BIRATIONAL PROPERTIES OF PENCILS OF DEL PEZZO SURFACES OF DEGREE 1 AND 2.

MIKHAIL GRINENKO

Abstract. In this paper we study the birational rigidity problem for smooth Mori fibrations on del Pezzo surfaces of degree 1 and 2. For degree 1 we obtain a complete description of rigid and non-rigid cases.

1. Notions and results.

Due to progress in the Minimal Model Program, the classification problem in the modern birational geometry for varieties of negative Kodaira dimension can be formulated as follows: given a class of birational equivalency, to describe all Mori fibrations and birational maps between them.

Recall that a normal variety V with at most $\mathbb{Q}$-factorial terminal singularities is called *a Mori fibration* if there exists an extremal contraction $\varphi : V \to S$ of fibering type, i.e.

1) φ is a morphism with connected fibers onto a normal variety S and $\dim S < \dim V$;
2) $-K_V$ is φ-ample and the relative Picard number is equal to 1: $\rho(V/S) = \rho(V) - \rho(S) = 1$.

Note that often comparing birational classes of varieties, we suffice to know Mori structures rather than Mori fibrations themselves (roughly speaking, Mori fibrations modulo birational maps over the base). From this viewpoint, the following class of varieties is rather important (and simple for describing):

Definition 1.1. *A Mori fibration V/S is said to be birationally rigid, if any birational map $\chi : V \dashrightarrow V'$ onto another Mori fibration V'/S' is birational over the base ("square"), i.e., there exists a birational map*

This work was partially supported by the grants RFBR no. 99–01–01132, Grant of Leading Scientific Schools no. 96–15–96146, and INTAS-OPEN 97/2072.

$\psi : S \dashrightarrow S'$ making the following diagram to be commutative:

$$\begin{array}{ccc} V & \overset{\chi}{\dashrightarrow} & V' \\ \downarrow & & \downarrow \\ S & \overset{\psi}{\dashrightarrow} & S' \end{array}$$

Now let $\rho : V \to S$ be a Mori fibration. Then any non-empty linear system $\mathcal{D}$ on V is a subsystem of $|a(-K_V) + \rho^*(A)|$, where $a = \mu(V, \mathcal{D}) \geq 0$ is the so-called *quasi-effective threshold*, and A is a divisor on S. The behaviour of linear systems under birational maps between Mori fibrations is given by the following theorem ([1], theorem 4.2):

PROPOSITION 1.2. *Let $\rho : V \to S$ and $\rho' : V' \to S'$ be Mori fibrations, $\chi : V \dashrightarrow V'$ a birational map, $\mathcal{D}' = |n'(-K_{V'}) + \rho'^*(A')|$ a very ample linear system on V', where A' is an ample divisor on S'. Denote $\mathcal{D} = \chi_*^{-1}\mathcal{D}' \subset |n(-K_V) + \rho^*(A)|$. Then:*

(i) *$n \geq n'$ always, and if $n = n'$ then χ is square;*
(ii) *if a log pair $K_V + \frac{1}{n}\mathcal{D}$ is canonical and numerically effective, then χ is an isomorphism inducing also an isomorphism between the bases.*

The statement (i) of the proposition allows us to describe various Mori structures on V, while (ii) can be used to get Mori models (i.e., various Mori fibrations birational to V).

There is another important number arising from the maximal singularities method (now playing the key role in the birational classification). This is so-called *the adjunction threshold.*

DEFINITION 1.3. *A non-negative number $\alpha(V, \mathcal{D})$ is called the adjunction threshold of a pair $(V, \mathcal{D})$, where $\mathcal{D}$ is a non-empty linear system without fixed components, V is projective and non-singular in codimension 1, if $\alpha(V, \mathcal{D})$ is the smallest number such that for any positive integers m and n with $\frac{m}{n} > \alpha(V, \mathcal{D})$ the linear system $|nD + mK_V|$ is empty, where D is an element of $\mathcal{D}$.*

The following statement clarifies the role of the adjunction threshold ([10], 2.1.):

PROPOSITION 1.4. *Let $\chi : V \to V'$ be a birational map between terminal $\mathbb{Q}$-factorial varieties, $\mathcal{D}'$ a non-empty linear system on V' without fixed components, and $\mathcal{D} = \chi_*^{-1}\mathcal{D}'$ its strict transform. If $\alpha(V, \mathcal{D}) > \alpha(V', \mathcal{D}')$, then a pair $K_V + \frac{1}{\alpha(V,\mathcal{D})}\mathcal{D}$ is not canonical (in other words, $\mathcal{D}$ has a maximal singularity)*

This condition "to be not canonical" is strong enough, and we can often either prove its impossibility, or decrease the adjunction threshold applying an "untwisting" birational automorphism or jumping to another Mori model (as in the Sarkisov program). A relation between both the thresholds is given by the following obvious lemma:

LEMMA 1.5. *Let $\mathcal{D} \subset |n(-K_V) + \rho^*(A)|$ be a non-empty linear system without fixed components on a Mori fibration $\rho : V \to S$. Always $n = \mu(V, \mathcal{D}) \geq \alpha(V, \mathcal{D})$, and if A is effective, then the equality holds; if $C \circ \rho^*(A) < 0$ for a general curve C covering the base, then $\mu(V, \mathcal{D}) > \alpha(V, \mathcal{D})$.*

The key idea is that the relation between these thresholds determines whether a variety is rigid. For Mori fibrations that are pencils of del Pezzo surfaces, it looks as follows:

CONJECTURE 1.6. *Let $V/\mathbb{P}^1$ be a pencil of del Pezzo surfaces of degree 1,2 or 3 (we assume it to be a Mori fibration). $V/\mathbb{P}^1$ is birationally rigid if and only if $\alpha(V, \mathcal{D}) = \mu(V, \mathcal{D})$ for any linear system without fixed components (i.e., a linear system $|n(-K_V) - F|$ either is empty, or has a base divisor; here F is the class of a fiber).*

Let us note that "nearly all" (smooth) pencils of del Pezzos of degree 1, 2 or 3 are sufficiently "twisted" along the base, and in [9] their rigidity is proved. In this paper, we deal with all cases of degree 1 (theorem 2.6) and advance in degree 2 (theorem 3.1). In particular, conjecture 1.6 turns out to be true for smooth cases of degree 1.

The paper is based on using the maximal singularities method ([4], [6]) in its most perfect kind [10].

Everywhere in this paper the characteristic of the ground field is assumed to be 0.

The author would like to thank the Directors and the stuff of the Max-Planck Institute for Mathematics in Bonn for hospitality and excellent conditions during the work.

2. SMOOTH VARIETIES WITH A PENCIL OF DEL PEZZO SURFACES OF DEGREE 1.

2.1. **The essential construction.** We follow to the idea given in [5].

Let $\rho : V \to \mathbb{P}^1$ be a smooth Mori fibration on del Pezzo surfaces of degree 1, $\operatorname{Pic} V = \mathbb{Z}[-K_V] \oplus \mathbb{Z}[F]$, where F is the class of a fiber. By the Grauert theorem, $\rho_*\mathcal{O}(-2K_V + mF)$ is a vector bundle of rank 4 on $\mathbb{P}^1$. We may choose m such that

$$\mathcal{E} = \rho_*\mathcal{O}(-2K_V + mF) \simeq \mathcal{O} \oplus \mathcal{O}(n_1) \oplus \mathcal{O}(n_2) \oplus \mathcal{O}(n_3),$$

where $0 \leq n_1 \leq n_2 \leq n_3$. Suppose $b = n_1 + n_2 + n_3$.

Note that there exists a special section s_B:

$$s_B = Bas| - K_V + kF|$$

for $k \gg 0$. Obviously, any fiber $S \in |F|$ is smooth at the point $s_B \cap S$, which is also a unique base point of $| - K_S|$.

