Necessary Conditions for the Existence of Local Rules for Quasicrystals

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Abstract

We prove that if the class of tilings obtained by embedding a 2-dimensional $\mathbb{R}^2$ into $\mathbb{R}^4$ by the projection method admits local rule then the embedding must be based on quadratic irrationality, and non-degenerate in some sense. Absense of local rule for the class quasiperiodic tilings having 12-fold symmetry is proved.

Introduction

In this paper we study the local structure of quasiperiodic tiling obtained from the projection method. When **E** is a k-dimensional suspace of an Euclidean space $\mathbb{R}^n$ equipped with a base there is a class of tilings associate with **E**. The question is when this class of tilings has *local rule*, or in the terminology of De Bruijn,*local criteria.* This problem has been investigated by many authors (cf. [L],[B],[dB1],[dB2],[K],[IS][LPS1],[LPS2]). In particular, Levitov in [L] introduced the definition of local rules and found a necessary condition for the existence of local rules, called the SI condition. From the SI conditions it follows that the class of planar quasiperiodic tilings having m-fold symmetry has local rule only if $m = 5, 8, 10, 12$. Absence of local rules for the class of tilings having 8-fold symmetry is established by Burkov [B] and de Bruijn [dB2]. For the class of quasiperiodic tilings having 5-fold and 10-fold symmetry existence of local rules is proved by De Bruijn (cf.[dB],[Le1]).There remains the 12-fold symmetry case. In this paper we prove that this class of tilings does not have local rules, even "weak local rules" in the sense of Levitov.

In the case $n = 4, k = 2$ Levitov conjectured that if the class of tilings associate with $\mathbf{E}$ has local rule then $\mathbf{E}$ is based on quadratic irrationality and non-degenerate. Here we prove this hypothesis. In particular the case of 8-fold symmetry follows at once from this theorem.

The paper is organized as follow.In §1 we introduce definitions and preliminary facts about the strip method and the cut method. Here we follow the wonderful paper [ODK], some new facts are presented in §1.5. In §2 we reprove a Levitov's theorem about the SI conditions by a different method and discuss different definitions of the SI condition. §3 is the chief part, we consider the case $k = 2, n = 4$. In §4 we treat the class of quasiperiodic tilings having 12-fold symmetry.

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1 Basic definitions and preliminary facts

1.1 Tilings and Local rules

A tiling of $\mathbb{R}^k$ is a family of k-dimensional polyhedra which covers $\mathbb{R}^k$ without holes and overlaps such that up to translations there are only a finite number of polyhedra in this family.A polyhedron of this family is called *its tile.*Two polyhedra are *congruent* if the second is a translate of the first. The classes of congruent tiles are called prototiles. A tiling is called *special* if the intersection of every two its polyhedra is a common facet of lower dimension, if not empty. In this paper tilings are always assumed to be special unless the case of family $\mathcal{O}$ and its refinements that appear later. *An r-map* is a family of polyhedra lying inside a ball with radius r. *Two r-maps are congruent* if the second is a translate of the first. If T is a tiling then *an r-map* of T is the subfamily of tiles lying inside some ball with radius r.

A finite set of r-maps is called *a local rule of radius* r, or simply a local rule when we do not care about the radius of this local rule. *A tiling* T *satisfies a local rule* $\mathcal{A}$ *of radius* r if every r-map of T is congruent to one of $\mathcal{A}$. A local rule is called *quasiperiodic* if it is not trivial , i.e. at least one tiling satisfies it , and every tiling satisfying it has to be quasiperiodic. The exact definition of quasiperiodicity will be given later in section 1.3.

A family $\mathcal{T}$ of tilings *admits a local rule* $\mathcal{A}$ *of radius* r if $\mathcal{T}$ is the set of all tilings satisfying this local rule.

Two tilings have the same local structure if every r-map of the first is congruent to an r-map of the second and conversly, every r-map of the second is congruent to an r-map of the first. This must holds for evevry $r > 0$ (while in the definition of local rule a radius r is fixed).

If $\mathcal{T}$ is a set of tilings the *closure* $\overline{\mathcal{T}}$ of $\mathcal{T}$ is the set of all tilings T such that every r-map of T is congruent to an r-map of a tiling from $\mathcal{T}$, for every $r > 0$.A set $\mathcal{T}$ of tilings is *closed* if $\overline{\mathcal{T}} = \mathcal{T}$. It is evident from definitions that if $\mathcal{T}$ admits a local rule then $\mathcal{T}$ must be closed.For a set $\mathcal{T}$ of tilings denote by $\mathcal{T}(r)$ the set of all tilings T such that every r-map of T is congruent to an r-map of a tiling from $\mathcal{T}$. Then $\mathcal{T} \subset \mathcal{T}(r) \subset \mathcal{T}(r')$ for every $0 < r' < r$, and $\overline{\mathcal{T}} = \cap_{0<r\in R}\mathcal{T}(r)$. Also note that $\mathcal{T}(r) = \overline{\mathcal{T}}(r)$ and a set of tiling $\mathcal{T}$ admits a local rule of radius r iff $\mathcal{T} = \overline{\mathcal{T}} = \mathcal{T}(r)$.

