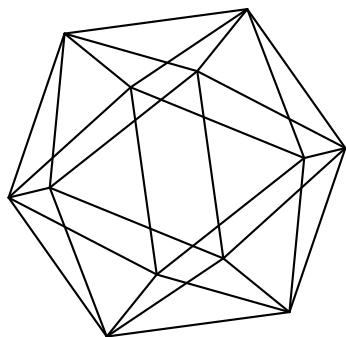


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by

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# ELLIPTIC HYPERGEOMETRIC FUNCTIONS

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**Introduction.** The wonderful book by Andrews, Askey, and Roy [2] is mainly devoted to special functions of hypergeometric type – to the plain hypergeometric series and integrals and their  $q$ -analogues. Shortly before its publication there appeared first examples of hypergeometric functions of a new type related to elliptic curves. A systematic theory of elliptic hypergeometric functions was constructed in 2000-2004 over a short period of time. The present complement reviews briefly the status of this theory by the fall of 2011. It repeats where possible the structure of the book [2], and it is substantially based on author’s thesis [66] and survey [70].

The theory of quantum and classical completely integrable systems played a crucial role in the discovery of these new special functions. An elliptic extension of the terminating very-well-poised balanced  $q$ -hypergeometric series  ${}_{10}\varphi_9$  with discrete values of parameters appeared for the first time in elliptic solutions of the Yang-Baxter equation [28] associated with the exactly solvable models of statistical mechanics [19]. The same terminating series with arbitrary parameters appeared in [78] as a particular solution of a pair of linear finite difference equations, the compatibility condition of which yields the most general known  $(1+1)$ -dimensional nonlinear integrable chain analogous to the discrete time Toda chain. An elliptic analogue of Euler’s gamma function depending on two bases  $p$  and  $q$  of modulus less than 1, which already appeared in Baxter’s eight vertex model [7], was investigated in [56], and in [64] a modified elliptic gamma function was constructed for which one of the bases may lie on the unit circle. General elliptic hypergeometric functions are defined by the integrals discovered in [61], which qualitatively differ from the terminating elliptic hypergeometric series. The appearance of such mathematical objects was quite unexpected, since no handbook or textbook of special functions contained any hint of their existence. However, the generalized gamma functions related to elliptic gamma functions and forming one of the key ingredients of the theory were constructed long ago by Barnes [6] and Jackson [34]. The most important known application of the elliptic hypergeometric integrals was found quite recently – they emerged in the description of topological characteristics of four-dimensional supersymmetric quantum field theories [23, 30, 74, 75].

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This is a complement to the book by G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Encyclopedia of Math. Appl. **71**, Cambridge Univ. Press, Cambridge, 1999, written for its Russian edition (to be published by the Independent University press, Moscow, 2012).

**Generalized gamma functions.** In the beginning of XXth century Barnes [6] constructed the following multiple zeta function:

$$\zeta_m(s, u; \omega) = \sum_{n_1, \dots, n_m \in \mathbb{Z}_{\geq 0}} \frac{1}{(u + \Omega)^s}, \quad \Omega = n_1 \omega_1 + \dots + n_m \omega_m, \quad \mathbb{Z}_{\geq 0} = 0, 1, \dots,$$

where  $u, \omega_j \in \mathbb{C}$ . This series converges for  $\operatorname{Re}(s) > m$  provided all  $\omega_j$  lie on one side of a line passing through the point  $u = 0$  (this forbids accumulation points of the  $\Omega$ -lattice in compact domains). Using an integral representation for analytical continuation of  $\zeta_m$  in  $s$ , Barnes also defined the multiple gamma function  $\Gamma_m(u; \omega) = \exp(\partial \zeta_m(s, u; \omega) / \partial s)|_{s=0}$ . It has the infinite product representation

$$\frac{1}{\Gamma_m(u; \omega)} = e^{\sum_{k=0}^m \gamma_{mk} \frac{u^k}{k!}} u \prod'_{n_1, \dots, n_m \in \mathbb{Z}_{\geq 0}} \left(1 + \frac{u}{\Omega}\right) e^{\sum_{k=1}^m (-1)^k \frac{u^k}{k \Omega^k}}, \quad (1)$$

where  $\gamma_{mk}$  are some constants analogous to Euler's constant (in [6], the normalization  $\gamma_{m0} = 0$  was used). The primed product means that the point  $n_1 = \dots = n_m = 0$  is excluded from it. The function  $\Gamma_m(u; \omega)$  satisfies  $m$  finite difference equations of the first order

$$\Gamma_{m-1}(u; \omega(j)) \Gamma_m(u + \omega_j; \omega) = \Gamma_m(u; \omega), \quad j = 1, \dots, m, \quad (2)$$

where  $\omega(j) = (\omega_1, \dots, \omega_{j-1}, \omega_{j+1}, \dots, \omega_m)$  and  $\Gamma_0(u; \omega) := 1/u$ . The function  $\Gamma_1(\omega_1 x; \omega_1)$  essentially coincides with the Euler gamma function  $\Gamma(x)$ . The plain,  $q$ -, and elliptic hypergeometric functions are connected to  $\Gamma_m(u; \omega)$  for  $m = 1, 2, 3$ , respectively.

Take  $m = 3$  and assume that  $\omega_{1,2,3}$  are pairwise incommensurate quasiperiods. Then define three base variables:

$$\begin{aligned} q &= e^{2\pi i \frac{\omega_1}{\omega_2}}, & p &= e^{2\pi i \frac{\omega_3}{\omega_2}}, & r &= e^{2\pi i \frac{\omega_3}{\omega_1}}, \\ \tilde{q} &= e^{-2\pi i \frac{\omega_2}{\omega_1}}, & \tilde{p} &= e^{-2\pi i \frac{\omega_2}{\omega_3}}, & \tilde{r} &= e^{-2\pi i \frac{\omega_1}{\omega_3}}, \end{aligned}$$

where  $\tilde{q}, \tilde{p}, \tilde{r}$  denote the  $\tau \rightarrow -1/\tau$  modular transformed bases. For  $|p|, |q| < 1$ , the infinite products

$$(z; q)_\infty = \prod_{j=0}^{\infty} (1 - zq^j), \quad (z; p, q)_\infty = \prod_{j,k=0}^{\infty} (1 - zp^j q^k)$$

are well defined. It is easy to derive equalities [34]

$$\frac{(z; q)_\infty}{(qz; q)_\infty} = 1 - z, \quad \frac{(z; q, p)_\infty}{(qz; q, p)_\infty} = (z; p)_\infty, \quad \frac{(z; q, p)_\infty}{(pz; q, p)_\infty} = (z; q)_\infty. \quad (3)$$

The odd Jacobi theta function (see formula (10.7.1) in [2]) can be written as

$$\begin{aligned} \theta_1(u|\tau) &= -i \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i \tau (n+1/2)^2} e^{\pi i (2n+1)u} \\ &= ip^{1/8} e^{-\pi i u} (p; p)_\infty \theta(e^{2\pi i u}; p), \quad u \in \mathbb{C}, \end{aligned}$$

where  $p = e^{2\pi i\tau}$ . The modified theta function (see Theorem 10.4.1 in [2])

$$\theta(z; p) := (z; p)_\infty (pz^{-1}; p)_\infty = \frac{1}{(p; p)_\infty} \sum_{k \in \mathbb{Z}} (-1)^k p^{k(k-1)/2} z^k \quad (4)$$

plays a crucial role in the following. It obeys the following properties:

$$\theta(pz; p) = \theta(z^{-1}; p) = -z^{-1} \theta(z; p) \quad (5)$$

and  $\theta(z; p) = 0$  for  $z = p^k$ ,  $k \in \mathbb{Z}$ . We denote

$$\theta(a_1, \dots, a_k; p) := \theta(a_1; p) \cdots \theta(a_k; p), \quad \theta(at^{\pm 1}; p) := \theta(at; p)\theta(at^{-1}; p).$$

Then the Riemann relation for products of four theta functions takes the form

$$\theta(xw^{\pm 1}, yz^{\pm 1}; p) - \theta(xz^{\pm 1}, yw^{\pm 1}; p) = yw^{-1} \theta(xy^{\pm 1}, wz^{\pm 1}; p) \quad (6)$$

(the ratio of the left- and right-hand sides is a bounded function of the variable  $x \in \mathbb{C}^*$ , and it does not depend on  $x$  due to the Liouville theorem, but for  $x = w$  the equality is evident).

Euler's gamma function can be defined as a special meromorphic solution of the functional equation  $f(u + \omega_1) = uf(u)$ . Respectively,  $q$ -gamma functions are connected to solutions of the equation  $f(u + \omega_1) = (1 - e^{2\pi i u / \omega_2})f(u)$  with  $q = e^{2\pi i \omega_1 / \omega_2}$ . For  $|q| < 1$ , one of the solutions has the form  $\Gamma_q(u) = 1/(e^{2\pi i u / \omega_2}; q)_\infty$  defining the standard  $q$ -gamma function (it differs from function (10.3.3) in [2] by the substitution  $u = \omega_1 x$  and some elementary multiplier). The modified  $q$ -gamma function ("the double sine", "non-compact quantum dilogarithm", "hyperbolic gamma function"), which remains well defined even for  $|q| = 1$ , has the form

$$\gamma(u; \omega) = \exp \left( - \int_{\mathbb{R}+i0} \frac{e^{ux}}{(1 - e^{\omega_1 x})(1 - e^{\omega_2 x})} \frac{dx}{x} \right), \quad (7)$$

where the contour  $\mathbb{R} + i0$  coincides with the real axis deformed to pass clockwise the point  $x = 0$  in an infinitesimal way. If  $\operatorname{Re}(\omega_1), \operatorname{Re}(\omega_2) > 0$ , then the integral converges for  $0 < \operatorname{Re}(u) < \operatorname{Re}(\omega_1 + \omega_2)$ . Under appropriate restrictions on  $u$  and  $\omega_{1,2}$ , the integral can be computed as a convergent sum of the residues of poles in the upper half plane. When  $\operatorname{Im}(\omega_1/\omega_2) > 0$ , this yields the expression  $\gamma(u; \omega) = (e^{2\pi i u / \omega_1} \tilde{q}; \tilde{q})_\infty / (e^{2\pi i u / \omega_2}; q)_\infty$ , which can be extended analytically to the whole complex  $u$ -plane. This function, serving as a key building block of the  $q$ -hypergeometric functions for  $|q| = 1$ , was not considered in [2] and [31]; for its detailed description see [24, 35, 38, 56, 81] and the literature cited therein.

In an analogous manner, elliptic gamma functions are connected to the equation

$$f(u + \omega_1) = \theta(e^{2\pi i u / \omega_2}; p)f(u). \quad (8)$$

Using the factorization (4) and equalities (3), it is easy to see that the ratio

$$\Gamma(z; p, q) = \frac{(pqz^{-1}; p, q)_\infty}{(z; p, q)_\infty} \quad (9)$$

satisfies the equations

$$\Gamma(qz; p, q) = \theta(z; p)\Gamma(z; p, q), \quad \Gamma(pz; p, q) = \theta(z; q)\Gamma(z; p, q).$$

Therefore the function  $f(u) = \Gamma(e^{2\pi i u/\omega_2}; p, q)$  defines a solution of equation (8) valid for  $|q|, |p| < 1$ , which is called the (standard) elliptic gamma function [56]. It can be defined uniquely as a meromorphic solution of three equations: equation (8) and

$$f(u + \omega_2) = f(u), \quad f(u + \omega_3) = \theta(e^{2\pi i u/\omega_2}; q) f(u)$$

with the normalization  $f(\sum_{m=1}^3 \omega_m/2) = 1$ , since non-trivial triply periodic functions do not exist. The reflection formula has the form  $\Gamma(z; p, q)\Gamma(pq/z; p, q) = 1$ . For  $p = 0$ , we have  $\Gamma(z; 0, q) = 1/(z; q)_\infty$ .

