting of the complexified cotangent bundle

$$T^*\otimes\mathbf{C} =\Omega^{1,0}(M)\dot{+}\overline{\Omega^{1,0}}(M).$$

Definition 1.7.2.

An almost complex structure on M we will call an almost complex structure on M with the following propertiy: for every $m\in M$ there exists an open neighborhood $m\in U_m$ and C^∞ functions $z^1,...,z^n:U_m\to\mathbf{C}$ such that

$$\{dz^1,...,dz^n\}$$

spann $\Gamma(U_m,\Omega^{1,0}|_{U_m})$. (Remember that $\dim_{\mathbb{R}}M=2n$.)

Definition 1.7.3.

Suppose that

$$\Omega_o^{1,0}\dot{+}\overline{\Omega_o^{1,0}} = \Omega_1^{1,0}\dot{+}\overline{\Omega_1^{1,0}} = T^*\otimes\mathbf{C}$$

are two diiferent almost complex structures on M such that at each point $m\in M$ we have:

$$\Omega_o^{1,0}\bigcap \overline{\Omega_o^{1,0}} = \Omega_1^{1,0}\bigcap \overline{\Omega_1^{1,0}} =0$$

Let

$$Pr^o\colon T^*\otimes\mathbf{C}\to\Omega_o^{1,0} \text{ and } Pr^1\colon T^*\otimes\mathbf{C}\to\Omega_1^{1,0}$$

are projections with respect to $\Omega_o^{0,1}$ and $\Omega_1^{0,1}$. Let

$$\phi\in\Gamma(M,Hom(\Omega_o^{1,0},\Omega_o^{0,1}))$$

be defined as follows:

$$\phi:\Omega_o^{1,0}\overset{(Pr^o)^{-1}}{\to}\Omega_1^{1,0}\overset{Pr^1}{\to}\Omega_o^{0,1}$$

then we call ϕ the Beltrami differential.

Remark 1.7.3.1.

Since $\Gamma(M,Hom(\Omega_o^{1,0},\Omega_o^{0,1})) \simeq\Gamma(M,\Theta\otimes\Omega_o^{0,1})$, where $\Theta:=(\Omega_o^{0,1})^*$. Then ϕ can be written in the following way:

$$\phi|_U=\sum_{\alpha,\mu} \phi^{\mu}_{\overline{\alpha}}\overline{dz^{\alpha}}\otimes\frac{\partial}{\partial z^{\mu}}$$

1.7.4.

If

$$\Omega_o^{1,0}\dot{+}\overline{\Omega_o^{1,0}} = T^*\otimes\mathbb{C}$$

is an integrable complex structure, then

$$\Omega_1^{1,0}\dot{+}\overline{\Omega_1^{1,0}} = T^*\otimes\mathbb{C}$$

is an integrable complex structure if and only if

$$\bar{\partial}\phi=\frac{1}{2}[\phi,\phi]$$

Trivial lemma 1.7.5.

Let

$$\{dz^1,...,dz^n\}$$

be a basis of $\Omega_o^{1,0}$, then

$$\{\theta^1,..,\theta^n\}$$

where

$$\theta^i=dz^i+\phi(dz^i)=dz^i+\sum_{\mu} \phi^i_{\overline{\mu}}\ \overline{dz^{\mu}}$$

is a basis for $\Omega_1^{1,0}$.

1.7.5. The non-obstruction Theorem in [Ti] and [T] .

THEOREM.

Let (M,g) be a Calabi-Yau manifold with a Ricci flat metric g. Let $\mathbb{H}^1(M,\Theta)$ be the harmonic part of $\Gamma(M,\Omega^{0,1}\otimes\Theta)$ and $\{\phi_1,\ldots,\phi_N\}$ be a basis of $\mathbf{H}^1(M,\Theta)$, then there exist

$$\phi(t_1,..,t_N)=\sum_{N}^{i=1}\phi_i t_i + \sum_{i_1+\ldots+i_N\geq 2}\phi_{i_1,\ldots,i_N}(t_1)^{i_1}\ldots(t_{i_N})^{i_N}$$

and $\epsilon>0$ such that if for each i=1,..,N $|t_i|<\epsilon$ $\phi(t_1,..,t_N)$ is a convergent power series and $\phi(t_1,..,t_N)\in\Gamma(M,\Omega^{0,1}\otimes\Theta)$ has the following properties:

a) $\bar\partial\phi(t_1,..,t_N)=\frac{1}{2}[\phi(t_1,..,t_N),\phi(t_1,..,t_N)]$

b) $\bar\partial^*\phi(t_1,..,t_N)=0$

c) $\phi_{i_1,\ldots,i_N}\lrcorner\omega_M(n,0)=\partial\Psi_{i_1,\ldots,i_N}$ for all $\phi_{i_1,\ldots,i_N}$ for which $i_1+..+i_N\geq 2$.

1.7.7. Let

$$U_\epsilon\subset\mathbf{C}^N:=\{(t_1,..,t_N)|\ |t_i|<\epsilon\}$$

Then the Kuranishi family $\mathfrak{X}\rightarrow U_\epsilon$ is defined as follows:

Let $\{\mathcal{W}_i\}$ be a covering of M and let

$$\phi(t_1,..,t_N)=\sum\phi_i(t_1,..,t_N)^{\mu}_{\bar\alpha}\overline{dz^\alpha}\otimes\frac{\partial}{\partial z_i^\mu}$$

then on U_ϵxM we will define the complex structure in the following way:

Let $\zeta_i^\mu(t_1,\ldots,t_N)$ be solutions of the system of differential equations:

$$(*)\qquad \frac{\bar\partial\zeta_i^\mu}{\overline{\partial z^\alpha}}=\sum\phi^\nu_{\bar\alpha}\frac{\partial\zeta_i^\mu}{\partial z^\nu}\ (\mu=1,\ldots,n;\ \alpha=1,..,n)$$

By Newlander-Nirenberg THEOREM (*) has solutions and so

$$\{\zeta_i^1((t_1,\ldots,t_N),\ldots,\zeta_i^n(t_1,\ldots,t_N),t_1,\ldots,t_N\}$$

will be holomorphic local coordinates in $\mathcal{W}_i$xU_ϵ.(See [N-N].)

1.8. Weil-Petersson metric on the moduli space of polarized CALABI-YAU manifolds.

Definition 1.8.1.

Let (M,L) be a polarized Calabi-Yau manifold and let $\mathfrak{X} \to U_\epsilon$ be the Kuranishi family of M. For each $t \in U_\epsilon$ for ϵ small enouph, L will define on $M_t = \pi^{-1}(t)$ a Kähler metric and so by [Y] a Calabi-Yau metric $g_{\alpha,\overline{\beta}}(t)$ such that $[\operatorname{Im} g_{\alpha,\overline{\beta}}(t)]=L$. So we can identify the tangent space

$$T_{t,U_\epsilon}=\mathbf{H}^1(M_t,\Theta_y)$$

where $\mathbf{H}^1(M_t,\Theta_y)$ is the harmonic part of $\Gamma(X_t,\Theta_t\otimes\Omega_t^{1,0})$ with respect to $g_{\alpha,\overline{\beta}}(t)$, so if ϕ_t and $\psi_t \in \mathbf{H}^1(M_t,\Theta_y)=T_{t,U_\epsilon}$, then

(1.8.1.1.) $$\langle\phi_t,\psi_t\rangle_{W.P.}=\int_M \phi^{\mu}_{t,\overline{\alpha}}\overline{\psi^{\nu}_{t,\overline{\beta}}}g^{\beta\overline{\alpha}}(t)g_{\mu\overline{\nu}}(t)\operatorname{vol}(g_{\alpha,\overline{\beta}}(t))$$

In [T] the following formula is proved:

(1.8.2.) $$\langle\phi_t,\psi_t\rangle_{W.P.}=(-1)^{\frac{n(n+1)}{2}}(i)^{n-2}\int_M[\phi_t\lrcorner\omega_t(n,0)]\wedge\overline{[\psi_t\lrcorner\omega_t(n,0)]}$$

where $\omega_t(n,0)$ is a holomorphic n-form on M_t such that

$$\omega_t(n,0)\wedge\overline{\omega_t(n,0)}=\operatorname{vol}(g_{\alpha,\overline{\beta}})$$

and $[\phi_t\lrcorner\omega_t(n,0)]$ is a class of cohomology of type $(n-1,1)$.
(See also [Ti])

(1.8.3.a.) Let M_O be a Calabi-Yau manifold. We can choose a covering $\{\mathcal{W}_i\}$ of M_O and the local coordinates $(z_i^1,\ldots,z_i^n)$ in such way that

$$\omega_O(n,0)|_{\mathcal{W}_i}=dz_i^1\wedge\ldots\wedge dz_i^n$$

(1.8.3.b.) For each $t=(t_1,\ldots,t_N)\in U_\epsilon$ we define $\mathbf{A}_t\in\Gamma(M_O,\operatorname{Hom}(T^*,T^*))$ in the following way:

$$\mathbf{A}_t(dz^i)=\Theta_t=dz^i+\sum_{\alpha=1}^{\alpha}\phi^i_{\overline{\alpha}}\,\overline{dz^\alpha}\ \&\ \mathbf{A}_t(\overline{dz^i})=\overline{dz^i}+\sum_{\alpha=1}^{\alpha}\phi^i_{\overline{\alpha}}\,dz^\alpha$$

Let A_t be the matrix of this operator in the basis $\{dz^1,\ldots,dz^n,\overline{dz^1},\ldots,\overline{dz^n}\}$

1.8.5. Trivial lemma.

If I_t is the complex structure defined by the Beltrami diferential $\phi(t)$, then

$$I_t=A_tI_oA_t^{-1}$$

(1.8.5.) Important Lemma.

In [T] the following Lemma is proved:

Lemma: (See [T])

$$1)\quad \Theta_t^1\wedge..\wedge\Theta_t^n|_{\mathcal{W}_i}=dz_i^1\wedge..\wedge dz_i^n+\sum_{k=1}^{n}(-1)^{\frac{k(k-1)}{2}}\wedge^k(\phi(t)\perp(dz_i^1\wedge..\wedge dz_i^n))$$

where

$$\wedge^k\phi(t)\in\Gamma(M_o,\mathrm{Hom}(\wedge^k\Omega_o^{1,0},\wedge^k\Omega_o^{0,1}))$$

and

$$\wedge^k\phi(t)(u^1\wedge..\wedge u^n):=\phi(t)(u^1)\wedge..\wedge\phi(t)(u^k)$$

$$2)\quad \Theta_t^1\wedge..\wedge\Theta_t^n|_{\mathcal{W}_i}=dz_i^1\wedge..\wedge dz_i^n+\sum_{k=1}^{n}(-1)^{\frac{k(k-1)}{2}}\wedge^k(\phi(t)\perp(dz_i^1\wedge..\wedge dz_i^n))$$

is a globally defined form of type (n,0) on M_t.

3) $\omega_t(n,0)|_{\mathcal{W}_i}=\Theta_t^1\wedge..\wedge\Theta_t^n|_{\mathcal{W}_i}$ is a closed form and so it is a holomorphic n-form on M_t.

DEFINITION 1.8.6.

Let

$$\Psi(t_1,...,t_n,\overline{t_1},...,\overline{t_n}):=(-1)^{\frac{n(n+1)}{2}}(i)^n\int_{M_o}\omega_t(n,0)\wedge\overline{\omega_t(n,0)}$$

where $\omega_t(n,0)$ is defined in Lemma (1.8.5.) 3).

THEOREM. 1.8.7. (For the proof see [T].)

1) $(h_{\alpha,\overline{\beta}})=-(\frac{\partial^2\log\Psi}{\partial t_\alpha\partial\overline{t}_\beta})$ is the Weil-Petersson metric on U_ϵ.

$$2)\ R_{\alpha\overline{\beta},\mu\overline{\nu}}=-\frac{\partial^4\log\Psi}{\partial t_\alpha\partial\overline{t}_\beta\partial t_\mu\partial\overline{t}_\nu}|_{t=o}=(-1)^{\frac{n(n+1)}{2}}(i)^{n-1}\int_{M_o}[\phi_\alpha\wedge\phi_\mu\perp\omega_t(n,0)]\wedge\overline{[\phi_\mu\wedge\phi_\nu\perp\omega_t(n,o)]}$$

$$R_{\alpha\overline{\beta},\alpha\overline{\nu}}=2(-1)^{\frac{n(n+1)}{2}}(i)^{n-1}\int_{M_o}[\wedge^2\phi_\alpha\perp\omega_t(n,0)]\wedge\overline{[\phi_\mu\wedge\phi_\nu\perp\omega_t(n,o)]}$$

$$R_{\alpha\overline{\beta},\alpha\overline{\beta}}=4(-1)^{\frac{n(n+1)}{2}}(i)^{n-1}\int_{M_o}[\wedge^2\phi_\alpha\perp\omega_t(n,0)]\wedge\overline{[\wedge^2\phi_\beta\perp\omega_t(n,0)]}$$

where $[\phi_\alpha\wedge\phi_\beta\perp\omega_t(n,0)]$ means the class of cohomology of $\mathbb{H}(\phi_\alpha\wedge\phi_\beta\perp\omega_t(n,0))$.