1.2 k-planes

In the Euclidean space $\mathbb{R}^n$ with origin $\mathbf{0}$ we fix a standard basis $\varepsilon_1,..., \varepsilon_n$. Let $\mathbf{Z}^n$ be the integer lattice. For a set of vectors $v_1,...,v_m$ from $\mathbb{R}^n$ let

$$Pol(v_1,...,v_m) = \left\{\sum_{i=1}^{i=m} \lambda_i v_i \mid \lambda_i \in [0,1]\right\}$$

The set $\gamma = Pol(\varepsilon_1,...,\varepsilon_n)$ is called *the unit cube.* Let M_j be the set of multi-indices $(i_1,...,i_j)$ such that $1 \leq i_1 < i_2 < ... < i_j \leq n$. If $I \in M_j$ let I^c be the multi-index of M_{n-j} such that $I \cup I^c$ is $\{1,2,...,n\}$. For $I = (i_1,...,i_j) \in M_j$ the set $\gamma_I = Pol(\varepsilon_{i_1},\ldots,\varepsilon_{i_j})$ and its translates by integer vectors (i.e. vectors from $\mathbf{Z}^n$) are called j-facets of the lattice $\mathbf{Z}^n$.

Suppose $\mathbf{E}$ is a k-dimensional subspace of $\mathbb{R}^n$. $\mathbf{E}$ is called *totally irrational* if there are no integer points lying on $\mathbf{E}$ except $\mathbf{0}$. Let $\mathbf{E}^\perp$ be the $(n-k)$-dimensional subspace perpendicular to $\mathbf{E}$ and $\mathbf{p}$ be the projector along $\mathbf{E}^\perp$ on $\mathbf{E}$, $\mathbf{p}^\perp$ be the projection on $\mathbf{E}^\perp$ along $\mathbf{E}$. Put $e_i = \mathbf{p}(\varepsilon_i)$, $e_i^\perp = \mathbf{p}^\perp(\varepsilon_i)$, $i = 1,...,n$. A set X is called a $\mathbf{p}$-prism (or simply a prism) if $X = \mathbf{p}(X)+\mathbf{p}^\perp(X)$.

$\mathbf{E}$ is called *rational (resp. quadratic)* if it is spanned by k vectors $v_1,...,v_k$ with coordinates belonging to $\mathbf{Q}$ (resp. $\mathbf{Q}\sqrt{d}$, where d is a natural number).

A k-plane means a k-dimensional plane. Denote U_r the ball in $\mathbf{E}$ with center at $\mathbf{0}$ and radius r.

1.3 The projection method

Let us briefly recall the projection method used to construct quasiperiodic tilings. The reader is referred to [dB1,ODK,GR] for full expositions on these subjects.

Suppose $\mathbf{E}$ be a k-dimensensional subspace in $\mathbb{R}^n$. For simplicity we assume the following condition of genericity holds true :

(*) Every k vectors from $\{e_1,\ldots,e_n\}$ are linear independent.

This condition is equivalent to :

(*')Every $n-k$ vectors from $\{e_1^\perp, \dots, e_n^\perp\}$ are linear independent (cf [ODK]).

In fact this condition is not essential but it will be more convenient for us. The set of all subspaces **E** satisfying this condition form a dense open subset of the Grassmanian $G_{n,k}$.

We obtain a strip in $\mathbb{R}^n$ by shifting the cell γ along an affine k-plane parallel to **E**:

$$\mathbf{S}_\alpha = \mathbf{E} + \gamma + \alpha \, , \alpha \in \mathbf{E}^\perp$$

It is proved in [ODK] that for translation α s.t. the boundary of the strip does not contain any point of $\mathbf{Z}^n$ (in this case α is called *regular*) the strip contains exactly a unique k-dimensional surface built up of k-facets of the lattice $\mathbf{Z}^n$ lying in $\mathbf{S}_\alpha$. This surface goes through all the vertices of the lattice $\mathbf{Z}^n$ falling inside $\mathbf{S}_\alpha$ and has an obvious polyhedral structure. By projecting along $\mathbf{E}^\perp$ on **E** this polyhedral structure we get a tiling T_α of **E**. Note that there are no overlaps: the restriction of **p** on this surface is one-to-one. The prototiles are the projections $\mathbf{p}(\gamma_I)$ of k-dimensional facets of the lattice $\mathbf{Z}^n$. A point $\alpha \in \mathbf{E}^\perp$ is called *irregular* if it is not regular. Denote **Ir** the set of all irregular points. This set is of measure 0 and is described below in 1.4.