Let $\pi : X \to \mathbb{P}^1$ be a natural projection for $X \stackrel{\text{def}}{=} \mathbf{Proj}\ \mathcal{E}$, $\varphi : V \to X$ a morphism defined by $| - 2K_V + mF|$. Then $\rho = \pi \circ \varphi$. It is easy to see that if $Q \subset X$ is the image of φ, then $\varphi : V \to Q$ is a degree 2 finite morphism branched along a smooth divisor $R_Q = R \cap Q$ that does not intersect $t_B \stackrel{\text{def}}{=} \varphi(s_B)$, where $R \subset X$ is a divisor.

Q is restricted to any fiber T of $X/\mathbb{P}^1$ as a non-degenerated quadratic cone with $T \cap t_B$ as its vertex, and $R|_T$ is a cubic surface. So if M is the class of the tautological bundle on X and L is the class of a fiber, we have

$$Q = 2M + aL; \quad R = 3M + cL.$$

Let t_0 be a section corresponding to the surjection $\mathcal{E} \to \mathcal{O} \to 0$, l the class of a line in $L \cong \mathbb{P}^3$; set $t_B \sim t_0 + \varepsilon l$, where ε is obviously nonnegative.

Choose a very ample divisor $H \sim M + \beta l$, $\beta > 0$. So $D = H \cap Q$ is a conic bundle with $N = t_B \circ H$ degenerations. We may suppose that all fibers of D are reduced and all its singular points are du Val $\mathbf{A_1}$. It is easy to compute that $K_D^2 = 8 - 2b - 2\beta - 3a$. Let us blow up all singular points of D and then contract all (-1)-curves (there will be $2N$ of them) onto a ruled surface D'. We have

$$K_D^2 + 2N = K_{D'}^2 = 8,$$

hence

$$N = b + \beta + \frac{3}{2}a.$$

Note that a must be even. Set $a = 2a'$. Then,

$$N = t \circ H = (t_0 + \varepsilon l) \circ (M + \beta l) = \beta + \varepsilon,$$

and we obtain $\varepsilon = b + 3a'$. Since $R|_Q \cap t_B = \emptyset$, we have $0 = (t_0 + \varepsilon l) \circ (3M + cL) = 3\varepsilon + c$. So, we have proved the following lemma:

Lemma 2.1. *The following equalities hold:*

$$\begin{aligned} &Q = 2M - \tfrac{2}{3}(b - \varepsilon)L, \\ &t_B = t_0 + \varepsilon l, \\ &R = 3M - 3\varepsilon L. \end{aligned}$$

In the sequel we will always assume $b > 0$. Really, if $b = 0$, then $Q = 2M$ and $\varepsilon = 0$, whence $V \cong \mathbb{P}^1 \times S$ for some smooth del Pezzo surface S. So V is rational and even not a Mori fibration because of the Picard group rank.

LEMMA 2.2. *The only following two cases may occur:*

1) $\varepsilon = 0$, $2n_2 = n_1 + n_3$, n_1 *and* n_3 *are even; so we have*

$$Q = 2M - 2n_2L, \quad t_B = t_0, \quad R = 3M;$$

2) $\varepsilon = n_1 > 0$, $n_3 = 2n_2$, n_1 *is even and* $n_2 \geq 3n_1$*; in this case*

$$Q = 2M - 2n_2L, \quad t_B = t_0 + n_1 l, \quad R = 3M - 3n_1L.$$

PROOF. Suppose first $\varepsilon > 0$. Then $\varepsilon \geq n_1$ because of irreducibility of t_B. Since R is irreducible and curves of the class $t_0 + n_2 l$ sweep any effective divisor of the class $M - n_3L$, we have $(t_0 + n_2 l) \circ R = 3n_2 - 3\varepsilon \geq 0$, i.e., $\varepsilon \leq n_2$.

Let $n_1 = n_2 = n_3$. Since $t_B \circ (M - n_1L) = 0$, the linear system $|M - n_1L|$ is two-dimensional, and any of its elements is irreducible, there exists at least one-dimensional subsystem in it that contains t_B. Let $C \in |M - n_1L - t_B|$ be a general element, then

$$Q \circ C = S + S' \sim 2M^2 - 2\left(\frac{1}{3}(b - \varepsilon) + n_1\right) ML,$$

where S and S' are ruled surfaces covering the base. Note that there exists only a unique curve of the class t_0, and $t_B \cap t_0 = \emptyset$. At least one of these surfaces, say, S, containes both t_0 and t_B. It is easy to see that S is of the class $M^2 - 2n_1ML$, whence $S' \sim M^2 - \frac{4}{3}n_1ML$. Since $S \neq S'$, then $t_0 \not\subset S'$, and denoting by P the intersection of two general elements of $|M - n_1L|$, we can see that $\dim(S' \cap P) = 0$ (note that $\operatorname{Bas}|M - n_1L| = t_0$). We get a contradiction:

$$0 \leq S' \circ P = (M^2 - \frac{4}{3}n_1ML) \circ (M^2 - 2n_1ML) = -\frac{n_1}{3} < 0.$$

So if $\varepsilon \geq n_1$ then $n_1 < n_3$; moreover, $n_1 > 0$ because of irreducibility of R_Q (otherwise, if $n_1 = 0$, there should be at least 1-dimensional family of lines of the class t_0 sweeping a surface, and R_Q should contain it).

Let G be an irreducible ruled surface that is the complete intersection of general elements of $|M - n_2L|$ and $|M - n_3L|$. Let us show that $t_B \subset G$. Indeed, G has the type $\mathbf{F}_{\mathbf{n_1}}$ with t_0 as its exceptional section. Assume the converse, i.e., $t_B \not\subset G$. Then, for a general fiber $S \in |L|$, the line $G|_S$ does not lie on $Q|_S$. So $G \not\subset Q$, and we may suppose that

$G \circ Q = t_0 + C$, where C is a curve with no t_0 as a component. It is easy to compute that

$$C \sim t_0 + 2\left(b - n_2 - n_3 - \frac{b-\varepsilon}{3}\right) l,$$

and

$$2\left(b - n_2 - n_3 - \frac{b-\varepsilon}{3}\right) \geq n_1,$$

which, taking into account $\varepsilon \leq n_2$, gets a contradiction: $n_1 > \frac{4}{3}n_1$.

So $t_B \subset G$, hence $n_2 < n_3$. In fact, the equality should led $\dim|M - n_3L|$, thus $\operatorname{Bas}|M - n_3L| = t_0$, and we could always find G such that $t_B \not\subset G$.

By T we denote a unique effective element of $|M - n_3L|$. We show that $T|_Q = 2G$. Indeed, assuming the converse, i.e., $t|_Q = G + G'$, $G \neq G'$, and taking into account that $t_B \subset G \cap G'$ and $t_0 \subset G$, we see $t_0 \not\subset G'$. Then

$$G'|_G = t_0 + \#(t_0 \cap t_B = \varepsilon - n_1) \text{ fibers}.$$

Choose a general $S \in |M - n_1L|$. It is clear that $S|_T$ is a ruled surface of the type $\mathbf{F}_{\mathbf{n_2}}$ with the exceptional section t_0, and $G \cap S|_T = t_0$. Let $C = G' \cap S|_T$ be an irreducible curve. We have $C \circ t_0 = \varepsilon - n_1$, $Q \circ T \circ S = t_0 + C$, whence

$$C \sim t_0 + 2\left(n_2 - \frac{b-\varepsilon}{3}\right) l.$$

This yields $C \circ t_0 = n_2 - \frac{2}{3}(b - \varepsilon) = \varepsilon - n_1$, and we get a contradiction again:

$$\varepsilon = n_1 + n_2 - 2n_3 < 0.$$

So we have

$$T \circ Q = 2M^2 - 2\left(n_3 + \frac{b-\varepsilon}{3}\right) ML = 2G = 2(M^2 - (n_2 + n_3)ML),$$

thus $2n_2 = n_1 + n_3 - \varepsilon$. If ε were greater than n_1, then R_Q should contain G, because G is covered by curves of the class $t_0 + n_1 l$. This contradicts to irreducibility of R_Q. So $\varepsilon = n_1$ and then $2n_2 = n_3$. In order to show $n_2 \geq 3n_1$, we note that $S_Q = S \cap Q$ is a conic bundle without degenerations, i.e., a ruled surface of the type $\mathbf{F}_{\mathbf{n_2}}$. Then R cuts off an effective curve on S_Q that does not contain t_0, and t_0 itself. We get $3n_3 - 3n_1 \geq 5n_2$, which yields $n_2 \geq 3n_1$.