3) For each $\phi_\alpha,\phi_\beta\in\mathbb{H}^1(M_o,\Theta_o)$, $[\phi_\alpha\wedge\phi_\beta\perp\omega_t(n,0)]$ is a primitive class of cohomology.

1.8.7.1. Cor.

Weil-Petersson metric has a negative sectional curvature and moreover the holomorphic sectional curvature is bounded away from zero.

#2.CONSTRUCTION OF THE TEICHMÜLLER SPACE OF CALABI-YAU MANIFOLDS.

Definition 2.1.

A pair $(M;\gamma_1,...,\gamma_{b_n})$ will be called a marked Calabi-Yau manifold if

a) M is a Calabi-Yau manifold.

b) $\gamma_1,...,\gamma_{b_n}$ is a basis for $H_n(M,\mathbf{Z})/\mathrm{Tor}H_n(M,\mathbf{Z})$.

Lemma 2.1.1. Let

$$\begin{array}{ccc} M & \subset & \mathfrak{X} \\ \downarrow & & \downarrow \\ o & \in & \mathcal{K} \end{array}$$

be the Kuranishi family of marked Calabi-Yau manifold $(M;\gamma_1,...,\gamma_{b_n})$ then $\mathfrak{X}\to\mathcal{K}$ is the local universal family of marked Calabi-Yau manifold $(M;\gamma_1,...,\gamma_{b_n})$.

Remark. Since as C^{∞} manifold $\mathfrak{X}\simeq M_o \times \mathcal{K}$, then if we fix a basis $(\gamma_1,...,\gamma_{b_n})$ in $H_n(M,\mathbf{Z})/\mathrm{Tor}H_n(M,\mathbf{Z})$ it means that in a canonical way we fixed a basis in $H_n(M,\mathbf{Z})/\mathrm{Tor}H_n(M,\mathbf{Z})$ for every $t\in\mathcal{K}$. So we have a marked family $\mathfrak{X}\to\mathcal{K}$.

Proof of 2.1.1. The proof is based on the following proposition.

Proposition 2.1.1.1. Let $f:M_o\to M_o$ be a holomorphic automorphism of M_o and suppose that $f^*=id$, where

$$f^*:H^n(M_o,\mathbf{Z})\to H^n(M_o,\mathbf{Z})$$

then f induced the identity map on the Kuranishi space of M_o, i.e. on $\mathcal{K}$.

Proof of 2.1.1.1.: For the proof of this proposition see [B].

Q.E.D.

The end of the proof of lemma 2.1.1.

We need to prove that if

$$\begin{array}{ccc} M_o & \subset & \mathcal{Y} \\ \downarrow & & \downarrow \\ w_o & \in & W \end{array}$$

is a family of marked Calabi-Yau manifolds, where W is a "small" polycylinder, then there exists a unique map of families:

$$\mu: \begin{array}{ccc} \mathcal{Y} & \to & \mathfrak{X} \\ \downarrow & & \downarrow \\ W & \to & \mathcal{K} \end{array}$$

such that:

a) $\mu(w_o)=o$ and $\mu:M_o \to M_o$ is an isomorphism of marked Calabi-Yau manifolds.

b) The family $\mathcal{Y} \to W$ is the pullback via μ of the Kuranishi family.

Since there are no obstructions to deformations of Calabi-Yau manifodls, it follows that the Kuranishi family is complete. This means that threre exists a holomorphic map μ

$$\begin{array}{ccc} \mathcal{Y} & \to & \mathfrak{S} \\ \mu: \downarrow & & \downarrow \\ W & \to & \mathfrak{K} \end{array}$$

such that

a) $\mu(w_o)=o$ and $\mu:M_o \to M_o$ is an isomorphism of marked Calabi-Yau manifolds.

b) The family $\mathcal{Y} \to W$ is the pullback via μ of the Kuranishi family.

(See [K], [KM} and [KNS].)

Let μ' be another map that fulfills the conditions a) and b) as for the map μ. From [K] it follows that we have

$$\mu(w)=\sigma(\mu'(w)) \text{ for } \forall w \in W$$

where σ is an isomorphism of the Kuranishi family such that

$$\sigma:M_o \to M_o$$

preserve the marking, i.e.

$$\sigma^*=\text{id on } H_n(M_o,\mathbb{Z})/\text{Tor}H_n(M_o,\mathbb{Z}).$$

From (2.1.1.) it follows that σ=id on $\mathfrak{K}$ and so $\mu=\mu'$.

Q.E.D.

2.2. The construction of TEICHMÜLLER SPACE of marked CALABI-YAU manifold.

Definition 2.2.1.

Let $I(M_o)$ be the set of all integrable complex structures on M_o, then by definition the Teichmüller space of M_o is $\mathfrak{T}(M_o):=I(M_o)/\text{Diff}_o(M_o)$, where $\text{Diff}_o(M_o)$ is the group of diffeomorphisms of M_o isotopic to the identity.

THEOREM 2.2.2.

a) $\mathfrak{T}(M_o)$ is a comlex analytic manifold of dimension equal to $\dim_{\mathbb{C}} H^1(M_o,\Theta_o)$.

b) There exists a universal family $\mathfrak{X} \to \mathfrak{T}(M_o)$ of marked Calabi-Yau manifolds.

PROOF: Let $\{\mathfrak{S}\rightarrow\mathfrak{U}\}$ be all possible Kuranishi families of marked Calabi-Yau manifolds. From 2.1.1.1. it follows that we can glue all families $\{\mathfrak{S}\rightarrow\mathfrak{U}\}$ by identifying all isomorphic marked Calabi-Yau manifolds. Take the component that contains $(M_O:\gamma_1,\dots,\gamma_{b_n})$. In such a way we will get the universal family $\mathfrak{X}\rightarrow\mathfrak{T}(M_O)$ of marked Calabi-Yau manifolds that fulfills *a)* and *b)*.

Q.E.D.

Remark 2.2.3.a.

Exactly in the same way will get the Teichmüller space of all marked Calabi-Yau manifolds, i.e. except the marking $(\gamma_1,\dots,\gamma_{b_n})$, we are fixing

$$L\in H^2(M_O,\mathbf{Z})\cap H^{1,1}(M_O,\mathbf{R})$$

and L is the imaginary class of a cohomology of the part imaginary of a Käler metric on M_O.

2.2.3.b:

Let $\mathfrak{T}(M_O)_{(\gamma_1,\dots,\gamma_{b_n})}$ be the Teichmüller space of all marked polarized Calabi-Yau manifolds, then on $\mathfrak{T}_{(M_O,\gamma_1,\dots,\gamma_{b_n})}$ we can define the Weil-Petersson metric.

2.2.3.c.

From now on we wil consider only marked polarized Calabi-Yau manifolds. The marking class $L\in H^2(M_O,\mathbf{Z})\cap H^{1,1}(M_O,\mathbf{R})$.

#3. TOTALLY GEODESIC TWO DIMENSIONAL SUBMANIFOLDS IN $\tilde{\mathfrak{T}}(M_o)$.

DEFINITION 3.1.

Let M be a Riemannian manifold and let S be a connected submanifold of M. Let $p \in M$. The submanifold S is said to be geodesic at p if each M-geodesic which is tangent to S at p is a curve in S. The submanifold S is called totally geodesic if it is geodesic at each of its point.

DEFINITION 3.2.

Let $(M_o,(g_{\alpha\overline{\beta}}))$ be a Calabi-Yau manifold with a Ricci flat matric $(g_{\alpha\overline{\beta}})$ such that

$$[\operatorname{Im}(g_{\alpha\overline{\beta}})]=L\in H^{1,1}(M_o,\mathbf{Z})$$

and L is a fixed class of cohomology.

Let $\gamma\in\mathbf{H}^1(M_o,\Theta_o)$. From 1.7.6. it follows that γ defines an one-dimensional submanifold

$$\gamma(t)\subset\mathfrak{T}(M_o)_{(\delta_1,\ldots,\delta_{b_n};\ L)}$$

where $(\delta_1,\ldots,\delta_{b_n})$ is a basis of $H_n(M_o,\mathbf{Z})/\mathrm{Tor}$. The point $\gamma(0)=o\in\mathfrak{T}(M_o)_{(\delta_1,\ldots,\delta_{b_n};\ L)}$ corresponds to M_o. The complex structure for each $t\in\gamma(t)$ is defined by

(3.2.1.a.) $\gamma(t)=\gamma_1 t+\gamma_2 t^2+\ldots.+\gamma_n t^n+..\in\Gamma(M_o,\Omega_o^{0,1}\otimes\Theta_o)$

(3.2.1.b.) $\overline{\partial}\gamma(t)=\frac{1}{2}[\gamma(t),\gamma(t)]$

(3.2.1.c.) $\gamma(t)=\gamma_1 t+\frac{1}{2}G\overline{\partial}^*[\gamma(t),\gamma(t)]$; $\gamma_1\in\mathbf{H}^1(M_o,\Theta_o)$

(3.2.1.d.) $\gamma_i\perp\omega_{M_o}(n,0)=\partial\psi_i$ for $i>1$ (See [Ti] and [T].)

From Kuranishi theory it follows that $\gamma(t)$ is uniquely defined. (See [KM].)

THEOREM 3.

$\gamma(t)$ is a totally geodesic submanifold in $\mathfrak{T}(M_o)_{(\delta_1,\ldots,\delta_{b_n};\ L)}$, where $\gamma(t)$ is defined as in (3.2.1.a.), (3.2.1.b.), (3.2.1.c.) & (3.2.1.d.)

PROOF OF THEOREM 3:

Let D be the Levi-Chevita connection of the Weil-Petersson metric on $\mathfrak{T}(M_o)_{(\delta_1,\ldots,\delta_{b_n};\ L)}$. Since the Weil-Petersson metric is a Kähler one it implies that

(3.3.) $$D=D^{1,0}+\overline{\partial}$$

where $\overline{\partial}$ is the usual $\overline{\partial}$-operator.

REMARK 3.4.

$D_{\dot{\gamma}(t)}\dot{\gamma}(t)=0$ implies that $\gamma(t)$ is a totally geodesic submanifold in $\mathfrak{T}(M_o)_{(\delta_1,\ldots,\delta_{b_n};\ L)}$.

REMARK 3.5.

Since $\gamma(t)$ is a complex analytic submanifold in $\mathfrak{T}(M_O)_{(\delta_1,\ldots,\delta_{b_n};\ L)}$ it follows that

(3.5.1.) $$D_{\dot\gamma(t)}\dot\gamma(t)=0 \Leftrightarrow D^{1,0}_{\dot\gamma(t)}\dot\gamma(t)=0$$

LEMMA 3.6. $D^{1,0}_{\dot\gamma(t)}\dot\gamma(t)=0$ for $\gamma(t)$ defined as above.

Proof of 3.6.:

The proof of LEMMA 3.6. is based on the following formula:

(3.6.1.) $$\frac{d^2}{dtd\bar t}\|\dot\gamma(t)\|^2=\|D^{1,0}_{\dot\gamma(t)}\dot\gamma(t)\|^2-2R_{\dot\gamma(t),\dot{\bar\gamma}(t),\dot\gamma(t),\dot{\bar\gamma}(t)}\|\dot\gamma(t)\|^2$$

where $R_{\dot\gamma(t),\dot{\bar\gamma}(t),\dot\gamma(t),\dot{\bar\gamma}(t)}$ is the holomorphic sctional curvature of the Weil-Petersson metric. (For the proof of (3.6.1.) see [G].)

