On finest and modular t-stabilities

by A. L. Gorodentsev[1] and S. A. Kuleshov[2]

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Abstract.

This paper extends our previous work [GKR]. We interpret the Harder – Narasimhan filtrations as random processes over a set of slopes (which plays the role of time) and discuss the functorial properties of Harder – Narasimhan filtrations. We construct the finest refinement for t-stability on Abelian category (satisfying some natural finiteness conditions). We introduce modular t-stability (whose semistable subcategories can be separated on the level of K_0) and give a complete description of finest and modular t-stabilities on a category generated by an exceptional pair. The description of semistable classes is quite parallel in this case to the geometric theory of continuous fractions. For categories generated by exceptional collections of higher length we construct a huge family of modular t-stabilities depending on a choice of irrational curve on appropriate Grassmannian.

Contents

[1]Institute of Theoretical and Experimental Physics (Moscow) & Independent University of Moscow
[2]Zhukovsky Air Force Engineering Academy (Moscow) & Independent University of Moscow

§1. Introduction.

1.1. t-stabilities. Recall (see [GKR]) that *a t-stability* on a triangulated category T is a totally ordered collection $\Phi = \{\Pi_\varphi\}_{\varphi\in\Phi}$ of non empty strictly full extension closed[1] subcategories $\Pi_\varphi \subset \mathrm{T}$ such that

- the grading shift functor $\Pi_\varphi \mapsto \Pi_\varphi[1]$ is correctly acting on Φ by some non decreasing bijection $\Phi \xrightarrow[\sim]{\tau} \Phi$;
- $\mathrm{Hom}^{\leqslant 0}(\Pi_\psi, \Pi_\varphi) = 0$ for all $\psi > \varphi$;
- each non zero object $X \in \mathrm{T}$ is fitted into a diagram (finite Postnikov tower):

$$\begin{tikzcd}[column sep=small]
 & X_{\varphi_0} \arrow[dr, dashed] & & X_{\varphi_1} \arrow[dr, dashed] & & & & X_{\varphi_n} \arrow[dr, dashed] & \\
X = F^0X & & F^1X \arrow[ll, "p_1"'] \arrow[ul, "q_0"] & & F^2X \arrow[ll, "p_2"'] \arrow[ul, "q_1"] & \cdots \arrow[l] & F^nX \arrow[l] \arrow[ur, "q_n"] & & F^{n+1}X = 0 \arrow[ll, "p_{n+1}"']
\end{tikzcd} \tag{1}$$

in which all $X_{\varphi_i} \in \Pi_{\varphi_i}$ are non zero, $\varphi_i < \varphi_{i+1}$ for all i, and all the triangles are distinguished.

The subcategories $\Pi_\varphi \subset \mathrm{T}$ are called *semistable* categories, indexes φ are called (generalized) *slopes*, objects $G \in \Pi_\varphi$ are called semistable objects of slope φ, and diagram (1) is called the *Harder – Narasimhan filtration* of X w. r. t. t-stability Φ. There are several reasons for why t-stabilities are interesting.

On the one hand, t-stability formalizes what one would like to understand as 'functorial filtration' defined for all objects of a category. For example, canonical filtrations associated with t-structures (e.g. canonical filtrations of derived categories), Beilinson-type decompositions w. r. t. semiorthogonal generators, the classical Harder – Narasimhan filtration and Grothendieck's filtration by torsion sheaves in categories of coherent sheaves in algebraic geometry — these all are particular cases of t-stabilities. Roughly speaking, t-stability is a way for canonical presentation of objects as consequent extensions of some standard, semistable, indecomposable (in some sense) objects.

On the other hand, the set of isomorphism classes of objects of a semistable category gives a first coarse approximation to what one would like to understand as 'moduli space' of objects having some 'prescribed topological type'. There is a hope that, if the semistable categories are small enough (i. e. t-stability is sufficiently fine), then some rough topological invariants of the corresponding moduli spaces could be read from the Hochschild complexes of the corresponding semistable categories.

1.2. Fine and coarse t-stabilities. We say that t-stability Ψ is *coarser* than Φ (respectively, Φ is *finer* than Ψ), if Ψ is obtained from Φ by decomposing Φ into a disjoint union of totally ordered segments and a fusion of all semistable subcategories in each segment.

This relation provides the set of all t-stabilities with a partial ordering and can be used for building new stability data from given ones. In particular, each t-stability Φ defines a family

[1] a subcategory $\mathrm{A} \subset \mathrm{T}$ is called *extension closed*, if for any $A, C \in \mathrm{A}$ the existence of a distinguished triangle $A \to B \to C \to A[1]$ forces $B \in \mathrm{A}$

of the t-structures[2] on $\mathcal{T}$ that consists off all possible decompositions $\Phi = \bigsqcup_{m\in\mathbb{Z}} \tau^m\widehat{\Phi}$, where $\widehat{\Phi} \subset \Phi$ is any connected[3] fundamental domain for the shifting automorphism $\Phi \xrightarrow{\tau} \Phi$ from n° 1.1. This is quite similar to Dedekind's approach to the real numbers.

For example, the standard Mumford stability on the derived category of coherent sheaves on elliptic curve C provides this category with the set of slopes $\Phi = \bigsqcup_{m\in\mathbb{Z}} \tau^m\widehat{\mathbb{Q}}$, where $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$, (see [GKR]) and leads to a family of t-structures parameterized precisely by the Dedekind reals and the cores of irrational t-structures in these family can be comprehend, at request, as categories of 'holomorphic bundles on non commutative torus' (see [PS]).

Although the finest t-stabilities seem to be accumulating the most deep information about a triangulated category, a good framework for handling them is rather non clear. The problem is that any finest set of slopes is extremely huge even in the simplest cases. Say, the finest refinement of the standard Mumford slope on elliptic curve contains the curve itself[1]. In general case, the finest refinement of a given t-stability should encode, besides other things, at least all possible Jordan – Hölder data on all Π_φ (see [J] to gain some insight about how could it look like).

1.3. Modular t-stabilities and T. Bridgeland's approach. A natural posterization level for a set of slopes is to restrict ourself by collections of semistable categories $\Pi_\varphi \subset \mathcal{T}$ that are uniquely (up to a shift) recovered from their images in the Grothendieck group $K_0(\mathcal{T})$. Let us say that a t-stability $\{\Pi_\psi\}_{\psi\in\Psi}$ on $\mathcal{T}$ is *modular*, if a coincidence of some classes of semistable objects $X \in \Pi_\psi$, $X' \in \Pi_{\psi'}$ in $K_0(\mathcal{T})$ always implies that $\Pi_{\psi'} = \Pi_\psi[2m]$ for some $m \in \mathbb{Z}$. A finest modular t-stability seems to be a good formalization for intuitive concept of 'fixation of topological invariants[2]' and should be considered as a starting point for building the moduli spaces.

T. Bridgeland was the first who has introduced the concept of stability data in the context of triangulated categories (see [Br1], [Br2]). He deals with modular t-stabilities of special type, namely, Bridgeland's slope set $\Phi \subset \mathbb{R}$ is real and is considered together with a concrete inclusion into $\mathbb{R}$, which should be induced by an additive homomorphism[3] $Z : K_0(\mathcal{T}) \longrightarrow \mathbb{C}$. A good feature of this approach is that it leads to a nice finite dimensional topological spaces, which parameterize pairs (Φ, Z). Moreover, the projection $(\Phi, Z) \mapsto Z$ realizes such a space as a topological cover of some domain inside $\mathrm{Hom}(K_0(\mathcal{T}), \mathbb{C})$. But semistable categories arising in this story are too coarse for serving deep geometry: say, Giesecker stabilities and fine exceptional stabilities, which all are modular, lay outside Bridgeland's theory. It seems also that Bridgeland's moduli spaces of pairs (Φ, Z) are mostly moduli of inclusions $\Phi \hookrightarrow \mathbb{R}$ (and, as a consequence, moduli of some t-structures coming from *the same* fine t-stability) than the modules of fine semistable subcategories themselves: we can always take some fixed family Φ of fine enough semistable categories (in fact, even the modular ones) in such a way that the variation of Bridgeland's central charge does nothing but serves inclusions $\Phi \hookrightarrow \mathbb{R}$

[2] the t-structures are just the coarsest t-stabilities, certainly

[3] i. e. such that for any $\varphi_1, \varphi_2 \in \widehat{\Phi}$, $\varphi_1 < \varphi_2$, the whole segment $[\varphi_1, \varphi_2] \overset{\text{def}}{=} \{\, \varphi \in \Phi \mid \varphi_1 < \varphi < \varphi_2 \,\}$ is also contained in $\widehat{\Phi}$

[1] see [GKR] for precise description of this t-stability)

[2] in fact, only 'in the first order' before the higher K-theory

[3] more precisely, a slope $\varphi(X)$ of all objects from a given semistable subcategory Π_φ has to be equal to $\tan(\mathrm{Arg}Z)$, where $Z : K_0(\mathcal{T}) \longrightarrow \mathbb{C}$ is the additive homomorphism in question

together with appropriate decompositions of Φ into coarser subsets.

1.4. What is this paper about. We are sure that the set Φ of fine enough semistable categories should be considered separately as a deep invariant of the category T. In §2 we show that the Harder – Narasimhan filtration of X is in fact *functorial* in X being considered as a result of some random processes in T over a time set Φ. This simple remark leads to a decomposition of the identity functor Id_{T} into a kind of 'direct path integral' over Φ, where the set of semistable categories should be considered as a kind of 'discrete measure' along these passes. We finish §2 with some speculations in this direction and with some program of further investigations as well.

In §3 we prove the existence of the finest refinement of any t-structure on a category satisfying some natural finiteness conditions. This result finalizes some considerations from our previous paper [GKR] and indicates that there should be some canonical family of fine slope sets and these family should be intrinsically recovered from the category itself.

The rest of paper deals with some precisely computable examples which illustrate some of our ideas in very simple but not completely trivial demonstrative cases. In §4 we recall what is category generated by an exceptional pair and in §5 and §6 we precisely describe the set Φ of all finest modular semistable subcategories (which are the categories of semistable Kronecker modules of given slope) and all t-structures obtained by their posterization (the Abelian cores of this t-structures are categories of coherent sheaves on weighted projective line). In §7 we make some remarks towards modular t-structures on categories generated by exceptional collections of higher length. The set of fine semistable slopes becomes much more complicated here and requires some 'coherent' ordering of all rational geodesic arcs inside a triangle on a sphere. Such an ordering is given by an irrational convex curve.

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§2. Positive processes in triangulated categories.

2.1. Notations and terminology. Let us fix some infinite totally ordered set Φ called *a time* (if you are 'a physicist') or *a set of slopes* (if you are 'a mathematician'). Elements $\varphi \in \Phi$ are called either *moments of time* or *slopes*. We consider Φ as a category with one arrow $\varphi_1 \longleftarrow \varphi_2$ for each inequality $\varphi_1 < \varphi_2$. A functor $\Phi \xrightarrow{F} \mathsf{A}$, to any other category A, is then a family of objects $F^\varphi \in \mathsf{A}$ labelled by $\varphi \in \Phi$ and equipped with maps $F^\varphi \xleftarrow{\overleftarrow{F}_{\varphi\psi}} F^\psi$ for all $\varphi < \psi$.