There are many definitions of quasiperiodic tilings but perhaps all the authors agree that T_α is quasiperiodic for regular α. One wishes to see wether the set $\mathcal{T}_E$ of all tilings T_α for regular α admits local rule or not. Unfortunately the set $\mathcal{T}_E$ is never closed unless **E** is rational. So the question should be formulated as follow: when the closure $\overline{\mathcal{T}_E}$ admits a local rule of some radius? A trivial example is when **E** is rational then $\overline{\mathcal{T}_E} = \mathcal{T}_E$ and $\overline{\mathcal{T}_E}$ admits local rules.

For the same reason we will consider a tiling T *quasiperiodic* if T is congruent to a tiling from $\overline{\mathcal{T}_E}$ for some **E**. How big is the closure $\overline{\mathcal{T}_E}$ is discussed in the following sections.

1.4 The cut method

Let's now consider another construction of these tilings, known as the cut method [ODK].This construction is essential for us.

Put $P_I = \mathbf{p}(\gamma_I), P_I^\perp = -\mathbf{p}^\perp(\gamma_{I^c})$ and $C_I = P_I + P_I^\perp, C_{I,\xi} = C_I + \xi, I \in M_k$.

Each $C_{I,\xi}$ is a prism. If a k-plane $\mathbf{E}+\alpha$ intersects with a prism $C_{I,\xi}$ then the intersection is congruent to P_I. For a prism X we define $\partial^{\|}(X) = \mathbf{p}(X) + \partial(\mathbf{p}^\perp(X))$ and $\partial^\perp(X) = \partial(\mathbf{p}(X)) + \mathbf{p}^\perp(X)$ where ∂Y is the boundary of the set Y in **E** or in $\mathbf{E}^\perp$. The sets $\partial^{\|}(X)$ and $\partial^\perp(X)$ are called resp. the parallel and the perpendicular boundaries of prism X. The parallel (resp. perpendicular) boundary of a family of prisms ,by definition, is the union of the parallel (resp. perpendicular) boundaries of all the prisms of this family.

Consider the family $\mathcal{O} = \{ C_{I,\xi}, I \in M_k, \xi \in \mathbf{Z}^n\}$. Its parallel and perpendicular boundaries are denoted respectively by **B** and $\mathbf{B}^\perp$. This family covers the whole $\mathbb{R}^n$ without overlaps and holes, i.e. it is a partition of $\mathbb{R}^n$.

This partition is called "oblique periodic tiling" of $\mathbb{R}^n$ in [ODK] because it is invariant under translations from $\mathbf{Z}^n$. The union of $\binom{n}{k}$ prisms C_I, $I \in M_k$ is a fundamental domain of the group $\mathbf{Z}^n$. One can regard this union as a rearrangement of the unit cell γ. Every k-plane $\mathbf{E}+\alpha$,where $\mathbf{E}+\alpha$ does not meet $\mathbf{B}$, inherits a unique tiling from the family $\mathcal{O}$. The equivalence between the cut method and the projection method is now stated as follow: $\alpha \in \mathbf{E}^\perp$ is regular if and only if $\mathbf{E}+\alpha$ does not meet the parallel boundary $\mathbf{B}$, in this case the intersections of $\mathbf{E}+\alpha$ with prisms from $\mathcal{O}$ define a tiling on $\mathbf{E}+\alpha$ and by projecting on $\mathbf{E}$ we get the tiling T_α.

The set **Ir** of irregular points is $\mathbf{p}^\perp(\mathbf{B})$. Let $f_J^\perp$ for $J=(j_1,\ldots,j_{n-k-1}) \in M_{n-k-1}$ be the $(n-k-1)$-plane spanned by $e_{j_1}^\perp,\ldots,e_{j_{n-k-1}}^\perp$. Then the set of irregular points in $\mathbf{E}^\perp$ is the union of $\binom{n}{n-k-1}$ families of parallel $(n-k-1)$-planes, each of the form $f_J^\perp + \mathbf{p}^\perp(\mathbf{Z}^n)$ (cf.[ODK]). Each family is dense in $\mathbf{E}^\perp$ but the union of its members has measure 0.