If we prove that at each point $t\in\gamma(t)$ we have

(3.6.2.) $$\frac{d^2}{dtd\bar t}\|\dot\gamma(t)\|^2=-2R_{\dot\gamma(t),\dot{\bar\gamma}(t),\dot\gamma(t),\dot{\bar\gamma}(t)}\|\dot\gamma(t)\|^2$$

then Lemma 3.6. follows and hence Theorem 3 will be proved.

Proof of 3.6.2.:

The proof will be based on some results proved in [Ti] and [T]. We will remind that in [T] it was proved that:

(3.6.2.1.) $$\|\dot\gamma(t)\|^2=\frac{d^2}{dtd\bar t}\psi(t,\bar t)$$

where

(3.6.2.2.) $$\psi(t,\bar t):=(-1)^{\frac{n(n+1)}{2}}(i)^{n-1}\int_{M_O}\omega_t(n,0)\wedge\overline{\omega_t(n,0)}$$

We need to remind the definition of $\omega_t(n,0)$. We know that (1.8.5.) implies that we have a family of Calabi-Yau manifolds $\mathfrak{X}_{\gamma(t)}\rightarrow\gamma(t)$. According to [T] we have also a family of holomorphic n-forms $\omega_t(n,0)\in H^0(\mathfrak{X}_{\gamma(t)},\Omega^n_{\mathfrak{X}_{\gamma(t)}/\gamma(t)})$ and locally :

(3.6.2.3.) $\omega_t(n,0)|_{U_i}=(\mathcal{A}_t dz^1)\wedge\ldots\wedge(\mathcal{A}_t dz^n)=$

$$dz^1\wedge..\wedge dz^n+\sum_{k=1}^{n}(-1)^{\frac{k(k-1)}{2}}\left(\wedge^k\gamma(t)\perp(dz^1\wedge\ldots\wedge dz^n)\right)$$

where

$$\omega_o(n,0)|_{U_i}=dz^1\wedge\ldots\wedge dz^n;\ \gamma(t)=\sum_{n=1}^{\infty}\gamma_n t^n\in\Gamma(M_o,\mathrm{Hom}(\Omega_o^{1,0},\Omega_o^{0,1}))$$

$$\gamma(t)=\gamma_1 t+\tfrac{1}{2}\overline{\partial}^*G[\gamma(t),\gamma(t)]\ ,\ \gamma_1\in\mathbf{H}^1(M_o,\Theta_o)$$

$$\wedge^k\gamma(t)\in\Gamma(M_o,\mathrm{Hom}(\wedge^k\Omega_o^{1,0},\wedge^k\Omega_o^{0,1}))$$

$\mathcal{A}_t$ is defined as follows: $\mathcal{A}_t:=id+\gamma(t)\in\Gamma(M_o,\mathrm{Hom}(T^*M_o\otimes\mathbf{C},T^*M_o\otimes\mathbf{C}))$.

PROPOSITION 3.6.2.4.(See [T]).

Let $\gamma(t)$ be defined as follows:

(3.2.1.a.) $\gamma(t)=\gamma_1 t+j_2 t^2+\ldots.+\gamma t^n+..\in\Gamma(M_o,\Omega_o^{0,1}\otimes\Theta_o)$

(3.2.1.b.) $\overline{\partial}\gamma(t)=\frac{1}{2}[\gamma(t),\gamma(t)]$

(3.2.1.c.) $\gamma(t)=\gamma_1 t+\frac{1}{2}G\overline{\partial}^*[\gamma(t),\gamma(t)]\ ;\ \gamma_1\in\mathbf{H}^1(M_o,\Theta_o)$

Then

1) $\|\dot\gamma(t)\|^2=\frac{d^2}{dtd\overline{t}}\psi(t,\overline{t})$ for all t for which $\gamma(t)$ is defined.

2) $\frac{d^2}{dtd\overline{t}}\|\dot\gamma(t)\|^2|_{t=0}=-2R_{\dot\gamma(0),\dot{\overline{\gamma}}(0),\dot\gamma(0),\dot{\overline{\gamma}}(0)}\|\dot\gamma(0)\|^2$.

REMARK.3.6.2.5.

A) It was proved in [Ti] and [T] that if $\gamma(t)=\gamma_1 t+\frac{1}{2}G\overline{\partial}^*[\gamma(t),\gamma(t)]\ ;\ \gamma_1\in\mathbf{H}^1(M_o,\Theta_o)$ then

$$\overline{\partial}\lambda(t)-\tfrac{1}{2}[\lambda(t),\lambda(t)]=0$$

B) In [T] it is proved that

$$\frac{d^2}{dtd\overline{t}}\|\dot\gamma(t)\|^2|_{t=0}=-2R_{\dot\gamma(0),\dot{\overline{\gamma}}(0),\dot\gamma(0),\dot{\overline{\gamma}}(0)}\|\dot\gamma(0)\|^2$$

is equivalent to the following fact: In the Taylor expension of the cohomology class $[\omega_t(n,0)]$:

(*) $[\omega_t(n,0)]=[\omega_o(n,0)]+t[\omega_o(n\text{-}1,1)]+t^2[\omega_o(n\text{-}2,2)]+\ldots$

the coeficient in front of t^2 does not contain classes of cohomology of type (n-1,1).

PROPOSITION I. Let $\gamma(t)=\sum_{k=1}^{\infty}\gamma_k t^k \in \Gamma(M_o,\Omega_0^{0,1}\otimes\Theta_o)$ be such that:

a) $\overline{\partial}\gamma(t)=\frac{1}{2}[\gamma(t),\gamma(t)]$, b) $\overline{\partial}^*\gamma(t)=0$, c) $\mathbf{H}\gamma(t)=\gamma_1 t$. Let $\omega_t(n,0)=(\mathcal{A}_t dz^1)\wedge\ldots\wedge(\mathcal{A}_t dz^n)$ be the holomorphic n-form defined as in (3.6.2.1.). Then we have the following formulas:

$$[\omega_t(n,0)]=[\omega_o(n,0)]+\sum_{k=a}^{n}(-1)^{\frac{k(k-1)}{2}}t^k[\wedge^k\gamma_1\perp\omega_o(n,0)]$$

where if ω is a closed form on M_o, then $[\omega]$ denote the class of cohomology of ω.

Proof of Proposition I:

It was proved in [Ti] and [T] that if conditions a), b) and c) are fulfilled then for $k\geq 2$ the following formula is true:

(I.a) $$\gamma_k\perp\omega_o(n,0)=\partial\psi_k$$

On the other hand from the definition of $\mathcal{A}_t$ it follows that we have on the leval of forms:

(I.b) $$\omega_t(n,0)=\omega_o(n,0)+\sum_{k=a}^{\infty}t^k\Big(\sum_{i_1+..+i_k=k}^{\infty}(-1)^{\frac{k(k-1)}{2}}(\gamma_{i_1}\wedge..\wedge\gamma_{i_k})\perp\omega_o(n,0)\Big)$$

Sublemma. Suppose that among $i_1,\ldots,i_k$ there exists $1\leq j\leq k$ such that $i_j\geq 2$ then $\mathbf{H}((\gamma_{i_1}\wedge..\wedge\gamma_{i_k})\perp\omega_o(n,0))=0$.

Proof of the sublemma:

We know that if $i_j\geq 2$ then $\gamma_{i_j}\perp\omega_o(n,0)=\partial\psi_{i_j}$. From here our sublemma follows directly.

Q.E.D.

Proposition I follows directly from the sublemma, the condition $\gamma_1\in\mathbf{H}^1(M_o,\Theta_o)$ and formula (I.b.).

Q.E.D.

The end of the proof of the THEOREM:

PROPOSITION II. In the Taylor expension:

(3.6.2.6.) $$[\omega_t(n,0)]=[\omega_{t_o}(n,0)]+\sum_{k=1}^{n}(t-t_o)^k[\omega_{t_o}^{(k)}]$$

the coefficient $[\omega_{t_o}^{(2)}]$ in front of $(t-t_o)^2$ does not contain classes of cohomology on M_{t_o} of type (n-1,1), where $\omega_t(n,0)$ is defined as in (3.6.2.1.).

If we prove PROPOSITION II then THEOREM 3 will be proved.

PROOF OF PROPOSITION II:

In order to prove PROPOSITION II we need to prove the following fact:

FACT: For $k\geq 2$ $[\omega^k_{t_o}]\cap H^{n-1,1}(M_{t_o},C)=0$, where $[\omega^k_{t_o}]$ is defined in (3.6.2.6.).

PROOF OF THE FACT:

Suppose that $[\omega^k_{t_o}]\cap H^{n-1,1}(M_{t_o},C)\neq 0$. Let $[\omega^k_{t_o}]=[\omega^k_{t_o}(n\text{-}1,1)]+..$on M_{t_o}. Let

$$\mathbf{H}[\omega^k_{t_o}(n\text{-}1,1)]|_U=\sum_{i,j}\psi_{i,j,k,t_o}\Theta^1_{t_o}\wedge..\wedge\hat{\Theta}^i_{t_o}\wedge..\wedge\Theta^n_{t_o}\wedge\overline{\Theta}^j_{t_o}$$

where $\Theta^i_{t_o}=\mathcal{A}_{t_o}dz^i$ and $\omega_o(n,0)|_U=dz^1\wedge...\wedge dz^n$. In the appendix of [T] the following Lemama was proved:

LEMMA ([T].). Let $\omega_o(n\text{-}1,1)=\sum_{i,j}\phi_{i,j}dz^1\wedge..\wedge d\hat{z}^i\wedge..\wedge dz^n\wedge\overline{dz}^j$ be a harmonic form, then

$$\omega_{t_o}(n\text{-}1,1)=\sum_{i,j}\phi_{i,j}(\mathcal{A}_{t_o}dz^1)\wedge..\wedge(\mathcal{A}_{t_o}d\hat{z}^i)\wedge..\wedge(\mathcal{A}_{t_o}dz^n)\wedge\overline{\mathcal{A}_{t_o}dz}^j$$

represent a non-zero class of cohomology on M_{t_o}. (Here $\mathcal{A}_{t_o}$ is defined as above.)

From this LEMMA it follows that if we consider the form $\mathbf{H}[\omega^k_{t_o}(n\text{-}1,1)]$ on M_o, then we have

(s.3.2.) $$\mathbf{H}[\omega^k_{t_o}(n\text{-}1,1)]=\omega_o(n\text{-}1,1)+....\text{ on } M_o$$

where $\omega_{t_o}(n\text{-}1,1)\neq 0$. From (s.3.2.) it follows that on M_o we have:

(s.3.3.) $$\frac{d^k}{dt^k}[\omega_t(n,0)]=[\omega_{t_o}(n\text{-}1,1)]+O(t-t_o),\text{ where } [\omega_{t_o}(n\text{-}1,1)]\neq 0$$

(s.3.3.) follows from the formula:

(s.3.4.) $$\frac{d^k}{dt^k}[\omega_t(n,0)]=[\omega^k_{t_o}(n\text{-}1,1)]+O(t-t_o).$$

Formula (s.3.3.) contradicts PROPOSITION I. So PROPOSITION II is proved.

Q.E.D.

Notice that from REMARK 3.6.2.5.B it follows that PROPOSITION II implies the following formula:

(**) $$\|\omega_{t_o}(n,0)\|^{-2}\,\frac{d^4\psi(t,\bar{t})}{dt^2 d\bar{t}^2}\Big|_{t=t_o}=\|\omega_{t_o}(n,0)\|^{-2}\,\frac{d^2\|\dot{\gamma}(t)\|^2}{dt d\bar{t}}\Big|_{t=t_o}$$

$$=-2R_{\dot{\gamma}(t_o),\bar{\dot{\gamma}}(t_o),\dot{\gamma}(t_o),\bar{\dot{\gamma}}(t_o)}\|\dot{\gamma}(t_o)\|^2$$

and it was already pointed out (See (3.6.1.)) that (**) implies that for each point $t_o\in\gamma(t)$ we have

$$D^{1,0}_{\dot{\gamma}(t_o)}\dot{\gamma}(t_o)=0.$$

So $\gamma(t)$ is a totally geodesic submanifold in $\mathfrak{X}(M_o)_{(\delta_1,..,\delta_{b_n};L)}$.

Q.E.D.

#4 THE COMPLETENESS OF THE WEIL-PETERSSON METRIC.