A functor $\Phi \xrightarrow{F} \mathsf{A}$ is called *an elementary process*, if it has a finite image and is locally constant and semicontinuous from the left. In other words, each elementary process F defines and is uniquely defined by a finite chain of *events* $\varphi_0 < \cdots < \varphi_n$ in Φ and a finite chain of maps

$$F^0 \xleftarrow{f_0} F^1 \xleftarrow{f_1} \cdots \xleftarrow{f_n} F^n \tag{2}$$

in $\mathcal{A}$ such that

$$F^{\gamma} = F^{i}\,, \quad \text{for} \quad \varphi_{i-1} < \gamma \leqslant \varphi_i$$
$$\overleftarrow{F}_{\alpha\beta} = f_k \circ f_{k+1} \circ \cdots \circ f_m\,, \quad \text{for} \quad \varphi_{k-1} < \alpha \leqslant \varphi_k \quad \& \quad \varphi_m < \beta \leqslant \varphi_{m+1}$$

(assuming $F^{n+1} = 0$, $\varphi_{n+1} = +\infty$, $\varphi_{-1} = -\infty$). An object F^0 (isomorphic to F^{φ} for all $\varphi \leqslant \varphi_0$) is called *a result of the process*.

The elementary processes form a full subcategory $\mathcal{F}(\Phi,\mathcal{A}) \subset \mathcal{F}un(\Phi,\mathcal{A})$ in the category of all functors $\Phi \to \mathcal{A}$. The arrows in $\mathcal{F}(\Phi,\mathcal{A})$ are the natural transformations of functors: an arrow $F_1 \xrightarrow{\eta} F_2$ is a family of maps[1] $F_1^{\varphi} \xrightarrow{\eta^{\varphi}} F_2^{\varphi}$ in $\mathcal{A}$ such that for any $\alpha < \beta$ we have a commutative square

$$\begin{tikzcd} F_1^{\alpha} \arrow[r, "\overleftarrow{F}_{\alpha\beta}"] \arrow[d, "\eta^{\alpha}"'] & F_1^{\beta} \arrow[d, "\eta^{\beta}"'] \\ F_2^{\alpha} \arrow[r, "\overleftarrow{F}_{\alpha\beta}"] & F_2^{\beta}\,. \end{tikzcd}$$

Certainly, sending a process F to its result F^0, we get a functor

$$\mathcal{F}(\Phi,\mathcal{A}) \xrightarrow{\text{ev}} \mathcal{A}\,. \tag{3}$$

2.2. Increments and positivity. Let $\mathcal{A} = \mathcal{T}$ be a triangulated category. Then we can associate with any elementary process F a collection of its cones G^{φ}. By the definition, $G^{\varphi} = 0$ for $\varphi \notin \{\varphi_0, \varphi_1, \dots, \varphi_n\}$. For the events, that is, for $\varphi = \varphi_i$, a cone G^{φ_i} is defined by distinguished triangles

$$G^{\varphi_i} \longleftarrow F^{\varphi_i} \longleftarrow F^{\varphi_{i+1}} \longleftarrow G^{\varphi_i}[-1]\,.$$

These non zero cones (defined up to non unique non canonical isomorphisms) are called *increments* of the process. An elementary process F is called *positive*, if its increments satisfy the conditions

$$\mathrm{Hom}^{\leqslant 0}(G^{\psi}, G^{\varphi}) = 0 \text{ for all } \varphi < \psi \tag{4}$$

Let for any ψ from some subset $\Psi \subset \Phi$ a full faithful (possibly, zero) subcategory $\Pi_{\psi} \subset \mathcal{T}$ is given. Such a collection of subcategories is called *positive*, if $\mathrm{Hom}^{\leqslant 0}(\Pi_{\beta}, \Pi_{\alpha}) = 0$ for all $\alpha < \beta$. Given a collection of subcategories $\{\Pi_{\psi}\}_{\psi\in\Psi} \subset \mathcal{T}$, where $\Psi \subset \Phi$, let us write

$$\mathcal{P}ro_{\Psi}(\Pi_{\psi}) \subset \mathcal{F}(\Phi,\mathcal{A})$$

for a full subcategory of elementary processes whose events are in Ψ and increments belong to Π_{ψ}.

2.2.1. LEMMA. *For any positive collection of categories $\{\Pi_{\psi}\}_{\psi\in\Psi}$ the restriction of the evaluation functor* (3) *onto a $\mathcal{P}ro_{\Psi}(\Pi_{\psi}) \subset \mathcal{F}(\Phi,\mathcal{A})$ gives a full faithful embedding:*

$$\mathcal{P}ro_{\Psi}(\Pi_{\psi}) \xhookrightarrow{\text{ev}} \mathcal{T}\,.$$

[1]labelled by $\varphi \in \Phi$

Moreover, the collection of increments $G^\psi \in \Pi_\psi$ for a process $F \in \mathrm{P}\,ro_\Psi(\Pi_\psi)$ with a result $X = F^{-\infty} \in \mathrm{T}$ is functorial in X.

Proof. Let $X_1, X_2 \in \mathrm{T}$ be the results of two processes F_1, F_2 with events in Ψ and increments in Π_ψ's. We have to show that each map $X_1 \xrightarrow{f} X_2$ is uniquely lifted to a natural transformation of processes $F_1^\psi \xrightarrow{f^\psi} F_2^\psi$ accompanied by compatible transformation of increments $G_1^\psi \xrightarrow{g^\psi} G_2^\psi$. We will construct such a lifting inductively by running over all the consequent events from left to right. Let at some moment ψ at least one of the processes undergoes a non trivial event. Then we have a diagram of distinguished triangles

$$\begin{tikzcd}
F_1^{>\psi}[1] & G_1^\psi \arrow[l] \arrow[d, dashed, "g^\psi"'] & F_1^\psi \arrow[l] \arrow[d, "f^\psi"] \arrow[dl, dotted, "f'"] & F_1^{>\psi} \arrow[l] \arrow[d, dashed, "f^{>\psi}"] \arrow[dl, dotted, "f''"] & \\
& G_2^\psi & F_2^\psi \arrow[l] & F_2^{>\psi} \arrow[l] & G_2^\psi[-1] \arrow[l]
\end{tikzcd} \tag{5}$$

where the middle solid vertical arrow is given by the inductive assumption and two dashed vertical arrows should be constructed in such a way that the diagram becomes commutative. This construction is essentially the same as in [BBD]: straightforward compositions f', f'' (dotted skew lines on diagram (5)) can be lifted to required maps g^ψ, $f^{>\psi}$, because the obstruction space (which is inside $\mathrm{Hom}^0(F_1^{>\psi}, G_2^\psi)$ in the both cases) vanishes. These liftings are unique, because the ambiguity space (which is inside $\mathrm{Hom}^{-1}(F_1^{>\psi}, G_2^\psi)$ in the both cases) vanishes as well. The both vanishing conditions

$$\mathrm{Hom}^0(F_1^{>\psi}, G_2^\psi) = \mathrm{Hom}^{-1}(F_1^{>\psi}, G_2^\psi) = 0$$

follow from the fact that $F_1^{>\psi}$ is a result of a process whose increments G_1^φ have slopes $\varphi > \psi$, i. e. satisfy $\mathrm{Hom}^{\leqslant 0}(G_1^\varphi, G_2^\psi) = 0$. $\square$

2.2.2. COROLLARY. *If categories $\{\Pi_\psi\}_{\psi\in\Psi}$ form a positive collection, then each Π_ψ is extension closed, i. e. any distinguished triangle*

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1] \tag{6}$$

with $A, C \in \Pi_\psi$ forces $B \in \Pi_\psi$ as well.

Proof. Let B be the result of a process $F \in \mathrm{P}\,ro_\Psi(\Pi_\psi)$. Then the distinguished triangle (6) is lifted to the natural transformation of processes $A \longrightarrow F \longrightarrow B$, where A, B are considered as trivial processes with the only event at ψ with the increments A, B respectively. This extension leads for all $\alpha < \psi$ and $\beta > \psi$ to the following commutative diagrams of distinguished triangles:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & F^{>\alpha} \arrow[r] \arrow[d] & C \arrow[d, equal] \\
A \arrow[r] \arrow[d] & F^{\alpha} \arrow[r] \arrow[d] & C \arrow[d] \\
0 \arrow[r] & G^{\alpha} \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & F^{>\beta} \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\
0 \arrow[r] \arrow[d, equal] & F^{\beta} \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\
0 \arrow[r] & G^{\beta} \arrow[r] & 0
\end{tikzcd}$$

So, all $G^\alpha = 0$ for $\alpha < \psi$, $F^\alpha = B$ for $\alpha \leqslant \psi$, and all $F^\beta = G^\beta = 0$ for $\beta > \psi$. At the moment ψ the above diagrams turn to:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & F^{>\psi} \arrow[r] \arrow[d] & 0 \arrow[d] \\
A \arrow[r] \arrow[d, equal] & F^{\psi} \arrow[r] \arrow[d] & C \arrow[d, equal] \\
A \arrow[r] & G^{\psi} \arrow[r] & C
\end{tikzcd}$$

which shows that $B = F^\psi = G^\psi \in \Pi_\psi$. □

2.3. Some speculations: t-stabilities, positive *K*-theory and integrable systems. Two lemmas of the previous section show that a t-stability (as it was defined in n° 1.1) is nothing but a positive collection of subcategories $\{\Pi_\varphi\}_{\varphi\in\Phi}$ over appropriate slope set (= time) Φ such that it contains all the shifts $\Pi_\varphi[m]$ together with any Π_φ. Moreover, the Harder – Narasimhan filtrations of objects are automatically functorial w. r. t. the objects under this approach. In this langauge, the refinement of t-stability means just a non decreasing epimorphism $\Phi \twoheadrightarrow \Psi$ equipped with appropriate positive subdivision of positive subcategories Π_ψ. In the next section we show how this ideology can be used to construct the finest refinement of a given t-stability on an Abelian category.

It seems that one can associate with a given triangulated category $\mathcal{T}$ an intrinsic collection of time sets Ψ such that each of out of them admits a natural positive collection of semistable categories Π_ψ intrinsically coming with Ψ. For example, one can start with *a positive nerve* of $\mathcal{T}$, whose simpleces are elementary positive processes over $[0, 1]$ like it was described in [Df] for the case of 'ordinary' Waldhausen's K-theory. Then one can equip this simplicial space by a local system of subcategories $\Pi \in \mathcal{T}$ obtained by appropriate limit procedure. We hope to clarify this picture in the next paper, which is in preparation now.

Another productive viewpoint on a positive collection of subcategories comes from the theory of integrable systems on symplectic manifolds. A collection of increments of a positive processes can be considered as a discrete measure, that is, a distribution of 1-forms supported along subcategories Π_φ, which are analogs of special Lagrangian cycles in a sense of [Ty], i. e. those cycles that the universal Maslov phase ψ is constant along them. A differentiable (= non discrete) approximation of this story should be served by appropriate DG-enhancement of $\mathcal{T}$ that emulates a choice of canonical bundle and a Levi – Civita connection on it. We are planing to develop the corresponding DG-formalism in the next papers as well.

After this stream of consciousness, we spend the rest of the paper on some concrete examples, which have putted the above ideology into our mind. All computations below will be based rather on our initial understanding of t-stabilities explained in the introduction and coming from [GKR].