The modified elliptic gamma function, which is well defined for  $|q| = 1$ , has the form [64]

$$G(u; \omega) = \Gamma(e^{2\pi i \frac{u}{\omega_2}}; p, q)\Gamma(re^{-2\pi i \frac{u}{\omega_1}}; \tilde{q}, r). \quad (10)$$

It yields the unique solution of three equations: equation (8) and

$$f(u + \omega_2) = \theta(e^{2\pi i u/\omega_1}; r) f(u), \quad f(u + \omega_3) = e^{-\pi i B_{2,2}(u; \omega)} f(u)$$

with the normalization  $f(\sum_{m=1}^3 \omega_m/2) = 1$ . Here

$$B_{2,2}(u; \omega) = \frac{u^2}{\omega_1 \omega_2} - \frac{u}{\omega_1} - \frac{u}{\omega_2} + \frac{\omega_1}{6\omega_2} + \frac{\omega_2}{6\omega_1} + \frac{1}{2}$$

denotes the second order Bernoulli polynomial appearing in the modular transformation law for the theta function

$$\theta\left(e^{-2\pi i \frac{u}{\omega_1}}; e^{-2\pi i \frac{\omega_2}{\omega_1}}\right) = e^{\pi i B_{2,2}(u; \omega)} \theta\left(e^{2\pi i \frac{u}{\omega_2}}; e^{2\pi i \frac{\omega_1}{\omega_2}}\right). \quad (11)$$

One can check [22] that the same three equations and normalization are satisfied by the function

$$G(u; \omega) = e^{-\frac{\pi i}{3} B_{3,3}(u; \omega)} \Gamma(e^{-2\pi i \frac{u}{\omega_3}}; \tilde{r}, \tilde{p}), \quad (12)$$

where  $|\tilde{p}|, |\tilde{r}| < 1$ , and  $B_{3,3}(u; \omega)$  is the third order Bernoulli polynomial

$$B_{3,3}\left(u + \sum_{m=1}^3 \frac{\omega_m}{2}; \omega\right) = \frac{u(u^2 - \frac{1}{4} \sum_{m=1}^3 \omega_m^2)}{3\omega_1 \omega_2 \omega_3}.$$

The functions (10) and (12) therefore coincide, and their equality defines one of the laws of the  $SL(3; \mathbb{Z})$ -group of modular transformations for the elliptic gamma function [26]. From expression (12), the function  $G(u; \omega)$  is seen to remain meromorphic when  $\omega_1/\omega_2 > 0$ , i.e.  $|q| = 1$ . The reflection formula for it has the form  $G(a; \omega)G(b; \omega) = 1$ ,  $a + b = \sum_{k=1}^3 \omega_k$ . In the regime  $|q| < 1$  and  $p, r \rightarrow 0$  (i.e.,  $\text{Im}(\omega_3/\omega_1), \text{Im}(\omega_3/\omega_2) \rightarrow +\infty$ ), expression (10) obviously degenerates to the modified  $q$ -gamma function  $\gamma(u; \omega)$ . Representation (12) yields an alternative way of reduction to  $\gamma(u; \omega)$ ; a rigorous limiting connection of such a type was built for the first time in a different way by Ruijsenaars [56].

As shown by Barnes, the  $q$ -gamma function  $1/(z; q)_\infty$  where  $z = e^{2\pi i u/\omega_2}$  and  $q = e^{2\pi i \omega_1/\omega_2}$ ,  $\text{Im}(\omega_1/\omega_2) > 0$ , equals the product  $\Gamma_2(u; \omega_1, \omega_2)\Gamma_2(u - \omega_2; \omega_1, -\omega_2)$  up to the exponential of a polynomial. Similarly, the modified  $q$ -gamma function  $\gamma(u; \omega)$  equals up to an exponential factor to the ratio  $\Gamma_2(\omega_1 + \omega_2 - u; \omega)/\Gamma_2(u; \omega)$ .

Since  $\theta(z; q) = (z; q)_\infty (qz^{-1}; q)_\infty$ , the  $\Gamma_2(u; \omega)$ -function represents (in the sense of the number of divisor points) “a quarter” of the  $\theta_1(u/\omega_2 | \omega_1/\omega_2)$  Jacobi theta function. Correspondingly, one can consider equation (8) as a composition of four equations for  $\Gamma_3(u; \omega)$  with different arguments and quasiperiods and represent the elliptic gamma functions as ratios of four Barnes gamma functions of the third order with some simple exponential multipliers [29, 64]. For some other important results for the generalized gamma functions, see [42, 47].

**The elliptic beta integral.** It is convenient to use the compact notation

$$\begin{aligned}\Gamma(a_1, \dots, a_k; p, q) &:= \Gamma(a_1; p, q) \cdots \Gamma(a_k; p, q), \\ \Gamma(tz^{\pm 1}; p, q) &:= \Gamma(tz; p, q)\Gamma(tz^{-1}; p, q), \quad \Gamma(z^{\pm 2}; p, q) := \Gamma(z^2; p, q)\Gamma(z^{-2}; p, q)\end{aligned}$$

for working with elliptic hypergeometric integrals. We start consideration from the elliptic beta integral discovered by the author in [61].

**Theorem 1.** *Take eight complex parameters  $t_1, \dots, t_6$ , and  $p, q$ , satisfying the constraints  $|p|, |q|, |t_j| < 1$  and  $\prod_{j=1}^6 t_j = pq$ . Then the following equality is true*

$$\kappa \int_{\mathbb{T}} \frac{\prod_{j=1}^6 \Gamma(t_j z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z} = \prod_{1 \leq j < k \leq 6} \Gamma(t_j t_k; p, q), \quad (13)$$

where  $\mathbb{T}$  denotes the positively oriented unit circle and  $\kappa = (p; p)_\infty (q; q)_\infty / 4\pi i$ .

The first proof of this formula was based on the elliptic extension of Askey’s method [3]. A particularly short proof was given in [67]. It is based on the partial  $q$ -difference equation

$$\begin{aligned}\rho(z; qt_1, t_2, \dots, t_5; p, q) - \rho(z; t_1, \dots, t_5; p, q) \\ = g(q^{-1}z)\rho(q^{-1}z; t_1, \dots, t_5; p, q) - g(z)\rho(z; t_1, \dots, t_5; p, q),\end{aligned}$$

where  $\rho(z; t; p, q)$  is the integral kernel divided by the right-hand side expression in equality (13) with  $t_6$  replaced by  $pq/t_1 \cdots t_5$  and

$$g(z) = \frac{\prod_{k=1}^5 \theta(t_k z; p)}{\prod_{k=2}^5 \theta(t_1 t_k; p)} \frac{\theta(t_1 \prod_{j=1}^5 t_j; p)}{\theta(z^2, z \prod_{j=1}^5 t_j; p)} \frac{t_1}{z}.$$

A similar  $p$ -difference equation is obtained after permutation of  $p$  and  $q$ . Jointly they show that the integral  $\int_{\mathbb{T}} \rho(z; t; p, q) dz/z$  is a constant independent of the parameters  $t_j$  (to see this just integrate the equations). Taking a special limit of parameters  $t_j$  such that integral’s value is asymptotically given by the sum of residues of a fixed pair of poles (see below), one finds this constant.

The elliptic beta integral (13) defines the most general known univariate exact integration formula generalizing Euler’s beta integral. For  $p \rightarrow 0$ , one obtains the Rahman integral [43] (see Theorem 10.8.2 in [2]), which reduces to the well known Askey-Wilson  $q$ -beta integral [4] (see Theorem 10.8.1 in [2]) if one of the parameters vanishes. The binomial theorem  ${}_1F_0(a; x) = (1-x)^{-a}$  (see formula (2.1.6) in [2]) was proved by Newton. The  $q$ -binomial theorem  ${}_1\varphi_0(t; q; x) = (tx; q)_\infty / (x; q)_\infty$  (see

Ch. 10.2 in [2]) was established by Gauss and several other mathematicians. These formulas represent the simplest plain and  $q$ -hypergeometric function identities. At the elliptic level, this role is played by the elliptic beta integral evaluation, i.e. formula (13) can be considered as an elliptic binomial theorem.

Replace in formula (13)  $\mathbb{T}$  by a contour  $C$  which separates sequences of the integrand poles converging to zero along the points  $z = t_j q^k p^m$ ,  $k, m \in \mathbb{Z}_{\geq 0}$ , from their reciprocals obtained by the change  $z \rightarrow 1/z$ , which go to infinity. This allows one to lift the constraints  $|t_j| < 1$  without changing the right-hand side of formula (13). Substitute now  $t_6 = pq/A$ ,  $A = \prod_{k=1}^5 t_k$ , and suppose that  $|t_m| < 1$ ,  $m = 1, \dots, 4$ ,  $|pt_5| < 1 < |t_5|$ ,  $|pq| < |A|$ , and the arguments of  $t_1, \dots, t_5$ , and  $p, q$  are linearly independent over  $\mathbb{Z}$ . Then the following equality takes place [20]:

$$\kappa \int_C \Delta_E(z, \underline{t}) \frac{dz}{z} = \kappa \int_{\mathbb{T}} \Delta_E(z, \underline{t}) \frac{dz}{z} + c_0(\underline{t}) \sum_{|t_5 q^n| > 1, n \geq 0} \nu_n(\underline{t}), \quad (14)$$

where  $\Delta_E(z, \underline{t}) = \prod_{m=1}^5 \Gamma(t_m z^{\pm 1}; p, q) / \Gamma(z^{\pm 2}, A z^{\pm 1}; p, q)$  and

$$c_0(\underline{t}) = \frac{\prod_{m=1}^4 \Gamma(t_m t_5^{\pm 1}; p, q)}{\Gamma(t_5^{-2}, A t_5^{\pm 1}; p, q)}, \quad \nu_n(\underline{t}) = \frac{\theta(t_5^2 q^{2n}; p)}{\theta(t_5^2; p)} \prod_{m=0}^5 \frac{\theta(t_m t_5)_n}{\theta(q t_m^{-1} t_5)_n} q^n.$$

We have introduced here a new parameter  $t_0$  with the help of the relation  $\prod_{m=0}^5 t_m = q$  and used the elliptic Pochhammer symbol

$$\theta(t)_n = \prod_{j=0}^{n-1} \theta(t q^j; p) = \frac{\Gamma(t q^n; p, q)}{\Gamma(t; p, q)}, \quad \theta(t_1, \dots, t_k)_n := \prod_{j=1}^k \theta(t_j; p)_n$$

(the indicated ratio of elliptic gamma functions defines  $\theta(t)_n$  for arbitrary  $n \in \mathbb{C}$ ). The multiplier  $\kappa$  is absent in the coefficient  $c_0$  due to the relation  $\lim_{z \rightarrow 1} (1 - z) \Gamma(z; p, q) = 1/(p; p)_\infty (q; q)_\infty$  and doubling of the number of residues because of the symmetry  $z \rightarrow z^{-1}$ .

In the limit  $t_5 t_4 \rightarrow q^{-N}$ ,  $N \in \mathbb{Z}_{\geq 0}$ , the integral over the contour  $C$  (equal to the right-hand side of equality (13)) and the multiplier  $c_0(\underline{t})$  in front of the sum of residues diverge, whereas the integral over the unit circle  $\mathbb{T}$  remains finite. After dividing all the terms by  $c_0(\underline{t})$  and going to the limiting relation, we obtain the Frenkel-Turaev summation formula

$$\sum_{n=0}^N \nu_n(\underline{t}) = \frac{\theta(q t_5^2, \frac{q}{t_1 t_2}, \frac{q}{t_1 t_3}, \frac{q}{t_2 t_3})_N}{\theta(\frac{q}{t_1 t_2 t_3 t_5}, \frac{q t_5}{t_1}, \frac{q t_5}{t_2}, \frac{q t_5}{t_3})_N}, \quad (15)$$

which was established for the first time in [28] by a completely different method. For  $p \rightarrow 0$  and fixed parameters, formula (15) reduces to the Jackson sum for a terminating  ${}_8\varphi_7$ -series (see Ex. 16 in Ch. 10 and formula (12.3.5) in [2]). We stress that all terminating elliptic hypergeometric series identities like identity (15) represent relations between ordinary elliptic functions, i.e. they do not involve principally new special functions in contrast to the elliptic hypergeometric integral identities.