4.1. LEMMA.

Let $\gamma(t)$ be defined as follows:

(3.2.1.a.) $\gamma(t)=\gamma_1 t+j_2 t^2+....+\gamma t^n+..\in\Gamma(M_O,\Omega_O^{0,1}\otimes\Theta_O)$

(3.2.1.b.) $\overline{\partial}\gamma(t)=\frac{1}{2}[\gamma(t),\gamma(t)]$

(3.2.1.c.) $\gamma(t)=\gamma_1 t+\frac{1}{2}G\overline{\partial}^*[\gamma(t),\gamma(t)]$; $\gamma_1\in\mathbf{H}^1(M_O,\Theta_O)$

Then $\gamma(t)$ is defined for all $t\in\mathbf{C}$ if M_O is a Calabi-Yau manifold, i.e. $\gamma(t)\in\Gamma(M_O,\Omega_O^{0,1}\otimes\Theta_O)$ for all $t\in\mathbf{C}$.

PROOF OF 4.1.:

The proof of this lemma is based on the proof of the convergence of the series:

$$\gamma(t)=\gamma_1 t+j_2 t^2+....+\gamma t^n+..$$

in the Holder norm for $|t|<\epsilon$. We will recall that proof.

DEFINITION 4.1.1. Let $\left\{U_j\right\}$ be a covering of M_O such that $\left\{z_j^1,..,z_j^n\right\}$ are local coordinates in U_j and $\omega_O(n,0)|_{U_j}=dz_j^1\wedge..\wedge z_j^n$. Let $\phi\in\Gamma(M_O,\Omega_O^{0,1}\otimes\Theta_O)$, i.e.

$$\phi=\sum_{j=1}^{n}\phi_j^\lambda\left(\frac{\partial}{\partial z_j^\lambda}\right),\ \phi_j^\lambda=\sum_{\alpha=1}^{n}\phi_{j,\overline{\alpha}}^\lambda\overline{dz}^\alpha$$

Let $k\in\mathbf{Z}$, $k>0$, $\alpha\in\mathbf{R}$, $0<\alpha<1$. Let $h=(h_1,...,h_n)\in\mathbf{Z}^{2n}$, $h_i\geq0$, $\sum_{i=1}^{2n}h_i=|h|$, where $n=\dim_{\mathbf{C}}M_O$. Then denote

$$D_j^h=\left(\frac{\partial}{\partial x_j^1}\right)^{h_1}...\left(\frac{\partial}{\partial x_j^n}\right)^{h_n},\ x_j^\alpha=x_j^{2\alpha-1}+ix_j^\alpha$$

Then the Hölder norm is defined as follows:

$$\|\phi\|_{k,\alpha}=\max_{\alpha,j,\beta}\left(\sum_{|h|<k}\left(\sup_{z\in U_j}|D_j^h\phi_{j,\overline{\beta}}^\lambda(z)|\right)+\right.$$

$$\sup_{\substack{y,z\in U,\beta,\lambda\\|h|=k}}\frac{|D_j^h\phi_{j,\overline{\beta}}^\lambda(y)-D_j^h\phi_{j,\overline{\beta}}^\lambda(z)|}{|y-z|^\alpha}$$

From the following a priori estimates:

$$\|[\phi,\psi]\|_{k,\alpha}\leq C\|\phi\|_{k+1,\alpha}\|\psi\|_{k+1,\alpha}$$
$$\|G\phi\|_{k,\alpha}\leq C\|\phi\|_{k-2,\alpha}$$
$$\|\overline{\partial}^*\phi\|\leq C\|\phi\|_{k+1,\alpha}$$

where C depenends only on k and α but not on ϕ and ψ. Kuranishi obtained that if

$$\gamma(t)=\sum_{n=0}^{\infty}\gamma_n t^n,\text{ where }\overline{\partial}\gamma(t)=\tfrac{1}{2}[\gamma(t),\gamma(t)]\text{ and }\overline{\partial}^*\gamma(t)=0,$$

then $\gamma(t)$ converges in the norm $\|\ \|_{k,\alpha}$. It is immediate that $\gamma(z,t)$ is C^k since the series converge in the norm $\|\ \|_{k,\alpha}$. To obtain that $\gamma(z,t)$ is a C^∞ one refers to the regularity theorem for quasi-elliptic operator of Douglis and Nirenberg. (See [DN].). More precisely Kuranishi proved that

$$\|\gamma(t)\|_{k,\alpha}\ll\frac{\beta}{16\lambda}\sum_{\mu=1}^{\infty}\frac{\lambda^\mu}{\mu^2}t^\mu$$

where the sign $\ll$ means that

(4.1.2.) $$\|\gamma_n\|_{k,\alpha}\leq\frac{\beta}{16\lambda}\frac{\lambda^n}{n^2}$$

δ and λ are some positive constants which depend in our case on the Calabi-Yau metric.

(4.1.3.)

Let

$$f(t)=\frac{\beta}{16\lambda}\sum_{n=1}^{\infty}\frac{\lambda^n}{n^2}t^n,$$

then f(t) has an analytic continuation for all $t\in\mathbf{C}\backslash\{S\}$, where S is a finite set in **C**. (The continuation is as a multivalued complex analytic function and it is univalent in $\mathbf{C}\backslash(1.\infty)$)

Proof of 4.1.3.:

Let $x=\lambda t$ and let $g(x)=\frac{\beta}{16\lambda}\sum_{n=1}^{\infty}\frac{x^n}{n^2}$. Clearly

$$g'(x)=\frac{\beta}{16\lambda}\left(1+\frac{x}{2}+\frac{x^2}{3}+..+\frac{x^{m-1}}{m}+..\right)$$

On the other hand we have $xg'(x)=\frac{\beta}{16\lambda}\left(x+\frac{x^2}{2}+\frac{x^3}{3}+..+\frac{x^m}{m}+..\right)$, so $x(xg'(x))'=\frac{\beta}{16\lambda}(\frac{1}{1-x}-1)$. The last equality implies that

$$x^2g''(x)+xg'(x)-\frac{\beta}{16\lambda}(\frac{1}{1-x}-1)=0$$

From the theory of differential equations of second order we get that g(x) is a complex analytic function on $\mathbf{C}\setminus\{S\}$, where S is a finite set of complex numbers. So this is true for $f(t)=g(\lambda t)$. (See [WW].) Q.E.D.

From (4.1.3.) and the fact that

$$\sum_{n=1}^{\infty}\|\gamma_n\|_{k,\alpha}t^n \ll f(t)$$

it follows that:

4.3.1.1. Cor.: $\gamma(t)\in C^{\infty}(M_o,\Omega_o^{0,1}\otimes\Theta_o)$ for $t\in\mathbf{C}\setminus\{S\}$ where S is the set of the singular points of the equarion

$$(*)\qquad x(xg'(x))'-\frac{\beta}{16\lambda}\left(\frac{1}{1-x}\right)=0$$

End of the proof of 4.1.:

From (*) it follows that the set of the singular points of the differential equation (*) consists of at most two points x=0 and x=1. When x=0 then t=0 since $x=\lambda t$ so $\lambda(t)$ is a well defined section of $\Gamma(M_o,\mathrm{Hom}(\Omega_o^{0,1},\Omega_o^{1,0}))$ namely $\gamma(0)=0$. On the other hand we know that

$$g(x)=\frac{\beta}{16\lambda}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\beta}{16\lambda}\zeta(2)$$

and so $\gamma(t)$ is well defined at all points $t\in\mathbf{C}$. 4.1. is proved.

Q.E.D.

REMARK. From 4.1. it follows that $\gamma(t)$ is defined for all $t\in\mathbf{C}$.

4.2. LEMMA.

Let $\mathfrak{X}_{\gamma(t)}\xrightarrow{\pi}\gamma(t)$ be the family of the integrable structures on $\mathbf{C}\mathrm{x}M_o$ defined by $\gamma(t)$. Then on $\pi^{-1}(t)=M_t$ for all $t\in\mathbf{C}$ there exists a holomorphic n-form $\omega_t(n,o)$ which has no zeroes.

PROOF OF 4.2.: In the proof of 3.6.2. we introduce an operator

$$\mathcal{A}_t=id+\gamma(t)\in C^{\infty}(M_o,\mathrm{Hom}(T^*\otimes\mathbf{C},T^*\otimes\mathbf{C})$$

and then according to [T] $\mathcal{A}_t$ defines a family of holomorphic n-forms $\omega_t(n,o)$ where

$$\omega_t(n,o)|_U=(\mathcal{A}_t dz^1)\wedge\ldots\wedge(\mathcal{A}_t dz^n)$$

It is easy to see that $(\mathcal{A}_t dz^1)\wedge\ldots\wedge(\mathcal{A}_t dz^n)$ is zero at some point $z\in U$ iff the vectors $\{\mathcal{A}_t dz^i, i=1,..,n\}$ are linearly independent. But this is impossible since the operator $\mathcal{A}_t=id+\gamma(t)$ has rank 2n.

(4.2.*.) So $\omega_t(n,0)$ is different from zero for all $t \in \mathbb{C}$.

So in order to finish the proof of the LEMMA we need to prove the following PROPOSITION:

PROPOSITION 4.3. Let $A_t := \begin{bmatrix} id & \gamma(t) \\ \overline{\gamma(t)} & id \end{bmatrix}$, where $\gamma(t)$ is defined as in PROPOSITION I on p. 17, then

a.) det A_t is a real analytic function of Ret and Imt, i.e. it does not depend on $z \in M_o$,

b) for each $t \in \mathbb{C}$ $\det A_t \neq 0$.

PROOF OF PROPOSITION 4.3.a.:

IDEA OF THE PROOF:

We will prove that

$$\frac{d(\mathrm{vol}(g_{\alpha,\overline{\beta}}(t))}{dt} = 0, \text{ where } (g_{\alpha,\overline{\beta}}(t)) \text{ is the CALABI-YAU metrics}$$

such that $\mathrm{Im}(g_{\alpha,\overline{\beta}}(t)) = L$. The proof of the above formula is based on some results proved in [S] and [N].

Next we will prove that for Ricci flat metrics we have

$$\mathrm{vol}(g_{\alpha,\overline{\beta}}(t)) = \mu_t(n,0) \wedge \overline{\mu_t(n,0)} \text{ , where } \mu_t(n,0) \text{ is some holomorphic n-form on } M_t.$$

Combining these two formulas we get that $\det(A_t)$ does depends only on Ret and Imt.

Step 1. From the definition of $\det A_t$ and since $\left\{\Theta_t^i := A_t dz^i\right\}$ we get that:

(4.3.1.) $$\omega_t(n,0) \wedge \overline{\omega_t(n,0)} = \det(A_t)\omega_o(n,0) \wedge \overline{\omega_o(n,0)}$$

This formula is true since $\omega_t(n,0) = \Theta_t^1 \wedge \ldots \wedge \Theta_t^n$. Remember that $\omega_t(n,0)$ is a globaly defined holomorphic n-form on M_t if $\omega_o(n,0)_{|U} := dz^1 \wedge .. \wedge dz^n$.

Step 2. Let $\left\{g_{\alpha,\overline{\beta}}(t)\right\}$ be the Calabi-Yau metric on M_t such that $[\mathrm{Im}\ g_{\alpha,\overline{\beta}}(t)] := L$, then

(4.3.1.) $$\mathrm{vol}(g_{\alpha,\overline{\beta}}(t)) = \frac{1}{\|\omega_t(n,0)\|^2}\omega_t(n,0) \wedge \overline{\omega_t(n,0)}$$

PROOF OF STEP 2:

From the Bochener principle, i.e. $\nabla_t \omega_t(n,0)=0$, where ∇_t is the Levi-Chevita connection of the Calabi-Yau metric $\left\{g_{\alpha,\overline{\beta}}(t)\right\}$ and from $\nabla_t \mathrm{vol}(g_{\alpha,\overline{\beta}}(t))=0$ we get (4.3.1.).

Q.E.D.

Step 3.

For each t we can find a basis $\left\{\Theta_t^i\right\}$ of $\Omega_t^{1,0}$ such that:

a) $<\Theta_t^i,\Theta_t^j>=c(t)\delta_{i,\overline{j}}$, where < , > is the scalar product defined by the Calabi-Yau metric $\left\{g_{\alpha,\overline{\beta}}(t)\right\}$.

b) $\Theta_t^1\wedge\ldots\wedge\Theta_t^n=\omega_t(n,0)$

PROOF OF STEP 3: Step 3 follows directly from Step 2.

Q.E.D.