§3. Finest refinement of t-stabilities on Abelian categories.

3.1. t-stabilities that refine t-structures. In this section we construct the finest t-stability on a triangulated category $\mathcal{T}$ which admits a non-degenerate t-structure whose Abelian core $\mathcal{A}$ satisfies some natural finiteness conditions. In particular, this result will imply the existence of the finest refinements for all modular t-stabilities and the standard t-stabilities on derived categories of coherent sheaves on smooth algebraic curves as well as the derived categories of quiver representations.

Our construction of the finest refinement splits into a series of reductions described in [GKR, §1]. For convenience of reading, let us recall them here briefly.

3.2. Abelian stability data. First of all, there is the standard procedure (see [GKR], [Br1]) for inducing t-stabilities on $\mathcal{T}$ from stability data defined on $\mathcal{A}$, that is, from a family $\{\Pi_\varphi\}_{\varphi\in\Phi}$ of extension closed subcategories $\Pi_\varphi \subset \mathcal{A}$ labelled by some totally ordered set Φ such that

$$\mathrm{Hom}_{\mathcal{A}}(\Pi_{\varphi'}, \Pi_{\varphi''}) = 0 \quad \forall\, \varphi' > \varphi''$$

and any non zero object $X \in \mathcal{A}$ has the Harder – Narasimhan filtration

$$\begin{tikzcd}
X = F^0X \arrow[d] & F^1X \arrow[l, hook'] \arrow[d] & F^2X \arrow[l, hook'] \arrow[d] & \cdots \arrow[l, hook'] & F^nX \arrow[l, hook'] \arrow[d] & F^{n+1}X = 0, \arrow[l, hook'] \\
G_0 & G_1 & G_2 & & G_n &
\end{tikzcd}$$

where $G_i = F^iX/F^{i+1}X \in \Pi_{\varphi_i}$ and $\varphi_i < \varphi_j$ for all $i < j$.

In its own turn, the Abelian stability data can be constructed via [Ru] by equipping the set of objects $Ob\mathcal{A}$ with a pre-ordering that satisfies the following conditions:

- (the seesaw property) the middle term of any exact sequence of nonzero objects

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

either lays strictly between A, C, or[1] $A \sim B \sim C$.

$$\begin{aligned} &\text{either } A < B \Leftrightarrow A < C \Leftrightarrow B < C,\\ &\text{or } A > B \Leftrightarrow A > C \Leftrightarrow B > C,\\ &\text{or } A = B \Leftrightarrow A = C \Leftrightarrow B = C.\end{aligned}$$

- (finiteness conditions) there are no infinite proper chains:

$$\begin{aligned} &A_1 \supset A_2 \supset A_3 \supset \ \ldots, \text{ with } A_1 \leqslant A_2 \leqslant A_3 \leqslant \ \ldots\\ &A_1 \subset A_2 \subset A_3 \subset \ \ldots \subset A, \text{ with } A_1 < A_2 < A_3 < \ \ldots\\ &A_1 \subset A_2 \subset A_3 \subset \ \ldots \subset A, \text{ with } A_1 > A_2 > A_3 > \ \ldots\end{aligned}$$

[1] we write '$\sim$' for an equivalence relation on $\mathcal{A}$ induced by the preordering in question, that is, $A \sim B$ means the both $A \leqslant B$ and $A \geqslant B$

The corresponding stability data are defined now as follows. Call an object $A \in \mathcal{A}$ *R-semistable* (resp. *R-stable*), if $B \leqslant A$ (resp. $B < A$) for each proper subobject $B \subset A$. Write Φ for a set of equivalence classes of objects of $\mathcal{A}$ w. r. t. the preorder in question and for each $\varphi \in \Phi$ denote by Π_φ the set of all semistable objects in the class φ. Rudakov has proven in [Ru] that

3.2.1. LEMMA. *Subcategories $\{\Pi_\varphi\}_{\varphi\in\Phi}$ provide $\mathcal{A}$ with the Abelian stability data. These stability data have the following extra properties:*

(1) *a semistable subcategory Π_φ is Abelian;*

(2) *for each $\varphi \in \Pi_\varphi$ there exists a stable object $A \in \Pi_\varphi$;*

(3) *a stable object of a semistable subcategory is irreducible in the sense that it has no proper subobject of the same subcategory;*

(4) *two stable objects of Π_φ either are isomorphic or have no nonzero morphisms from one to another;*

(5) *each nonzero object of Π_φ has a finite Jordan – Hölder with stable quotients.*

(6) *a semistable subcategory Π_φ is Noetherian and Artinian.*

We call the stability data of this sort *R-stability* for short.

A preorder required to produce R-stability can be defined using the vector-slope technique. Namely, given an ordered collection[1] of integer additive functions on $\mathcal{A}$: l, $\{d_i\}_{i\in I}$ such that for any $X \in \mathcal{A}$ we have $d_i(X) = 0$ for all i outside some finite subset $I(X) \subset I$ and $l(X) > 0$ for each non zero object $X \in \mathcal{A}$, let us form *a slope function* λ on $\mathcal{A}$ by prescription

$$\lambda(X) = \frac{1}{l(X)} \sum_{i\in I} d_i(A)\, A$$

So, the slopes are vectors with rational coordinates. Comparing them lexicographically, we define a pre-ordering on $\mathcal{A}$ by prescription $X < Y \overset{\text{bydef}}{\iff} \lambda(X) < \lambda(Y)$. It is easy to see that if $\mathcal{A}$ is Noetherian and Artinian[2], then this preorder provides $\mathcal{A}$ with R-stability.

3.3. Main theorem. The rest of this section is devoted to the proof of the following theorem.

3.3.1. THEOREM. *Any R-stability $\{\Pi_\varphi\}_{\varphi\in\Phi}$ on Abelian category $\mathcal{A}$ has a finest refinement.*

PROOF. We a going to construct an infinite chain

$$\Phi = L_0 \succ L_1 \succ \cdots \succ L_k \succ L_{k+1} \cdots$$

of consequent refinements of the given stability data such that

$$\begin{aligned} &L_k\text{-stable object is } L_{k+1}\text{-stable;} \\ &L_{k+1}\text{-semistable object is } L_k\text{-semistable.} \end{aligned} \tag{7}$$

[1] maybe infinite

[2] i. e. for each $A \in \mathcal{A}$ there are no infinite proper chains $A \supset A_1 \supset A_2 \supset \ldots,\ A_1 \subset A_2 \subset \ldots \subset A$

After that we consider the set Stab of all L_k-stable objects (for all $k \in \mathbb{Z}_{\geqslant 0}$) and equip it with total order such that if $A, B \in$ Stab and $A > B$, then $\mathrm{Hom}(A, B) = 0$. Finally we give the finest stability data on $\mathbb{A}$ with $\Psi =$ Stab as the slope set.

The first refinement L_1 is constructed as follows. Let us fix some $\varphi \in \Phi$. By our assumptions, the corresponding semistable category Π_φ is of finite length. We denote the length function on Π_φ by l_0. This is non-negative additive function vanishing only at $X = 0$. Write Stab^0_φ for the set of all Φ-stable objects in Π_φ. Then for any $A, B \in \mathrm{Stab}^0_\varphi$ either $A \simeq B$ or

$$\mathrm{Hom}_{\mathbb{A}}(A, B) = \mathrm{Hom}_{\mathbb{A}}(B, A) = 0\,.$$

Let us fix some total order on each set Stab^0_φ. Then, combining these orders with the order on the set Φ, we get a total order on the set $\mathrm{Stab}^0 = \bigcup\limits_{\varphi\in\Phi} \mathrm{Stab}^0_\varphi$ such that

$$\forall\, A, B \in \mathrm{Stab}^0 \quad A > B \Rightarrow \mathrm{Hom}_{\mathbb{A}}(A, B) = 0\,.$$

Further, for each $A \in \mathrm{Stab}^0_\varphi$ let

$$\deg_A : \Pi_\varphi \longrightarrow \mathbb{Z}_{\geqslant 0}$$

be an additive function that takes X to the number of Jordan–Holder quotients of X isomorphic to A. For each nonzero $X \in \Pi_\varphi$ we introduce the slope function $\lambda^1_\varphi(X)$ by the following rule

$$\lambda^1_\varphi(X) = \sum_{A\in\mathrm{Stab}^0_\varphi} \frac{\deg_A(X)}{l_0(X)} \cdot A\,.$$

Since an object $X \in \Pi_\varphi$ has the finite Jordan – Hölder filtration, we see that this sum is actually finite and equals

$$\lambda^1_\varphi(X) = \sum_{i=1}^{n} \frac{\deg_{A_i}(X)}{l_0(X)} \cdot A_i\,,$$

where $\mathrm{Simpl}^1(X) = \{A_1,\, A_2,\, \dots,\, A_n\}$ is the set of all pairwise non-isomorphic stable quotients of X.

Let $X, Y \in \Pi_\varphi$ and $\{A_1,\, A_2,\, \dots,\, A_n\} = \mathrm{Simpl}^1(X) \cup \mathrm{Simpl}^1(Y)$. Without loss of generality, we can assume that $A_1 > A_2 > \cdots > A_n$. We rewrite the slopes $\lambda^1_\varphi(X)$ and $\lambda^1_\varphi(Y)$ as follows:

$$\lambda^1_\varphi(X) = \left(\frac{\deg_{A_1}(X)}{l_0(X)} \cdot A_1, \frac{\deg_{A_2}(X)}{l_0(X)} \cdot A_2, \dots, \frac{\deg_{A_n}(X)}{l_0(X)} \cdot A_n\right)$$
$$\lambda^1_\varphi(Y) = \left(\frac{\deg_{A_1}(Y)}{l_0(Y)} \cdot A_1, \frac{\deg_{A_2}(Y)}{l_0(Y)} \cdot A_2, \dots, \frac{\deg_{A_n}(Y)}{l_0(Y)} \cdot A_n\right)$$

Comparing them lexicographically, we get an R-stability on the Abelian category Π_φ. Now we apply this procedure to each subcategory Π_φ and consider new vector slope $\lambda^1 = (\varphi, \lambda^1_\varphi)$ on whole the category $\mathbb{A}$:

$$\lambda^1(X) = \begin{cases} (\varphi(X), \lambda^1_\varphi(X)), & \text{if } X \in \Pi_\varphi, \\ (\varphi(X), 0), & \text{if } X \notin \bigcup\limits_{\varphi\in\Phi} \Pi_\varphi. \end{cases}$$

It produces the next R-stability L_1 refining $L_0 = \Phi$. Note that preorder on $\mathcal{A}$ induced by the slope function λ^1 is compatible with the initial one coming with R-stability: $\lambda^1(A) < \lambda^1(B)$ for all $A < B$.

For better understanding what was really happening during the previous refinement, consider a toy example: let an object $X \in \Pi_\varphi$ have $\lambda_0(X) = 2$, then there are three possibilities for the Jordan – Hölder filtration of X: either $X \rightsquigarrow (A, A)$, or $X \rightsquigarrow (A, B)$ with $A > B$, or $X \rightsquigarrow (B, A)$ with $A > B$. Clearly, in the first case $\lambda^1(X) = (\varphi, A)$ and any proper subobject $A \subset X$ has $\lambda^1(A) = (\varphi, A)$. Therefore, X is L_1-semistable but is not L_1-stable. In the second case $\lambda^1(X) = (\varphi, \frac{A}{2}, \frac{B}{2})$. If the exact triple $0 \longrightarrow B \longrightarrow X \longrightarrow A \longrightarrow 0$ does not split, then the object X has a unique nontrivial subobject, namely B with $\lambda^1(B) = (\varphi, 0, \frac{B}{1}) < \lambda^1(X)$. Hence X is λ^1-stable. In the last case $\lambda^1(X) = (\varphi, \frac{A}{2}, \frac{B}{2})$, but X has the subobject A with $\lambda^1(A) = (\varphi, A, 0) > \lambda^1(X)$. So, such X is unstable.