It remains to show that n_1 is even. Let $C \sim t_0 + n_1 l$ be a general irreducible curve on G, and C_V its inverse image on V. Then C_V covers C with the ramification divisor of degree $t_B \circ C = n_1$. Clearly, this degree must be even. This completes the case $\varepsilon > 0$.

Now let $\varepsilon = 0$. We may assume $n_1 < n_3$. Take $T \in |M - n_3L|$. It is easy to observe that $T = \mathbf{Proj}\ \mathcal{E}_T$, where

$$\mathcal{E}_T = \mathcal{O} \oplus \mathcal{O}(n_1) \oplus \mathcal{O}(n_2).$$

Since T can be covered by curves of the class $t_0 + n_2 l$ and $Q \sim 2M - \frac{2}{3}bL$ is irreducible, then $n_2 \geq \frac{b}{3}$, i.e., $2n_2 \geq n_1 + n_3$, and $n_2 > n_1$ since $n_3 > n_1$.

Further, let $C \sim t_0 + n_1 l$ be irreducible. Since $b = n_1 + n_2 + n_3 > 3n_1$, then $C \circ Q = 2(n_1 - \frac{b}{3}) < 0$, and therefore $C \subset Q$. A ruled surface $G \sim (M - n_2L) \circ (M - n_3L) = M^2 - (n_2 + n_3)ML$ has the type $\mathbf{F_{n_2}}$ and can be covered by curves of the same class as C, so $G \subset Q$. Therefore, $T|_Q = G + G'$ for some G'.

Let $M_T = M|_T$ be the tautological divisor on T, $L_T = L|_T$. Then G has the class $M_T - n_2L_T$ in T, $Q|_T \sim 2M_T - \frac{2}{3}bL_T$, whence $G' \sim M_T - (\frac{2}{3}b - n_2)L_T$. Note that $\frac{2}{3}b - n_2 > n_1$ since $n_1 < n_2$.

But T has only one divisor of the form $M_T - xL_T$ for $x > n_1$: this is $M_T - n_2L_T$. So $\frac{2}{3}b - n_2 = n_2$, hence $n_1 + n_3 = 2n_2$.

Now, let $C \subset Q$ be a general irreducible curve of the class $t_0 + n_1 l$, C_V its inverse image on V. Then C_V is the double cover of C branched along a divisor of degree $R \circ C = 3n_1$, whence n_1 and n_3 are even. Lemma 2.2 is proved.

Such a construction of V allows us to deal completely with intersections of cycles on V. Denote G_V, F, and H the inverse images of $G \sim (M - n_2L) \circ (M - n_3L)$, $L_Q = L|_Q$, and $M_Q = M|_Q$ on V respectively. Then, let $2s_0$ be the pull back of t_0, and $2f$ the pull back of a line in a fiber of L_Q.

LEMMA 2.3. *$\overline{\mathbf{NE}}(V) = \mathbf{NE}(V) = \mathbb{R}_+[s_0] \oplus \mathbb{R}_+[f]$, and:*

1) *if $\varepsilon = 0$, then $K_V = -G_V + (\frac{1}{2}n_1 - 2)F$, $H = 2G_V + n_3F$, $K_V^2 = s_0 + (4 - n_2)f$, $s_0 \circ F = f \circ G_V = 1$, $s_0 \circ G_V = -\frac{1}{2}n_3$, $f \circ F = 0$.*
2) *if $\varepsilon = n_1 > 0$, then $K_V = -G_V - (\frac{1}{2}n_1 + 2)F$, $H = 2(G_V + n_2F)$, $s = s_0 + \frac{1}{2}n_1 f$, $K_V^2 = s_0 + (4 + \frac{3}{2}n_1 - n_2)f$, $s_0 \circ F = f \circ G_V = 1$, $s_0 \circ G_V = -\frac{1}{2}n_3 = -n_2$, $f \circ F = 0$.*

PROOF. It can be easily checked by the following way. First blow up t_B on Q with an exceptional divisor S; then take the double cover branched along a smooth divisor composed from the pre-image of R_Q and S; after that, contract the pre-image of S onto s_B.

LEMMA 2.4. *Let $\varepsilon = 0$ (i.e., $s_B = s_0$). If $|n(-K_V) - mF|$, $m > 0$, has no fixed components, then $n_3 = 2$.*

Proof. Let $D \in |n(-K_V) - mF|$. Since

$$D|_{G_V} \sim ns_0 - \left(m + n\left(\frac{n_1}{2} + n_3 - n_2 - 2\right)\right) f$$

is an effective curve, we obtain

$$0 < \frac{m}{n} \leq 2 + n_2 - n_3 - \frac{n_1}{2} = 2 - \frac{n_3}{2}.$$

It only remains to take into account that n_3 is positive and even. The lemma is proved.

Lemma 2.5. *Let $\varepsilon = n_1 > 0$ (so $s_B \neq s_0$). If $|n(-K_V) + mF|$ has no fixed components, then $m > 0$.*

Proof. Let $D \in |n(-K_V) + mF|$. Since

$$D|_{G_V} \sim ns_0 + [m + n(n_1 - n_2 + 2)]f$$

is an effective curve, we have $2 + n_1 - n_2 + \frac{m}{n} \geq 0$. Using $n_2 \geq 3n_1$, we see that $2 + n_1 - n_2 \leq 2 - 2n_1 < -2$, whence $m > 0$. The lemma is proved.

2.2. **Results about rigidity.**

Theorem 2.6. *Conjecture 1.6 is true for smooth Mori fibrations on del Pezzo surfaces of degree 1 (over $\mathbb{P}^1$).*

In order to prove this theorem, we shall use the following proposition:

Proposition 2.7. *Let $V/\mathbb{P}^1$ be a smooth Mori fibration on del Pezzo surfaces of degree 1, V'/S' another Mori fibration, $\chi : V \dashrightarrow V'$ a birational map. Suppose that*

$$\mathcal{D}' = |n'(-K_{V'}) + \textit{pull back of an ample divisor from the base}\ |$$

is a very ample linear system. Suppose aslo

$$\mathcal{D} = \chi_*^{-1}\mathcal{D}' \subset |n(-K_V) + mF|$$

for some $m \geq 0$.

Then $n = n'$ (so χ is birational over the base), possibly except the case $\varepsilon = n_1 = 0$, $n_2 = 1$, $n_3 = 2$.

Proof of the proposition. By lemma 1.5, $\mu(V', \mathcal{D}') = n' = \alpha(V', \mathcal{D}')$ and $\mu(V, \mathcal{D}) = n = \alpha(V, \mathcal{D})$. If $n = n'$, the assertion follows from proposition 1.2.

Suppose $n > n'$. Then a log pair $K_V + \frac{1}{n}\mathcal{D}$ is not canonical, i.e., the linear system $\mathcal{D}$ has a maximal singularity over some point. But using [9], we may say a little more.