4.3.2. REMARK.

Since $\gamma(t)\in\Gamma(M_o, \mathrm{Hom}(\Omega_o^{1,0}, \Omega_o^{0,1}))$ and $\gamma(t)=\gamma_1 t+..+\gamma_n t^n+\ldots$ is such that a) $\overline{\partial}^*\gamma(t)=0$ & b) $\mathbb{H}(\gamma(t))=\gamma_1 t$ we will suppose that

(4.3.2.a.) $$\gamma_n=\sum_0^\infty \gamma_n{}^i_{\overline{j}}\overline{\Theta}^j\otimes(\Theta^i)^*$$

4.3.3. DEFINITION. We will define $B_t\in\Gamma(M_o,\mathrm{Hom}(T^*(M_o),T^*(M_o))$ as follows:

$$B_t(\Theta_o^i)=\Theta_t^i \text{ for } i=1,\ldots,n$$

where Θ_t^i are defined as in Step 3.

Step 4. $$B_t=\begin{bmatrix} \mathrm{id} & t\gamma_1 \\ \overline{t\gamma_1} & \mathrm{id}\end{bmatrix}+0(t^2)$$

PROOF OF STEP 4: In [T] the following formula was proved:

(4.3.4.) $$\omega_t(n,0)=B_t(\Theta_o^1)\wedge..\wedge B_t(\Theta_o^n)=\Theta_o^1\wedge..\wedge\Theta_o^n+(-1)^i t(\sum_{i,j}\gamma^i_{1\overline{j}}\overline{\Theta_o^j}\wedge\Theta_o^1\wedge..\wedge\hat{\Theta}_o^i..\wedge\Theta_o^n)+0(t^2)$$

Step 4 follows directly from (4.3.4.) and the fact that $B_o=\mathrm{id}$.

Q.E.D.

STEP 5: The followimg formula is true:

(4.3.5.) $\omega_t(n,0)=B_t(\Theta_o^1)\wedge..\wedge B_t(\Theta_o^n)=$

$$=\Theta_{t_o}^1\wedge..\wedge\Theta_{t_o}^n+(-1)^i(t-t_o)(\sum_{i,j}\gamma_{1\bar{j}}^i\overline{\Theta_{t_o}^j}\wedge\Theta_{t_o}^1\wedge..\wedge\hat{\Theta}_{t_o}^i..\wedge\Theta_{t_o}^n)+0(t^2)$$

where $\gamma_{1,j}^i$ are the same as in (4.3.4.).

PROOF OF STEP 5:

$$\omega_t(n,0)=B_t(B_{t_o}^{-1}\Theta_{t_o}^1)\wedge..\wedge B_t(B_{t_o}^{-1}\Theta_{t_o}^n)$$

Elementary straitforward caculations show that in the basis $\left(\Theta_{t_o}^1,..,\Theta_{t_o}^n,\overline{\Theta_{t_o}^1},..,\overline{\Theta_{t_o}^n}\right)$ the matrix of the operator $B_{t-t_o}=B_tB_{t_o}^{-1}$ is given by:

(4.3.5.a.) $$B_{t-t_o}(\Theta_{t_o}^i)=\Theta_{t_o}^i+(t-t_o)(\sum_{i,j}\gamma_{1\bar{j}}^i\overline{\Theta_{t_o}^j})+\text{higher order terms.}$$

(4.3.5.) follows directly from (4.3.5.a.) and (4.3.4.).

Q.E.D.

STEP 6.

The form

$$\omega_{t_o}(n-1,1):=\sum_{i,j}(-1)^{i-1}\gamma_{1\bar{j}}^i\overline{\Theta_{t_o}^j}\wedge\Theta_{t_o}^1\wedge..\wedge\hat{\Theta}_{t_o}^i..\wedge\Theta_{t_o}^n)$$

is a harmonic form of type $(n-1,1)$ on M_{t_o} with respect to the Calabi-Yau metric $(g_{\alpha,\bar{\beta}}(t_o))$.

PROOF OF STEP 6:

We need to prove that

a) $d(\omega_{t_o}(n-1,1))=0$ and

b) $\overline{\partial}_{t_o}^*(\omega_{t_o}(n-1,1))=0$, where $\overline{\partial}_{t_o}^*$ is the conjugate of ∂_{t_o} with respect to the Calabi-Yau metric $(g_{\alpha,\bar{\beta}}(t_o))$.

Proof of a:

Since $\omega_t(n,0)=\omega_{t_o}(n,0)+(t-t_o)(\omega_{t_o}(n-1,1))+O(t^2)$ and $\omega_{t_o}(n,0)$ are closed froms it follows that $d(\omega_{t_o}(n-1,1))=0$

Q.E.D.

Proof of Step 6. b:

The proof of b) is based on the following two facts:

Fact 1.(For the proof see [T])

Suppose that M is a Calabi-Yau manifold with a Calabi-Yau metric $(g_{\alpha,\overline{\beta}})$. Let $\Gamma(M,\Omega^{1,0}\otimes\Theta)$ and $\Gamma(M,\Omega^{n-1,1})$ be the Hilbert spaces with the induced L^2 norm by $(g_{\alpha,\overline{\beta}})$. .Let $\omega(n,0)\neq 0$ be the holomorphic form such that $\|\omega(n,0)\|^2=1$. Let

$$i:\Gamma(M,\Omega^{1,0}\otimes\Theta)\rightarrow\Gamma(M,\Omega^{n-1,1})$$

be defined as follows $i(\phi):=\phi\lrcorner\omega(n,0)$, then i is an isometry of the Hilbert spaces which preserve the Hodge decompositions of both spaces, i.e.

$$i(\mathbf{H}^1(M,\Theta))=\mathbf{H}^{n-1,1}(M),\ (\overline{\partial}\Gamma(M,\Omega^{0,0}\otimes\Theta))=\overline{\partial}\Gamma(M,\Omega^{n-1,0}),\ i(\overline{\partial}^*\Gamma(M,\Omega^{0,2}\otimes\Theta))=\overline{\partial}^*\Gamma(M,\Omega^{n-1,2}).$$

Fact 2.

The following formula holds on Kähler manifolds:

$$i\overline{\partial}^*_{t_o}=\partial_{t_o}\wedge_{t_o}-\wedge_{t_o}\partial_{t_o}$$

where $\wedge_{t_o}\omega:=L^*\lrcorner\omega$ and

$$L^*:=\frac{i}{C(t_o)}\sum_{\alpha=1}^{n}(\Theta^{\alpha}_{t_o})^*\wedge\overline{(\Theta^{\alpha}_{t_o})^*}$$

where $(\Theta^{\alpha}_{t_o})^*$ are dual to $\Theta^{\alpha}_{t_o}$ and $\left\{\Theta^{\alpha}_{t_o}\right\}$ are defined as in Step 3.(See [KN].)

From the definition of $\gamma(t)$ it follows that γ_1 is a harmonic form with coefficients in Θ on M_o. From Fact 1 it follows that

$$\omega_o(n-1,1):=\gamma_1\lrcorner\omega_o(n,0)=\sum_{i,j}(-1)^{i-1}\gamma^i_{1\overline{j}}\overline{\Theta^j_o}\wedge\Theta^1_o\wedge..\wedge\hat{\Theta}^i_o..\wedge\Theta^n_o)$$

is a harmonic form of type (n-1,1). Moreover $\omega_o(n-1,0)$ is a primitive form on M_o. (See [T].) This means that

(4.6.1.) $$\wedge_o\omega_o(n-1,1)=0\Leftrightarrow\sum_{i<j}(-1)^{i-j}\gamma^i_{1\overline{j}}+\sum_{i>j}(-1)^{i-j-1}\gamma^i_{1\overline{j}}=0(\text{ for j fixed})$$

From the definition of $\wedge_{t_o}\omega_{t_o}(n-1,1)$, the fact that

$$<\Theta^{\alpha}_{t_o},\Theta^{\beta}_{t_o}>=C(t_o)\delta_{\alpha,\overline{\beta}}$$

and the explicit formula in Step 5 for $\omega_{t_o}(n-1,1)$, i.e.

$$\omega_{t_o}(n-1,1):=\sum_{i,j}(-1)^{i-1}\gamma^{i}_{1,\overline{j}}\overline{\Theta^{j}_{t_o}}\wedge\Theta^{1}_{t_o}\wedge..\wedge\hat{\Theta}^{i}_{t_o}..\wedge\Theta^{n}_{t_o})$$

(4.6.2) $\quad \wedge_{t_o}\omega_{t_o}(n-1,1)=0\Leftrightarrow\sum_{i<j}(-1)^{i-j}\gamma^{i}_{1,\overline{j}}+\sum_{i>j}(-1)^{i-j-1}\gamma^{i}_{1,\overline{j}}=0(\text{ for j fixed})$

The last equality follows from Step 5 and (4.6.1.). (4.6.2.) Shows that $\overline{\partial}^{*}_{t_o}\omega_{t_o}(n-1,1)=0$.

Q.E.D.

Step 7.

Let $(g_{\alpha,\overline{\beta}}(t))$ be the Calaby-Yau metrics on $\mathfrak{S}_{\gamma(t)}\rightarrow\gamma(t)$ such that $[\mathrm{Im}(g_{\alpha,\overline{\beta}}(t))]=L$ then

$$\mathrm{vol}(\ g_{\alpha,\overline{\beta}}(t))=\mathrm{vol}(\ g_{\alpha,\overline{\beta}}(0)=\omega_o(n,0)\wedge\overline{\omega_o(n,0)}$$

Proof of Step 7:

In [N] it is proved that if M is a Calabi-Yau manifold and $(g_{\alpha,\overline{\beta}})$ is a Calabi-Yau metric on M and $\gamma_1\in\mathbf{H}^1(M,\Theta)$, then the following formulas hold;

(4.7.1.) $$\gamma_{1,\overline{i}\,\overline{j}}=\gamma_{1,\overline{j}\,\overline{i}}$$

From (4.7.1.)it follows immediately that

(4.7.2.) $$\frac{d}{dt}\mathrm{vol}(g_{\alpha,\overline{\beta}}(t))|_{t=t_o}=0$$

for each t_o. (See [N] or [S].). From here and using Step 1 we obtain that $\det(A_t)$ depends only on t, i.e. $\det(A_t)$ is a function only on t and $\overline{t}$. Indeed from (4.7.2) it follows that

(4.7.3) $$\mathrm{vol}(\ g_{\alpha,\overline{\beta}}(t))=\mathrm{vol}(\ g_{\alpha,\overline{\beta}}(0)=\omega_o(n,0)\wedge\overline{\omega_o(n,0)}$$

and so since $\omega_t(n,0)\wedge\omega_t(0,n)=C(t)\mathrm{vol}(\ g_{\alpha,\overline{\beta}}(t))$, where C(t) depends only on t. From (4.7.3.) we see that C(t) is equal to det (A_t).

Q.E.D.

The end the proof of part a:

From Step 1 it follows that $\det(A_t)$ is a function only on t and $\overline{t}$ for $|t|<\epsilon$. Since as it was proved that $\gamma(t)$ is defined for all $t\in\mathbb{C}$ we get that $\det(A_t)$ is defined for all $t\in\mathbb{C}$. On the other hand we know that $\overline{\partial}^{*}\gamma(t)=0$. From one of the definitions of $\overline{\partial}^{*}=(-i)(\wedge\partial-\partial\wedge)$ and the

4.5.2. Proposition.

The map $\exp:\mathbf{C}\to\gamma(t)$, where $\exp(t)=\gamma(t)$ is an one to one map between $\mathbf{C}$ and $\gamma(t)\subset\tilde{\mathfrak{T}}(M_o)$.

Proof of 4.5.2.:

We have already shown that for each $t\in\mathbf{C}$ $\gamma(t)\in C^{\infty}(M_o,\mathrm{Hom}(\Omega_o^{1,0},\Omega_o^{0,1})$ and it defines a new integrable complex structure on M_o. From the formula for the class of the cohomology of $\omega_t(n,o)$ defined in 3.6.2.1.

$$[\omega_t(n,o)]=[\omega_o(n,o)]+\sum_{k=1}^{n} t^k[\wedge^k\gamma_1\perp\omega_o(n,o)]$$

we get that if $t_1\neq t_2$, then $[\omega_{t_1}(n,o)]\neq[[\omega_{t_2}(n,o)]$.

So the map exp is one to one an etale. Here we used the local Torelli theorem.

Q.E.D.

4.5.2.1.