Returning to the proof, write L^1_φ for the image of the slope function λ^1_φ, denote by $\mathrm{Stab}^1_{\lambda^1_\varphi}$ the set of λ^1-stable objects in $\Pi_{\lambda^1_\varphi}$ whose λ^1-slope equals λ^1_φ, and fix a total order on each $\mathrm{Stab}^1_{\lambda^1_\varphi}$. Then we get a totaly ordered set $\mathrm{Stab}^1 = \bigcup\limits_{\varphi\in\Phi} \bigcup\limits_{\lambda^1_\varphi \in L^1_\varphi} \mathrm{Stab}^1_{\lambda^1_\varphi}$. We have

$$\forall\, A, B \in \mathrm{Stab}^1 \quad A > B \Rightarrow \mathrm{Hom}_{\mathcal{A}}(A, B) = 0\,.$$

Moreover, the stabilities L_0 and L_1 satisfy condition (7). Hence $\mathrm{Stab}^0 \subset \mathrm{Stab}^1$.

Now suppose that we have a finite chain of R-stabilities $L_0 \succ L_1 \succ \cdots \succ L_k$ satisfying condition (7) and such that for each $i > 0$ the stability L_i is induced by a slope function λ^i and has the slope set L_i. Then we get also a chain of totaly ordered sets $\mathrm{Stab}^0 \subset \mathrm{Stab}^1 \subset \cdots \subset \mathrm{Stab}^k$ of λ^i-stable objects of fixed slope λ^i such that $A > B \Rightarrow \mathrm{Hom}(A, B) = 0$ for all $A, B \in \mathrm{Stab}^k$. We are going construct a next R-stability L_{k+1} extending this chain.

For each L_k-semistable Abelian subcategory Π_{λ^k} and any non zero object $X \in \Pi_{\lambda^k}$ let $l_k(X)$ be a length of the Jordan – Hölder filtration of X with λ^k-stable quotients in Π_{λ^k}. Further, for each λ^k-stable object $A \in \Pi_{\lambda^k}$ define an additive function $\deg_A$ on Π_{λ^k} whose value at an object $X \in \Pi_{\lambda^k}$ is equal to the number Jordan – Hölder quotients of X isomorphic to A. Let us form a new vector slope function $\lambda^{k+1}_{\lambda^k}$ on Π_{λ^k} by prescription

$$\lambda^{k+1}_{\lambda^k}(X) = \sum_{A\in \mathrm{Stab}^k_{\lambda^k}} \frac{\deg_A(X)}{l_k(X)} \cdot A,$$

where $\mathrm{Stab}^k_{\lambda^k} = \mathrm{Stab}^k \cap \Pi_{\lambda^k}$, and extend it onto the whole of $\mathcal{A}$ as

$$\lambda^{k+1}(X) = \begin{cases} (\lambda^k(X), \lambda^{k+1}_{\lambda^k}(X)), & \text{if } X \in \Pi_{\lambda^k}, \\ (\lambda^k(X), 0), & \text{if } X \notin \bigcup\limits_{\lambda^k\in L_k} \Pi_{\lambda^k}. \end{cases}$$

Ordering the values of $\lambda^{k+1}(X)$ lexicographically, we get the next R-stability L_{k+1} on $\mathcal{A}$, which is finer than L_k and satisfies (7). Now write $\mathrm{Stab}^{k+1}_{\lambda^{k+1}}$ for a set of all λ^{k+1}-stable objects in $\Pi_{\lambda^{k+1}}$, fix an arbitrary total order on it, and consider a totaly ordered set $\mathrm{Stab}^{k+1} = \bigcup \mathrm{Stab}^{k+1}_{\lambda^{k+1}}$. Of course, $A > B \Rightarrow \mathrm{Hom}(A, B) = 0$ for all $A, B \in \mathrm{Stab}^{k+1}$.

The inductive process just described produces an infinite chain R-stabilities

$$L_0 \succ L_1 \succ \cdots L_k \succ \cdots$$

on $\mathcal{A}$ and a chain of totaly ordered sets of stable objects $\mathrm{Stab}^0 \subset \mathrm{Stab}^1 \subset \cdots \subset \mathrm{Stab}^k \subset \cdots$. The finest stability data L we are constructing is nothing but the projective limit of L_i's. More precisely, take $\mathrm{Stab} = \bigcup_k \mathrm{Stab}^k$ as the slope set of our the finest stability. This is a totaly ordered set such that $A > B \Rightarrow \mathrm{Hom}(A, B) = 0$ for all $A, B \in \mathrm{Stab}$. The semistable subcategory Π_A of slope $A \in \mathrm{Stab}$ is defined as with $\Pi_A = \langle A \rangle$.

To check that L really serves the stability data, it remains to construct a finite Harder – Narasimhan filtration for any non zero object $X \in \mathcal{A}$. As the first approximation to it, take the Harder – Narasimhan filtration of X w. r. t. L_0. Then replace all quotients $X_i \notin \mathrm{Stab}^0$ by their Harder – Narasimhan filtrations w. r. t. L_1, after that replace there all L_2-unctable quotients e. t. c.This procedure is finite, because all X_i are of finite length. $\square$

3.3.2. COROLLARY. *Let $\{\Pi_\psi\}_{\psi\in\Psi}$ be a t-stability a triangulated category $\mathcal{T}$. Suppose that there exist a subset $\Phi \subset \Psi$ and a t-structure $(\mathcal{T}^{\geqslant 0}, \mathcal{T}^{\leqslant 0})$ with the core $\mathcal{A}$ such that*

(1) *the restriction of $\{\Pi_\psi\}_{\psi\in\Psi}$ onto $\mathcal{A}$ gives there R-stability data $\{\Pi_\varphi\}_{\varphi\in\Phi}$;*

(2) *for each $\psi \in \Psi$ there exist $\varphi \in \Phi$ and $n \in \mathbb{Z}$ such that $\Pi_\psi = \Pi_\varphi[n]$;*

Then there exists a finest t-stability $\{\Pi_\psi\}_{\psi\in\Psi}$ that refines $\Psi \preceq \Phi$.

§4. Toy example: category $\mathcal{P}_h$ generated by an exceptional Hom-pair.

4.1. Preliminary remarks and notations. Let $\mathcal{P}_h$ be $\mathbb{C}$-linear triangulated category with finite dimensional Hom's[1] generated (as a triangulated category) by a pair of objects (E_0, E_1) such that

$$\mathrm{Hom}^0(E_0, E_0) = \mathrm{Hom}^0(E_1, E_1) = \mathbb{C}\,, \tag{8}$$

$$\mathrm{Hom}^k(E_1, E_0) = 0 \text{ for all } k \in \mathbb{Z}\,, \tag{9}$$

$$\mathrm{Hom}^k(E_0, E_0) = \mathrm{Hom}^k(E_1, E_1) = \mathrm{Hom}^k(E_0, E_1) = 0 \text{ for all } k \neq 0\,, \tag{10}$$

$$\mathrm{Hom}^0(E_0, E_1) = H\,, \quad \dim H = h \geqslant 2\,. \tag{11}$$

We call any such a pair (E_0, E_1) *an exceptional* Hom-*pair*.

It is well known (see [Bo1], [Bo2]) that $\mathcal{P}_h$ is equivalent to the bounded derived category of Kronecker modules over H^*. By the definition, such a module is a representation of the quiver with two vertices [0], [1] and h-dimensional space of arrows $[0] \to [1]$, which is isomorphic to H^*. Equivalently, a Kronecker module is given by a linear map $V_1 \xrightarrow{\psi} H \otimes V_2$, where V_1, V_2 are arbitrary vector spaces. Actually, only the number of arrows, that is, $\dim H$ is essential here, because any linear isomorphism $H_1 \xrightarrow{\sim} H_2$ induces an equivalence of the categories of the Kronecker modules over H_1^* and over H_2^*. In particular, the categories of the Kronecker

[1] this means that $\mathrm{Hom}^\bullet(X, Y) = \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}^k(X, Y) = \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{P}_h}(X, Y[k])$ is a finite dimensional graded vector space over $\mathbb{C}$ for any pair of objects X, Y

modules over H and over H^* are equivalent. The equivalence between $\mathcal{P}_h$ and the derived category of the Kronecker modules takes two irreducible 1-dimensional modules $\mathbb{C} \xrightarrow{0} H\otimes 0$ and $0 \xrightarrow{0} H\otimes\mathbb{C}$ to E_0 and $E_1[-1]$ respectively.

For $d=2$ the category $\mathcal{P}_2$ is also identified with the bounded derived category of coherent sheaves on the projective line $\mathbb{P}_1$ via taking E_0, E_1 to be the invertible sheaves $\mathcal{O}$, $\mathcal{O}(1)$.

We write M for the lattice $K_0(\mathcal{P}_h) \simeq \mathbb{Z}^2$ equipped with the (non symmetric) bilinear form

$$\chi([E],[F]) \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}} (-1)^k \dim \operatorname{Hom}^k(E,F)\,.$$

If we write e_0, e_1 for the classes $[E_0]$, $[E_1]$, then the scalar product of $x = x_0e_0 + x_1e_1$ and $y = y_0e_1 + y_1e_1$ is written as

$$\chi(x,y) = x_0y_0 + h\,x_0y_1 + x_1y_1\,.$$

4.2. Orthogonal geometry of M. It is a paraphrase of the Peell equation theory that all the vectors $e\in M$ with $\chi(e,e)=1$ can be arranged into two chains $\{e_i\}_{i\in\mathbb{Z}}$ and $\{-e_i\}_{i\in\mathbb{Z}}$ formed by consequent integer points laying on two distinct branches of the hyperbola[1]

$$x^2 + h\,xy + y^2 = 1 \tag{12}$$

(see fig. A). All e_i's are recovered recursively from any two consequent elements e_0, e_1 by the equation

$$e_{i-1} + e_{i+1} = h\,e_i \tag{13}$$

(we refer to [GoKu] for the details).