Namely, let D_1, D_2 be general elements of $\mathcal{D}$. We observe that

$$D_1 \circ D_2 = \begin{cases} n^2 s_0 + ((4-n_2)n^2 + 2mn)f, & \text{if } \varepsilon = 0 \\ n^2 s_0 + ((4+\frac{3}{2}n_1 - n_2)n^2 + 2mn)f, & \text{if } \varepsilon = n_1 > 0 \end{cases}$$

Put down $D_1 \circ D_2 = Z^h + Z^v$, where Z^h and Z^v are horizontal and vertical cycles respectively. Then there exist a geometric discrete valuation ν centered at a point B_0 over some point $t \in \mathbb{P}^1$, and a positive number $e = \nu(\mathcal{D}) - n\delta$, where δ is the canonical multiplicity with respect to ν, such that for the component $Z^v_t \subset F_t$ of the cycle Z^v in the fiber F_t over the point t we have

$$\deg Z^v_t < \begin{cases} (4-n_2)n^2 + 2n\frac{e}{\nu(F_t)}, & \text{if } \varepsilon = 0 \\ (4+\frac{3}{2}n_1 - n_2)n^2 + 2n\frac{e}{\nu(F_t)}, & \text{if } \varepsilon = n_1 > 0, \end{cases}$$

where $\deg Z^v_t \stackrel{\text{def}}{=} Z^v_t \circ (-K_V)$ (this is so-called *the supermaximal singularity condition*, in the notions of [9]). Besides, there exist positive numbers Σ_0, $\Sigma'_0 \le \Sigma_0$, and Σ_1 such that multiplicities m^h and m^v of the cycles Z^h and Z^v_t at B_0 satisfy

$$(\Sigma_0 + \Sigma_1)(m^h \Sigma_0 + m^v \Sigma'_0) > (2n\Sigma_0 + n\Sigma_1 + e)^2.$$

Note that $m^h \le \deg Z^h \stackrel{\text{def}}{=} Z^h \circ F = n^2$ and $m^v \le 2 \deg Z^v_t$. So, in the case $\varepsilon = n_1 > 0$ we have

$$(\Sigma_0 + \Sigma_1)((9 + 3n_1 - 2n_2)n^2\Sigma_0 + 4ne) > (2n\Sigma_0 + n\Sigma_1 + e)^2,$$

hence

$$(2n_2 - 3n_1 - 5)\Sigma_0(\Sigma_0 + \Sigma_1) + (n\Sigma_1 - e)^2 < 0,$$

i.e., $2n_2 < 5 + 3n_1$. Taking into account the condition $n_2 \ge 3n_1$, we obtain $n_1 \le 1$, which is impossible because n_1 must be even.

Now let $\varepsilon = 0$. Then

$$(\Sigma_0 + \Sigma_1)((8 - 2n_2)n^2\Sigma_0 + 4ne) > (2n\Sigma_0 + n\Sigma_1 + e)^2,$$

whence

$$(2n_2 - 5)\Sigma_0(\Sigma_0 + \Sigma_1) + (n\Sigma_1 - e)^2 < 0.$$

So $n_2 < \frac{5}{2}$, so the only three cases are possible:

$n_1 = 0$, $n_2 = 2$, $n_3 = 4$;
$n_1 = 2$, $n_2 = 2$, $n_3 = 2$;
$n_1 = 0$, $n_2 = 1$, $n_3 = 2$.

We show that the first two cases can not occur. Suppose first that there are no sections of the class s_0 through the point B_0. Let Z^h_1 be the sum of all horizontal cycles through B_0. So $Z_1^h \sim As_0 + Bf$ for some $B \geq A$. Since $n_2 = 2$ in the considering cases, we have

$$\sum_{p\in\mathbb{P}^1} \deg Z_p^v \leq 2n^2 + 2mn - B \leq 2n^2 + 2mn - A,$$

which implies the following inquality for the supermaximal singularity:

$$m^v \leq 2\deg Z_t^v < 4n^2 + 4n\frac{e}{\nu(F_t)} - 2A.$$

But we know that $m^h \leq A$, and then

$$(\Sigma_0 + \Sigma_1)(A\Sigma_0 + (4n^2 - 2A)\Sigma_0 + 4ne) > (2n\Sigma_0 + n\Sigma_1 + e)^2, \tag{2.1}$$

so we get a contradiction:

$$A\Sigma_0(\Sigma_0 + \Sigma_1) + (n\Sigma_1 - e)^2 < 0.$$

It remains to prove that there are no maximal singularities over points that lie on sections of the class s_0.

Consider the case $n_1 = n_2 = n_3 = 2$. Note that $s_B \sim s_0$ is a unique section of such a class in this case. If $\mathcal{D}$ has a maximal singularity over the point $B_0 \in s_B$, then for a general $D \in \mathcal{D}$ it holds $\nu_1 = \operatorname{mult}_{B_0} D > n$, and for a general curve $C \sim f$ in the fiber through B_0 we get $n = D \circ C \geq \nu_1 > n$, which is impossible.

Now let $n_1 = 0$, $n_2 = 2$, $n_3 = 4$. Proving of this case consists of two lemmas below and an observation that any curve of the class s_0 lies on the divisor G_V. Recall that the valuation ν can be realized as a chain of blow-ups centered in nonsingular centers $B_0, B_1, \ldots$; a general $D \in \mathcal{D}$ has multiplicities $\nu_1, \nu_2, \ldots$ in these centers respectively.

Lemma 2.8. *Let $\mathcal{D} \subset |n(-K_V) + mF|$, $m \geq 0$, has a maximal singularity over a point $B_0 \in S$, where S is a fiber of $V/\mathbb{P}^1$. By $l \in |-K_S|$ we denote the curve through B_0. Then*

a) *S is smooth at B_0;*
b) *B_0 is a double point of l;*
c) *ν defines an infinitely near singularity, and if $B_1 \cap S^1 \neq \emptyset$ (upper indices denote the strict transform on the correspondig floor of the chain of blow-ups realizing ν), then $B_1 \cap l^1 \neq \emptyset$ and $\nu_1 + \nu_2 \leq 3n$.*

Proof. If S is singular at B_0, then a general curve $C \in |2(-K_S) - B_0|$ is also singular at this point, so we get a contradiction:

$$2n = D \circ C \geq 2\nu_1 > \nu_1.$$

Further, suppose l is nonsingular at B_0. Set

$$D|_S \sim kl + C,$$

where $l \not\subset \operatorname{Supp} C$. We have

$$n < \nu_1 \leq \operatorname{mult}_{B_0} D|_S = k + \operatorname{mult}_{B_0} C.$$

But $\operatorname{mult}_{B_0} C \leq C \circ l = (n(-K_S) - kl) \circ l = n - k$, and then

$$n < \nu_1 \leq k + n - k = n.$$

So l has a double point at B_0. Now we show that ν defines an infinitely near singularity. Assuming the converse, we get $\nu_1 > 2n$. Supposing $D|_S \sim kl + C$, we see that

$$2 \operatorname{mult}_{B_0} C \leq C \circ l = n - k,$$

whence

$$2n < \nu_1 \leq 2k + \operatorname{mult}_{B_0} C \leq \frac{3}{2}k + \frac{1}{2}n,$$

i.e., $k > n$, a contradiction.

Now let $B_1 \cap S^1 \neq \emptyset$, but $B_1 \cap l^1 = \emptyset$. Denote E_1 the exceptional divisor of the blow-up of B_0. Then

$$D^1|_{S^1} = (D \circ S)^1 + mE_1|_{S^1},$$

and for the decomposition $D|_S \sim kl + C$ we get

$$\tilde{\nu}_1 + \tilde{\nu}_2 \leq 2k + \operatorname{mult}_{B_0} C + \operatorname{mult}_{\tilde{B}_1} C^1 + m,$$

where $\tilde{B}_1 = B_1 \cap S^1$, $\tilde{\nu}_1 = \operatorname{mult}_{B_0} D|_S$, $\tilde{\nu}_2 = \operatorname{mult}_{\tilde{B}_1} (D|_S)^1$. Then,

$$\operatorname{mult}_{\tilde{B}_1} C^1 \leq \operatorname{mult}_{B_0} C \leq \frac{1}{2}(n(-K_S) - kl) \circ l = \frac{1}{2}(n - k),$$

so

$$\tilde{\nu}_1 + \tilde{\nu}_2 \leq 2k + 2\frac{n-k}{2} + m = n + k + m.$$

But $\tilde{\nu}_1 \geq \nu_1 + m$ and $\tilde{\nu}_2 \geq \nu_2$, thus

$$2n < \nu_1 + \nu_2 \leq n + k,$$

i.e., $k > n$, which is impossible.