**General elliptic hypergeometric functions.** Definitions of the general elliptic hypergeometric series and integrals were given and investigated in detail in [62] and [64], respectively. So, a formal series  $\sum_{n \in \mathbb{Z}} c_n$  is called an elliptic hypergeometric series if  $c_{n+1} = h(n)c_n$ , where  $h(n)$  is some elliptic function of  $n \in \mathbb{C}$ . This definition is contained implicitly in the considerations of [78]. It is well known [5] that an arbitrary elliptic function  $h(u)$  of order  $s+1$  with the periods  $\omega_2/\omega_1$  and  $\omega_3/\omega_1$  can be represented in the form

$$h(u) = y \prod_{k=1}^{s+1} \frac{\theta(t_k z; p)}{\theta(w_k z; p)}, \quad z = q^u. \quad (16)$$

The equality  $h(u + \omega_2/\omega_1) = h(u)$  is evident, and the periodicity  $h(u + \omega_3/\omega_1) = h(u)$  brings in the balancing condition  $\prod_{k=1}^{s+1} t_k = \prod_{k=1}^{s+1} w_k$ . Because of the factorization of  $h(n)$ , in order to determine the coefficients  $c_n$  it suffices to solve the equation  $a_{n+1} = \theta(tq^n; p) a_n$ , which leads to the elliptic Pochhammer symbol  $a_n = \theta(t)_n a_0$ . The explicit form of the bilateral elliptic hypergeometric series is now easily found to be

$${}_{s+1}G_{s+1} \left( \begin{matrix} t_1, \dots, t_{s+1} \\ w_1, \dots, w_{s+1} \end{matrix}; q, p; y \right) := \sum_{n \in \mathbb{Z}} \prod_{k=1}^{s+1} \frac{\theta(t_k)_n}{\theta(w_k)_n} y^n,$$

where we have chosen the normalization  $c_0 = 1$ . By setting  $w_{s+1} = q$ ,  $t_{s+1} =: t_0$ , we obtain the one sided series

$${}_{s+1}E_s \left( \begin{matrix} t_0, t_1, \dots, t_s \\ w_1, \dots, w_s \end{matrix}; q, p; y \right) := \sum_{n \in \mathbb{Z}_{\geq 0}} \frac{\theta(t_0, t_1, \dots, t_s)_n}{\theta(q, w_1, \dots, w_s)_n} y^n. \quad (17)$$

For fixed  $t_j$  and  $w_j$ , the function  ${}_{s+1}E_s$  degenerates in the limit  $p \rightarrow 0$  to the basic  $q$ -hypergeometric series  ${}_{s+1}\varphi_s$  satisfying the condition  $\prod_{k=0}^s t_s = q \prod_{k=1}^s w_s$ . There are some problems with the convergence of the infinite series (17), and we therefore assume its termination due to the condition  $t_k = q^{-N} p^M$  for some  $k$  and  $N \in \mathbb{Z}_{\geq 0}$ ,  $M \in \mathbb{Z}$ . The additive system of notation for these series (see, e.g., Ch. 11 in [31] or [66]) is more convenient for consideration of certain questions, but we skip it here.

The series (17) is called well-poised if  $t_0 q = t_1 w_1 = \dots = t_s w_s$ . In this case the balancing condition takes the form  $t_1 \cdots t_s = \pm q^{(s+1)/2} t_0^{(s-1)/2}$ , and the functions  $h(u)$  and  ${}_{s+1}E_s$  become invariant under the changes  $t_j \rightarrow pt_j$ ,  $j = 1, \dots, s-1$ , and  $t_0 \rightarrow p^2 t_0$ . For odd  $s$  and balancing condition of the form  $t_1 \cdots t_s = +q^{(s+1)/2} t_0^{(s-1)/2}$ , one has the symmetry  $t_0 \rightarrow pt_0$  and  ${}_{s+1}E_s$  becomes an elliptic function of all free parameters  $\log t_j$ ,  $j = 0, \dots, s-1$ , with equal periods (such functions were called in [62, 66] totally elliptic functions). Under the four additional constraints  $t_{s-3} = q\sqrt{t_0}$ ,  $t_{s-2} = -q\sqrt{t_0}$ ,  $t_{s-1} = q\sqrt{t_0/p}$ ,  $t_s = -q\sqrt{pt_0}$ , connected to doubling of the argument of theta functions, the series are called very-well-poised. In [63], a special notation was introduced for the very-well-poised

elliptic hypergeometric series:

$$\begin{aligned} {}_{s+1}E_s & \left( \begin{matrix} t_0, t_1, \dots, t_{s-4}, q\sqrt{t_0}, -q\sqrt{t_0}, q\sqrt{t_0/p}, -q\sqrt{pt_0} \\ qt_0/t_1, \dots, qt_0/t_{s-4}, \sqrt{t_0}, -\sqrt{t_0}, \sqrt{pt_0}, -\sqrt{t_0/p} \end{matrix}; q, p; -y \right) \\ & = \sum_{n=0}^{\infty} \frac{\theta(t_0 q^{2n}; p)}{\theta(t_0; p)} \prod_{m=0}^{s-4} \frac{\theta(t_m)_n}{\theta(qt_0 t_m^{-1})_n} (qy)^n =: {}_{s+1}V_s(t_0; t_1, \dots, t_{s-4}; q, p; y), \end{aligned} \quad (18)$$

where the balancing condition has the form  $\prod_{k=1}^{s-4} t_k = \pm t_0^{(s-5)/2} q^{(s-7)/2}$ , and for odd  $s$  we assume the positive sign choice for preserving the symmetry  $t_0 \rightarrow pt_0$ . If  $y = 1$ , then  $y$  is omitted in the series notation. Summation formula (15) gives thus a closed form expression for the terminating  ${}_{10}V_9(t_0; t_1, \dots, t_5; q, p)$ -series.

A contour integral  $\int_C \Delta(u) du$  is called an elliptic hypergeometric integral if its kernel  $\Delta(u)$  satisfies the system of three equations

$$\Delta(u + \omega_k) = h_k(u) \Delta(u), \quad k = 1, 2, 3, \quad (19)$$

where  $\omega_{1,2,3} \in \mathbb{C}$  are some pairwise incommensurate parameters and  $h_k(u)$  are some elliptic functions with periods  $\omega_{k+1}, \omega_{k+2}$  (we set  $\omega_{k+3} = \omega_k$ ). One can weaken the requirement (19) by keeping only one equation, but then there appears a functional freedom in the choice of  $\Delta(u)$ , which should be fixed in some other way.

Omitting the details of such considerations from [64, 66], we present the general form of permissible functions  $\Delta(u)$ . We suppose that this function satisfies the equations (19) for  $k = 1, 2$ , where

$$h_1(u) = y_1 \prod_{j=1}^s \frac{\theta(t_j e^{2\pi i u/\omega_2}; p)}{\theta(w_j e^{2\pi i u/\omega_2}; p)}, \quad h_2(u) = y_2 \prod_{j=1}^{\ell} \frac{\theta(\tilde{t}_j e^{-2\pi i u/\omega_1}; r)}{\theta(\tilde{w}_j e^{-2\pi i u/\omega_1}; r)},$$

$|p|, |r| < 1$  and  $\prod_{j=1}^s t_j = \prod_{j=1}^s w_j$ ,  $\prod_{j=1}^{\ell} \tilde{t}_j = \prod_{j=1}^{\ell} \tilde{w}_j$ . If we take  $|q| < 1$ , then the most general meromorphic  $\Delta(u)$  has the form

$$\Delta(u) = \prod_{j=1}^s \frac{\Gamma(t_j e^{2\pi i \frac{u}{\omega_2}}; p, q)}{\Gamma(w_j e^{2\pi i \frac{u}{\omega_2}}; p, q)} \prod_{j=1}^{\ell} \frac{\Gamma(\tilde{t}_j e^{-2\pi i \frac{u}{\omega_1}}; \tilde{q}, r)}{\Gamma(\tilde{w}_j e^{-2\pi i \frac{u}{\omega_1}}; \tilde{q}, r)} \prod_{k=1}^m \frac{\theta(a_k e^{2\pi i \frac{u}{\omega_2}}; q)}{\theta(b_k e^{2\pi i \frac{u}{\omega_2}}; q)} e^{cu+d}, \quad (20)$$

where the parameters  $d \in \mathbb{C}$  and  $m \in \mathbb{Z}_{\geq 0}$  are arbitrary, and  $a_k, b_k, c$  are connected with  $y_1$  and  $y_2$  by the relations  $y_2 = e^{c\omega_2}$  and  $y_1 = e^{c\omega_1} \prod_{k=1}^m b_k a_k^{-1}$ . It appears that the function  $h_3(u)$  cannot be arbitrary – it is determined from expression (20).

For  $|q| = 1$  it is necessary to choose  $\ell = s$  in formula (20) and fix parameters in such a way that the  $\Gamma$ -functions are combined to the modified elliptic gamma function  $G(u; \omega)$  (it is precisely in this way that this function was built in [64]):

$$\Delta(u) = \prod_{j=1}^s \frac{G(u + g_j; \omega)}{G(u + v_j; \omega)} e^{cu+d}, \quad (21)$$

where the parameters  $g_j, v_j$  are connected to  $t_j, w_j$  by the relations  $t_j = e^{2\pi i g_j/\omega_2}$ ,  $w_j = e^{2\pi i v_j/\omega_2}$ , and  $y_{1,2} = e^{c\omega_{1,2}}$ . The integrals  $\int_C \Delta(u) du$  with kernels of the

indicated form define elliptic analogues of the Meijer function. For even more general theta hypergeometric integrals, see [64].

We limit consideration to the case when both  $\ell$  and  $m$  in (20) are equal to zero. The corresponding integrals are called well-poised, if  $t_1 w_1 = \dots = t_s w_s = pq$ . The additional condition of very-well-poisedness fixes eight parameters  $t_{s-7}, \dots, t_s = \{\pm(pq)^{1/2}, \pm q^{1/2}p, \pm p^{1/2}q, \pm pq\}$  and doubles the argument of the elliptic gamma function:  $\prod_{j=s-7}^s \Gamma(t_j z; p, q) = 1/\Gamma(z^{-2}; p, q)$ . The most interesting are the very-well-poised elliptic hypergeometric integrals with even number of parameters

$$I^{(m)}(t_1, \dots, t_{2m+6}) = \kappa \int_{\mathbb{T}} \frac{\prod_{j=1}^{2m+6} \Gamma(t_j z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z}, \quad \prod_{j=1}^{2m+6} t_j = (pq)^{m+1}, \quad (22)$$

with  $|t_j| < 1$  and “correct” choice of the sign in the balancing condition. They represent integral analogues of the  $s+1 V_s$ -series with odd  $s$ , “correct” balancing condition and the argument  $y = 1$ , in the sense that such series appear as residue sums of particular pole sequences of the kernel of  $I^{(m)}$ . Note that  $I^{(0)}$  coincides with the elliptic beta integral.

Properties of the elliptic functions explain the origins of hypergeometric notions of balancing, well-poisedness, and very-well-poisedness. However, strictly speaking these notions are consistently defined only at the elliptic level, because there are limits to such  $q$ -hypergeometric identities in which they are not preserved any more [45, 63, 84]! The fact of unique determination of the balancing condition for series (18) with odd  $s$  and integrals (22) (precisely these objects emerge in interesting applications) illustrates a deep internal tie between the “elliptic” and “hypergeometric” classes of special functions. Multivariable elliptic hypergeometric series and integrals are defined analogously to the univariate case – it is necessary to use systems of finite difference equations for kernels with the coefficients given by elliptic functions of all summation or integration variables [62, 64], which is a natural generalization of the approach of Pochhammer and Horn to functions of hypergeometric type [2, 32].

**An elliptic analogue of the Euler-Gauss hypergeometric function.** Take eight parameters  $t_1, \dots, t_8 \in \mathbb{C}$  and two base variables  $p, q \in \mathbb{C}$  satisfying the constraints  $|p|, |q| < 1$  and  $\prod_{j=1}^8 t_j = p^2 q^2$  (the balancing condition). For all  $|t_j| < 1$  an elliptic analogue of the Euler-Gauss hypergeometric function  ${}_2F_1(a, b; c; x)$  (see Ch. 2 in [2]) is defined by the integral [66]

$$V(t) \equiv V(t_1, \dots, t_8; p, q) := \kappa \int_{\mathbb{T}} \frac{\prod_{j=1}^8 \Gamma(t_j z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z}, \quad (23)$$

i.e. by the choice  $m = 1$  in expression (22). Note that it can be reduced to both Euler and Barnes type integral representations of  ${}_2F_1$ -series. For other admissible values of parameters, the  $V$ -function is defined by the analytical continuation of expression (23). From this continuation one can see that the  $V$ -function is meromorphic for all values of parameters  $t_j \in \mathbb{C}^*$  when the contour of integration is not pinched.