A section Ω will be referred to as a k-dimensional surface in $\mathbb{R}^n$ such that the restriction $\mathbf{p}_{|\Omega} : \Omega \mapsto \mathbf{E}$ is a homeomorphism. If the section Ω does not meet the parallel boundary $\mathbf{B}$ then by projecting the intersection of Ω with the perpendicular boundary $\mathbf{B}^\perp$ we get a tiling T_Ω of $\mathbf{E}$. Let $\mathcal{S}_\Omega$ be the set of all prisms from $\mathcal{O}$ meeting Ω then T_Ω is simply the projection of $\mathcal{S}_\Omega$ on $\mathbf{E}$.

An important example is the case when $\Omega = E+\alpha$ where α is regular. Then $\mathcal{S}_\Omega$ is the set of all prisms from $\mathcal{O}$ which meets $\mathbf{E}+\alpha$. If $C_1,\ldots,C_m$ is a finite collection of prisms from $\mathcal{S}_\Omega$ then the projections on $\mathbf{E}^\perp$ of $C_1,\ldots,C_m$ are $(n-k)$-dimensional polyhedra having non-empty interior intersection,i.e. their interiors have non-empty intersection.

1.5 More about the cut method

Denote $K = -\mathbf{p}^\perp(\gamma)$. It is an $(n-k)$-dimensional polyhedron. Then all $P_I^\perp$ are contained in K. Suppose T is a tiling of $\mathbf{E}$ with prototiles $P_I, I \in M_k$. *A lift* of T is a map l:{tiles of T}→{prisms from $\mathcal{O}$} such that for every tile P of T we have $\mathbf{p}(l(P)) = P$. If $\eta \in \mathbf{E}^\perp \cap \mathbf{Z}^n, \eta \neq \mathbf{0}$ and $\mathbf{p}(C) = P$ then $\mathbf{p}(C+\eta) = P$. Hence if $\mathbf{E}^\perp$ is not totally irrational and T has a lift then it has many lifts.

Suppose l is a lift of T and P_1, P_2 are two neighboring tiles of T sharing a common $(k-1)$-dimensional facet. Then $C_1 = l(P_1), C_2 = l(P_2)$ have non-empty intersection if and only if $\mathbf{p}^\perp(C_1), \mathbf{p}^\perp(C_2)$ have non-empty intersection. Both $\mathbf{p}^\perp(C_1), \mathbf{p}^\perp(C_2)$ are $(n-k)$-dimensional polyhedra lying in $\mathbf{E}^\perp$.

Lemma 1.5.1:*If $\mathbf{p}^\perp(C_1)$ has non-empty intersection with the interior of $Q = \mathbf{p}^\perp(C_2)$ then $C_1+\eta$ does not meet C_2 for every $\eta \in \mathbf{E}^\perp \cap \mathbf{Z}^n$ and $\eta \neq \mathbf{0}$.*

Proof: Suppose $C_1+\eta$ meets C_2 then $\mathbf{p}^\perp(C_1)+\eta$ meets Q. It follows that $Q^0+\eta$ meets Q^0 where Q^0 is the interior of Q. Since Q is congruent to $P_I^\perp$ for some I, which , in turn, is contained in K we conclude that $K^0+\eta$ has non-empty intersection with K. Lemma V. 2 of [ODK] asserts that $\eta = \mathbf{0}$.□

A lift l of a tiling T is called *connected* if for every pair of tiles P_1, P_2 sharing a common $(k-1)$-facet the polyhedra $\mathbf{p}^\perp(l(P_1)), \mathbf{p}^\perp(l(P_2))$ have non-empty interior intersection. Of course in this case $l(P_1), l(P_2)$ must have non-empty intersection. The following is a consequence of lemma 1.5.1.

Proposition 1.5.2:*If two connected lift l_1, l_2 of a tiling T coincide at some tile, i.e. $l_1(P) = l_2(P)$ for some tile P of T then they are equal, $l_1 = l_2$.*

Suppose X is an open subset of $\mathbf{E}$. Then $\mathbf{p}^{-1}(X)$ is the sum $X + \mathbf{E}^\perp$. Consider the set of all prisms of the family $\mathcal{O}$ lying inside $X + \mathbf{E}^\perp$. By projecting the parallel boundaries of these prisms on $\mathbf{E}^\perp$ we get a subset $\mathbf{Ir}(X)$ of $\mathbf{Ir}$. If α and β are regular and both belong to the same connected component of $\mathbf{E}^\perp \setminus \mathbf{Ir}(X)$ then T_α coincides with T_β inside X. In the case when X is bounded, say $X = U_r$, the set $\mathbf{Ir}(X)$ is a closed subset of $\mathbf{E}^\perp$ of codimension 1 and $\mathbf{E}^\perp \setminus \mathbf{Ir}(X)$ consists of $(n-k)$-dimensional polyhedra without boundary.