So $B_1 \cap l^1 \neq \emptyset$. In order to show that $\nu_1 + \nu_2 \leq 3n$, we argue as before, only taking into account that

$$\tilde{\nu}_1 + \tilde{\nu}_2 \leq 3k + \operatorname{mult}_{B_0} C + \operatorname{mult}_{\tilde{B}_1} C^1 + m.$$

We have $\nu_1 + \nu_2 \leq 2k + n \leq 3n$. The lemma is proved.

LEMMA 2.9. *Let $\varepsilon = 0$, $n_1 = 0$. Then $G_V \cong C \times \mathbb{P}^1$, where C is an elliptic curve.*

PROOF. We use the notation of section 2.1. It is obvious that $G_V \cong C \times \mathbb{P}^1$. We shall prove the smoothness of C.

Notice that $R|_G \sim 3t_0$. Let B be an irreducible component of $\operatorname{Supp} R|_G$. Clearly, $B \sim t_0$. Let $\psi : \tilde{V} \to V$ be the blow-up of B with the exceptional divisor E. E is a ruled surface of the type $\mathbf{F_{n_2}}$. Denote t_E and l_E the classes of the exceptional section and a fiber of E respectively. Then for strict transforms of the divisors G and R on $\tilde{V}$ we have (using their smoothness along B)

$$\tilde{G}|_E \sim t_E, \quad \tilde{R}|_E \sim t_E + n_2 l_E,$$

hence $\tilde{G}|_E \cap \tilde{R}|_E = \emptyset$. So any fiber of $G \cong \mathbb{P}^1 \times \mathbb{P}^1$ meets R transversally, i.e., at three different points. Thus, C is nonsingular.

The lemma is proved. This completes also the proof of proposition 2.7.

REMARK 2.10. In fact, proposition 2.7 works also in the case $n_1 = 0$, $n_2 = 1$, $n_3 = 2$, but the proof is more complicated. Since this case is non-rigid, satisfies theorem 2.6, and can not be advanced to a more complete description yet (see below), we omit proving of it.

COROLLARY 2.11. *For $\varepsilon = n_1 > 0$, all smooth Mori fibrations on del Pezzo surfaces of degree 1 over $\mathbb{P}^1$ are rigid over the base; for $\varepsilon = 0$, all cases are also rigid, except the following:*

a) $n_1 = n_2 = n_3 = 2$;
b) $n_1 = 0$, $n_2 = 1$, $n_3 = 2$.

PROOF. Rigidity follows from proposition 2.7, using lemma 2.4 in the case $\varepsilon = 0$ or lemma 2.5 when $\varepsilon = n_1 > 0$.

We will deal with non-rigid cases after the following remark.

REMARK 2.12. In [9] it was proved that (smooth) pencils of del Pezzo surfaces (of degree 1, 2 or 3) are all rigid, if the so-called $\mathbf{K^2}$-condition holds:

cycles $aK^2 - bf$ are not effective for any $a, b > 0$.

For degree 1, this condition is satisfied exactly when $n_2 \geq 4$ (for $\varepsilon = 0$) or $n_2 \geq 4 + \frac{3}{2} n_1$ (for $\varepsilon = n_1 > 0$). The second is always true except $n_1 = 2$, $n_2 = 6$, $n_3 = 12$.

Another sufficient condition of rigidity was proposed in [5] (conjecture 1.2):

$$(-K_V)^3 + m_0 + 1 \leq 2,$$

where

$$m_0 = \min\{r \in \mathbb{Q} : (-K_V) + rF \text{ is nef}\}.$$

It holds exactly when $n_2 \geq 3$ ($\varepsilon = 0$) or $n_2 \geq 2 + \frac{3}{2}n_1$ ($\varepsilon = n_1 > 0$). Notice that the second is always true.

Thus, the second sufficient condition is more exact. Both these conditions do not include the rigid case $n_1 = 0$, $n_2 = 2$, $n_3 = 4$ for $\varepsilon = 0$.

Now we consider the non-rigid cases. We start with the case $n_1 = 0$, $n_2 = 1$, $n_3 = 2$. It was already considered in some works (see [5], section 2, §2, and [7]). It is easy to see that a unique effective divisor of the class G_V is the direct product of $\mathbb{P}^1$ and an elliptic curve, the linear system $\mathcal{H} = |2G_V + 2F| = |2(-K_V) - 2F|$ is base points free and defines a morphism contracting G_V along $\mathbb{P}^1$ onto a curve l on a Fano variety U of index 2 and degree 5 (this is so-called the double cone over the Veronese surface). Indeed, the linear system $|M_Q|$ maps Q onto a cone $\tilde{Q} \in \mathbb{P}^6$ over the Veronese surface in $\mathbb{P}^5$, and U is obtained by taking the double cover branched along a cubic section that does not pass through the cone vertex. l covers one of the generators of $\tilde{Q}$. Let

$$\mu_l : V = V_l \to U$$

be such a contraction. Notice that $-K_V \sim G_V + 2F$, and V is a Fano itself. Thus, $V/\mathbb{P}^1$ is not rigid, and $1 = \alpha(V, \mathcal{H}) < \mu(V, \mathcal{H}) = 2$, i.e., the conditions of conjecture 1.6 hold. Notice that U has a lot of such structures. Unfortunately, the technique of the maximal singularities method is not enough yet to deal completely with this case (for example, it fails when l has a double point; see also remark 2.10). Nevertheless, we can formulate the following conjecture, which seems to be true:

CONJECTURE 2.13. *Any smooth Mori fibration in the class of birational equivalency of U is biregular to either U or V_l for some l.*

The remaining case is $\varepsilon = 0$, $n_1 = n_2 = n_3 = 2$. Let $V/\mathbb{P}^1$ be the corresponding pencil of del Pezzo surfaces of degree 1. Notice that

$$\mathcal{N}_{s_0|V} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$$

and $|mH|$ for $m \gg 0$ gives a contraction of s_0. So there exists a flop

$$\mu : V \dashrightarrow U$$

centered at s_0. Then, $\dim |G_V| = 1$:

$$2 \dim |G_V| = 2 \dim |G| \leq \dim |2G| = \dim |(M - 2L)|_Q| = 2.$$

Let $G_U = \overline{\mu(F)}$, and F_U the class of a fiber of $U/\mathbb{P}^1$. Obviously, $\mu^{-1}|G_U| = |F|$ and $\mu^{-1}|F_U| = |G_V|$. Moreover, $U/\mathbb{P}^1$ is a pencil of del Pezzo surfaces of degree 1 with the same structure parameters $n_1 = n_2 = n_3 = 2$.

PROPOSITION 2.14. *Let $\chi : V \dashrightarrow W$ be a birational map onto a Mori fibration $\gamma : W \to S$. Then either χ or $\chi \circ \mu^{-1} : U \dashrightarrow W$ are birational over the base. Moreover, U and V are unique smooth Mori models in their class of birational equivalency; $Bir(V) = Aut(V)$, and in general case, $Bir(V) =< \sigma >_2$, where σ is the double cover involution.*

PROOF. Let $\mathcal{D}_W = |n'(-K_W) + \gamma^*(A)|$ be a very ample linear system, where A is an ample divisor on S, and $\mathcal{D}_V = \chi_*^{-1}\mathcal{D}_W \subset |n(-K_W) + mF|$.

Suppose $m \geq 0$. Then from proposition 1.2 it follows that $n = n'$ and χ is birational over the base. So we may assume $m = -l < 0$. Denote $\mathcal{D}_U = \mu^{-1}\mathcal{D}_V$. Since $-K_U = G_U + F_U$, we have

$$\mathcal{D}_U \subset |(n-l)(-K_U) + lF_U|,$$

and $\chi \circ \mu^{-1}$ is birational over the base by proposition 2.7.