To see this, compute residues of the integrand poles and define the analytically continued function as a sum of the integral over some fixed contour and residues of the poles crossing this contour. More precisely,  $\prod_{1 \leq j < k \leq 8} (t_j t_k; p, q)_\infty V(\underline{t})$  is a holomorphic function of parameters [44]. As shown in [69], the  $V$ -function has delta-function type singularities at certain values of  $t_j$ 's.

The first nontrivial property of function (23) consists in its reduction to the elliptic beta integral under the condition for a pair of parameters  $t_j t_k = pq$ ,  $j \neq k$  (expression (13) appears from  $t_7 t_8 = pq$ ). The  $V$ -function is evidently symmetric in  $p$  and  $q$ . It is invariant also under the  $S_8$ -group of permutations of parameters  $t_j$  isomorphic to the Weyl group  $A_7$ . Consider the double integral

$$\kappa \int_{\mathbb{T}^2} \frac{\prod_{j=1}^4 \Gamma(a_j z^{\pm 1}, b_j w^{\pm 1}; p, q) \Gamma(c z^{\pm 1} w^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}, w^{\pm 2}; p, q)} \frac{dz}{z} \frac{dw}{w},$$

where  $a_j, b_j, c \in \mathbb{C}$ ,  $|a_j|, |b_j|, |c| < 1$ , and  $c^2 \prod_{j=1}^4 a_j = c^2 \prod_{j=1}^4 b_j = pq$ . Using formula (13) for integration over  $z$  or  $w$  (the permutation of the order of integrations is permitted), we obtain the following transformation formula:

$$V(\underline{t}) = \prod_{1 \leq j < k \leq 4} \Gamma(t_j t_k, t_{j+4} t_{k+4}; p, q) V(\underline{s}), \quad (24)$$

where  $|t_j|, |s_j| < 1$ , and

$$\begin{cases} s_j = \rho^{-1} t_j, & j = 1, 2, 3, 4 \\ s_j = \rho t_j, & j = 5, 6, 7, 8 \end{cases} ; \quad \rho = \sqrt{\frac{t_1 t_2 t_3 t_4}{pq}} = \sqrt{\frac{pq}{t_5 t_6 t_7 t_8}}.$$

This fundamental relation was derived by the author in [64], where the function  $V(\underline{t})$  appeared for the first time. It represents an elliptic analogue (moreover, integral generalization) of Bailey's transformation for four non-terminating  ${}_{10}\varphi_9$ -series [31].

Repeat transformation (24) once more with the parameters  $s_{3,4,5,6}$ , playing the role of  $t_{1,2,3,4}$ , and permute parameters  $t_3, t_4$  with  $t_5, t_6$  in the resulting expression. This yields the relation

$$V(\underline{t}) = \prod_{j,k=1}^4 \Gamma(t_j t_{k+4}; p, q) V(T^{\frac{1}{2}}/t_1, \dots, T^{\frac{1}{2}}/t_4, U^{\frac{1}{2}}/t_5, \dots, U^{\frac{1}{2}}/t_8), \quad (25)$$

where  $T = t_1 t_2 t_3 t_4$ ,  $U = t_5 t_6 t_7 t_8$  and  $|T|^{1/2} < |t_j| < 1$ ,  $|U|^{1/2} < |t_{j+4}| < 1$ ,  $j = 1, 2, 3, 4$ . Now equating the right-hand sides of relations (24) and (25), and expressing parameters  $t_j$  in terms of  $s_j$ , one obtains the third relation

$$V(\underline{s}) = \prod_{1 \leq j < k \leq 8} \Gamma(s_j s_k; p, q) V(\sqrt{pq}/s_1, \dots, \sqrt{pq}/s_8), \quad (26)$$

where  $|pq|^{1/2} < |s_j| < 1$  for all  $j$ .

Consider the Euclidean space  $\mathbb{R}^8$  with the scalar product  $\langle x, y \rangle$  and an orthonormal basis  $e_i \in \mathbb{R}^8$ ,  $\langle e_i, e_j \rangle = \delta_{ij}$ . The root system  $A_7$  consists of the vectors  $v = \{e_i - e_j, i \neq j\}$ . Its Weyl group consists of the reflections  $x \rightarrow S_v(x) = x - 2v\langle v, x \rangle / \langle v, v \rangle$  acting in the hyperplane orthogonal to the vector  $\sum_{i=1}^8 e_i$  (i.e.,

the coordinates of the vectors  $x = \sum_{i=1}^8 x_i e_i$  satisfy the constraint  $\sum_{i=1}^8 x_i = 0$ , and it coincides with the permutation group  $S_8$ .

Connect parameters of the  $V(\underline{t})$ -function to the coordinates  $x_j$  as  $t_j = e^{2\pi i x_j} (pq)^{1/4}$ . Then the balancing condition assumes the form  $\sum_{i=1}^8 x_i = 0$ . The first  $V$ -function transformation (24) is now easily seen to correspond to the reflection  $S_v(x)$  for the vector  $v = (\sum_{i=5}^8 e_i - \sum_{i=1}^4 e_i)/2$  having the canonical length  $\langle v, v \rangle = 2$ . This reflection extends the group  $A_7$  to the exceptional Weyl group  $E_7$ . Relations (25) and (26) were proved in a different fashion by Rains in [44], where it was indicated that these transformations belong to the group  $E_7$ .

Denote by  $V(qt_j, q^{-1}t_k)$  elliptic hypergeometric functions contiguous to  $V(\underline{t})$  in the sense that  $t_j$  and  $t_k$  are replaced by  $qt_j$  and  $q^{-1}t_k$ , respectively. The following contiguous relation for the  $V$ -functions is valid

$$t_7 \theta(t_8 t_7^{\pm 1}/q; p) V(qt_6, q^{-1}t_8) - (t_6 \leftrightarrow t_7) = t_7 \theta(t_6 t_7^{\pm 1}; p) V(\underline{t}), \quad (27)$$

where  $(t_6 \leftrightarrow t_7)$  denotes the permutation of parameters in the preceding expression (such a relation was used already in [61]). Indeed, for  $y = t_6$ ,  $w = t_7$ , and  $x = q^{-1}t_8$  the Riemann relation (6) is equivalent to the  $q$ -difference equation for  $V$ -function's integrand  $\Delta(z, \underline{t}) = \prod_{k=1}^8 \Gamma(t_k z^{\pm 1}; p, q)/\Gamma(z^{\pm 2}; p, q)$  coinciding with (27) after replacement of  $V$ -functions by  $\Delta(z, \underline{t})$  with appropriate parameters. Integration of this equation over the contour  $\mathbb{T}$  yields formula (27). Substitute now the symmetry transformation (26) in (27) and obtain the second contiguous relation

$$t_6 \theta\left(\frac{t_7}{qt_8}; p\right) \prod_{k=1}^5 \theta\left(\frac{t_6 t_k}{q}; p\right) V(q^{-1}t_6, qt_8) - (t_6 \leftrightarrow t_7) = t_6 \theta\left(\frac{t_7}{t_6}; p\right) \prod_{k=1}^5 \theta(t_8 t_k; p) V(\underline{t}).$$

An appropriate combination of these two equalities yields the equation

$$\mathcal{A}(\underline{t}) \left( U(qt_6, q^{-1}t_7) - U(\underline{t}) \right) + (t_6 \leftrightarrow t_7) + U(\underline{t}) = 0, \quad (28)$$

where we have denoted  $U(\underline{t}) = V(\underline{t})/\Gamma(t_6 t_8^{\pm 1}, t_7 t_8^{\pm 1}; p, q)$  and

$$\mathcal{A}(\underline{t}) = \frac{\theta(t_6/qt_8, t_6 t_8, t_8/t_6; p)}{\theta(t_6/t_7, t_7/qt_6, t_6 t_7/q; p)} \prod_{k=1}^5 \frac{\theta(t_7 t_k/q; p)}{\theta(t_8 t_k; p)}. \quad (29)$$

Substituting  $t_j = e^{2\pi i g_j/\omega_2}$ , one can check that the potential  $\mathcal{A}(\underline{t})$  is a modular invariant elliptic function of the variables  $g_1, \dots, g_7$ , i.e. it does not change after the replacements  $g_j \rightarrow g_j + \omega_{2,3}$  or  $(\omega_2, \omega_3) \rightarrow (-\omega_3, \omega_2)$ .

Now denote  $t_6 = cx$ ,  $t_7 = c/x$ , and introduce new variables

$$\varepsilon_k = \frac{q}{ct_k}, \quad k = 1, \dots, 5, \quad \varepsilon_8 = \frac{c}{t_8}, \quad \varepsilon_7 = \frac{\varepsilon_8}{q}, \quad c = \frac{\sqrt{\varepsilon_6 \varepsilon_8}}{p^2}.$$

In terms of  $\varepsilon_k$  the balancing condition takes the standard form  $\prod_{k=1}^8 \varepsilon_k = p^2 q^2$ . After the replacement of  $U(\underline{t})$  in formula (28) by some unknown function  $f(x)$ ,

we obtain a  $q$ -difference equation of the second order which is called *the elliptic hypergeometric equation*:

$$A(x)(f(qx) - f(x)) + A(x^{-1})(f(q^{-1}x) - f(x)) + \nu f(x) = 0, \quad (30)$$

$$A(x) = \frac{\prod_{k=1}^8 \theta(\varepsilon_k x; p)}{\theta(x^2, qx^2; p)}, \quad \nu = \prod_{k=1}^6 \theta\left(\frac{\varepsilon_k \varepsilon_8}{q}; p\right). \quad (31)$$

We have already one functional solution of this equation

$$f_1(x) = \frac{V(q/c\varepsilon_1, \dots, q/c\varepsilon_5, cx, c/x, c/\varepsilon_8; p, q)}{\Gamma(c^2 x^{\pm 1}/\varepsilon_8, x^{\pm 1}\varepsilon_8; p, q)}, \quad (32)$$

where it is necessary to impose the constraints (in the previous parametrization)  $\sqrt{|pq|} < |t_j| < 1$ ,  $j = 1, \dots, 5$ , and  $\sqrt{|pq|} < |q^{\pm 1}t_6|, |q^{\pm 1}t_7|, |q^{\pm 1}t_8| < 1$ , which can be relaxed by analytical continuation. Other independent solutions can be obtained by the multiplication of one of the parameters  $\varepsilon_1, \dots, \varepsilon_5$ , and  $x$  by powers of  $p$  or by permutations of  $\varepsilon_1, \dots, \varepsilon_5$  with  $\varepsilon_6$ .

Denote  $\varepsilon_k = e^{2\pi i a_k / \omega_2}$ ,  $x = e^{2\pi i u / \omega_2}$ , and  $F_1(u; \underline{a}; \omega_1, \omega_2, \omega_3) := f_1(x)$ . Then one can check that equation (30) is invariant with respect to the modular transformation  $(\omega_2, \omega_3) \rightarrow (-\omega_3, \omega_2)$ . Therefore one of the linear independent solutions of (30) has the form  $F_2(u; \underline{a}; \omega_1, \omega_2, \omega_3) := F_1(u; \underline{a}; \omega_1, -\omega_3, \omega_2)$ . The same solution would be obtained if we repeat the derivation of equation (30) and its solution (32) after replacing  $\Gamma$ -functions by the modified elliptic gamma function  $G(u; \omega)$ . This shows that  $F_2$ -function is well defined even for  $|q| = 1$ . Different limiting transitions from the  $V$ -function and other elliptic hypergeometric integrals to  $q$ -hypergeometric integrals of the Mellin-Barnes or Euler type are described in [66, 68] and much more systematically in [11, 15, 17, 47].