Proposition 1.5.3:*If Ω is a section not meeting $B + U_{2r}$ then the tiling T_Ω belongs to $\mathcal{T}_E(r)$. Other words, every r-map of T_Ω is a translate of an r-map of a tiling from $\mathcal{T}_E$.*

Proof: Let $X = U_r + x$ where x is a point of $\mathbf{E}$. We have to prove that there is a tiling T_α for regular α such that $T_\Omega = T_\alpha$ inside X. Consider $\mathbf{Ir}(X)$. If $y \in \mathbf{B} \cap (\mathbf{E}^\perp + X)$ then $y + U_{2r} \subset \mathbf{B} + U_{2r}$, hence one sees easily that $\mathbf{Ir}(X) + X \subset \mathbf{B} + U_{2r}$. As $\mathbf{Ir}(X)$ divides $\mathbf{E}^\perp$ into many connected components, the set $\mathbf{Ir}(X) + X$ divides $\mathbf{E}^\perp + X$ into connected components which project (by $\mathbf{p}^\perp$) on the corresponding connected components of $\mathbf{E}^\perp$ divided by $\mathbf{Ir}(X)$, and $\Omega \cap (\mathbf{E}^\perp + X)$ must lie in one of these connected component. If we take α be a regular point lying inside the projection on $\mathbf{E}^\perp$ of this connected component then obviously $T_\Omega = T_\alpha$ inside X. □

A sequence of tilings $T_1, T_2, \ldots$ of $\mathbf{E}$ *converges to a tiling* T if for every $r > 0$ there is a natural number N such that for $i > N$ the tiling T_i coincides with T inside the ball U_r.

If $\alpha \in \mathbf{Ir}$ is an irregular point then there are several hyperplanes (i.e. planes of codimension 1 in $\mathbf{E}^\perp$) from $\mathbf{Ir}$ going through α. They divide $\mathbf{E}^\perp$ into many parts. If $\alpha_1, \alpha_2, \ldots$ are regular, belong to one part and the sequence α_i converges to α then it is easy to see that T_{α_i} converge to a tiling, called *the quasiperiodic tiling defined by α and this part.* This tiling depends only on the part containing α_i but not on concrete points α_i. A rigorous proof of this fact is presented in [LPS2].

Remark: One can prove that two diferent parts define different tilings.

When we say that a quasiperiodic tiling defined by an irregular α we mean that it is defined by α and one part of $\mathbf{E}^\perp$ divided by hyperplanes from $\mathbf{Ir}$ going through α. We see that every regular α defines a unique quasiperiodic tiling while an irregular α defines many, more than one but a finite number, of quasiperiodic tilings. If T is a quasiperiodic tiling defined by some α (regular or not) then T has a connected lift and every connected lift l of T has the following property: For every finite number of tiles $P_1, P_2, \ldots, P_m$ of T the polyhedra $\mathbf{p}^\perp(l(P_1)), \mathbf{p}^\perp(l(P_2)), \ldots, \mathbf{p}^\perp(l(P_m))$ have non-emty interior intersection. In particular, all the projections $\mathbf{p}^{\perp}(P)$ where P are tiles of T have a common point, this common point is unique.

A section Ω is reduced to planar section if there is $\alpha \in \mathbf{E}^{\perp}$ if for every $x \in \mathbf{E}$ the segment $[\Omega(x), x+\alpha]$ does not meet the parallel boundary $\mathbf{B}$ or meets $\mathbf{B}$ only at point $x+\alpha$. Here $\Omega(x)$ is the point of Ω lying upon x,i.e. $\Omega(x) = \Omega \cap \mathbf{p}^{-1}(x)$. A section Ω is reduced to planar section if and only if the projections of all the prisms from $\mathcal{S}_{\Omega}$ on $\mathbf{E}^{\perp}$ have a common point.

Proposition 1.5.4: *If T is a tiling from $\overline{\mathcal{T}_E}$ then after a shift T is coincident with a quasiperiodic tiling defined by $\alpha \in \mathbf{E}^{\perp}$, not necessarily regular.*

Proof: After a shift we may assume that $\mathbf{0}$ is a vertex of T. We can choose regular α_i in K such that T_{α_i} coincides with T inside the ball U_r with radius $r = i$. Because K is a compact set, after selecting a subsequence, we may assume that the sequence of points α_i converges to a point α in K. Then obviously $T = \lim T_{\alpha_i}$. If α is regular then $T = T_{\alpha}$. If α is not regular, again by selecting a subsequence we may assume that all α_i belong to the same part of $\mathbf{E}^{\perp}$ divided by hyperplanes from $\mathbf{Ir}$ going through α. Then T_{α_i} converge to the quasiperiodic tiling defined by this part. □

As a consequence of propositions 1.5.2 and 1.5.4 we get the following.