Since exp is an one to one complex analytic map between $\mathbf{C}$ and $\gamma(t)$ we get from the completeness of $\mathbf{C}$ that $\gamma(t)$ is also complete. Here we use (ii) of 4.5.1.. Let α be any real tangent vector to $\tilde{\mathfrak{T}}(M_o)$ at the point o corresponding to M_o. Let I_o be the complex structure operator on $\tilde{\mathfrak{T}}(M_o)$, then we will have that

$$\alpha+iI\alpha\in\mathbf{H}^1(M_o,\Theta_o)$$

Since $\mathbf{H}^1(M_o,\Theta_o)$ can be identified with the complex tangent space at $o\in\tilde{\mathfrak{T}}(M_o)$. Let $\gamma_1=\alpha+iI\alpha$ and let

$$\gamma(t)=\sum_{n=1}^{\infty}\gamma_n t^n,\ \gamma(t)=\gamma_1 t+\tfrac{1}{2}\overline{\partial}^*G[\gamma(t),\gamma(t)].$$

We proved that $\gamma(t)$ is a totally geodesic complete submanifold in $\tilde{\mathfrak{T}}(M_o)$ so the geodesic $\alpha(t)$ in $\tilde{\mathfrak{T}}(M_o)$ corresponding to the direction α will be for all t in $\gamma(t)$. Since $\gamma(t)$ is complete with respect to the Weil-Petersson metric we get that $\alpha(t)$ has an infinite length. This is condition (iii) of 4.1..

Q.E.D.

defines a new complex structure on M_0. All these complex structures we will denote by $\tilde{\mathfrak{T}}(M_0)$ and we will call it the extended Teichnüller space.

REMARK 4.4.1..

We have a family of marked complex analytic manifolds $\pi:\mathfrak{X}\rightarrow\tilde{\mathfrak{T}}(M_0)$ and each fibre has the properties that it has a holomorphic n-form without zeroes. It is not very difficult to prove that there exists an open and everywhere dense subset U in $\tilde{\mathfrak{T}}(M_0)$ such that for each $t\in U$ $\pi^{-1}(t)=M_t$ is a Calabi-Yau manifold.

REMARK 4.4.2..

The function $\Psi(t_1,..,t_N,\overline{t}_1,..,\overline{t}_N)$ defined in (1.8.6.) is defined on the whole $\tilde{\mathfrak{T}}(M_0)$ and moreover it is real analytic in

$$(|t_1|^2,...,|t_N|^2).$$

This follows from the definition of $\Psi(t_1,..,t_N,\overline{t}_1,..,\overline{t}_N)$. From Theorem 1.8.7. it follows that the Weil-Petersson metric is defined on the whole $\tilde{\mathfrak{T}}(M_0)$, since from [T] it follows that

$$\log\Psi(t_1,..,t_N,\overline{t}_1,..,\overline{t}_N)$$

is the potential of the Weil-Petersson metric.

THEOREM 4.5.

The extended Teichmüller space $\tilde{\mathfrak{T}}(M_0)$ is complete with respect to the Weil-Petersson metric.

PROOF OF THEOREM 4.5.:

The proof is based on the following theorem proved in [H]:

THEOREM 4.5.1.:

Let M be a Riemannian manifold. The following conditions are equivelent:

(i) M is complete, (ii) Each bounded closed subset is compact, (iii) Each maximal geodesic in M has the form $\gamma_M(t), -\infty<t<\infty$. (Has an infinite length.)

We will use condition (iii) of 4.5.1.. If we take any $\gamma_1\in\mathbf{H}^1(M_0,\Theta_0)$ and we define

$$\gamma(t)=\sum_{n=1}^{\infty}\gamma_n t^n, \text{ where } \gamma(t)=\gamma_1 t+\frac{1}{2}\,\overline{\partial}^*G[\gamma(t),\gamma(t)]$$

We know that $\gamma(t)$ is a totally geodesic submanifold defined for all $t\in\mathbf{C}$. We will prove that $\gamma(t)$ is a complete with respect to the Weil-Petersson metric on $\gamma(t)$.

THEOREM proved in [DK] which state that if $(g_{\alpha,\overline{\beta}})$ is a Calabi-Yau metric then with respect to any local holomorphic coordinates $(z^1,...,z^n)$ of an open subset $U\subset M_O$ $g_{\alpha,\overline{\beta}}$ is a real analytic functions with respect to $(Rez^1, Imz^1,..,Rez^n, Imz^n)$. So from this result and the standart **Caushy-Covalevskaya theorem** we get that all the elements of $\gamma(t)$ are real analytic functions of $(Rez^1, Imz^1,..,Rez^n, Imz^n, Ret, Imt)$. So from here we get that $\det(A_t)$ is a real analytic function of $(Rez^1, Imz^1,..,Rez^n, Imz^n, Ret, Imt)$. This shows that on UxC $\det(A_t)$ depends only of Ret and Imt since this is true in an open subset of UxC. So part a). is proved.

Q.E.D.

PROOF of b:

The proof of b is based on the following trivial remark:

REMARK: Suppose that for each $t\in\mathbf{C}$ we have:

$$<\omega_t(n,0),\omega_t(0,n)>:=(-1)^{\frac{n(n-1)}{2}}(i)^n\int_{M_O}\omega_t(n,0)\wedge\omega_t(0,n)>0 \tag{4.7.4.}$$

then we will have that $\det(A_t)\neq 0$.

PROOF of 4.7.4.: In order to prove (4.7.4.) we will define

$$\Omega:=\left\{u\in PH^n(M_O,\mathbf{C})|\ <u,u>=0 \text{ and } <u,\overline{u}>>0\right\}$$

Then we can define a map $\pi:\gamma(t)\to\Omega$ in the following manner: for each $t\in\gamma(t)$ $\pi(t)$ will be the line in $H^n(M_O,C)$ spanned by $\omega_t(n,0)$. We already know that $[\omega_t(n,0)]\neq 0$. For example this follows from the formula already proved in PROPOSITION I on page 17:

$$[\omega_t(n,o)]=[\omega_o(n,o)]+\sum_{k=a}^{n}(-1)^{\frac{k(k-1)}{2}}t^k[\wedge^k\gamma_1\perp\omega_o(n,o)]:$$

From this formula we get immediately that the map π is an injection and since $\frac{d\pi}{dt}\neq 0$ for each $t\in\gamma(t)$ we get that $\pi(\gamma(t))\subset\Omega$. So this proves b.

Q.E.D.

4.4. DEFINITION.

Let $\left\{\phi_i\right\}$ for i=1,..,N be an orthonormal basis in $\mathbf{H}^1(M_O,\Theta_O)$. From LEMMA 4.2. it follows that for each $(t_1,...,t_N)\in\mathbf{C}^N$

$$\phi(t_1,..,t_N)=\sum_{i=1}^{N}\phi_i t_i\ +\tfrac{1}{2}\overline{\partial}^*G[\phi(t_1,..,t_N,\phi(t_1,..,t_N)]$$

defines a new complex structure on M_0. All these complex structures we will denote by $\tilde{\mathfrak{T}}(M_0)$ and we will call it the extended Teichnüller space.

REMARK 4.4.1..

We have a family of marked complex analytic manifolds $\pi:\mathfrak{X}\rightarrow\tilde{\mathfrak{T}}(M_0)$ and each fibre has the properties that it has a holomorphic n-form without zeroes. It is not very difficult to prove that there exists an open and everywhere dense subset U in $\tilde{\mathfrak{T}}(M_0)$ such that for each $t\in U$ $\pi^{-1}(t)=M_t$ is a Calabi-Yau manifold.

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The function $\Psi(t_1,..,t_N,\overline{t}_1,..,\overline{t}_N)$ defined in (1.8.6.) is defined on the whole $\tilde{\mathfrak{T}}(M_0)$ and moreover it is real analytic in

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We will use condition (iii) of 4.5.1.. If we take any $\gamma_1\in\mathbb{H}^1(M_0,\Theta_0)$ and we define

$$\gamma(t)=\sum_{n=1}^{\infty}\gamma_n t^n, \text{ where } \quad \gamma(t)=\gamma_1 t+\frac{1}{2}\,\overline{\partial}^*G[\gamma(t),\gamma(t)]$$

We know that $\gamma(t)$ is a totally geodesic submanifold defined for all $t\in\mathbf{C}$. We will prove that $\gamma(t)$ is a complete with respect to the Weil-Petersson metric on $\gamma(t)$.

4.5.2. Proposition.

The map $\exp:\mathbf{C}\to\gamma(t)$, where $\exp(t)=\gamma(t)$ is an one to one map between $\mathbf{C}$ and $\gamma(t)\subset\tilde{\mathfrak{T}}(M_o)$.

Proof of 4.5.2.:

We have already shown that for each $t\in\mathbf{C}$ $\gamma(t)\in C^{\infty}(M_o,\mathrm{Hom}(\Omega_o^{1,0},\Omega_o^{0,1})$ and it defines a new integrable complex structure on M_o. From the formula for the class of the cohomology of $\omega_t(n,o)$ defined in 3.6.2.1.

$$[\omega_t(n,o)]=[\omega_o(n,o)]+\sum_{k=1}^{n}t^k[\wedge^k\gamma_1\lrcorner\omega_o(n,o)]$$

we get that if $t_1\neq t_2$, then $[\omega_{t_1}(n,o)]\neq[[\omega_{t_2}(n,o)]$.

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$$\alpha+iI\alpha\in\mathbb{H}^1(M_o,\Theta_o)$$

Since $\mathbb{H}^1(M_o,\Theta_o)$ can be identified with the complex tangent space at $o\in\tilde{\mathfrak{T}}(M_o)$. Let $\gamma_1=\alpha+iI\alpha$ and let

$$\gamma(t)=\sum_{n=1}^{\infty}\gamma_n t^n,\ \gamma(t)=\gamma_1 t+\tfrac{1}{2}\bar{\partial}^*G[\gamma(t),\gamma(t)].$$

We proved that $\gamma(t)$ is a totally geodesic complete submanifold in $\tilde{\mathfrak{T}}(M_o)$ so the geodesic $\alpha(t)$ in $\tilde{\mathfrak{T}}(M_o)$ corresponding to the direction α will be for all t in $\gamma(t)$. Since $\gamma(t)$ is complete with respect to the Weil-Petersson metric we get that $\alpha(t)$ has an infinite length. This is condition (iii) of 4.1..

Q.E.D.

#5. TORELLI PROBLEM FOR CALABI-YAU MANIFOLDS.

5.1. Variations of HODGE structures. (See [G].)

Let X and S be complex manifolds and let $f:X\to S$ be a complex analyric map between those two manifolds. We will consider both $f:X\to S$ as analytic fibre bundle and as topological fibre bundle. Fix a base point $s_0\in S$ and consider the action of $\pi_1(S)$ on the cohomology $H^n(S,\mathbf{Q})$. If $L\in H^2(M_{s_0},\mathbf{Q})$ is the cohomology class of the hyperplane section relative to the given projective embedding:

$$\begin{array}{ccc} X & \subset & \mathbf{P}^N x X \\ \downarrow & & \downarrow \\ S & \equiv & S \end{array}$$

then L will be invariant under $\pi_1(S)$. Thus for $n\leq m=\dim_{\mathbf{C}}M_{s_0}$ we may define the primitive cohomology $P^n(M_{s_0},\mathbf{Q})$ to be the kernal of

$$L^{r+1}:H^{m-r}(M_{s_0},\mathbf{C})\to H^{m+r+2}(M_{s_0},\mathbf{C})$$

Because of Lefschetz decomposition:

(4.1.1.) $$H^n(M_{s_0},\mathbf{C})=\bigoplus_{k=0}^{[\frac{n}{2}]}L^kP^{n-2k}(M_{s_0},\mathbf{C}) \text{ (See [G].)}$$

which is $\pi_1(S)$-invariant direct sum decomposition we need to consider only the primitive cohomology.