Any triple of consequent elements e_{i-1}, e_i, e_{i+1} provides M with a pair of dual integer semiorthonormal bases

$$\{e,f\} = \{e_i, e_{i+1}\} \quad\text{and}\quad \{f^\vee, e^\vee\} = \{e_{i-1}, e_i\}$$

satisfying the properties $\chi(e,e^\vee)=1$, $\chi(f,f^\vee)=-1$, $\chi(e,f^\vee)=\chi(f,e^\vee)=0$. In particular, the coordinates of any vector $v = x\,e + y\,f$ can be computed as $x = \chi(v,e^\vee)$, $y = -\chi(v,f^\vee)$. Moreover, all the semiorthonormal bases[2] (e,f) for M are exhausted by ones of the form $(e,f) = (\pm e_i, \pm e_{i+1})$. The Gram matrix of such a basis always equals

$$\begin{pmatrix} 1 & \pm h \\ 0 & 1 \end{pmatrix},$$

where the plus sign appears for the basises (e_i, e_{i+1}) and $(-e_i, -e_{i+1})$ formed by consequent unit vectors laying at the same branch of hyperbola (12). It is

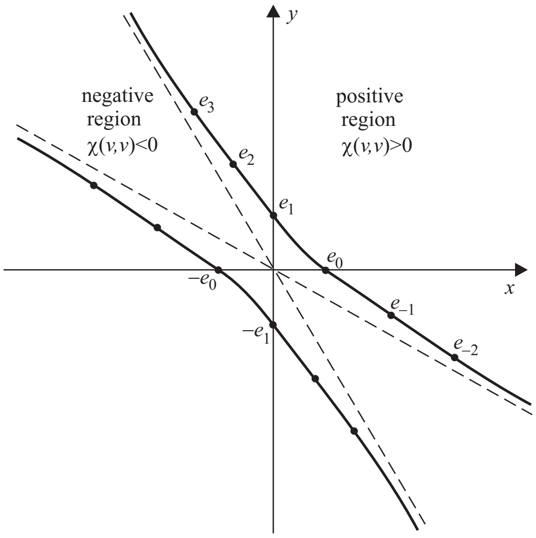

Fig. A. The unit vectors of M.

[1] for $h=2$ this hyperbola degenerates to a pair of lines $x+y=\pm 1$

[2] i. e. such that $\chi(e,e)=\chi(f,f)=1$ and $\chi(f,e)=0$

clear from fig. A that for any positive vector $v \in M$ (i. e. such that $\chi(v, v) > 0$) there exists a unique base of this form (i. e. with positive Gram matrix) such that v expands through this basis with positive coefficients.

4.3. Exceptional objects of $\mathcal{P}_h$. It follows immediately from the previous arithmetical analysis that all exceptional objects[3] of $\mathcal{P}_h$ are exhausted by $E_i[m]$, $i, m \in \mathbb{Z}$, which are the shifts of consequent left and right mutations of the initial exceptional generators E_0, E_1.

Recall, that the mutations in exceptional pair (E, F) are defined by the distinguished triangles

$$\begin{gathered} L_E F \longrightarrow \operatorname{Hom}^\bullet(E,F)\otimes E \longrightarrow F \longrightarrow L_E F[1]\,, \\ R_F E[-1] \longrightarrow E \longrightarrow \operatorname{Hom}^{\times\,\bullet}(E,F)\otimes F \longrightarrow R_F E\,, \end{gathered} \tag{14}$$

induced by the canonical contraction maps

$$\begin{gathered} \operatorname{Hom}^\bullet(E,F)\otimes E \longrightarrow F\,, \\ E \longrightarrow \operatorname{Hom}^{\times\,\bullet}(E,F)\otimes F\,. \end{gathered}$$

It is easy to check (see [GoKu, §2]) that $(L_E F, E)$ and $(F, R_F E)$ are exceptional Hom-pairs generating $\mathcal{P}_h$ as soon as (E, F) is such a pair. Furthermore, the left and right mutations are inverse to each other.

So, if we put recursively $E_{i-1} \overset{\text{def}}{=} L_{E_i} E_{i+1}$, $E_{i+2} \overset{\text{def}}{=} R_{E_{i+1}} E_i$ for all $i \in \mathbb{Z}$ starting from $i = 0$, then we get a series of exceptional objects whose classes $e_i = [E_i] = [E_i[2m]] \in M$ coincide with the unit vectors sitting on the right upper branch of the hyperbola (12) considered above. The odd shifts $E_i[2m+1]$ are sitting on the down left branch.

It is easy to check that the Hom-spaces between exceptional objects E_i (forming one orbit w. r. t. the mutations (14)) are controlled as follows:

$$\begin{aligned} &\operatorname{Hom}^0(E_i,E_j)\neq 0\,, \quad \text{iff } i\leqslant j\,, \\ &\operatorname{Hom}^1(E_i,E_j)\neq 0\,, \quad \text{iff } i> j+1\,, \\ &\operatorname{Hom}^k(E_i,E_j)= 0\;\forall\, i,j \;\text{ for } k\neq 0,1\,. \end{aligned} \tag{15}$$

This implies immediately the following inequalities between slopes of exceptional objects (assuming that they are semistable).

4.3.1. LEMMA. *If $i \leqslant j$, then $\varphi(E_i) \leqslant \varphi(E_j)$ for any t-stability Φ such that E_i, E_j are Φ-semistable.*

□

4.4. Orthogonal decompositions in $\mathcal{P}_h$. It is a consequence of the Beilinson type theorem for an exceptionally generated triangulated category (see [GoKu, §2]) that any object X of $\mathcal{P}_h$ is fitted into distinguished triangle

$$X \longrightarrow \bigoplus_i U_0^i\otimes E_0[-i] \xrightarrow{\ \delta\ } \bigoplus_i U_1^i\otimes E_1[-i] \longrightarrow X[1]\,, \tag{16}$$

which is functorial in X and has

$$\begin{aligned} U_0^i &= \operatorname{Hom}^{i+1}({}^\vee E_1, X) = \operatorname{Hom}^{-i}(X, E_1^\vee)^*\,, \\ U_1^i &= \operatorname{Hom}^{i+1}({}^\vee E_0, X) = \operatorname{Hom}^{-i}(X, E_0^\vee)^*\,, \end{aligned} \tag{17}$$

[3] i. e. E such that $\operatorname{Hom}^\bullet(E,E)$ is the one dimensional algebra $\mathbb{C}$ situated at degree 0

where ${}^\vee E_1 = R_{E_1}E_0 = E_2$, ${}^\vee E_0 = E_1$, $E_1^\vee = E_0$, $E_0^\vee = L_{E_0}E_1 = E_{-1}$. Usually, the triangle (16) is abbreviated to

$$X \longrightarrow U_0^\bullet \otimes E_0 \longrightarrow U_1^\bullet \otimes E_1$$

assuming that $U_0^\bullet$, $U_1^\bullet$ are graded vector spaces and the tensor product of a graded vector space with an object is defined as $V^\bullet \otimes E = \bigoplus\limits_{k\in\mathbb{Z}} V^k \otimes E[-k]$.

Since (E_0, E_1) is an exceptional Hom-pair, the map δ in (16) splits into the direct sum of maps

$$U_0^{-i} \otimes E_0[i] \xrightarrow{\delta_i} U_1^{-i} \otimes E_1[i]\,.$$

Hence, any object X canonically splits into a direct sum $X = \bigoplus\limits_{i\in\mathbb{Z}} X_i$, where X_i comes from the distinguished triangle

$$X_i \longrightarrow U_0^{-i} \otimes E_0[i] \longrightarrow U_1^{-i} \otimes E_1[i] \longrightarrow X_i[1]\,. \tag{18}$$

We will call such X_i *a pure object of level* i w. r. t. the base (E_0, E_1). The direct computation using the orthogonality conditions (8)–(10) shows that $\mathrm{Hom}^0(X_i, X_j)$ can be non zero only if $j - i = 0, 1$.

So, if we write $\mathrm{P}_h{}^{\leqslant n}$ for a full extension closed subcategory of P_h generated by all pure objects of level[1] $i \geqslant n$ and write $\mathrm{P}_h{}^{\geqslant n}$ for similar subcategory spanned by all pure objects of level $i \leqslant n$, then the pair $(\mathrm{P}_h{}^{\leqslant 0}, \mathrm{P}_h{}^{\geqslant 1})$ provides P_h with a t-structure whose core $\mathrm{P}_h{}^0 = \mathrm{P}_h{}^{\leqslant 0} \cap \mathrm{P}_h{}^{\geqslant 0}$ consists of all X in P_h fitted into distinguished triangle

$$X \longrightarrow V_0 \otimes E_0 \longrightarrow V_1 \otimes E_1 \longrightarrow X[1]$$

Since the morphisms between such objects are represented by morphisms of triangles

$$\begin{tikzcd} Y_0 \arrow[r] & V_0 \otimes E_0 \arrow[r] & V_1 \otimes E_1 \\ X_0 \arrow[r] \arrow[u, "f"] & U_0 \otimes E_0 \arrow[r] \arrow[u, "f_0\otimes id"] & U_1 \otimes E_1 \arrow[u, "f_1\otimes id"] \end{tikzcd}$$

the category $\mathrm{P}_h{}^0$ is nothing but the Abelian category of the Kronecker modules over H^* (in particular, the homological dimension of $\mathrm{P}_h{}^0$ equals 1, because of $\mathrm{Hom}^k(X, Y) = 0$ for $k \neq 0, 1$ if $X, Y \in \mathrm{P}_h{}^0$).

4.4.1. LEMMA. *Let X be an object of P_h such that its class $x = [X] \in M$ is positive, i. e. $\chi(x, x) > 0$. Then X is indecomposable iff X is exceptional.*

PROOF. Since $\chi(x, x) > 0$, there are two integer points on the hyperbola (12), say e_0, e_1, such that $x = v_0e_0 + v_1e_1$, where the integer coefficients v_0, v_1 have the same sign or, more precisely, satisfy inequality $v_0 \cdot v_1 \geqslant 0$. On the other hand, since X is indecomposable, only one out of its components (18) is non zero, i. e. after appropriate shift $X \to X[m]$ we can assume that the orthogonal decomposition of X through E_0, E_1 is given by distinguished triangle

$$X \longrightarrow V_0 \otimes E_0 \longrightarrow V_1 \otimes E_1 \longrightarrow X[1]$$

for some (non graded) vector spaces V_0, V_1. Hence, in M we have $v_0 = \dim V_0$, $v_1 = -\dim V_1$, which is compatible with $v_0 \cdot v_1 \geqslant 0$ only if one of V_0, V_1 vanishes. Since X is indecomposable, the remaining non zero space $V_i = \mathbb{C}$. $\square$

[1] inconsistency of inequalities comes from the traditional definition of t-structures, see [BBD], [GeMa]

§5. Exceptional t-stability on P_h.

5.1. Definition of exceptional t-stability. Let (F_0, F_1) be an exceptional pair generating P_h. Then there is *an exceptional t-stability* $\{\Pi_\varepsilon\}$ on P_h constructed as follows (see [GKR] for details). The set of slopes $\{\varepsilon\}$ is in 1–1 correspondence with the set of exceptional objects of the form $F_i[k]$, where $i = 0, 1$, $k \in \mathbb{Z}$. It is totally ordered by prescriptions $F_0[k] < F_1[m]$ for all $m, k \in \mathbb{Z}$ and $F_i[k] < F_i[m]$ for all $k < m$ and each $i = 0, 1$. For a slope ε corresponding to $F_i[k]$, the semistable category Π_ε is equivalent to the category of vector spaces and consists of all objects of the form $V \otimes F_i[k]$, where V is (non graded) vector space. In other words, the semistable objects of the exceptional stability built from (F_0, F_1) are exhausted by the direct powers $F_i[k]^{\oplus d}$.

Actually, in [GKR] some collection of such exceptional t-stabilities was constructed for a fixed exceptional pair (F_0, F_1). They all have the same set of the semistable subcategories and differ from each other only in a choice of the total ordering on this set. This ordering is described by an integer $p \in [1, +\infty]$ and lines up the exceptional objects as

$$\cdots < F_1[-2] < F_0[p-1] < F_1[-1] < F_0[p] < F_1 < F_0[p+1] < F_1[1] < F_0[p+2] < \cdots$$

We will consider here only one ordering corresponding to $p = +\infty$ and described above.