**Biorthogonal functions of the hypergeometric type.** In analogy with the residue calculus for the elliptic beta integral (14), one can consider the sum of residues for a particular geometric progression of poles of the  $V$ -function kernel for one of the parameters. This leads to the very-well-poised  ${}_{12}V_{11}$ -elliptic hypergeometric series the termination of which is guaranteed by a special discretization of the chosen parameter. In this way one can rederive contiguous relations for the terminating  ${}_{12}V_{11}$ -series of [78, 79] out of the contiguous relations for the  $V$ -function, which we omit here. For instance, this yields the following particular solution of the elliptic hypergeometric equation (30):

$$R_n(x; q, p) = {}_{12}V_{11}\left(\frac{\varepsilon_6}{\varepsilon_8}; \frac{q}{\varepsilon_1 \varepsilon_8}, \frac{q}{\varepsilon_2 \varepsilon_8}, \frac{q}{\varepsilon_3 \varepsilon_8}, \frac{qp}{\varepsilon_4 \varepsilon_8}, \frac{qp}{\varepsilon_5 \varepsilon_8}, \varepsilon_6 x, \frac{\varepsilon_6}{x}; q, p\right), \quad (33)$$

where  $pq/\varepsilon_4 \varepsilon_8 = q^{-n}$ ,  $n \in \mathbb{Z}_{\geq 0}$  (we recall that  $\prod_{k=1}^8 \varepsilon_k = p^2 q^2$ ). Properties of the  $R_n$ -function were described in [64], whose notation passes to ours after the replacements  $t_{0,1,2} \rightarrow \varepsilon_{1,2,3}$ ,  $t_3 \rightarrow \varepsilon_6$ ,  $t_4 \rightarrow \varepsilon_8$ ,  $\mu \rightarrow \varepsilon_4 \varepsilon_8 / pq$ , and  $A\mu / qt_4 \rightarrow pq / \varepsilon_5 \varepsilon_8$ .

Equation (30) is symmetric in  $\varepsilon_1, \dots, \varepsilon_6$ . The series (18) is elliptic in all parameters, therefore function (33) is symmetric in  $\varepsilon_1, \dots, \varepsilon_5$  and each of these variables can be used for terminating the series. A permutation of  $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_5$  with  $\varepsilon_6$  yields

$R_n(z; q, p)$  up to some multiplier independent on  $x$  due to an elliptic analogue of the Bailey transformation for terminating  ${}_{12}V_{11}$ -series [28], which can be obtained by degeneration from equality (24).

The same contiguous relations for the  ${}_{12}V_{11}$ -series yield the following three term recurrence relation for  $R_n(x; q, p)$  in the index  $n$ :

$$(z(x) - \alpha_{n+1})\rho(Aq^{n-1}/\varepsilon_8)(R_{n+1}(x; q, p) - R_n(x; q, p)) + (z(x) - \beta_{n-1}) \\ \times \rho(q^{-n})(R_{n-1}(x; q, p) - R_n(x; q, p)) + \delta(z(x) - z(\varepsilon_6))R_n(x; q, p) = 0, \quad (34)$$

where

$$z(x) = \frac{\theta(x\xi^{\pm 1}; p)}{\theta(x\eta^{\pm 1}; p)}, \quad \alpha_n = z(q^n/\varepsilon_8), \quad \beta_n = z(Aq^{n-1}), \\ \rho(t) = \frac{\theta\left(t, \frac{\varepsilon_6}{\varepsilon_8 t}, \frac{q\varepsilon_6}{\varepsilon_8 t}, \frac{qt}{\varepsilon_1 \varepsilon_2}, \frac{qt}{\varepsilon_2 \varepsilon_3}, \frac{qt}{\varepsilon_1 \varepsilon_3}, \frac{q^2 t \eta^{\pm 1}}{A}; p\right)}{\theta\left(\frac{q t^2 \varepsilon_8}{A}, \frac{q^2 t^2 \varepsilon_8}{A}; p\right)}, \\ \delta = \theta\left(\frac{q^2 \varepsilon_6}{A}, \frac{q}{\varepsilon_1 \varepsilon_8}, \frac{q}{\varepsilon_2 \varepsilon_8}, \frac{q}{\varepsilon_3 \varepsilon_8}, \varepsilon_6 \eta^{\pm 1}; p\right).$$

Here  $A = \varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_6 \varepsilon_8$ , and  $\xi$  and  $\eta$  are arbitrary gauge parameters,  $\xi \neq \eta^{\pm 1} p^k$ ,  $k \in \mathbb{Z}$ . The initial conditions  $R_{-1} = 0$  and  $R_0 = 1$  guarantee that all the dependence on the variable  $x$  enters only through  $z(x)$ , and that  $R_n(x)$  is a rational function of  $z(x)$  with poles at the points  $\alpha_1, \dots, \alpha_n$ .

The elliptic hypergeometric equation for the  $R_n$ -function can be rewritten in the form of a generalized eigenvalue problem  $\mathcal{D}_1 R_n = \lambda_n \mathcal{D}_2 R_n$  for some  $q$ -difference operators of the second order  $\mathcal{D}_{1,2}$  and discrete spectrum  $\lambda_n$  [64]. We denote by  $\phi_\lambda$  solutions of an abstract spectral problem  $\mathcal{D}_1 \phi_\lambda = \lambda \mathcal{D}_2 \phi_\lambda$ , and by  $\psi_\lambda$  solutions of the equation  $\mathcal{D}_1^T \psi_\lambda = \lambda \mathcal{D}_2^T \psi_\lambda$ , where  $\mathcal{D}_{1,2}^T$  are the operators conjugated with respect to some inner product  $\langle \psi | \phi \rangle$ , i.e.  $\langle \mathcal{D}_{1,2}^T \psi | \phi \rangle = \langle \psi | \mathcal{D}_{1,2} \phi \rangle$ . Then  $0 = \langle \psi_\mu | (\mathcal{D}_1 - \lambda \mathcal{D}_2) \phi_\lambda \rangle = (\mu - \lambda) \langle \mathcal{D}_2^T \psi_\mu | \phi_\lambda \rangle$ , i.e. the function  $\mathcal{D}_2^T \psi_\mu$  is orthogonal to  $\phi_\lambda$  for  $\mu \neq \lambda$ . As shown by Zhedanov [86] (see also [78, 79]), this simple fact can be used for a formulation of the theory of biorthogonal rational functions generalizing orthogonal polynomials. Analogues of the functions  $\mathcal{D}_2^T \psi_\mu$  for  $R_n(z; q, p)$  have the form

$$T_n(x; q, p) = {}_{12}V_{11}\left(\frac{A\varepsilon_6}{q}; \frac{A}{\varepsilon_1}, \frac{A}{\varepsilon_2}, \frac{A}{\varepsilon_3}, \varepsilon_6 x, \frac{\varepsilon_6}{x}, \frac{qp}{\varepsilon_4 \varepsilon_8}, \frac{qp}{\varepsilon_5 \varepsilon_8}; q, p\right), \quad (35)$$

which are rational functions of  $z(x)$  with poles at the points  $\beta_1, \dots, \beta_n$ .

Denote  $R_{nm}(x) := R_n(x; q, p)R_m(x; p, q)$  and  $T_{nm}(x) := T_n(x; q, p)T_m(x; p, q)$ , where all the  ${}_{12}V_{11}$ -series terminate simultaneously because of the modified termination condition  $\varepsilon_4 \varepsilon_8 = p^{m+1} q^{n+1}$ ,  $n, m \in \mathbb{Z}_{\geq 0}$ . The functions  $R_{nm}$  now solve not one but two generalized eigenvalue problems which differ from each other by the permutation of  $p$  and  $q$ .

**Theorem 2.** *The following two-index biorthogonality relation is true:*

$$\kappa \int_{C_{mn,kl}} T_{nl}(x) R_{mk}(x) \frac{\prod_{j \in S} \Gamma(\varepsilon_j x^{\pm 1}; p, q)}{\Gamma(x^{\pm 2}, Ax^{\pm 1}; p, q)} \frac{dx}{x} = h_{nl} \delta_{mn} \delta_{kl}, \quad (36)$$

where  $S = \{1, 2, 3, 6, 8\}$ ,  $C_{mn,kl}$  denotes the contour separating sequences of points  $x = \varepsilon_j p^a q^b$  ( $j = 1, 2, 3, 6$ ),  $\varepsilon_8 p^{a-k} q^{b-m}, p^{a+1-l} q^{b+1-n}/A$ ,  $a, b \in \mathbb{Z}_{\geq 0}$ , from their  $x \rightarrow x^{-1}$  reciprocals, and the normalization constants have the form

$$\begin{aligned} h_{nl} &= \frac{\prod_{j < k, j, k \in S} \Gamma(\varepsilon_j \varepsilon_k; p, q)}{\prod_{j \in S} \Gamma(A\varepsilon_j^{-1}; p, q)} h_n(q, p) \cdot h_l(p, q), \\ h_n(q, p) &= \frac{\theta(A/q\varepsilon_8; p)\theta(q, q\varepsilon_6/\varepsilon_8, \varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_3, \varepsilon_2\varepsilon_3, A\varepsilon_6)_n q^{-n}}{\theta(Aq^{2n}/q\varepsilon_8; p)\theta(1/\varepsilon_6\varepsilon_8, \varepsilon_1\varepsilon_6, \varepsilon_2\varepsilon_6, \varepsilon_3\varepsilon_6, A/q\varepsilon_6, A/q\varepsilon_8)_n}. \end{aligned}$$

This theorem was proved in [64] by direct computation of the integral in the left-hand side with the help of formula (13). The appearance of the two-index orthogonality relations for functions of one variable is a new phenomenon in the theory of special functions. It should be remarked that only for  $k = l = 0$  there exists the limit  $p \rightarrow 0$  and the resulting functions  $R_n(x; q, 0)$ ,  $T_n(x; q, 0)$  coincide with Rahman's family of continuous  ${}_{10}\varphi_9$ -biorthogonal rational functions [43]. A special limit  $\text{Im}(\omega_3) \rightarrow \infty$  in the modular transformed  $R_{nm}$  and  $T_{nm}$  leads to the two-index biorthogonal functions which are expressed as products of two modular conjugated  ${}_{10}\varphi_9$ -series [66]. A special restriction for one of the parameters in  $R_n(x; q, p)$  and  $T_n(x; q, p)$  leads to the biorthogonal rational functions of a discrete argument derived by Zhedanov and the author in [78] which generalizes Wilson's functions [85]. All these functions are natural generalizations of the Askey-Wilson polynomials [4].

Note that  $R_{nm}(x)$  and  $T_{nm}(x)$  are meromorphic functions of the variable  $x \in \mathbb{C}^*$  with essential singularities at  $x = 0, \infty$  and only for  $k = l = 0$  or  $n = m = 0$  do they become rational functions of some argument depending on  $x$ . The continuous parameters biorthogonality relation for the  $V$ -function itself was established in [69]. The biorthogonal functions generated by the three-term recurrence relation (34) after shifting  $n$  by an arbitrary (complex) number are not investigated yet. A generalization of the described "classical" biorthogonal functions to the "semiclassical" level associated with the higher order elliptic beta integrals (22) was suggested by Rains in [49].

**Elliptic beta integrals on root systems.** Define a  $C_n$  (or  $BC_n$ ) root system analogue of the constant  $\kappa$ :  $\kappa_n = (p; p)_\infty^n (q; q)_\infty^n / (2\pi i)^n 2^n n!$ . Describe now a  $C_n$ -elliptic beta integral representing a multiparameter generalization of integral (13), which was classified in [21] as an integral of type I.

**Theorem 3.** Take  $n$  variables  $z_1, \dots, z_n \in \mathbb{T}$  and complex parameters  $t_1, \dots, t_{2n+4}$  and  $p, q$  satisfying the constraints  $|p|, |q|, |t_j| < 1$  and  $\prod_{j=1}^{2n+4} t_j = pq$ . Then

$$\begin{aligned} \kappa_n \int_{\mathbb{T}^n} \prod_{1 \leq j < k \leq n} \frac{1}{\Gamma(z_j^{\pm 1} z_k^{\pm 1}; p, q)} \prod_{j=1}^n \frac{\prod_{m=1}^{2n+4} \Gamma(t_m z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \frac{dz_1}{z_1} \dots \frac{dz_n}{z_n} \\ = \prod_{1 \leq m < s \leq 2n+4} \Gamma(t_m t_s; p, q). \end{aligned} \quad (37)$$

Formula (37) was suggested and partially confirmed by van Diejen and the author in [21]. It was proved by different methods in [44, 50, 66, 67]. It reduces to one of Gustafson's integration formulas [33] in a special  $p \rightarrow 0$  limit.