Proposition 1.5.5: *If a tiling T_{Ω} of a section Ω not meeting B belongs to $\overline{\mathcal{T}_E}$ then Ω is reduced to a planar section.*

To prove that Local rules do not exist by the above propositions it suffices to prove that for every $r > 0$ there is a section not meeting $\mathbf{B} + U_r$ and not reduced to planar section. This is the main idea for all the proves below. Every proof is based on this idea.

2 The SI-condition

2.1 On the set $\mathbf{B} + U_r$

Let h_J for $J = (j_1, \ldots, j_{n-k-1}) \in M_{n-k-1}$ be the $(n-k-1)$-dimensional subspace spanned by $\varepsilon_{j_1}, \ldots, \varepsilon_{j_{n-k-1}}$. Of course h_J is a rational subspace and $\mathbf{p}^{\perp}(h_J) = f_J^{\perp}$. The set $h_J + \mathbf{Z}^n$ is a locally discrete family of parallel $(n-k-1)$-planes in $\mathbb{R}^n$. Here locally discrete means that every compact meets only a finite number of $(n-k-1)$-planes from this family. This follows from the rationality of h_J. A set of the type $h_J + U_r + \xi$ for $\xi \in \mathbf{Z}^n$ is called *a wall of width* r. Each wall is a set of dimension $n-1$ and is contained in a unique $(n-1)$-plane. The set $\mathcal{W}_J(r) = h_J + U_r + \mathbf{Z}^n$ is a family of walls. There are $\binom{n}{n-k-1}$ families of walls. Denote $\mathcal{W}$(r) the set of all walls, $\mathcal{W}(r) = \bigcup \mathcal{W}_J(r), J \in M_{n-k-1}$, it is a closed subset of $\mathbb{R}^n$. The following proposition is very important for us.

Proposition 2.1.1: *For every $r > 0$ there is r' such that $\mathbf{B} + U_r$ is contained in the union of all the walls of width r', $\mathbf{B} + U_r \subset \mathcal{W}(r')$.*

Proof: Because $\mathcal{W}(r')$ is invariant under translations by $\mathbf{Z}^n$ it is sufficient to prove that there is r' such that for every $I \in M_k$ the set $\partial^{||}(C_I) + U_r$ is contained in $\mathcal{W}(r')$. Suppose Q is a facet of $P_I^{\perp}$ then Q is a polyhedron (in fact, Q is a parallelepiped) lying in a hyperplane $f_J^{\perp} + \mathrm{p}^{\perp}(\xi)$ for some $\xi \in \mathbf{Z}^n$ and $J \in M_{n-k-1}$. Since $\mathrm{p}^{\perp}(h_J) = f_J^{\perp}$, $\ker(\mathrm{p}^{\perp})= \mathbf{E}$ and Q is a compact set, there is $r_1 > 0$ such that $Q \subset h_J + \xi + U_{r_1}$.Because there is a finite number of C_I for $I \in M_k$ and each C_I has a finite number of facets, we can choose r_1 such that for every facet Q of a prism $C_I, I \in M_k$ we have $Q \subset \bigcup_{J \in M_{n-k-1}} (h_J + U_{r_1} + \mathbf{Z}^n)$.Thus we have $Q + U_r + P_I \subset \bigcup_{J \in M_{n-k-1}} (h_J + U_{r+r_1+r_2} + \mathbf{Z}^n)$ where $r_2 = \max_{I \in M_k}(\text{diameter of } P_I)$. This means that $\partial^{||}(C_I) + U_r$ is contained in $\mathcal{W}(r')$ where $r' = r + r_1 + r_2$.□

The set $\mathbf{B} + U_r$ is very complicated, but the set $\mathcal{W}(r)$ is more easily to deal with.

2.2 The case $n = k + 1$

In this case it is known that $\overline{\mathcal{T}_E}$ does not admit Local Rule (cf.[L]). We reprove this theorem in order to illustrate the method used here.