5.2. Characterization of the exceptional t-stabilities. The rest of this section is devoted to a proof of

5.2.1. PROPOSITION. *Let Ψ be a t-stability on P_h such that some exceptional object E is not Ψ-semistable. Then, up to a choice of the linear ordering on the set of slopes, Ψ is an exceptional t-stability built from some exceptional pair (F_0, F_1).*

PROOF. We fix the exceptional base (E_0, E_1) with $E_1 = E$ and let the Harder – Narasimhan filtration of E_1 start with an exact triangle

$$\begin{tikzcd} & Y \arrow[dr, dashed] & \\ E_1 \arrow[ur] & & X \arrow[ll] \end{tikzcd} \qquad (19)$$

where Y is Ψ-semistable and $\mathrm{Hom}^{\leqslant 0}(X, Y) = 0$. Let $Y = \oplus Y_i$ be the decomposition into direct sum of pure objects (18) w. r. t. the exceptional base (E_0, E_1). Then all Y_i except for Y_1 should be zero. Indeed, $\mathrm{Hom}^0(E_1, Y_i) = 0$ for $i \neq 1$ implies that $Y_i[-1]$ is a direct summand of X and this leads to $\mathrm{Hom}^{-1}(X, Y) \neq 0$.

So, Y is pure of level 1 and fits in the distinguished triangle

$$Y \longrightarrow U_0 \otimes E_0[1] \longrightarrow U_1 \otimes E_1[1] \,.$$

Then, by the octahedron axiom, X fits into diagram

$$\begin{tikzcd}
X \arrow[r] & U_0 \otimes E_0 \arrow[r] \arrow[d, equal] & U_1' \otimes E_1 \arrow[r] & X[1] \\
 & U_0 \otimes E_0 \arrow[r] & U_1 \otimes E_1 \arrow[r] \arrow[u] & Y \arrow[u] \\
 & & E_1 \arrow[r, equal] \arrow[u] & E_1 \arrow[u, "f"'] \\
 & & & X \arrow[u]
\end{tikzcd} \tag{20}$$

and is pure of level 0. In particular, $\mathrm{Hom}^i(X,X) = \mathrm{Hom}^i(Y,Y) = 0$ for $i \neq 0, 1$.

Let us show that $\mathrm{Hom}^1(X,X) = \mathrm{Hom}^1(Y,Y) = 0$ as well (this implies, in particular, that the both are positive, i. e. $\chi(X,X) > 0$, $\chi(Y,Y) > 0$). It follows from the first row of (20) that $\mathrm{Hom}^1(X,E_1) = 0$. So, applying $\mathrm{Hom}(X,*)$ to the right column of (20), we get exact triangle

$$\mathrm{Hom}^0(X,Y) \longrightarrow \mathrm{Hom}^1(X,X) \longrightarrow \mathrm{Hom}^1(X,E_1)\,,$$

which shows that $\mathrm{Hom}^1(X,X) = 0$. Similarly, $\mathrm{Hom}^1(E_1,Y) = 0$ by the middle row of (20) and, taking $\mathrm{Hom}(*,Y)$ of the right column in (20), we get the exact triangle

$$\mathrm{Hom}^0(X,Y) \longrightarrow \mathrm{Hom}^1(Y,Y) \longrightarrow \mathrm{Hom}^1(E_1,Y)\,,$$

which shows that $\mathrm{Hom}^1(Y,Y) = 0$.

Now it follows from asphericity[1] and positivity of X, Y that all their indecomposable direct summands are positive as well. Hence, by n° 4.4.1, X and Y are direct sums of some exceptional objects. Applying the asphericity once more, we see that $X = E_i[m]^{\oplus u_0} \oplus E_{i+1}^{\oplus u_0}[m]$ and $Y = E_j[k]^{\oplus v_0} \oplus E_{j+1}^{\oplus v_0}[k]$ for some integer i, j, k, m. Taking into account the conditions $\mathrm{Hom}^0(X,E_1) \neq 0$, $\mathrm{Hom}^0(E_1,Y) \neq 0$, and $\mathrm{Hom}^{\leqslant 0}(X,Y) = 0$, we see that either $m = 0$, $k = 1$, $i \leqslant 1$, $j \leqslant -1$ or $m = -1$, $k = 0$, $i \geqslant 2$, $j \leqslant 0$. Since $\mathrm{Hom}^{-1}(E_n, E_m[1]) \neq 0$ for $m \geqslant n$ and $\mathrm{Hom}^0(E_n, E_m[1]) \neq 0$ for $m \leqslant n-2$, there are only two possibilities for X, Y:

$$\text{either} \quad X = U \otimes E_i\,,\ Y = V \otimes E_{i-1}[1]\,, \quad \text{where } i < 1 \tag{21}$$
$$\text{or} \quad X = U \otimes E_i[-1]\,,\ Y = V \otimes E_{i+1}\,, \quad \text{where } i > 1\,. \tag{22}$$

In particular, X, $Y[-1]$ are the multiplicities of consequent exceptional objects forming an exceptional Hom-pair.

Let us assume that the first case (21) takes place and prove that Ψ should be the exceptional semistability build from the pair (E_{i-1}, E_i). Indeed, all the twists $E_{i-1}[k]$ are Ψ-semistable, because of Ψ-semistablity of Y. If we show that X is also Ψ-semistable, then we get Ψ-semistablity of all twists $E_i[m]$ as well. Since $\psi(E_i[m]) > \psi(E_{i-1}[k])$ for all m, k (because of our ordering agreement from n° 5.1), the t-stability Ψ has to be a refinement of the exceptional t-stability build from the pair (E_{i-1}, E_i) and we are done. So, it remains to check that X is Ψ-semistable.

[1] we will call an object X *aspherical*, if $\mathrm{Hom}^i(X,X) = 0$ for all $i \neq 0$

To this aim, let us continue the Harder – Narasimhan filtration (19) by the next triangle, which destabilizes X:

$$\begin{tikzcd}[column sep=small] & Y \arrow[dr, dashed] & & Y' \arrow[dr, dashed] & \\ E_1 \arrow[ur] & & X \arrow[ll] \arrow[ur] & & X' \arrow[ll] \end{tikzcd}$$

and satisfies

$$\mathrm{Hom}^{\leqslant 0}(X', Y') = \mathrm{Hom}^{\leqslant 0}(X', Y) = \mathrm{Hom}^{\leqslant 0}(Y', Y) = 0\,. \tag{23}$$

But the right triangle here is just a direct sum multiplicity of the first triangle of the Harder – Narasimhan filtration for E_i:

$$\begin{tikzcd}[column sep=small] & Y' \arrow[dr, dashed] & \\ X \arrow[ur] & & X' \arrow[ll] \end{tikzcd} = \left(\begin{tikzcd}[column sep=small] & Y'' \arrow[dr, dashed] & \\ E_i \arrow[ur] & & X'' \arrow[ll] \end{tikzcd} \right)^{\oplus N}$$

which is completely similar to (19). So, by the same reason as above,

$$\text{either} \quad X'' = U'' \otimes E_j\,,\ Y'' = V'' \otimes E_{j-1}[1] \quad \text{or} \quad X'' = U'' \otimes E_j[-1]\,,\ Y'' = V'' \otimes E_{j+1}\,.$$

But this contradicts to orthogonality conditions (23), because of the Hom-relations (15) between the exceptional objects of the same level. So, X should be stable. The second alternative case (22) is handled quite similarly. □

§6. Modular t-stability on P_h.

6.1. Definition of modular t-stability. Let us say that a t-stability $\{\Pi_\psi\}_{\psi\in\Psi}$ on P_h is *modular*, if all exceptional objects are semistable of distinct slopes and the slope function is factorized through $K_0(\mathrm{P}_h)$ in the following sense: if the images of semistable objects $X \in \Pi_\psi$, $X' \in \Pi_{\psi'}$ in $K_0(\mathrm{P}_h)$ coincide, then $\Pi_{\psi'} = \Pi_\psi[2m]$ for some $m \in \mathbb{Z}$.

6.2. Example: μ-stability. An example of modular t-stability comes from quiver representations theory. Let us fix an exceptional pair E_0, E_1 and consider the associated t-structure on P_h described in n° 4.4. Its core $\mathrm{P}_h{}^0 = \mathrm{P}_h{}^{\leqslant 0} \cap \mathrm{P}_h{}^{\geqslant 0}$ consists of all X in P_h fitted into distinguished triangle

$$X \longrightarrow U_0 \otimes E_0 \longrightarrow U_1 \otimes E_1 \longrightarrow X[1]\,,$$

where U_0, U_1 are (non graded) vector spaces. An ordered pair of additive functions

$$(u_1, u_0) = (\dim U_1, \dim U_0)$$

defines *a slope*

$$\mu = \frac{u_0}{u_1} \in \widehat{\mathbb{Q}} \overset{\text{def}}{=} \mathbb{Q}_{\geqslant 0} \cup \{\infty\}$$

on the Abelian category $\mathcal{A}$. In T. Bridgeland's terminology [Br1], this slope corresponds to the *central charge*

$$Z = -u_0 + iu_1 \;:\; K_0(\mathcal{P}_h) \longrightarrow \mathbb{C}\,.$$

centered at $\mathcal{P}_h{}^0$. An object X of $\mathcal{P}_h{}^0$ is called *μ-semistable*, if $\mu(Y) \leqslant \mu(X)$ for all proper subobjects $Y \subset X$ in $\mathcal{P}_h{}^0$. If for all proper subobjects $Y \subset X$ the strict inequality $\mu(Y) < \mu(X)$ holds, then X is called μ-stable. The semistable subcategory $\Pi_\mu \subset \mathcal{A}$ consists of all semistable objects X of slope μ. This stability data on $\mathcal{P}_h{}^0$ is extended a t-stability on $\mathcal{P}_h$ in the standard way (see [Br1], [GKR]) by declaring $\{\Pi_\mu[m]\}_{\mu\in\widehat{\mathbb{Q}},m\in\mathbb{Z}}$ to be the set of semistable subcategories ordered by $\Pi_\mu[m] < \Pi_\mu[n]$ for $m < n$. We call this t-stability on $\mathcal{P}_h$ the *μ-stability*. The fact that all exceptional objects are μ-stable goes back to J.-M. Drezet (see [Dz]).