Let $E=P^n(M_{s_0},\mathbf{C})$ and let $\mathbb{E}\to S$ be the complex vector bundle, with constant transition functions, associated with the action of $\pi_1(S)$ on E. There is the usual flat, holomorphic connection:

$$D:\mathcal{O}(\mathbb{E})\to\Omega^1_S(\mathbb{E})$$

which one has on any such vector bundle associated to a representation of the fundamental group. In fact we have a short exact sheaf sequence:

(5.1.2.) $$0\to\mathcal{C}(\mathbb{E})\to\mathcal{O}_S(\mathbb{E})\overset{D}{\to}\Omega^1_S(\mathbb{E})$$

where the sheaf $\mathcal{C}(\mathbb{E})$ is just the sheaf of locally constant sections of $\mathbb{E}$ and it has the following interpretation:

Let $R^nf_*(\mathbf{C})$ be the usual Leray cohomology sheaf of

$$f:X\to S,$$

which is the sheaf arrising from the presheaf:

$$U\to H^n(f^{-1}(U),\mathbf{C})$$

where U runs through the family of all open sets of S. We will define the Leray primitive cohomology sheaf to be $P^n_{f_*}(\mathbf{C})$ to be the kernal of:

$$L^{r+1}: R_{n-r} f_* \mathbf{C} \to R^{m+r+2} f_* \mathbf{C}$$

Then $\mathbb{C}(\mathbb{E})$ is just $P^n_{f_*}(\mathbf{C})$.

The fibre of $\mathbf{E}_S$ is the vector space $P^n(M_s,\mathbf{Q})$ and as such has the structure of the primitive cohomology space of a Kähler manifold. (See [G].) Translating this structure into on the flat bundle $\mathbf{E} \to S$, we find the following:

I) A flat conjugation $e \to \overline{e}$ $(e \in \mathbb{E})$

II) A flat non-degenerate bilinear form

(5.1.3.) $$Q: \mathbf{E} \otimes \mathbf{E} \to \mathbf{C},\ Q(e,e') = (-1)^n Q(e',e) = \int_{M_s} L^{m-n} \wedge e \wedge e'$$

called the Hodge bilinear form.

III) A filtration of $\mathbf{E}$ by holomorphic subbundles

(5.1.4.) $$\mathbf{F}^o(M_s) \subset \mathbf{F}^1(M_s) \subset \dots \subset \mathbf{F}^n(M_s) = \mathbf{E}$$

where

$$\mathbf{F}^q(M_s) = P^{n,0}(M_s) + \dots + P^{n-q,q}(M_s)$$

and

$$P^{n-q,q}(M_s) = H^{n-q,q}(M_s) \cap P^n(M_s,\mathbf{Q})$$

(IV) The Hodge filtration is isotropic, i.e.

(5.1.5.) $$(\mathbf{F}^q)^{\perp} = \mathbf{F}^{n-q-1}$$

where

$$(\mathbf{F}^q)^{\perp} = \{e \in \mathbb{E} \mid Q(e, \mathbf{F}^q) = 0\}$$

(V) The bilinear form (5.1.3.) is real (i.e., $\overline{Q} = Q$) and if we let

$$\mathbf{F}^{n-q,q} = \mathbf{F}^q \cap \overline{\mathbf{F}^{n-q}} = \mathbf{F}^q \cap (\overline{\mathbf{F}^{q-1}})^{\perp}$$

then we have the Hodge decomposition, which is a C^{∞} direct sum decomposition:

(5.1.6.) $$\mathbf{E} = \bigoplus_{q=0}^{n} \mathbf{F}^{q,n-q}$$

(VI) The Riemann-Hodge bilinear relations:

$$Q(\mathbf{F}^{n-q,q}, \overline{\mathbf{F}^{n-r,r}}) = 0 \quad (q \neq r)$$

(5.1.7.)

$$(-1)^{n+q} Q(\mathbf{F}^{n-q,q}, \overline{\mathbf{F}^{n-q,q}}) > 0$$

are valid.

(VII) The infinitesimal period relations. (See [G].)

(5.1.8.) $$D:\mathcal{O}_S(\mathbf{F}^q)\to\Omega^1_S(\mathbb{F}^{q+1})$$

holds.

Definition. 5.1.8.

We shall call a data given by $\mathbf{E}=(\mathbf{E},D,Q,\{\mathbf{F}^q\})$ and by I-VII a variation of Hodge structure.

5.2. Classifying spaces for HODGE STRUCTURES.

Let E be a complex vector space and let

$$0<h_o\leq h_1\leq\ldots\leq h_{n-1}<h_n=\dim_{\mathbf{C}}$$

be an increasing sequence of integers which is self-dual in the sence that

$$h_{n-q-1}=h_n-h_q$$

We also have a non-singular bilinear form

$$Q:E\otimes E\to\mathbf{C}\ Q(e,e')=(-1)^n Q(e',e)$$

Consider the set $\tilde{D}$ of all filtrations

$$F^o\subset F^1\subset F^2\subset\ldots\subset F^{n-1}\subset F^n=E,\ \dim_{\mathbf{C}}=h_q$$

which satisfy the first Riemann-Hodge bilinear relation

$$(F^q)^{\perp}=F^{n-q-1}$$

or equivelently

$$Q(F^q,F^{n-q-1})=0$$

We say that such filtrations are isotopic or self-dual. In [G] the following proposition is proved:

Proposition.

$\tilde{D}$ is in a natural way a projective and smooth complete algebraic variety which is a homogeneous space

$$\tilde{D}=G/B$$

of complex Lie group G divided by a parabolic subgroup B.

Suppose that $E=E_{\mathbf{R}}\otimes\mathbf{C}$ and Q is real on the real space $E_{\mathbf{R}}$. Define the Hermitian inner product $<\ ,\ >$ on E by

$$<e,e'>=(-1)^n Q(e,\overline{e'})\quad (e,e'\in E)$$

(5.2.1.) Definition.

$D \subset \tilde{D}$ will be called the period domain if D consists of all isotropic filtrations:

$$F^o \subset ... \subset F^n = E$$

which satisfy the second Riemann-Hodge bilinear relation:

$(-1)^n < , >: E^q x E^q \to \mathbf{C}$ is positive definite, where

$E^q := \{e \in F^q | <e, F^{q-1}> = 0\}$

In [G] the following Proposition is proved:

Proposition. D is an open complex submanifold of $\tilde{D}$ which is a homogeneous complex manifold

$$D = G/H$$

of a real, simple non-compact Lie group G divided by a compact subgroup H.

5.3. Notation.

The period domain of a Calabi-Yau manifold M_o will be denoted by $D(M_o)$.

5.3.1. Definition.

Let

$$\mathfrak{X} \to \mathfrak{T}(M_o)$$

be the universal family of marked, polarized Calabi-Yau manifolds. Let

$$p: \mathfrak{T}(M_o) \to D(M_o)$$

be the map which is defined as follows:

$$p(t) := (H^{n,0}(M_t)) \subset (H^{n,0}(M_t) + H^{n-1,1}(M_t)) \cap P^n(M_o, \mathbf{C}) \subset ... \subset P^n(M_o, \mathbf{C})$$

i.e. p(t) is just the Hodge strucrture on $P^n(M_o, \mathbf{C})$ defined by the complex structure on M_t. p will be called the period map.

5.4. THEOREM. $p: \mathfrak{T}(M_o) \to D(M_o)$ is an inclusion.

Proof: Suppose that the period map

$$p: \mathfrak{T}(M_o) \to D(M_o)$$

is not an inclusion, then we can find two points s_o and s_1 such that:

a) $s_o \neq s_1 \in \mathfrak{T}(M_o)$

b) $p(s_o) = p(s_1)$

We must get a contradiction. From THEOREM 3.1. we know that $\mathfrak{T}(M_o)$ can be embedded in $\tilde{\mathfrak{T}}(M_o)$, where the Weil-Petersson metric is a complete one. Since the curvature operator of the Weil-Petersson metric is a non-positive one we can joint s_o and s_1 by a unique geodesic. (See [T] and [H].) Call this geodesic $\mu(s)$.

Let

$$\mu(0)=s_O,\ \mu(1)=s_1 \text{ and } \dot\mu(0)\in T_{s_O,\mathfrak{T}(M_O)}(\text{real tangent space})$$

Let

$$\dot\gamma(0)=\dot\mu(0)+iI\dot\mu(0)$$

(I is the complex structure operator on $\mathfrak{T}(M_O)$)

Since

$$I(\dot\gamma(0))=i\dot\gamma(0)$$

we get that

$$\dot\gamma(0)\in \mathbf{H}^1(M_O,\Theta_O)$$

Let $\gamma(t)$ be the totally geodesic one-dimensional complex submanifold defined in #3. From the definition of totally geodesic submanifold it follows that $\mu(s)\subset\gamma(t)$. From [T] we get the following formula for $\omega_t(n,0)$ on M_t along $\gamma(t)$:

(Remember that we have a family along $\gamma(t)$)

(5.4.1.) $$\omega_t(n,0)=\omega_{s_O}(n,0)+\sum_{k=1}^{n} t^k(-1)^{\frac{k(k-1)}{2}}((\wedge^k\dot\gamma(0)\perp\omega_{s_O}(n,0))$$

From (5.4.1.) we get that if s_O and s_1 are two differnt points on $\gamma(t)$ then

$$[\omega_{s_O}(n,0)]\neq[\omega_{s_1}(n,0)] \text{ in } H^n(M_O,\mathbf{C})$$

This is so since we proved that all forms $\{(\wedge^k\dot\gamma(0)\perp\omega_{s_O}(n,0)\}$ for $k=1,\dots,n$ are harmonic forms with respect to the Calabi-Yau metric $(g_{\alpha,\overline{\beta}}(0))$ on M_O. So from the definition of the period map it follows that $p(s_O)\neq p(s_1)$.

Q.E.D.

5.5. The second version of TORELLI THEOREM.

5.5.1. Definition.

Let $E=P^n(M_O,\mathbf{C})$ be the primitive classes of cohomology on M_O, where $n=\dim_{\mathbf{C}}M_O$. Let $D'(M_O)\subset\mathbf{P}(E)$ be defined as follows:

$$D'(M_O):=\{\mu\in\mathbf{P}(E)|\ Q(\mu,\overline{\mu})>0 \text{ and } Q(\mu,\mu)=0\}$$

where

$$Q(\mu,\nu):=(-1)^{\frac{n(n+1)}{2}}(i)^n\int_{M_O}\mu\wedge\overline{\nu}$$

5.5.1. Definition.

The period map

$$p:\mathfrak{T}(M_0)\rightarrow D'(M_0)$$

is defined in the following way:

$$t\rightarrow p(t)=\left(\ldots,\int_{\gamma_i}\omega_t(n,0),\ldots\right)\in\mathbf{P}(E)$$

(Since each point $t\in\mathfrak{T}(M_0)$ corresponds to a marked Calabi-Yau manifold and $\omega_t(n,0)$ is defined up to a constant it follows that p is correctly defined.)

In [G] it is proved that p has a maximal rank and that p is a holomorphic map.

5.7. THEOREM. $p:\mathfrak{T}(M_0)\rightarrow D'(M_0)$ is an embedding.

Proof: The proof is exactly the same as the proof of THEOREM 5.4.

Q.E.D.

#6 THE FILLING IN PROBLEM FOR CALABI-YAU MANIFOLDS.

THEOREM 6.

Let $\pi^*:\mathcal{M}^*\to D^*$ a family of CALABI-YAU manifolds over the punctured disk $D^*:=\left\{t\in\mathbf{C}|0<|t|<1\right\}$ such that the monodtromy operator T which acts on the middlie homology group, i.e. on $H_n(M_t,\mathbf{Z})$ is trivial, i.e. T=id, then there exists a family over $D:=\left\{t\in\mathbf{C}|\ |t|<1\right\}$ $\pi:\mathcal{M}\to D$ of nonsingular complex manifolds such that on $M_o:=\pi^{-1}(0)$ there exists a holomorphic n-form $\omega_o(n,0)$ without zeroes.

PROOF OF THEOREM 5:

We have assumed that the family $\pi^*:\mathcal{M}^*\to D^*$ has a trivial monodromy. Using the fact that there exists an universal family $\mathfrak{X}\to\mathfrak{T}(M_o)$ of marked Calabi-Yau manifolds over the Teichmüller space we get that there exists a map $f:D^*\to\mathfrak{T}(M_o)$. Indeed since the monodromy operator T of the family is trivial, we see immediately that if we marked one fibre M_t, i.e. choose a basis $\{\delta_1,..,\delta_n\}$ of $H_n(M_t,\mathbf{Z})$, then we have marked all fibres. So from this trivial remark we get the existence of $f:D^*\to\mathfrak{T}(M_o)$. Using the fact that Global Torelli Theorem holds for Calabi-Yau manifolds, i.e. the period map $p:\mathfrak{T}(M_o)\to D(M_o)$ is an inclusion (See THEOREM 6.1.) and the fact that in our case the period map $p:D^*\to D(M_o)$ can be prolonged to a map $p:D\to D(M_o)$(See [G].) we get that f can be prolonged to a map $f:D\to\mathfrak{T}(M_o)$. Now our theorem follows THEOREM 4.1..