Now let $\chi : V \dashrightarrow W$ be a birational map onto a Mori fibration $W/\mathbb{P}^1$ with $-K_W$ to be *nef*. We may assume χ to be birational over the base. Suppose $\mathcal{Y} = |n(-K_W) + m'F_W|$ is a very ample linear system, $m' > 0$, and $\mathcal{D} = \chi_*^{-1}\mathcal{Y} \subset |n(-K_V) + mF|$ for some $m \geq 0$. If $K_V + \frac{1}{n}\mathcal{D}$ were canonical, i.e., D had no maximal singularities, then χ would be an isomorphism by proposition 1.2.

So let $\mathcal{D}$ have maximal singularities. We may assume that they are all infinitely near (see [9]). By Y and D we denote general elements of $\mathcal{Y}$ and $\mathcal{D}$ respectively. Choose a resolution of singularities of χ:

$$\begin{tikzcd} & Z \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ V \arrow[rr, dashed, "\chi"] & & W \end{tikzcd}$$

Notice that

$$nK_Z + \psi^{-1}Y = m'\psi^*(F_W) + \sum a_i E_i'$$

and

$$nK_Z + \varphi^{-1}D = m\varphi^*(F) + \sum b_i E_i,$$

where E_i', E_i are exceptional divisors, a_i are all positive. Since $\mathcal{D}$ has maximal singularities, then

$$\mathcal{M} = \{i : b_i < 0\}$$

is not empty. For a point $t \in \mathbb{P}^1$ we denote $\mathcal{M}_t = \{i : \varphi(E_i) \in F_t\}$.

Obviously, $\dim|nK_Z + \psi^{-1}Y| = m'$, so

$$\dim|m\varphi^*(F) + \sum b_i E_i| = \dim|(m' + m - m')\varphi^*(F) + \sum b_i E_i| = m'.$$

Denote $I = \{t \in \mathbb{P}^1 : \mathcal{M}_t \neq \emptyset\}$. Then there exists a split

$$m - m' = \sum_{t \in I} k_t,$$

where k_t are all positive, such that

$$\dim |\sum_{t\in I} k_t \varphi^{-1}(F_t) + \sum_{t \in I}\sum_{i\in\mathcal{M}_t} (k_t c_i - b_i)E_i| = 0,$$

where for $t \in I$ positive numbers c_i are defined by

$$\varphi^*(F_t) = \varphi_*^{-1}(F_t) + \sum_{i\in\mathcal{M}_t} c_i E_i.$$

So for any $t \in I$ and $i \in \mathcal{M}_t$ we have $k_t c_i - b_i \geq 0$, and, for every i, k_i is the smallest positive number with such a property. Thus

$$k_t = \max_{i\in\mathcal{M}_t} \ulcorner b_i c_i^{-1} \urcorner,$$

where $\ulcorner\ \urcorner$ is the round up.

Let $D_1, D_2 \in \mathcal{D}$ be general. We can put down

$$D_1 \circ D_2 = Z^h + Z^v \sim n^2 s + (2n^2 + 2mn)f,$$

where Z^v and Z^h are vertical and horizontal cycles. Since s_0 is unique in its class and centers of maximal singularities can not lie on s_0, then horizontal cycles being caught by at least one of these centers can put down as $As_0 + Bf$ for some $B \geq A$. Taking into account that

$$m \leq m' + \sum_{t\in I} k_t,$$

we get an estimation

$$\deg Z^v = \sum_{t\in\mathbb{P}^1} \deg Z_t^v \leq 2n^2 + 2mn - A \leq 2n^2 + 2m'n + 2n\sum_{t\in I} k_t - A$$

(the degrees of vertical cycles are estimated by intersection with $-K_V$). Thus there exist $p \in I$ and $j \in \mathcal{M}_p$ such that

$$\deg Z_p^v \leq 2n^2 + 2m'n + 2n\ulcorner b_j c_j^{-1}\urcorner - A.$$

As in [9], §4, there exist positive numbers Σ_0, Σ_1, and Σ_0', where $\Sigma_0 \geq \Sigma_0'$ and $c_j \geq \Sigma_0'$, such that the following inequality for multiplicities $m^h = \mathrm{mult}_{B_0} Z^h$ and $m^v = \mathrm{mult}_{B_0} Z_j^v$ of horizontal and vertical cycles at $B_0 = \varphi(E_j)$ holds:

$$(\Sigma_0 + \Sigma_1)(\Sigma_0 m^h + \Sigma_0' m^v) \geq (2n\Sigma_0 + n\Sigma_1 + b_j)^2. \tag{2.2}$$

Notice that $\ulcorner b_j c_j^{-1}\urcorner \Sigma_0 \leq b_j + \Sigma_0$. By lemma 2.8, $\mathcal{D}$ has an infinitely near singularity over the point $B_0 \in F_p$. Consider two cases.

1. $B_1 \cap F_p^1 = \emptyset$, i.e., the center of the second blow up in the chain realizing the maximal singularity, does not intersect the strict transform of the fiber. In this case, since B_1 is a point, we have $\Sigma_0' \leq \frac{1}{2}\Sigma_0$. Then

$$\Sigma_0' m^v \leq 2 \deg Z_p^v \frac{\Sigma_0}{2} \leq (2n^2 + 2m'n + 2n - A)\Sigma_0 + 2nb_j,$$

and, using (2.2) and an estimation $m^h \leq A$, we get

$$n^2 - 2m'n - 2n < 0,$$

i.e.,

$$\frac{m'}{n} > \frac{1}{2} - \frac{1}{n}. \tag{2.3}$$

2. Now let $B_1 \cap S^1 \neq \emptyset$. Here we can not state that $\Sigma_0' \leq \frac{1}{2}\Sigma_0$, but lemma 2.8 gives us a good estimation $\nu_1 + \nu_2 \leq 3n$.

We know that $m^h \leq A$ and

$$\Sigma_0' m^v \leq 2 \deg Z_p^v \Sigma_0 \leq (4n^2 + 4m'n + 4n - 2A)\Sigma_0 + 4nb_j.$$

Substituting this in (2.2), we obtain

$$(A - 4m'n - 4n)\Sigma_0(\Sigma_0 + \Sigma_1) + (n\Sigma_1 - b_j)^2 < 0.$$

Notice that a condition $b_j > 0$ is nothing but the Noether-Fano inequality ([10]): for some set of non-increasing numbers $\{r_i\}$, where $\sum r_i = \Sigma_0 + \Sigma_1$, it holds

$$b_j = \sum r_i \nu_i - 2n\Sigma_0 - n\Sigma_1 > 0.$$

Applying $\nu_1 + \nu_2 \leq 3n$, we get

$$(n\Sigma_1 - b_j)^2 \geq \frac{1}{2} n^2 \Sigma_0 (\Sigma_0 + \Sigma_1).$$

Thus

$$\frac{1}{2}n^2 + A - 4m'n - 4n < 0,$$

and since $A \geq 0$, than

$$\frac{m'}{n} > \frac{1}{8} - \frac{1}{n}. \tag{2.4}$$

Comparing (2.3) and (2.4), we may assume that (2.4) holds always. It only remains to use the condition that $-K_W$ is *nef*: we can choose $\mathcal{Y}$ such that n is big enough but $\frac{m'}{n}$ is as small as we want, and then get a contradiction because of (2.4).

Now, substituting U for W, we obtain the statment about the birational automorphisms group. The uniqueness of smooth models of U and V follows from the description of all smooth (rigid and non-rigid) Mori fibrations on del Pezzo surfaces of degree 1 given above.

Proposition 2.14 is proved. This also completes corollary 2.11 and theorem 2.6.