6.2.1. LEMMA. *All exceptional objects of $\mathcal{P}_h$ are μ-stable.*

PROOF. Since all the exceptional objects of $\mathcal{P}_h$ are exhausted by shifts of exceptional objects sitting in the core $\mathcal{P}_h{}^0$, that is by E_i with $i \leqslant 0$ and $E_j[-1]$ with $j \geqslant 1$, it is enough to check only the semistability of these objects. These exceptional objects are obtained by consequent left and right mutations in the pair E_0, $E_1[-1]$. In terms of the Abelian category $\mathcal{P}_h{}^0$, the first pair of mutations is given by the universal extensions

$$\begin{gathered} 0 \longrightarrow E_1[-1] \longrightarrow E_{-1} \longrightarrow \mathrm{Ext}^1_{\mathcal{P}_h{}^0}(E_0, E_1[-1]) \otimes E_0 \longrightarrow 0\,, \\ 0 \longrightarrow \mathrm{Ext}^1_{\mathcal{P}_h{}^0}(E_0, E_1[-1])^* \otimes E_1[-1] \longrightarrow E_2 \longrightarrow E_0 \longrightarrow 0 \end{gathered} \tag{24}$$

and all the other mutations are given by the universal exact triples

$$\begin{gathered} 0 \longrightarrow E_{i-1} \longrightarrow \mathrm{Hom}_{\mathcal{P}_h{}^0}(E_i, E_{i+1}) \otimes E_i \longrightarrow E_{i+1} \longrightarrow 0\,, \text{ for } i \leqslant -1\,, \\ 0 \longrightarrow E_{i-1}[-1] \longrightarrow \mathrm{Hom}_{\mathcal{P}_h{}^0}(E_{i-1}, E_i)^* \otimes E_i[-1] \longrightarrow E_{i+1}[-1] \longrightarrow 0\,, \text{ for } i \geqslant 2\,. \end{gathered} \tag{25}$$

Under the identification of $\mathcal{P}_h{}^0$ with the category of the Kronecker modules over

$$H^* = \mathrm{Hom}(E_0, E_1)^*$$

the initial objects E_0, $E_1[-1]$ go to the irreducible 1-dimensional modules of the dimensions $(1,0)$ and $(0,1)$, which are clearly μ-stable. Further, the exceptional objects E_{-1}, $E_2[-1]$, obtained by the mutations (24), go to the modules of traceless and diagonal endomorphisms of H:

$$\begin{gathered} \mathrm{ad}(H) \hookrightarrow H \otimes H^*\,, \\ \mathbb{C} \cdot \mathrm{Id}_H \hookrightarrow H \otimes H^*\,, \end{gathered}$$

which are clearly μ-stable as well. All the other mutations (25) go to the standard reflections of these semistable modules in the category of quiver representations[1]. Recall (see details in [Dz]) that such the reflection takes a μ-stable module presented by a linear map $U \xhookrightarrow{f} H \otimes V$ (necessary injective) to a module presented by the linear map $\mathrm{coker}(f)^* \hookrightarrow H^* \otimes V^*$ followed by an isomorphism $H^* \otimes V^* \xrightarrow{\sim} H \otimes V^*$ induced by some fixed linear isomorphism $H \xrightarrow{\sim} H^*$. It is well known that such the reflection preserves μ-stability. □

[1] taking the source vertex to the target one and vice versa

6.3. General remarks on modular t-stability. Let Ψ be a modular t-stability on P_h. Then all Ψ-semistable objects are pure and their slopes strictly increase with the level. Indeed, in the decomposition of a semistable object X into the direct sum $X = \oplus_i X_i$, of pure objects, all X_i should be semistable of the same slope $\psi(X_i) = \psi(X)$. But distinguished triangle (18)

$$U_1 \otimes E_1[i-1] \longrightarrow X_i \longrightarrow U_0 \otimes E_0[i]$$

implies inequality $\psi(E_1[i-1]) < \psi(X_i) < \psi(E_0[i])$.

Further, if X, Y are Ψ-semistable with slopes $\psi(X) < \psi(Y)$ and an object Z is fitted into destabilizing distinguished triangle

$$X \longrightarrow Z \longrightarrow Y \longrightarrow X[1]$$

then in the Harder – Narasimhan decomposition $Z \rightsquigarrow (G_0, G_1, \dots, G_n)$, of Z, the slopes of all factors lay between the slopes of X, Y:

$$\psi(X) \leqslant \psi(G_0) < \psi(G_1) < \dots < \psi(G_n) \leqslant \psi(Y), \quad \forall i,$$

because the opposite inequalities: either $\psi(G_0) < \psi(X)$ or $\psi(G_n) > \psi(Y)$, would prohibit non zero maps $Z \longrightarrow G_0$ and $G_n \longrightarrow Z$ coming from the Harder – Narasimhan filtration.

The rest of this section is devoted to the proof of the following

6.3.1. PROPOSITION. *A modular t-stability on P_h is unique and coincides with μ-stability described in n° 6.2.*

The proof will consist of two steps. First of all, in n° 6.4, we completely describe all classes $m \in M = K_0(\mathrm{P}_h)$ that are realized by μ-semistable objects. This description is quite parallel to the geometric theory of continuous fractions and is of some own interest. Then, in n° 6.5, we combain this description with the above remarks in order to compare the μ-stability with an arbitrary modular t-stability Ψ and verify that they should have the same collection of the semistable subcategories ordered in the same way.

6.4. Description of μ-semistable classes. Let us depict a vector $m = x\,e_0 + y\,e_1 \in M$, which corresponds to the class $x \cdot [E_0] + y \cdot [E_1]$ by the point with coordinates (x, y) on fig. A and call such an integer point *stable*, if the corresponding class in $K_0(\mathrm{P}_h)$ can be realized by some μ-semistable object of P_h.

It is also convenient to change the coordinates (x, y) by the coordinates $(u_1, u_0) = (-y, x)$ computed w. r. t. the basis $(-e_1, e_0)$ formad by the classes of $E_1[-1]$, E_0 and redraw fig. A as it is shown on fig. B in order to focus on the classes of elements from the core $\mathrm{P}_h{}^0$, which are given by non negative (u_1, u_0). On fig. B, the slope $\mu = u_0/u_1$ is nothing but the usual geometric slope of the vectors and the asymptotic directions of hyperbola are the quadratic irrationals

$$\lambda^{\pm 1} = (-h \pm \sqrt{h^2 - 4})/2\,.$$

.4.1. LEMMA. *All positive stable classes, i. e. stable classes m satisfying $\chi(m, m) > 0$, are exhausted by multiples of exceptional vectors: $m = t \cdot e_i$ for some $t, i \in \mathbb{Z}$.*

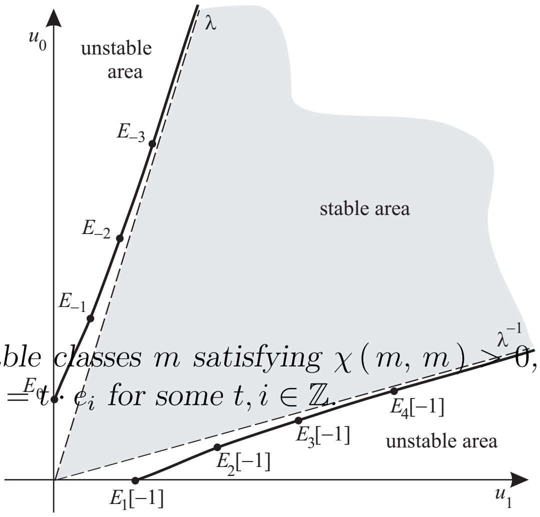

Fig. B. Description of stable classes.

Proof. Let X be μ semistable. Then X is pure. Moreover, we can assume that X is indecomposable, because of all direct summands of semistable objects should be also semistable of the same slope. By Lemma n° 4.4.1, all indecomposable objects in positive region are exhausted by exceptional objects. □

Now we are going to prove that all integer points inside the 'stable area' on fig. B can be realized by some μ-stable objects. It is enough realize only all irreducible integer vectors in the stable area (then their multiples will be realized by a direct sums of the representing semistable objects with themselves). It is well known that all irreducible vectors can be enumerated by walking through the vertexes of an infinite binary tree as follows.

Let us say that two irreducible integer vectors $u = (u_0, u_1)$, $v = (v_0, v_1)$ with non negative coordinates are *coterminous*, if

$$\det(u, v) \stackrel{\text{def}}{=} \det \begin{pmatrix} u_0 & v_0 \\ u_1 & v_1 \end{pmatrix} = -1\,.$$

We start with a triple of vectors $\{(0, 1), (1, 1), (1, 0)\}$ and will modify it by removing either the leftmost or the rightmost vector and taking the sum of two remaining vectors as the middle term of the resulting triple. Then we get a binary tree whose vertexes are labelled by all triples of irreducible vectors (u, v, w) with non negative coordinates such that $v = u + w$ and all three ordered pairs (u, v), (v, w), (u, w) are coterminous. The edges of tree correspond to the elementary transformations

$$\begin{tikzcd} & (u, v, w) \arrow[dl] \arrow[dr] & \\ (u, u + v, v) & & (v, v + w, w)\,. \end{tikzcd}$$

All the irreducible vectors with non negative coordinates will appear consequently as the middle terms of these triples. Those out of them that belong to the 'stable area' on fig. B can be inductively realized by μ-stable classes by means of the following

6.4.2. LEMMA. *Let $X, Y \in {\mathrm{P}_h}^0$ be μ-stable objects whose classes $([X], [Y])$ are coterminous. Then in any non trivial extension $0 \longrightarrow X \longrightarrow Z \longrightarrow Y \longrightarrow 0$ the middle term Z is μ-stable as well, classes $([X], [Z])$, $([Z], [Y])$ are coterminous, and both spaces $\mathrm{Hom}^0(Z, Y)$, $\mathrm{Hom}^0(X, Z)$ are non zero. Moreover, if (Z, Y) is not an exceptional pair, then $\mathrm{Hom}^1(Y, Z) \neq 0$, and if (X, Z) is not an exceptional pair, then $\mathrm{Hom}^1(Z, X) \neq 0$.*

Proof. Since $[Z] = [X] + [Y]$, the straightforward computation of the determinants shows that the pairs $([X], [Z])$, $([Z], [Y])$ are coterminous. This implies immediately that any destabilizing subobject of Z either coincides with Y or destabilizes Y (just imagine the corresponding picture), which implies that Z should be stable. Non-vanishing of the Hom^0-spaces is evident. To compute $\mathrm{Hom}^1(Y, Z)$, we apply $\mathrm{Hom}(Y, *)$ to our extension. Since $\mathrm{Hom}^0(Y, X) = \mathrm{Hom}^0(Y, Z) = 0$ by the μ-stability, we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}^0(Y, Y) \longrightarrow \mathrm{Hom}^1(Y, X) \longrightarrow \mathrm{Hom}^1(Y, Z) \longrightarrow \mathrm{Hom}^1(Y, Y) \longrightarrow 0$$

It easy to see that stable objects defined by a slope should be simple (see [Ru]). So, $\dim \mathrm{Hom}^0(Y, Y) = 1$ and $\mathrm{Hom}^1(Y, Z) = 0$ implies $\dim \mathrm{Hom}^1(Y, X) = 1$ and $\mathrm{Hom}^1(Y, Y) = 0$.

This is possible only if Y is exceptional and orthogonal to Z. Hence, Z is a multiple of an exceptional object, orthogonal to Y. Since Z is pure and simple, it is exceptional. So, (Z,Y) is an exceptional pair. $\square$

6.4.3. COROLLARY. *μ-semistable classes in $K_0(\mathbb{P}_h)$ are exhausted by the exceptional vectors and integer vectors $u=(u_0,u_1)$ satisfying inequality $\lambda^{-1}<\mu(u)=u_0/u_1<\lambda$, where $\lambda=(h+\sqrt{h^2-4})/2$. Moreover, for any distinct semistable vectors u, v with $\lambda^{-1}<\mu(u)\leqslant\mu(v)<\lambda$ there exists a chain of μ-semistable objects $(X_1,X_2,\dots,X_n)$ in the core $\mathbb{P}_h{}^0$ such that $[X_1]=u$, $[X_n]=v$, and $\mathrm{Hom}^0(X_i,X_{i+1})\neq 0$.*

6.5. Proof of the proposition n° 6.3.1. Let $\{\Pi_\psi\}_{\psi\in\Psi}$ be an arbitrary modular stability. First of all let us compare Ψ with the μ-stability on the Abelian core of $\mathbb{P}_h$.