**Theorem 4.** Take complex parameters  $t, t_1, \dots, t_6, p$  and  $q$  restricted by the conditions  $|p|, |q|, |t|, |t_m| < 1$  and  $t^{2n-2} \prod_{m=1}^6 t_m = pq$ . Then,

$$\begin{aligned} \kappa_n \int_{\mathbb{T}^n} \prod_{1 \leq j < k \leq n} \frac{\Gamma(t z_j^{\pm 1} z_k^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 1} z_k^{\pm 1}; p, q)} \prod_{j=1}^n \frac{\prod_{m=1}^6 \Gamma(t_m z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \frac{dz_1}{z_1} \dots \frac{dz_n}{z_n} \\ = \prod_{j=1}^n \left( \frac{\Gamma(t^j; p, q)}{\Gamma(t; p, q)} \prod_{1 \leq m < s \leq 6} \Gamma(t^{j-1} t_m t_s; p, q) \right). \end{aligned} \quad (38)$$

In order to prove formula (38), consider the following  $(2n-1)$ -tuple integral

$$\begin{aligned} \kappa_n \kappa_{n-1} \int_{\mathbb{T}^{2n-1}} \prod_{1 \leq j < k \leq n} \frac{1}{\Gamma(z_j^{\pm 1} z_k^{\pm 1}; p, q)} \prod_{j=1}^n \frac{\prod_{r=0}^5 \Gamma(t_r z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \\ \times \prod_{\substack{1 \leq j \leq n \\ 1 \leq k \leq n-1}} \frac{1}{\Gamma(t^{1/2} z_j^{\pm 1} w_k^{\pm 1}; p, q)} \prod_{1 \leq j < k \leq n-1} \frac{1}{\Gamma(w_j^{\pm 1} w_k^{\pm 1}; p, q)} \\ \times \prod_{j=1}^{n-1} \frac{\Gamma(w_j^{\pm 1} t^{n-3/2} \prod_{s=1}^5 t_s; p, q)}{\Gamma(w_j^{\pm 2}, w_j^{\pm 1} t^{2n-3/2} \prod_{s=1}^5 t_s; p, q)} \frac{dw_1}{w_1} \dots \frac{dw_{n-1}}{w_{n-1}} \frac{dz_1}{z_1} \dots \frac{dz_n}{z_n}, \end{aligned} \quad (39)$$

with the parameters  $p, q, t$  and  $t_r, r = 0, \dots, 5$ , lying inside the unit circle and such that  $t^{n-1} \prod_{r=0}^5 t_r = pq$ . Denote the integral in the left-hand side of equality (38) by  $I_n(t, t_1, \dots, t_5; p, q)$ . Integration over the variables  $w_j$  with the help of formula (37) brings expression (39) to the form  $\Gamma^n(t) I_n(t, t_1, \dots, t_5; p, q) / \Gamma(t^n)$  (after denoting  $t_6 = pq / t^{2n-2} \prod_{j=1}^5 t_j$ ). Because the integrand is bounded on the integration contour, we can change the order of integrations. As a result, integration over the variables  $z_j$  with the help of formula (37) brings expression (39) in the form  $\Gamma^{n-1}(t) \prod_{0 \leq r < s \leq 5} \Gamma(t_r t_s) I_{n-1}(t, t^{1/2} t_1, \dots, t^{1/2} t_5; p, q)$ , i.e. we obtain the following recurrence relation in the dimensionality of the integral of interest  $n$ :

$$I_n(t, t_1, \dots, t_5; p, q) = \frac{\Gamma(t^n; p, q)}{\Gamma(t; p, q)} \prod_{0 \leq r < s \leq 5} \Gamma(t_r t_s; p, q) I_{n-1}(t, t^{1/2} t_1, \dots, t^{1/2} t_5; p, q).$$

Iterating it with known initial condition (13) for  $n = 1$ , one obtains formula (38).

Integral (38) was constructed by van Diejen and the author in [20] and classified as of type II in [21] where from the described proof is taken. This proof models Anderson's derivation of the Selberg integral described in [2] (see Theorem 8.1.1 and Sect. 8.4). It also represents a direct generalization of Gustafson's method [33] of derivation of the multiple  $q$ -beta integral obtained from formula (38) after expressing  $t_6$  via other parameters, removing the multipliers  $pq$  with the help of the reflection formula for  $\Gamma(z; p, q)$ , and taking the limit  $p \rightarrow 0$ . A number of further limits in parameters leads to the Selberg integral – one of the most important known integrals because of many applications in mathematical physics [27]. Therefore formula (38) represents an elliptic analogue of the Selberg integral (an analogous extension of Aomoto's integral described in Theorem 8.1.2 of [2] is derived in [44]). It can be interpreted also as an elliptic extension of the  $BC_n$  Macdonald-Morris constant term identities.

In analogy with the one dimensional case [64], it is natural to expect that the multiple elliptic beta integrals define measures in the biorthogonality relations for some functions of many variables generalizing relations (36). In [44, 45], Rains has constructed a system of such functions on the basis of integral (38). These functions generalize also the Macdonald and Koornwinder orthogonal polynomials, as well as the interpolating polynomials of Okounkov. For a related work see also [18]. A systematic investigation of the limiting cases of univariate and multiple elliptic biorthogonal functions is performed in [16]. In this sense, the results obtained in [44, 45] represent to the present moment the top level achievements of the theory of elliptic hypergeometric functions of many variables. In particular, the following  $BC_n$ -generalization of transformation (24) was proved in [44]:

$$I_n(t_1, \dots, t_8; t; q, p) = I_n(s_1, \dots, s_8; t; q, p), \quad (40)$$

where

$$\begin{aligned} I_n(t_1, \dots, t_8; t; q, p) &= \kappa_n \prod_{1 \leq j < k \leq 8} \Gamma(t_j t_k; p, q, t) \\ &\times \int_{\mathbb{T}^n} \prod_{1 \leq j < k \leq n} \frac{\Gamma(t z_j^{\pm 1} z_k^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 1} z_k^{\pm 1}; p, q)} \prod_{j=1}^n \frac{\prod_{k=1}^8 \Gamma(t_k z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \frac{dz_j}{z_j}, \\ &\left\{ \begin{array}{l} s_j = \rho^{-1} t_j, \quad j = 1, 2, 3, 4 \\ s_j = \rho t_j, \quad j = 5, 6, 7, 8 \end{array} \right. ; \quad \rho = \sqrt{\frac{t_1 t_2 t_3 t_4}{p q t^{1-n}}} = \sqrt{\frac{p q t^{1-n}}{t_5 t_6 t_7 t_8}}, \quad |t|, |t_j|, |s_j| < 1, \end{aligned}$$

and  $\Gamma(z; p, q, t) = \prod_{j,k,l=0}^{\infty} (1 - z t^j p^k q^l)(1 - z^{-1} t^{j+1} p^{k+1} q^{l+1})$  is the elliptic gamma function of the higher level connected to the Barnes gamma function  $\Gamma_4(u; \omega)$ . In [72], this symmetry transformation is represented in the star-star relation form of solvable models of statistical mechanics, and equality (38) is represented in the star-triangle relation form which used an elliptic gamma function of even higher order related to  $\Gamma_5(u; \omega)$ -function.

There are about 10 proven exact evaluations of elliptic beta integrals on root systems. In particular, in [64] the author has constructed three different integrals for the  $A_n$  root system (two of them have different evaluation formulas for even and odd values of  $n$ ). In [76], Warnaar and the author have found one more  $A_n$ -integral which appeared to be new even after degeneration to the  $q$ - and plain hypergeometric levels. Another  $BC_n$ -integral has been constructed in [13, 48]. Very many new multiple elliptic beta integrals and symmetry transformations for their higher order generalizations were conjectured in [74, 75].

Let us describe a generalization of the elliptic beta integral (38). Take 10 parameters  $p, q, t, s, t_j, s_j, j = 1, 2, 3$ , of modulus less than 1 such that  $(ts)^{n-1} \prod_{k=1}^3 t_k s_k = pq$  and define the  $A_n$ -integral

$$\begin{aligned} I_n(t_1, t_2, t_3; s_1, s_2, s_3; t; s; p, q) &= \frac{(p; p)_\infty^n (q; q)_\infty^n}{(n+1)! (2\pi i)^n} \\ &\times \int_{\mathbb{T}^n} \prod_{1 \leq i < j \leq n+1} \frac{\Gamma(tz_i z_j, sz_i^{-1} z_j^{-1}; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \prod_{j=1}^{n+1} \prod_{k=1}^3 \Gamma(t_k z_j, s_k z_j^{-1}; p, q) \prod_{j=1}^n \frac{dz_j}{z_j}. \end{aligned} \quad (41)$$

Then for odd  $n$  one has

$$\begin{aligned} I_n(t_1, t_2, t_3; s_1, s_2, s_3; t; s; p, q) &= \Gamma(t^{\frac{n+1}{2}}, s^{\frac{n+1}{2}}; p, q) \\ &\times \prod_{1 \leq i < k \leq 3} \Gamma(t^{\frac{n-1}{2}} t_i t_k, s^{\frac{n-1}{2}} s_i s_k; p, q) \prod_{j=1}^{(n+1)/2} \prod_{i,k=1}^3 \Gamma((ts)^{j-1} t_i s_k; p, q) \\ &\times \prod_{j=1}^{(n-1)/2} \left( \Gamma((ts)^j; p, q) \prod_{1 \leq i < k \leq 3} \Gamma(t^{j-1} s^j t_i t_k, t^j s^{j-1} s_i s_k; p, q) \right), \end{aligned} \quad (42)$$

and for even  $n$  one has

$$\begin{aligned} I_n(t_1, t_2, t_3; s_1, s_2, s_3; t; s; p, q) &= \prod_{i=1}^3 \Gamma(t^{\frac{n}{2}} t_i, s^{\frac{n}{2}} s_i; p, q) \\ &\times \Gamma(t^{\frac{n}{2}-1} t_1 t_2 t_3, s^{\frac{n}{2}-1} s_1 s_2 s_3; p, q) \prod_{j=1}^{n/2} \left( \Gamma((ts)^j; p, q) \right. \\ &\times \left. \prod_{i,k=1}^3 \Gamma((ts)^{j-1} t_i s_k; p, q) \prod_{1 \leq i < k \leq 3} \Gamma(t^{j-1} s^j t_i t_k, t^j s^{j-1} s_i s_k; p, q) \right). \end{aligned} \quad (43)$$

These  $A_n$ -elliptic beta integrals were discovered by the author in [64]. As indicated in [74], the limit  $s \rightarrow 1$  reduces the odd  $n$  evaluation formula (42) to (38), i.e. we have a generalization of the elliptic Selberg integral of [20, 21]. The observation that the type II  $BC_n$ -hypergeometric identities can be obtained from the type II relations for  $A_{2n-1}$  and  $A_{2n}$  root systems was first made in [77] at the level of multiple  $q$ -hypergeometric series. It was also suggested there that the multiple elliptic biorthogonal rational functions associated with elliptic beta integrals (42)

and (43), the existence of which was conjectured by the author long ago [64], should also generalize the Rains biorthogonal functions [44, 45] to  $A_n$  root system. The possibilities for constructing these functions were discussed in [73].

In [74], the following symmetry transformation was conjectured for a two parameter extension of the  $A_{2n-1}$ -integral (41):

$$\begin{aligned} & \int_{\mathbb{T}^{2n-1}} \prod_{1 \leq j < k \leq 2n} \frac{\Gamma(tz_j z_k, sz_j^{-1} z_k^{-1}; p, q)}{\Gamma(z_j^{-1} z_k, z_j z_k^{-1}; p, q)} \prod_{j=1}^{2n} \prod_{k=1}^4 \Gamma(t_k z_j, s_k z_j^{-1}; p, q) \prod_{j=1}^{2n-1} \frac{dz_j}{z_j} \\ &= \prod_{1 \leq i < j \leq 4} \left( \Gamma(s^{n-1} s_i s_j, t^{n-1} t_i t_j; p, q) \prod_{m=0}^{n-2} \Gamma(t(st)^m s_i s_j, s(st)^m t_i t_j; p, q) \right) \quad (44) \\ & \times \int_{\mathbb{T}^{2n-1}} \prod_{1 \leq j < k \leq 2n} \frac{\Gamma(sz_j z_k, tz_j^{-1} z_k^{-1}; p, q)}{\Gamma(z_j^{-1} z_k, z_j z_k^{-1}; p, q)} \prod_{j=1}^{2n} \prod_{k=1}^4 \Gamma\left(\sqrt[4]{\frac{S}{T}} t_k z_j, \sqrt[4]{\frac{T}{S}} s_k z_j^{-1}; p, q\right) \prod_{j=1}^{2n-1} \frac{dz_j}{z_j}, \end{aligned}$$

where  $\prod_{j=1}^{2n} z_j = 1$ , the balancing condition reads  $(st)^{2n-2} ST = (pq)^2$ ,  $S = \prod_{k=1}^4 s_k$  and  $T = \prod_{k=1}^4 t_k$ , and  $|s|, |t|, |s_j|, |t_j|, |\sqrt[4]{T/S} s_j|, |\sqrt[4]{S/T} t_j| < 1$ . As shown in [74], for  $s \rightarrow 1$  this formula passes to the Rains transformation (40) and there are also two more similar symmetry transformations. Because the integrals in (44) have only  $S_4 \times S_4 \times S_2$  permutational symmetry in the parameters instead of the  $S_8$ -group of (38), these three Weyl group transformations lead not to the  $E_7$ -group, but to a much smaller group. Consideration of the analogous symmetry transformations for integrals on the root system  $A_{2n}$  has not been completed yet.