Proposition 2.2.1: *Suppose* $\mathbf{E}$ *is a totally irrational k-subspace of* $\mathbb{R}^n$ *with* $n = k + 1$. *Then* $\overline{\mathcal{T}_E}$ *does not admits local rule.*

Proof: When $n = k + 1$ proposition 2.1.1 states that $\mathbf{B} + U_r$ is contained in $\mathbf{Z}^n + U_{r'}$ for some r'. For a fixed $r > 0$ we have to find a section Ω not meeting $\mathbf{Z}^n + U_r$ and not reduced to planar section. $\mathbf{E}^{\perp}$ is a line and the set of regular points is dense in this line. Because $\mathbf{E}$ is totally irrational, two sets U_{r+1} and $U_{r+1} + \xi$ where $\xi \in \mathbf{Z}^n$ and $\xi \neq \mathbf{0}$ have no intersection. Hence there is a segment $V = [\alpha, \beta]$ in $\mathbf{E}^{\perp}$ containing $\mathbf{0}$ such that $V + U_{r+1}$ does not meet any $U_r + \xi$ for $\xi \in \mathbf{Z}^n$ and $\xi \neq \mathbf{0}$. We can choose α to be regular. The boundary of $V + U_{r+1}$ consists of two parts: $\alpha + U_{r+1}$ and its complement, denoted by Y. Consider the k-dimensional surface Ω' which coincides with $\mathbf{E} + \alpha$ outside $U_{r+1} + \mathbf{E}^{\perp}$, while inside $U_{r+1} + \mathbf{E}^{\perp}$ it coincides with Y(see fig.1)

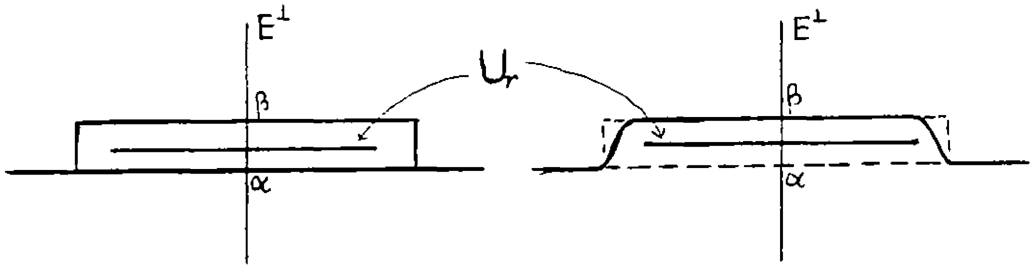

Fig.1

Then Ω' is not a section but it is easy to deform the part Y of Ω' inside $V + U_{r+1}$ such that the obtained surface Ω is a section not meeting U_r. Obviously Ω does not meet $\mathbf{Z}^n + U_r$ and Ω is not reduced to planar section.□

2.3 SI condition

Definition:E *satisfies the SI condition if for every $J \in M_{n-k-1}$ the space $h_J + \mathbf{E}$ contains a rational $(n-k)$-dimensional subspace.*

Note that $h_J + \mathbf{E}$ always contains a rational $(n-k-1)$-dimensional subspace, it is h_J. The reason why this condition is called SI (abriviation of *self-intersection*) is explained below.This was first introduced by Levitov [L] in another interpretation.

Proposition 2.3.1:*If* **E** *does not satisfy the SI condition then there exists $J \in M_{n-k-1}$ such that every two different walls from $\mathcal{W}_J(r)$ do not have intersection.*

Proof: Suppose $h_J + U_r + \xi$ intersects with $h_J + U_r$ for some $\xi \in \mathbf{Z}^n$. Then $\xi \in h_J - h_J + U_r - U_r = h_J + U_{2r} \subset h_J + \mathbf{E}$. If ξ does not belong to h_J then h_J, ξ span a rational $(n-k)$-dimensional subspace of $h_J + \mathbf{E}$, but if $\xi \in h_J$ then $h_J + \xi = h_J$ and the walls $h_J + U_r + \xi$,$h_J + U_r$ are the same.□

Lemma 2.3.2:*There is a 1-plane (i.e. a line) h in $\mathbf{E}^\perp$ not going through any intersection point of every two hyperplanes from* **Ir**.

Proof: Suppose X is the set of all points lying in at least two hyperplanes from **Ir**. Then X is the union of a countable number of planes of codimension 2 in $\mathbf{E}^\perp$. Choose an arbitrary 1-dimensional subspace h' of $\mathbf{E}^\perp$. Then $h' + X$ is the union of a countable number of planes of codimention 1. Hence (by Baire's category theorem or by counting Lebesgue measure) $h' + X$ can not cover the whole $\mathbf{E}^\perp$. Choose a point x in $\mathbf{E}^\perp$ not belonging to $h' + X$ then the line $h = x + h'$ is a line to find.□

Theorem 2.3.3:*If $\overline{\mathcal{T}_E}$ admits local rule then* **E** *satisfies the* SI *condition.*

This theorem was first proved by Levitov [L]. We present here another proof.

Proof: Suppose **E** does not satisfy the SI condition. For every $r > 0$ we will construct a section Ω not meeting $\mathcal{W}(\mathrm{r})$ and not reduced to a planar section.