6.5.1. LEMMA. *An object $X\in\mathbb{P}_h{}^0$ is Ψ-semistable iff it is μ-semistable. Moreover*

$$\psi(X)>\psi(Y)\Longleftrightarrow\mu(X)>\mu(Y)$$

for any pair of semistable objects $X,Y\in\mathbb{P}_h{}^0$.

PROOF. We will use the induction over the sum $s=u_0+u_1$ of coordinates of the classes $[X]$, $[Y]$, which lay inside the coordinate quadrant sown on fig. B. The base of the induction, when X, Y are among $\{E_1[-1]\,,\,,E_0\}$, is evident.

Let X be μ-semistable and $s([X])=m$. If X is not Ψ-semistable, then it has non trivial Harder – Narasimhan filtration $X\rightsquigarrow(G_0,\,G_1,\,\dots\,,\,G_n)$ w.r.t. Ψ. Since each G_i is Ψ-semistable, lies in the core, and $s([G_i])<s([X])$, we deduce from the inductive assumption that all G_i are μ-semistable and their μ-slopes are ordered precisely like their ψ-slopes. So, they give μ-destabilizing filtration for X as well/ This contradicts to μ-semistablity of X and proves that X is Ψ-semistable.

The same arguments show that any Ψ-semistable object X should be also μ-semistable. It remains to check that for any Ψ-semistable Y with $s([Y])<m$ the inequalities $\psi(Y)<\psi(X)$ and $\mu(Y)<\mu(X)$ are equivalent. Let $\mu(Y)<\mu(X)$. Consider a chain of μ-semistable objects $(X_1,X_2,\dots,X_n)$ from the Corollary n° 6.4.3. By the inductive assumption all X_i are Ψ-semistable. Then their ψ-slopes should be ordered in the same way as μ-slopes, because of $\mathrm{Hom}^0(X_i,X_{i+1})\neq 0$. Since Ψ is modular, we conclude that $\psi(Y)<\psi(X)$. $\square$

Let $\Psi'\subset\Psi$ be a collection formed by all Ψ-semistable subcategories Π_ψ laying inside the Abelian core $\mathbb{P}_h{}^0$ and all their shifts. It follows from the lemma that Ψ' coincides with the collection of all μ-semistable subcategories and is ordered in the same way. In fact, $\Psi'=\Psi$, because the collection Ψ' suffice to build the Harder – Narasimhan filtration for any non zero object X in $\mathbb{P}_h$. Indeed, since $X=\oplus_i X_i$, where X_i is pure of level i, it is enough to construct the Harder – Narasimhan filtration for X pure of level 0. But in this case μ-filtration gives what we need. This finishes the proof of Proposition n° 6.3.1.

6.6. Remark on geometric t-structures on $\mathbb{P}_h$. Finest modular t-stabilities on $\mathbb{P}_h$ described above lead to a family of t-structures on $\mathbb{P}_h$ obtained as follows: draw any line $u_0=\vartheta\,u_1$ on fig. B and fuse together all the semistable categories laying in the same half plane bounded by this line. Then we get an Abelian core of some t-structure on $\mathbb{P}_h$. We left to the reader to check that this Abelian core is equivalent to the category of coherent sheaves on appropriate weighted projective line[1] whose precise equation (in the notations of [GeLe]) depends on ϑ, λ.

It is also instructively to compare our description of *all* (fine enough) t-stabilities on $\mathcal{P}_h$ with 'moduli space of T. Bridgeland's stability data' described in a recent paper[1] [Ma]: we see immediately that Bridgeland's slope functions serve precisely the all possible order preserving inclusions of our semistable slope sets into $\mathbb{R}$ as well as their fusions via drawing a border line as above.

§7. Some remarks towards higher dimensions.

7.1. Categories generated by exceptional Hom-collections. We say that an ordered collection of objects $(E_0, E_1, E_2, \dots, E_n)$ is *an exceptional* Hom-*collection*, if any ordered pair (E_i, E_j) inside it is an exceptional Hom-pair.

Let $\mathcal{E}$ be a triangulated category generated by an exceptional Hom-collection

$$(E_0, E_1, E_2, \dots, E_n)\,.$$

There is a convenient t-structure on $\mathcal{E}$ coming from the quiver representation theory (comp. with [Bo1], [Bo2]). It is formed by a pair of full subcategories $(\mathcal{E}^{\leqslant 0}, \mathcal{E}^{\geqslant 0})$, where $\mathcal{E}^{\leqslant 0}$ is generated by all $E_i[m]$ with $m \geqslant -i$ (i. e. by E_0, $E_1[-1]$, … , $E_n[-n]$ and all their positive twists $E_i[-i+p]$, $p \geqslant 1$), and $\mathcal{E}^{\geqslant 0}$ is generated by all $E_i[m]$ with $m \leqslant -i$ (i. e. by the same E_0, $E_1[-1]$, … , $E_n[-n]$ and all their negative twists). The core of this t-structure is an Abelian category $\mathcal{A}$, of representations of the finite dimensional algebra $\operatorname{End}\left(\bigoplus_{i=0}^{n} E_i\right)$, or equivalently, the category of complexes

$$V_0 \otimes E_0 \longrightarrow V_1 \otimes E_1 \longrightarrow \cdots \longrightarrow V_n \otimes E_n \tag{26}$$

(thanks to the orthogonality conditions on E_i this complex has canonical convolution, i. e. gives an orthogonal decomposition of an object X fitted to this complex from the left:

$$X \longrightarrow V_0 \otimes E_0 \longrightarrow V_1 \otimes E_1 \longrightarrow \cdots \longrightarrow V_n \otimes E_n\,.$$

it is functorial in X). The morphisms in $\mathcal{A}$ are morphisms of complexes (commuting with the differentials).

This result of A. Bondal is especially demonstrative on the langauge of t-filtrations. Indeed, the only non obvious out of the t-structure axioms here is that any object $X \in \mathcal{E}$ is included into a distinguished triangle

$$X^{\leqslant 0} \longrightarrow X \longrightarrow X^{\geqslant 1} \longrightarrow X^{\leqslant 0}[1]\,.$$

[1] see [GeLe] for details of the corresponding theory

[1] which has appeared when this paper was almost prepared already

To check it, we start with an orthogonal decomposition of X w. r. t. to $(E_0, E_1, E_2, \dots, E_n)$ (see [GKR, §2]), which provides X with t-filtration

$$X \rightsquigarrow \left(\bigoplus_i V_0^i \otimes E_0[-i] \,,\ \dots\ ,\ \bigoplus_i V_0^i \otimes E_0[-i] \right) .$$

and then rearrange the quotients in accordance with the required ordering:

$$\dots E_0 < E_1[-1] < \dots < E_n[-n] < E_0[1] < E_1[0] \dots$$

This is possible, because of the orthogonality conditions

$$\mathrm{Hom}^1(E_i[k], E_j[m]) = 0 \quad \text{unless } i < j \text{ and } m - k = -1 \,.$$

7.2. R-stabilities on E . There is a huge family of fine modular t-stabilities on E induced by fine R-stabilities on the Abelian core A . Recall that by Rudakov's results [Ru] we can build t-stabilities from preorders on A which satisfy the seesaw and finiteness conditions listed in n° 3.2. On the level of K_0(E) such the preorderings have quite transparent geometric description we are going to explain now. For simplicity we will assume that $n = 2$, that is, E is generated by an exceptional triple E_0, E_1, E_2. The general case is completely similar and is obtained by replacing all 'triangles' by 'simplexes' and all 'lines' by 'hyperplanes'.

Write S for a sphere obtained by the factorization of real vector space $\mathbb{R} \underset{\mathbb{Z}}{\otimes} K_0$(E) through the natural action of multiplicative group $\mathbb{R}_{>0}$, of positive reals, by the scalar delatations. So, the shift functor $X \mapsto X[1]$ acts on S by the central symmetry $v \mapsto -v$. Then the classes of proper elements[1] of A are presented on S by the rational points of the triangle $\sigma = (e_0 e_1 e_2)$ spanned by the classes of exceptional generators E_0, E_1, E_2.

The seesaw property of a preorder on A means that the arcs cutted out of all rational geodesic cycles on S by σ should be ordered in such a way that this ordering induces a transitive relation on a set of rational points of σ. In other words, all the rational geodesic arcs inside σ should be equipped with an arrow (which goes along the arc in the increasing direction) in such a way that no cyclic triangles appear, i. e. all the rational geodesic triangles inside σ shoud have the maximal, the minimal and the middle vertex as on fig. C. Given such a rational triangle, let us call its edge joining the extreme vertexes (that is, the edge opposite to the middle vertex) *a base* of this triangle.

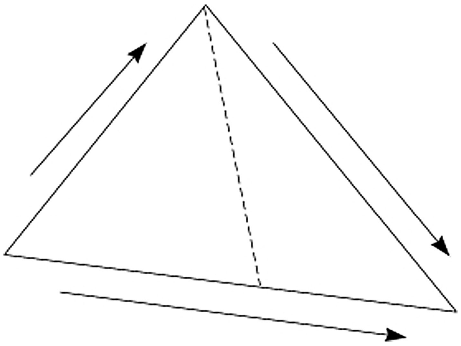

Fig. C. Oriented geodesic triangle.

It follows at once from the total transitivity that for each rational triangle inside σ there is an *irrational* line ℓ coming from the middle vertex to the base (the dashed line on fig. C) such that all the rational lines coming from the middle vertex to the base and laying in the left hyperplane bounded by ℓ are ordered like the left side of the triangle and all rational lines on the right side of ℓ are ordered like the right side of the triangle. Moreover, the transitivity implies also that these irrational lines have no intersection points inside σ.

[1] i. e. the actual complexes (26) but not their virtual differences

Vice versa, each continuous distribution of irrational lines without intersections inside σ provides all the rational geodesic arcs inside σ with a preorder suitable to define R-stability. Indeed, taking a continuous orientation of each irrational line of the family, we get well defined continuous notions of the right side and the left side of the line running through the family. Given a rational line ϱ, let us intersect it with some irrational line out of our family and order by drawing an arrow from left to right w. r. t. the chosen irrational line. Clearly, this orientation does not depend on a choice of an irrational line. And all together these orientations provide rational points of σ with transitive preorder.

In general case, the triangle σ should be replaced by the simplex spanned by the classes of $E_0, E_1, \dots, E_n$ and a continuous distribution of irrational lines should be replaced by a continuous distribution of irrational hyperplanes, that is by appropriate irrational curve on the real Grassmannian of codimension two subspaces in $\mathbb{R} \underset{\mathbb{Z}}{\otimes} K_0(\mathbb{E})$.

Let us remark finally that the Bridgeland's stabilities correspond to the families of irrational lines (hyperplanes) forming *a pencil* of lines (hyperplanes) passing through a fixed point outside σ. This just a quite thin subset in the whole set of fine t-stabilities on A.

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