**Integral analogues of the Bailey chains.** The Bailey chains, discovered by Andrews, serve as a powerful tool for building constructive identities for hypergeometric series (see Ch. 12 in [2]). They describe mappings of given sequences of numbers to other sequences with the help of matrices admitting explicit inversions. So, the most general Bailey chain for the univariate  $q$ -hypergeometric series suggested in [1] is connected to the matrix built from the  $s\varphi_7$  Jackson sum [10]. An elliptic generalization of this chain for the  $s+1V_s$ -series was built in [63], but we do not consider it here, as well as its complement described in [83]. Instead we present a generalization of the formalism of Bailey chains to the level of integrals discovered in [65].

We say that two functions  $\alpha(z, t)$  and  $\beta(z, t)$  form an elliptic integral Bailey pair with respect to the parameter  $t$ , if they are connected by the relation

$$\beta(w, t) = \kappa \int_{\mathbb{T}} \Gamma(tw^{\pm 1} z^{\pm 1}; p, q) \alpha(z, t) \frac{dz}{z}, \quad (45)$$

where  $|tw|, |t/w| < 1$  and  $\alpha(z, t)$  is analytical function of variable  $z$  near  $\mathbb{T}$ .

**Theorem 5.** *For a given elliptic integral Bailey pair  $\alpha(z, t)$  and  $\beta(z, t)$  with respect to the parameter  $t$ , the functions*

$$\begin{aligned}\alpha'(w, st) &= \frac{\Gamma(tuw^{\pm 1}; p, q)}{\Gamma(ts^2uw^{\pm 1}; p, q)} \alpha(w, t), \\ \beta'(w, st) &= \kappa \frac{\Gamma(t^2s^2, t^2suw^{\pm 1}; p, q)}{\Gamma(s^2, t^2, suw^{\pm 1}; p, q)} \int_{\mathbb{T}} \frac{\Gamma(sw^{\pm 1}x^{\pm 1}, ux^{\pm 1}; p, q)}{\Gamma(x^{\pm 2}, t^2s^2ux^{\pm 1}; p, q)} \beta(x, t) \frac{dx}{x},\end{aligned}$$

where  $w \in \mathbb{T}$ , form a new Bailey pair with respect to the parameter  $st$ , and the functions

$$\begin{aligned}\alpha'(w, t) &= \kappa \frac{\Gamma(s^2t^2, uw^{\pm 1}; p, q)}{\Gamma(s^2, t^2, w^{\pm 2}, t^2s^2uw^{\pm 1}; p, q)} \int_{\mathbb{T}} \frac{\Gamma(t^2sux^{\pm 1}, sw^{\pm 1}x^{\pm 1}; p, q)}{\Gamma(sux^{\pm 1}; p, q)} \alpha(x, st) \frac{dx}{x}, \\ \beta'(w, t) &= \frac{\Gamma(tuw^{\pm 1}; p, q)}{\Gamma(ts^2uw^{\pm 1}; p, q)} \beta(w, st)\end{aligned}$$

form a new Bailey pair with respect to the parameter  $t$ .

The proof is quite simple. In the first case, it is necessary to substitute the key relation for  $\beta(x, t)$  in the definition of  $\beta'(w, st)$ , to change the order of integrations and to take off one of the integrations with the help of the elliptic beta integral (there are certain restrictions on the parameters needed for the validity of this procedure). The second relation is proved in a similar manner. These chain substitution rules introduce two new parameters  $u$  and  $s$  at each step of their application. In fact, the described substitutions are related to one another by the inversion of the integral operator entering the definition of integral Bailey pairs [76].

This theorem is used analogously to the Bailey lemma for series [2]: one takes initial  $\alpha(z, t)$  and  $\beta(z, t)$ , found, say, from formula (13), and generates new pairs with the help of the described rules. Equality (45) for these pairs leads to a binary tree of identities for elliptic hypergeometric integrals of different multiplicities. As an illustration, we would like to give one nontrivial relation. With the help of formula (13), one can easily verify the validity of the following recurrence relation

$$\begin{aligned}I^{(m+1)}(t_1, \dots, t_{2m+8}) &= \frac{\prod_{2m+5 \leq k < l \leq 2m+8} \Gamma(t_k t_l; p, q)}{\Gamma(\rho_m^2; p, q)} \\ &\times \kappa \int_{\mathbb{T}} \frac{\prod_{k=2m+5}^{2m+8} \Gamma(\rho_m^{-1} t_k w^{\pm 1}; p, q)}{\Gamma(w^{\pm 2}; p, q)} I^{(m)}(t_1, \dots, t_{2m+4}, \rho_m w, \rho_m w^{-1}) \frac{dw}{w},\end{aligned}\tag{46}$$

where  $\rho_m^2 = \prod_{k=2m+5}^{2m+8} t_k / pq$  and the integral  $I^{(m)}$  was defined in (22). By an appropriate change of notation of the parameters, we obtain a concrete realization of the Bailey pairs:  $\alpha \propto I^{(m)}$  and  $\beta \propto I^{(m+1)}$ . For  $m = 0$ , substitution of the explicit expression (13) for  $I^{(0)}$  in the right-hand side of (46) yields identity (24). Other interesting consequences of the recursion (46) (an elliptic analogue of formula (2.2.2) in [2]) are considered in [66, 69]. Various generalizations of the integral transformation (45) to root systems and their inversions are described in [76].

**Connection to the representation theory.** Plain hypergeometric functions are connected to matrix elements of the representations of standard Lie groups (see, e.g., Sect. 9.14 in [2] where the Jacobi polynomials case is considered). Some of the  $q$ -special functions have been interpreted in a similar way in connection to quantum groups. Therefore it is natural to try to construct elliptic hypergeometric functions from the representations of “elliptic quantum groups”. The current top result along these lines was obtained in [54], where the terminating elliptic hypergeometric series of type I on the  $A_n$  root system was constructed as matrix elements for intertwiners between corepresentations of an elliptic quantum group. However, the whole construction is quite complicated and the elliptic hypergeometric integrals have not been treated in this way yet.

A qualitatively new group-theoretical interpretation of the elliptic hypergeometric functions has emerged, again, from mathematical physics (see [23, 30, 74, 75] and references therein). It directly connects the elliptic hypergeometric integrals to the representations of standard Lie groups. Take a Lie group  $G \times F$  and a set of its irreducible representations including the distinguished representation  $\text{adj}_G$ , adjoint for group  $G$  and trivial for  $F$  (the “vector” representation). Consider the following function of this group characters:

$$I(y; p, q) = \int_G d\mu(z) \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \text{ind}(p^n, q^n, z^n, y^n) \right), \quad (47)$$

where  $d\mu(z)$  is the  $G$ -group invariant (Haar) measure and

$$\begin{aligned} \text{ind}(p, q, z, y) &= \frac{2pq - p - q}{(1-p)(1-q)} \chi_{\text{adj}_G}(z) \\ &+ \sum_j \frac{(pq)^{r_j} \chi_{R_F, j}(y) \chi_{R_G, j}(z) - (pq)^{1-r_j} \chi_{R_F, j}(y) \chi_{R_G, j}(z)}{(1-p)(1-q)} \end{aligned} \quad (48)$$

with some fractional numbers  $r_j$ . Here  $\chi_{\text{adj}_G}(z)$  and  $\chi_{R_G, j}(z)$ ,  $\chi_{R_F, j}(y)$  are the characters of the “vector” and all other (“chiral”) representations, respectively. They depend on the maximal torus variables  $z_a$ ,  $a = 1, \dots, \text{rank } G$ , and  $y_k$ ,  $k = 1, \dots, \text{rank } F$ .

For  $G = SU(N)$  one has  $z = (z_1, \dots, z_N)$ ,  $\prod_{j=1}^N z_j = 1$ , and

$$\int_{SU(N)} d\mu(z) = \frac{1}{N!} \int_{\mathbb{T}^{N-1}} \Delta(z) \Delta(z^{-1}) \prod_{a=1}^{N-1} \frac{dz_a}{2\pi i z_a},$$

where  $\Delta(z) = \prod_{1 \leq a < b \leq N} (z_a - z_b)$ , and  $\chi_{SU(N), \text{adj}}(z) = (\sum_{i=1}^N z_i)(\sum_{j=1}^N z_j^{-1}) - 1$ .

For special sets of representations entering the sum  $\sum_j$  in (48) and some fractional numbers  $r_j$  formula (47) yields *all* known elliptic hypergeometric integrals with interesting properties. It has even deeper group-theoretical meaning in the context of the representation theory of superconformal group  $SU(2, 2|1)$ , where  $2r_j$  coincide with the eigenvalues of  $U(1)_R$ -subgroup generator (“ $R$ -charges”) and  $p, q$

are interpreted as group parameters for generators commuting with a distinguished pair of supercharges (see the next section).

Take the elliptic beta integral (13) and rewrite it as  $I_{\text{lhs}} = I_{\text{rhs}}$ , where  $t_k = (pq)^{1/6}y_k$ ,  $k = 1, \dots, 6$ . Then  $I_{\text{lhs}}$  is obtained from (47) for  $G = SU(2)$ ,  $F = SU(6)$  with two representations: the “vector” one  $(\text{adj}, 1)$  with  $\chi_{SU(2),\text{adj}}(z) = z^2 + z^{-2} + 1$  and the fundamental one  $(f, f)$  with  $\chi_{SU(2),f}(z) = z + z^{-1}$ ,  $r_f = 1/6$ , and

$$\chi_{SU(6),f}(y) = \sum_{k=1}^6 y_k, \quad \chi_{SU(6),\bar{f}}(y) = \sum_{k=1}^6 y_k^{-1}, \quad \prod_{k=1}^6 y_k = 1.$$

The latter constraint on  $y_k$  is nothing else than the balancing condition for the integral in appropriate normalization of parameters, i.e. this notorious condition is equivalent to the demand that the determinant of special unitary matrices is equal to 1. For  $I_{\text{rhs}}$  one has  $G = 1$ ,  $F = SU(6)$  with single representation  $T_A : \Phi_{ij} = -\Phi_{ji}$ ,  $i, j = 1, \dots, 6$ , with

$$\chi_{SU(6),T_A}(y) = \sum_{1 \leq i < j \leq 6} y_i y_j, \quad r_{T_A} = 1/3.$$

The elliptic beta integral evaluation formula thus proves the equality of two character functions on different groups with different sets of representations, which is a new type of group-theoretical duality. Since the elliptic hypergeometric integrals are expected to define automorphic functions in the cohomology class of the group  $SL(3, \mathbb{Z})$ , this could mean the equivalence of two differently defined automorphic functions. A physical interpretation of this construction is described in the next section.

**Applications in mathematical physics.** The most important known physical application of elliptic hypergeometric integrals has been found in four dimensional supersymmetric quantum field theories, where they emerge as superconformal indices.