3. Smooth varieties with a pencil of del Pezzo surfaces of degree 2.

3.1. **The essential construction.** In this section we study smooth Mori fibrations on del Pezzo surfaces of degree 2. Let $\rho : V \to \mathbb{P}^1$ be such a fibration. Denoting F the class of a fiber of ρ, we have

$$\mathrm{Pic}(V) = \mathbb{Z}[-K_V] \oplus \mathbb{Z}[F]$$

and $(-K_V)^2 \circ F = 2$. Since ρ is flat and $-K_V$ is ρ-ample, for some integer m we have

$$\rho_* \mathcal{O}(-K_V + mF) = \mathcal{E},$$

where $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(n_1) \oplus \mathcal{O}(n_2)$ is a vector bundle of rank 3 over $\mathbb{P}^1$, $0 \leq n_1 \leq n_2$. Set

$$X = \mathbf{Proj}\ \mathcal{E}$$

with a natural projection $\pi : X \to \mathbb{P}^1$. By M we denote the class of the tautological bundle on X, L the class of a fiber. Then

$$\mathrm{Pic}(X) = \mathbb{Z}[M] \oplus \mathbb{Z}[L].$$

It is easy to see that there exists a double cover $\varphi : V \to X$ branched along a smooth divisor R of the class $4M + 2aL$ such that

$$\rho = \pi \circ \varphi.$$

Further, let t_0 be the class of a section that corresponds to $\mathcal{E} \to \mathcal{O} \to 0$, l the class of a line in a fiber of π, $s_0 = \frac{1}{2}\varphi^*(t_0)$, and $f = \frac{1}{2}\varphi^*(l)$. Then

$$\overline{\mathbf{NE}}(X) = \mathbf{NE}(X) = \mathbb{R}_+[t_0] \oplus \mathbb{R}_+[l],$$

$$\overline{\mathbf{NE}}(V) = \mathbf{NE}(V) = \mathbb{R}_+[s_0] \oplus \mathbb{R}_+[f].$$

Denote $b = n_1 + n_2$, $H = \varphi^*(M)$; clearly, $F = \varphi^*(L)$. We have $M^3 = b$, $M^2 = t_0 + bl$, $K_X = -3M + (b-2)L$, $M \circ t_0 = L \circ l = 0$, $M \circ l = L \circ t_0 = 1$, V: $K_V = -H + (a+b-2)F$, $H^2 = 2s_0 + 2bf$, $H \circ F = 2f$, $H \circ s_0 = F \circ f = 0$, $H \circ f = F \circ s_0 = 1$, $(-K_V)^2 = 2s_0 + (8-4a-2b)f$, $H^3 = 2b$, $(-K_V)^3 = 12 - 6a - 4b$ (see [5]).

Such varieties with $K_V^2 = 2s_0 + \beta f$ for $\beta \leq 0$ were first studied in [9]. That was an exclusively important step in studying of geometry of Fano-fibered varieties.

3.2. **Results about rigidity.**

Theorem 3.1. *Let $V/\mathbb{P}^1$ be a smooth Mori fibration on del Pezzo surfaces of degree 2 over $\mathbb{P}^1$ with $K_V^2 = 2s_0 + \beta f$, where $\beta \leq 2$ (i.e., $2a + b \geq 3$). Then $V/\mathbb{P}^1$ is birationally rigid.*

Proof. Let $\chi : V \dashrightarrow V'$ be a birational map onto a Mori-fibration $\rho' : V' \to S'$, $\mathcal{D}' = | - n' K_{V'} + \rho'^* A'|$ a linear system as in proposition 1.2, and $\mathcal{D} = \chi_*^{-1}\mathcal{D}' \subset | - nK_V + mF|$. We will denote D a general element of $\mathcal{D}$. Obviously, $\mathcal{D}$ has no fixed components. If $n = n'$, the assertion follows from proposition 1.2. So let $n > n'$.

Lemma 3.2. *$m \geq 0$, i.e. $\mu(\mathcal{D}) = \alpha(\mathcal{D})$.*

Proof of the lemma. The cases $\beta \leq 0$ are proved in [9] (the so-called $\mathbf{K^2}$-condition holds in these cases). Let $\beta = 2$. Then $2a + b = 3$, b is odd, so $n_1 < n_2$. Clearly, $\mu(\mathcal{D}) = n$.

Assume the converse, i.e., $m < 0$. Consider a general curve $C \sim 2s_0 + 2n_1 f$ on V (there is at least 1-dimensional family of such curves). Then for general D we have $D \circ C \geq 0$, so

$$(nH - (\frac{n}{2}(b-1) - m)F) \circ (s_0 + n_1 f) \geq 0,$$

and we get a contradiction:

$$0 < \frac{-m}{n} \leq n_2 - \frac{b}{2} + \frac{1}{2} = \frac{n_1 - n_2 + 1}{2} \leq 0.$$

The lemma is proved.

By the lemma, we may assume $m \geq 0$. The following argumentation proving the theorem is nearly the same as in [9].

Obviously, $\mathcal{D}$ is not canonical, so it has maximal singularities. Let a curve B be the center of a maximal singularity. Then B is a section of ρ not lying on the ramification divisor, and we can "untwist" such a maximal singularity using the *Bertini* involution centered at B.

So we assume that the center of any maximal singularity is a point. Set $D_1 \circ D_2 = Z^h + Z^v$, where Z^h and Z^v are horizontal and vertical (effective) cycles. We see that

$$Z^h + Z^v = 2s_0 + (\beta n^2 + 4mn)f.$$

Denote $\deg Z^h \stackrel{\text{def}}{=} Z^h \circ F$ and $\deg Z^v \stackrel{\text{def}}{=} Z^v \circ (-K_V)$. Then, there exist a discrete valuation ν centered over a point B_0 in a fiber F_t (over a point $t \in \mathbb{P}^1$), a positive number $e = \nu(\mathcal{D}) - n\delta$, where δ is the canonical multiplicity with respect to ν, such that a component Z_t^v of Z^v lying in F_t has the degree

$$\deg Z_t^v < \begin{cases} \beta n^2 + 4n\frac{e}{\nu(F_t)}, & \text{if } \beta > 0; \\ 4n\frac{e}{\nu(F_t)}, & \text{if } \beta \leq 0. \end{cases}$$

Notice that $m^h \stackrel{\text{def}}{=} \operatorname{mult}_{B_0} Z^h \leq \deg Z^h = 2n^2$ and $m^v \stackrel{\text{def}}{=} \operatorname{mult}_{B_0} Z_t^v \leq \deg Z_t^v$. Then, there exist positive numbers $\Sigma_0 \geq \Sigma_0'$ and Σ_1 suct that the following inequality holds:

$$(\Sigma_0 + \Sigma_1)(m^h\Sigma_0 + m^v\Sigma_0') > (2n\Sigma_0 + n\Sigma_1 + e)^2.$$

It only remains to substitute the estimations of m^h and m^v:

$$(\Sigma_0 + \Sigma_1)((2 + \beta n^2)\Sigma_0 + 4ne) > (2n\Sigma_0 + n\Sigma_1 + e)^2,$$

i.e.,

$$(2 - \beta)n^2\Sigma_0(\Sigma_0 + \Sigma_1) + (n\Sigma_1 - e)^2 < 0,$$

which is impossible, if $\beta \leq 2$. Theorem 3.1 is proved.

Now we shall consider smooth Mori fibrations on del Pezzo surfaces of degree 2 for $2a + b \leq 2$. First, let $2a + b = 2$, i.e., $K_V^2 = 2s_0 + 4f$.

LEMMA 3.3. *For $2a + b = 2$, the only following cases may occur:*

1) $a = -1$, $n_1 = n_2 = 2$;
2) $a = 1$, $n_1 = n_2 = 0$;
3) $a = 0$, $n_1 = n_2 = 1$;
4) $a = 0$, $n_1 = 0$, $n_2 = 2$;
5) $a = -2$, $n_1 = 2$, $n_2 = 4$;
6) $a = -3$, $n_1 = 2$, $n_2 = 6$;
7) $a = -4$, $n_1 = 2$, $n_2 = 8$;

The first three cases are non-rigid.

PROOF. Since $b \geq 0$, the only cases 2), 3), and 4) are possible if $a \geq 0$.