Q.E.D.

#7. THE TEICHMULLER SPACE $\tilde{\mathfrak{T}}(M_O)$ IS A STEIN MANIFOLD.

7.1. THEOREM.

a) The Teichmüller space $\tilde{\mathfrak{T}}(M_O)$ is diffeomorphic to $\mathbf{R}^{2N}$, where $N=\dim\mathbf{H}^1(M_O,\Theta_O)$.

b) $\tilde{\mathfrak{T}}(M_O)$ is a Stein manifold.

PROOF: Let $\Psi(t_1,..,t_N,\bar{t}_1,..,\bar{t}_N)$ be the function defined in (1.8.6.). As it was shown in #3. $\Psi(t_1,..,t_N,\bar{t}_1,..,\bar{t}_N)$ is globally defined positive function on $\tilde{\mathfrak{T}}(M_O)$.

7.1.1. LEMMA.

a) $\log\Psi(t_1,..,t_N,\bar{t}_1,..,\bar{t}_N):\tilde{\mathfrak{T}}\rightarrow\mathbf{R}$ is a proper map.

b) $\log\Psi(t_1,..,t_N,\bar{t}_1,..,\bar{t}_N)$ is a non-degenerate Morse function and has a unique critical point in $\tilde{\mathfrak{T}}(M_O)$ and this critical point is a minimum.

c) $\log\Psi(t_1,..,t_N,\bar{t}_1,..,\bar{t}_N)$ is a holomorphically convex.

7.1.1.1. REMARK.

From Lemma 7.1.1. THEOREM 7.1. follows directly from well-known gradient deformations of Morse theory ([M]) and the results in [He].

Proof of *a)*:

It is enouph to prove that Ψ is a proper function, then $\log\Psi$ will be proper too. We need to prove that praimage of a compact set in $\mathbf{R}^1$ is a compact set in $\tilde{\mathfrak{T}}(M_O)$. Notice that for each $t\in\tilde{\mathfrak{T}}(M_O)$ $\Psi(t)\neq 0$. So from this ***remark*** we may suppose that K is a compact set in $\mathbf{R}^1$ that does not contains the point 0, i.e. we may suppose that $K=[a,b]$ and $0\notin[a,b]$. So we need to prove the following ***Proposition:***

7.1.1.a. Proposition.

If $\{t_n\}$ is any sequence of points in $\tilde{\mathfrak{T}}(M_O)$ such that

$$a\leq\Psi(t_n)\leq b \ \forall t_n \text{ and } 0\notin[a,b]$$

then there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that ***there exists***

$$\lim_{k\rightarrow\infty} t_{n_k}=t\in\tilde{\mathfrak{T}}(M_O)$$

Proof of 7.1.1.a.:

We know from #4 that $\tilde{\mathfrak{T}}(M_O)\subset D'(M_O)\subset Q\subset\mathbf{P}(E)$, where $E=P^n(M_O,C)$(primitive n-cohomology) and Q is defined as follows:

$$Q:=\{\omega\in E\subset H^n(M_O,C)| \int_{M_O}\omega\wedge\omega=0\}$$

$D'(M_o)$ is contained in an open in Q and $D'(M_o)$ is defined as follows:

$$D'(M_o) := \{\omega \in Q \mid (-1)^{\frac{n(n+1)}{2}} (i)^n \int_{M_o} \omega \wedge \overline{\omega} > 0\}$$

Since $\{t_n\} \in \tilde{\mathfrak{T}}(M_o) \subset D'(M_o) \subset Q$ and Q is a compact manifold, ***it follows that there exists a subsequence*** $\{t_{n_k}\}$ such that

(*) $\lim_{n_k \to \infty} t_{n_k} = t$ exists and $t \in Q \subset \mathbf{P}(H^n(M_o, \mathbf{C}))$

So since $t \in Q \subset \mathbf{P}(H^n(M_o, \mathbf{C}))$ t can be represented by some harmonic form $\omega_t \in \mathbb{H}^n(M_o, \mathbf{C})$ on M_o. From the conditions:

$$a < \Psi(t_n) = (-1)^{\frac{n(n+1)}{2}} (i)^{n-2} \int_{M_o} \omega_{t_n} \wedge \overline{\omega}_{t_n} < b, \ 0 \notin [a,b] \text{ and}$$

$$\lim_{n_k \to \infty} [\omega_{t_k}(n,0)] = [\omega_t]$$

we get that

$$(-1)^{\frac{n(n+1)}{2}} (i)^n \int_{M_o} \omega_t \wedge \overline{\omega}_t > 0$$

so

(**) $t \in \Omega$

Next we must prove that $t \in \tilde{\mathfrak{T}}(M_o)$. We know that $D'(M_o) = G/K$, where G is a semi-simple Lie group and K is a maximal compact subgroup in G. So on $D'(M_o)$ there exists a unique G-invariant metric h. $D'(M_o)$ is a complete Riemannian manifold with respect to h. Tian proved in [Ti] that the Weil-Petersson metric on $\tilde{\mathfrak{T}}(M_o)$ is just the restriction of h on $\tilde{\mathfrak{T}}(M_o)$. Since $\{t_{n_k}\}$ is Cauchy sequence on $D'(M_o)$ and so it is a Caushy sequence on $\tilde{\mathfrak{T}}(M_o)$. Because $t_{n_k} \in \tilde{\mathfrak{T}}(M_o)$ the definition of a complete Riemannian manifold and according to the results of #3., i.e $\tilde{\mathfrak{T}}(M_o)$ is complete Riemannian manifold with respect to the Weil-Petersson metric it follows that

$$\lim_{n_k \to \infty} t_{n_k} = t \in \tilde{\mathfrak{T}}(M_o)$$

So 7.1.1.a. is proved.

Q.E.D.

Proof of 7.1.1.*b. and c:*

From the definition of $\Psi(t_1,\ldots,t_N;\overline{t}_1,\ldots,\overline{t}_N)$ it follows that

$$\frac{\partial \log \Psi}{\partial t_i}=\frac{\partial \log \Psi}{\overline{\partial} \overline{t}_i}=0$$

at the point $t_o \in \tilde{\mathfrak{T}}(M_o)$ that corresponds to M_o.(See [T].) For any other point $t \neq t_o$ we have

$$\frac{\partial \log \Psi}{\partial t_i}=\frac{\overline{\partial} \log \Psi}{\overline{\partial} \overline{t}_i}\neq 0$$

In [T] it was proved that $\left(\frac{\partial^2 \log \Psi}{\partial t_i \overline{\partial} \overline{t}_j}\right)$ is the Weil-Petersson metric. So t_o is a unique

non-degenerate minimum of $\log\Psi(t_1,\ldots,t_N;\overline{t}_1,\ldots,\overline{t}_N)$ and it is holomorphically convex.

Q.E.D.

So THEOREM 7.1. is proved.

Q.E.D.

#8. NIELSON REALIZATION PROBLEM.

8.1. Definition.

Let $\mathrm{Diff}^{+}(M_O)$ be the group of diffeomorphisms of M_O that preserve the orientation. Let $\mathrm{Diff}_O^{+}(M_O)$ be the group of diffeomorphisms isotopic to the identity. It is easy to prove that $\mathrm{Diff}_O^{+}(M_O)$ is a normal subgroup in $\mathrm{Diff}^{+}(M_O)$. Let

$$\Gamma:=\mathrm{Diff}_O^{+}(M_O)/\mathrm{Diff}^{+}(M_O)$$

8.2. Proposition.

Γ acts discretely on the Teichmüller space $\tilde{\mathfrak{T}}(M_O)$ and preserve the Weil-Petersson metric.

Proof: If $g\in\Gamma$ and $t\in\tilde{\mathfrak{T}}(M_O)$ and t corresponds to M_t with a complex structure operator I_t *then* $g^*(t)$ corresponds to the manifold with an complex structure operator $g^*(I_t)$. So $g^*(t)\in\tilde{\mathfrak{T}}(M_O)$. If $t\in\tilde{\mathfrak{T}}(M_O)$ then we know that $t\in\mathfrak{K}$, where $\mathfrak{K}$ is the Kuranishi space of M_t. From [KM] it follows that all the points $t\in\mathfrak{K}$ corresponds to non-isomorphic complex manifold. From here it follows that Γ acts dicretely on $\tilde{\mathfrak{T}}(M_O)$.

Next we must prove that Γ preserve the Weil-Petersson metric. We know that we can identify the tangent space $T_{t,\tilde{\mathfrak{T}}(M_O)}$ at a point $t\in\tilde{\mathfrak{T}}(M_O)$ with $\mathbf{H}^1(M_O,\Theta_O)$, i.e.

$$T_{t,\tilde{\mathfrak{T}}(M_O)}\simeq\mathbf{H}^1(M_O,\Theta_O)\simeq\mathbf{H}^1(M_O,\Omega_O^{n-1})$$

and the last identification is given by

$$\phi\rightarrow\phi\perp\omega_t(n,0)$$

where $\omega_t(n,0)\wedge\omega_t(0,n)=\mathrm{vol}(g_{\alpha,\overline{\beta}})$ and $[\mathrm{Im}(g_{\alpha,\overline{\beta}})]=L$.

On the other hand we know that

$$<\phi_1,\phi_2>=(-1)^{\frac{n(n+1)}{2}}(i)^{n-2}\int_{M_O}[(\phi_1\perp\omega_t(n,0))]\wedge[\overline{\phi_2\perp\omega_t(n,0)}]=$$

$$<[(\phi_1\perp\omega_t(n,0))],[(\phi_2\perp\omega_t(n,0))]>$$

where $[(\phi_i\perp\omega_t(n,0))]\in H^n(M_O,\mathbf{C})$ for i=1,2. (See [T].) From this formula we get immediately that

$$<\phi_1,\phi_2>=<g^*(\phi_1),g^*(\phi_2)>$$

Q.E.D.

THEOREM 8.3.

If G is a finite subgroup of Γ, *then there exists a point* $x_o \in \tilde{\mathfrak{T}}(M_o)$ such that $g(x_o)=x_o$ $\forall$ $g\in G$.

Proof: We know from Lemma 5.1.1. that

$$\Psi(t_1,...,t_N;\overline{t}_1,...,\overline{t}_N):\tilde{\mathfrak{T}}(M_o)\to\mathbb{R}$$

is a proper C^∞ function, which is holomorphically convex and has a unique non-degenerate minimum. Let us define

$$\Psi_G:=\frac{1}{|G|}\sum g^*\Psi(t_1,...,t_N;\overline{t}_1,...,\overline{t}_N)$$

From the way we define Ψ_G it follows that it is a proper C^∞ function. We must prove that Ψ_G has a unique nondgenerate critical point. This fact will follow from the following *Proposition.*

8.3.1.Proposition.

$\Psi(t_1,...,t_N;\overline{t}_1,...,\overline{t}_N)$ is a convex function when it is restricted on a real geodesic $v(t)$ with respect to the Weil-Petersson metric.

Proof: We must prove that

$$\frac{d^2\Psi}{dt^2}>0 \text{ on } v(t)$$

From the definition of $\Psi:=<\omega_t,\omega_t>$, where ω_t is defined in (1.8.5.) it follows that

$$(*)\qquad \frac{d^2<\omega_t,\omega_t>}{dt^2}=2\frac{d(<\dot{\omega}_t,\omega_t>)}{dt}=2<\ddot{\omega}_t,\omega_t>+\|\dot{\omega}_t\|^2$$

From the definition of ω_t it follows that

$$\frac{d\omega_t}{dt}:=\dot{\omega}_t$$

is a tangent to the geodesic $v(t)$. (See [T]). From the definition of a geodesic with respect to the Weil-Petersson metric it follows that

$$\frac{d^2\omega_t}{dt^2}=\ddot{\omega} \text{ is a form of type (n-2,2)}$$

This was proved in #3. This yields

$$<\omega_{t_o},\ddot{\omega}_{t_o}>=0$$

So from (*) we obtain that

$$\frac{d^2 <\omega_t, \omega_t>}{dt^2} = \frac{\|\dot{\omega}_t\|^2}{\|\omega_t\|^2} > 0$$

Q.E.D.

From the ***Proposition 8.3.1.*** it follows that Ψ_G has a unique non-degenerate minimum x_o. Since Ψ_G is G invariant it follows that $g(x_o)=x_o \quad \forall g \in G$.

Q.E.D.

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