For  $\mathcal{N} = 1$  supersymmetric theories the full symmetry group is  $G_{\text{full}} = SU(2, 2|1) \times G \times F$ , where the space-time symmetry group is generated by  $J_i, \bar{J}_i$ ,  $i = 1, 2, 3$  ( $SU(2)$  subgroup generators, or  $SO(3, 1)$ -group Lorentz rotations),  $P_\mu, Q_\alpha, \bar{Q}_{\dot{\alpha}}$ ,  $\mu = 0, \dots, 3$ ,  $\alpha, \dot{\alpha} = 1, 2$  (supertranslations),  $K_\mu, S_\alpha, \bar{S}_{\dot{\alpha}}$  (special superconformal transformations),  $H$  (dilations), and  $R$  ( $U(1)_R$ -rotations);  $G$  is a local gauge invariance group and  $F$  is a global flavor symmetry group. The whole set of commutation relations between these operators can be found, e.g., in [74]. Choosing a particular pair of supercharges, say,  $Q = \bar{Q}_1$  and  $Q^\dagger = -\bar{S}_1$ , one obtains

$$QQ^\dagger + Q^\dagger Q = 2\mathcal{H}, \quad Q^2 = (Q^\dagger)^2 = 0, \quad \mathcal{H} = H - 2\bar{J}_3 - 3R/2. \quad (49)$$

Then the superconformal index (SCI) is defined by the following trace:

$$I(y; p, q) = \text{Tr} \left( (-1)^{\mathcal{F}} p^{\mathcal{R}/2+J_3} q^{\mathcal{R}/2-J_3} \prod_k y_k^{F_k} e^{-\beta \mathcal{H}} \right), \quad \mathcal{R} = H - R/2, \quad (50)$$

where  $\mathcal{F}$  is the fermion number operator ( $(-1)^{\mathcal{F}}$  is simply a  $\mathbb{Z}_2$ -grading operator in  $SU(2, 2|1)$ ),  $F_k$  are the maximal torus generators of the group  $F$ , and  $p, q, y_k, \beta$  are group parameters. The trace in (50) is taken over the Hilbert (Fock) space of quantum fields forming irreducible representations of the group  $G_{\text{full}}$ . Because operators  $\mathcal{R}, J_3, F_k, \mathcal{H}$  used in the definition of SCI commute with each other and with  $Q, Q^\dagger$ , non-zero contributions to the trace may come only from the space of zero modes of the operator  $\mathcal{H}$  (or the cohomology space of  $Q$  and  $Q^\dagger$  operators). Therefore there is no  $\beta$ -dependence. Computation of this trace leads to integral (47), where the integration over  $G$  reflects the gauge invariance of SCI. Function (48) is called the one-particle states index.

Some of the supersymmetric field theories are related to one another by the Seiberg electric-magnetic dualities [59], which are not proven yet despite of many convincing arguments. Equality of SCIs for such theories was conjectured by Römelsberger and proved in some cases by Dolan and Osborn [23] by identifying SCIs with the elliptic hypergeometric integrals. A related application to topological quantum field theories (which is using an elliptic hypergeometric integral identity of [12]) is discovered in [30]. In [74, 75] many new  $\mathcal{N} = 1$  supersymmetric dualities have been found and very many new integral identities have been conjectured, among which there are relations of a qualitatively new type (e.g., they involve higher order generalizations of integral (38) with  $t = (pq)^{1/K}$ ,  $K = 2, 3, \dots$ ).

We leave it as an exercise to determine what kind of transformation of elliptic hypergeometric integrals is hidden behind the equality of SCIs for the original Seiberg duality [59]. In this case one has two theories with  $F = SU(M)_l \times SU(M)_r \times U(1)$  (here  $U(1)$  is the baryon number preserving symmetry) and different gauge groups and representations. The “electric” theory has the group  $G = SU(N)$  and the set of representations described in the table below:

$SU(N)$	$SU(M)_l$	$SU(M)_r$	$U(1)$	$U(1)_R$
$f$	$f$	$1$	$1$	$\tilde{N}/M$
$\bar{f}$	$1$	$\bar{f}$	$-1$	$\tilde{N}/M$
adj	$1$	$1$	$0$	$1$

where  $\tilde{N} = M - N$ . The “magnetic” theory has the group  $G = SU(\tilde{N})$  with the representations described in the following table:

$SU(\tilde{N})$	$SU(M)_l$	$SU(M)_r$	$U(1)$	$U(1)_R$
$f$	$\bar{f}$	$1$	$N/\tilde{N}$	$N/M$
$\bar{f}$	$1$	$f$	$-N/\tilde{N}$	$N/M$
$1$	$f$	$\bar{f}$	$0$	$2\tilde{N}/M$
adj	$1$	$1$	$0$	$1$

The last columns of these tables contain the numbers  $2r_j$  – eigenvalues of the generator of  $U(1)_R$ -group  $R$ . The last rows correspond to the vector superfield representation, other rows describe chiral superfields. For  $N = 2, M = 3$  equality of SCIs is equivalent to the elliptic beta integral, as described in the previous section. For arbitrary  $N$  and  $M$  SCIs were computed in [23] (see also [74]). Physically, the

exact computability of SCIs describes a principally important physical phenomenon – the confinement of colored particles in supersymmetric theories of strong interactions. The equality of SCIs provides presently the most rigorous mathematical justification of the Seiberg dualities.

In [78], a discrete integrable system generalizing the discrete-time Toda chain has been constructed. A particular elliptic solution of this nonlinear chain equations has lead to the terminating  ${}_12V_{11}$ -series as a solution of the Lax pair equations. Derivation of this function from a similarity reduction of an integrable system equations reflects the essence of a powerful heuristic approach to all special functions of one variable (it was described in detail in [66] on the basis of a number of other new special functions constructed in this way). In [37], it was shown that the same  ${}_12V_{11}$ -series appears as a particular solution of the elliptic Painlevé equation discovered by Sakai [57]. An analogous role is played by the general solution of the elliptic hypergeometric equation [66, 68] and some multiple elliptic hypergeometric integrals [46, 49]. In [8], a different discrete integrable system was deduced from the semiclassical analysis of the elliptic beta integral.

The first physical interpretation of elliptic hypergeometric integrals was found in [66, 68], where it was shown that some of the  $BC_n$ -integrals describe either special wave functions or normalizations of wave functions in the Calogero-Sutherland type many body quantum mechanical models. The  $A_n$ -root system case was discussed in [73]. It is natural to expect that all superconformal indices are associated with such integrable systems [74].

Another rich field of applications of elliptic hypergeometric functions is connected with the exactly solvable models of statistical mechanics. As mentioned in the introduction, elliptic hypergeometric series showed up for the first time as solutions of the Yang-Baxter equation of IRF (interaction round the face) type. The vertex form of the Yang-Baxter equation naturally leads to the Sklyanin algebra [60]. Connection of the elliptic hypergeometric functions with this algebra is considered in [39, 45, 52, 53, 69]. In particular, in [53] Rosengren proved an old conjecture of Sklyanin on the reproducing kernel, and in [69] an elliptic generalization of the Faddeev modular double [25] was constructed.

Let us briefly describe how the  $V$ -function emerges in this context. The general linear combination of four Sklyanin algebra generators  $S_0, \dots, S_3$  can be represented in the form [45]

$$\Delta(\underline{a}) = \frac{\prod_{j=1}^4 \theta_1(a_j + u)}{\theta_1(2u)} e^{\eta \partial_u} + \frac{\prod_{j=1}^4 \theta_1(a_j - u)}{\theta_1(-2u)} e^{-\eta \partial_u},$$

where  $a_j$  and  $\eta$  are arbitrary parameters and  $e^{\pm \eta \partial_u} f(u) = f(u \pm \eta)$ . The Casimir operators take arbitrary continuous values, i.e. one deals in general with the continuous spin representations. The generalized eigenvalue problem

$$\Delta(a, b, c, d)f(u; \lambda, a, b; s) = \lambda \Delta(a, b, c', d')f(u; \lambda, a, b; s),$$

where  $s = a + b + c + d$  and  $c + d = c' + d'$ , is exactly solvable and  $f$  is given by a product of 8 elliptic gamma functions. Take the scalar product

$$\langle f, g \rangle = \kappa \int_{\mathbb{T}} \frac{f(u)g(u)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z}, \quad z = e^{2\pi i u}, \quad p = e^{2\pi i \tau}, \quad q = e^{4\pi i \eta},$$

and consider the conjugated generalized eigenvalue problem induced by it

$$\Delta^*(a, b, c, d)g(u; \mu, a, b; s) = \mu \Delta^*(a, b, c', d')g(u; \mu, a, b; s),$$

where  $\Delta^*$  is defined from the equality  $\langle \Delta f, g \rangle = \langle f, \Delta^* g \rangle$ . Then, the overlap of two dual bases with the same  $s$ -parameter  $\langle f, g \rangle$  is equal to  $V(t)$  for appropriately chosen parameters  $t_j$  [69].

In [8], it was shown that the elliptic beta integral can be rewritten as the star-triangle relation representing another form of the Yang-Baxter equation. This gave a new continuous spin solution of the latter relation and a new solvable model of statistical mechanics unifying many previously known cases. The Boltzmann weights appearing in this construction are identical with the kernel of integral transform (45). In [72], the most general general solution of the star-triangle relation connected to the hyperbolic beta integrals was found. Also, it was shown that the symmetry transformations for elliptic hypergeometric integrals can be rewritten as the “star-star” relation (an IRF type Yang-Baxter equation) leading to new checkerboard type solvable models of statistical mechanics and their multicomponent generalizations.

As a final example, we mention an interesting application of the discrete elliptic biorthogonal rational functions of [78] to random point processes related to the statistics of lozenge tilings of a hexagon (or plane partitions) described in [9].

**Conclusion.** The main part of the theory of plain hypergeometric functions has found a natural elliptic generalization, although the similarities start to show up for a rather large number of free parameters and structural restrictions. We would like to finish by listing some other achievements of the theory of elliptic hypergeometric functions. Multiple elliptic hypergeometric series were considered for the first time by Warnaar [82]. We described mostly properties of the elliptic hypergeometric integrals, since many results for the series represent their particular limiting cases being derivable via residue calculus. A combinatorial proof of the Frenkel-Turaev summation formula is given in [58]. Various generalizations of this sum to the root systems were found in [20, 51, 64, 82]. Multivariable analogues of the elliptic Bailey transformation for series were described in [18, 36, 44, 51, 82]. Expansions in partial fractions of the ratios of theta functions and the identities connected to them were considered by Rosengren in [51] (see also [21, 44, 50]). Such expansions played an important role in the method of proving elliptic beta integral evaluations considered in [67]. Some general properties of elliptic hypergeometric terms were described in [71].

The terminating continued fraction generated by the three term recurrence relation (34) and the Racah type termination condition was computed in [79]. The

raising and lowering operators connected with rational functions were discussed in [40, 41, 44, 80]. In particular, in [80] it was shown that the general lowering operator of the first order can exist only for the elliptic grids. A systematic investigation of the elliptic determinant formulas connected to the root systems is performed in the work of Rosengren and Schlosser [55]. Determinants of elliptic hypergeometric integrals were considered in [50, 64]. Elliptic Littlewood identities were discussed in [48] where many quadratic transformations for multiple elliptic hypergeometric functions were conjectured. Some of these conjectures were proved by van de Bult [14] (for quadratic transformations of the univariate series, see [62, 83]). In [88], the elliptic hypergeometric series  ${}_3E_2$  with arbitrary power counting argument was shown to describe some polynomials with a dense point spectrum. Connections to the Padé interpolation were analyzed in [40, 80, 87].

Solutions of various finite difference equations on the elliptic grid were considered by Magnus in [40, 41]. As shown in [75], reduction of elliptic hypergeometric integrals to the hyperbolic level leads to the state integrals for knots in three-dimensional space. The page limits of the present complement do not allow the author to cite a number of other interesting results, an essentially more complete review of the literature is given in papers [66, 70] and [74, 75].

To conclude, elliptic hypergeometric functions are universal functions with important applications in various fields of mathematics and theoretical physics. They unify special functions of elliptic and hypergeometric types under one roof and make them firm, unique, undefeatable objects living in the Platonic world of ideal bodies.

Despite of the very big progress in the development of the theory of elliptic hypergeometric functions, many open problems still remain. They include the proof of tens of existing conjectures on evaluations or symmetry transformations for integrals, a rigorous definition of infinite elliptic hypergeometric series, detailed investigation of the specific properties of functions when bases are related to roots of unity, computation of the nonterminating elliptic hypergeometric continued fraction, detailed analysis of the non-self-dual biorthogonal functions of [78] (still representing the most complicated known univariate special function of such a kind) and construction of their multivariable analogues, search of the number theoretic applications of these functions analogous to those considered in [89], investigation of their automorphic properties, higher genus Riemann surface generalizations, and so on.

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