Let $a < 0$. Then $b > 0$, and since $R \circ t_0 < 0$, any curve of the class s_0 lies on R. So $n_1 > 0$ because of irreducibility of R, and such a curve is unique on X. Further, let $\psi : \tilde{X} \to X$ be the blow-up of t_0 with E as an exceptional divisor. E is a ruled surface of the type $\mathbf{F_{n_2-n_1}}$. By t_E and l_E we denote the classes of an exceptional section and a fiber of E. Suppose $\tilde{R}$ is the strict transform of R, then

$$\tilde{R}|_E \sim t_E + (2a + n_2)l_E$$

is an irreducible curve on V because of the smoothness of R. So either $2a+n_2 = 0$, or $2a+n_2 \geq n_2-n_1$. It is easy to check that the second case is possible only if $n_1 = n_2 = 2$ and $a = -1$. Further, if $2a + n_2 = 0$, then $n_1 = 2$. Moreover, curves of the class $t_0 + n_1l$ lie on a unique effective divisor of the class $M - n_2L$, and since R is irreducible, we have $R \circ (t_0 + n_1l) \geq 0$, whence $a \geq -4$.

Now we show that the first three cases are non-rigid. Let $a = 1$, $n_1 = n_2 = 0$. We see that $X \cong \mathbb{P}^2 \times \mathbb{P}^1$, and V is a conic bundle with respect to the projection onto $\mathbb{P}^2$: double covers of curves of the class t_0 are conics since $R \circ t_0 = 2$. So V is non-rigid. Notice that this projection is given by a linear system $|-K_V - F|$, which is free from base points.

Let $a = -1$, $n_1 = n_2 = 2$. A unique curve of the class t_0 on X lies on the ramification divisor R, so s_0 is unique on V, too. Note that

$$\mathcal{N}_{s_0|V} = \mathcal{O}(-1) \oplus \mathcal{O}(-2)$$

and $|nH|$ for $n \geq 2$ gives a birational morphism contracting s_0. Then, the base set of the pencil $|M - L|$ is exactly s_0; all elements of this pencil are smooth and isomorphic to $\mathbf{F_2}$. For a general $S \in |M - L|$ the restriction $R|_S$ is composed from s_0 and some three-section that does not intersect s_0, so after taking the double cover and contracting the pre-image of s_0 the surface S becomes a del Pezzo surface of degree 1. This means that the anti-flip $V/\mathbb{P}^1 \dashrightarrow V'/\mathbb{P}^1$ (centered at s_0) gives us a Mori fibration on del Pezzo surfaces of degree 1 with a terminal singular point lying on the exceptional curve of V'. Thus, $V/\mathbb{P}^1$ is not rigid. Note that the pencil $|-K_V - F|$ has no fixed components.

Finally, for $a = 0$, $n_1 = n_2 = 1$ the linear system $|-2K_V|$ gives a small contraction onto the canonical model of V, which can be realized as double covering of a non-degenerated quadratic cone in $\mathbb{P}^4$ branched along a quartic section. If a curve of the class s_0 is unique on V (this means that t_0 lies on the ramification divisor), then s_0 is **-2**-curve of the width 2 (in the notions of [11]). Otherwise, there are two curves of the class s_0, which are disjoint and **-2**-curves of the width 1. In both the cases we obtain another structure of a smooth Mori fibration on del Pezzo surfaces of degree 2 after making a flop centered at these curves. Notice again that $|-K_V - F|$ has no fixed components. The case of two curves was studied in detail in [3].

Lemma 3.3 is proved.

REMARK 3.4. It is highly likely that cases 4) – 7) are all rigid. Unfortunately, the author can not prove it yet. As it often occur in the practice of the maximal singularity method, the problems are related to excluding of infinitely near singularities over points on some rational curves of a special kind.

Now we consider cases when $2a + b = 1$, i.e., $K_V^2 = 2s_0 + 6f$.

LEMMA 3.5. *If $2a + b = 1$, the only following three cases may occur:*

1) $a = 0$, $n_1 = 0$, $n_2 = 1$;
2) $a = -1$, $n_1 = 1$, $n_2 = 2$;

3) $a=-2$, $n_1=1$, $n_2=4$.

The first two are non-rigid.

PROOF. Arguing as before, it is easy to show that $a \geq -2$, so the only 1) – 3) are possible.

In order to show that the case 1) is non-rigid, note that V can be obtained by blowing up of an elliptic curve of degree 2 on a smooth Fano variety U of genus 9 and index 2 (this is so-called *the double space of index 2*, i.e., the double cover of $\mathbb{P}^3$ branched along a quartic). Observe that the birational morphism $V \to U$ is defined by the linear system $|-2K_V-2F|$. Some results (but very incomplete) about these varieties are contained in [8].

Consider the case $a=-1$, $n_1=1$, $n_2=2$. There exists a unique curve of the class s_0, and it is a **-2**-curve of the width 1. Let $V \dashrightarrow V^+$ be a flop centered at this curve. A linear system $|H-2F|$ contains a unique element, which we denote G_V. Its strict transform G_V^+ on V^+ is a surface that is isomorphic to the double cover of $\mathbb{P}^2$ branched along either a smooth conic or a couple of different lines. So G_V^+ is either $\mathbb{P}^1 \times \mathbb{P}^1$, or a quadratic cone in $\mathbb{P}^3$. Then, there exists an extremal contraction $V^+ \to U$ of G_V^+. U is a double cone over the Veronese surface, but with a quadratic singularity (arising from a quadratic singularity of the ramification divisor). We see also that U has (birationally) structures of fibrations on del Pezzo surfaces of degree 1 (see section 2, the case $n_1=0$, $n_2=1$, $n_3=2$). Notice that $|-K_V-F|$ has no fixed components.

The lemma is proved.

As after lemma 3.3, the author can say the same thing about the case 3): it should be rigid, but I can not prove it yet.

We complete this survey of del Pezzo fibrations by the following lemma:

LEMMA 3.6. *Cases $2a+b \leq 0$ can not occur.*

PROOF. The same reasons as above, except the case $a=0$. If $a=0$, then $b=0$, so V is isomorphic to the direct product of $\mathbb{P}^1$ and a smooth del Pezzo surface of degree 2. But in this case V is not a Mori fibration because of the relative Picard number (it is equal to 8). The lemma is proved.

REFERENCES

[1] Corti, A. *Factoring birational maps of threefolds after Sarkisov.* J. Alg. Geom. **4** (1995), 223-254

[2] Corti, A., *Del Pezzo surfaces over Dedekind schemes.* Ann. Math., II. Ser. 144, **3** (1996), 641-683

[3] Grinenko, M.M. *Birational automorphisms of a three-dimensional double cone.* Sb. Math. **189** (1998), No.7, 991-1007

[4] Iskovskikh, V.A.; Manin, Yu.I. *Three-dimensional quartics and counterexamples to the Lüroth problem.* Math. USSR, Sbornik **15** (1971), 141-166 (1972)

[5] Iskovskikh, V.A. *On the rationality of three-dimensional Del Pezzo fibrations.* Proc. Steklov Inst. Math. **208** (1995), 115-123

[6] Iskovskikh, V.A. *Birational automorphisms of three-dimensional algebraic varieties.* J. Sov. Math. **13** (1980), 815-868

[7] Khashin, S.I., *Birational automorphisms of a double Veronese cone of dimension three.* Mosc. Univ. Math. Bull. **39** (1984), No.1, 15-20

[8] Khashin, S.I., *Birational automorphisms of a index 2 Fano variety of degree 2.* MGU, M., 1985, pp. 41 (in russian)

[9] Pukhlikov, A.V. *Birational automorphisms of algebraic threefolds with a pencil of Del Pezzo surfaces.* Izv. Math. *62* (1998), No.1, 115-155

[10] Pukhlikov, A.V., *Essentials of the method of maximal singularities.* Warwick Preprint 31/1996

[11] Reid, M., *Minimal models of canonical 3-folds.* Algebraic varieties and analytic varieties, Proc. Symp., Tokyo 1981, Adv. Stud. Pure Math. **1** (1983), 131-180

Steklov Mathematical Institute

Max-Planck Institut für Mathematik
E-mail address: grin@mi.ras.ru / grinenko@mpim-bonn.mpg.de