Choose a line h as in lemma 2.3.2, we will find such a section in $F = E + h$. For this purpose consider the intersections of all the walls with F. For each $J \in M_{n-k-1}$ there is a family of walls $\mathcal{W}_J(r)$. If W_1, W_2 are two walls of different families then by lemma 2.3.2 the restrictions of these two walls on F do not have intersection: $(W_1 \cap F) \cap (W_2 \cap F) = \emptyset$. Now choose an index $J \in M_{n-k-1}$ as in proposition 2.3.1 and suppose W is a wall from $\mathcal{W}_J(r)$. Then $W \cap F$ is a compact set congruent to $\beta + U_r$ for some $\beta \in F$. By proposition 2.3.1 and lemma 2.3.2 the set $W \cap F$ does not meet the intersections of other walls with F. Because the union of all the walls is a closed subset of $\mathbf{E}^\perp$, there is a neighborhood of $W \cap F$ in F which does not meet the intersections of any other walls with F. We are in the situation like that of the case $n = k + 1$ and a trick like that of the proof of proposition 2.2.1 yields the result.□

2.4 Relation with the SI condition of Levitov

For each $i \in \{1, 2, \dots, n\}$ let L_i be the $(n-1)$-dimensional subspace spanned by $(n-1)$ vectors from $\varepsilon_1, \dots, \varepsilon_n$ without ε_i. Then the family $\mathcal{L}_i = L_i + \mathbf{Z}^n = L_i + m\varepsilon_i, m \in \mathbf{Z}$ is a family of equidistant $(n-1)$-planes of $I\!R^n$. The intersection of $\mathcal{L}_i$ with $\mathbf{E}$ is a family of equidistant parallel planes of codimension 1, called the i-th grid of $\mathbf{E}$. In general every k planes of codimension 1 have exactly one intersection point. Due to the genericity (*) we see that every k grids have intersection point different from $\mathbf{0}$. We say that $\mathbf{E}$ satisfies the Levitov SI condition iff every $(k+1)$ grids have intersection point different from $\mathbf{0}$, i.e. there are $(k+1)$ planes of codimension 1, one from each grid, having intersection point which is not $\mathbf{0}$.

$\mathbf{E}$ can be spanned by k vectors with coordinates :

$$v_1 = (v_{11}, v_{12}, \dots, v_{1n})$$
$$v_2 = (v_{21}, v_{22}, \dots, v_{2n})$$
$$\vdots$$
$$v_k = (v_{k1}, v_{k2}, \dots, v_{kn})$$

For $I = (i_1, \dots, i_k) \in M_k$ let A_I be the determinant of the matrix consiting of k columns $i_1, \dots, i_k$.

Proposition 2.4.1: *The following conditions are equivalent:*

a) $\mathbf{E}$ *satisfies the SI condition.*

b) $\mathbf{E}$ *satisfies the Levitov SI condition.*

c)For every $(k+1)$ indices $\{i_1, i_2, \dots, i_{k+1}\}$ from $\{1,2,\dots,n\}$, $(k+1)$ numbers A_{I-j}, $j = 1, \dots, k+1$ are linear dependent over $\mathbf{Q}$.

Here $I - j$ is the set of k indices from $\{i_1, i_2, \dots, i_{k+1}\}$ without i_j.

Proof: $1 \Leftrightarrow 2$. Consider for example $J = (k+2, \dots, n)$, then $J^c = (1, 2, \dots, k+1)$. There is a vector $v = (\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n) \in \mathbf{E}$ such that h_J and v span a rational space iff there are real numbers $a_{n-k+1}, \dots, a_n, b$ such that $a_{k+2}\varepsilon_{k+2} + \cdots + a_n\varepsilon_n + b(\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n)$ belongs to $\mathbf{Z}^n$, and b must not be zero. This holds iff $b\lambda_1, \dots, b\lambda_{k+1}$ are integers, that is, iff $\mathcal{L}_1, \dots, \mathcal{L}_{k+1}$ and $\mathbf{E}$ have a common point bv.

$3 \Leftrightarrow 2$. This follows from the definition of determinants.□

Note that the last condition is more convenient to verify when a concrete $\mathbf{E}$ is given.

2.5 Classes of tilings with the same local structure

Note that when $\mathbf{E}^\perp$ is not totally irrational then in $\mathcal{T}_E$ there are tilings which are not locally equivalent, that is, they have not the same local structure. To find a subclass of $\mathcal{T}_E$ having the same local structure one can proceed as follow. Let $Z(\mathbf{E})$ denote the smallest rational subspace of $I\!R^n$ containing $\mathbf{E}$. the following proposition is well-known (cf[L],[LPS1]):

Proposition 2.5.1: *Suppose α and β are regular and $(\alpha - \beta) \in \mathbf{p}^{\perp}(Z(\mathbf{E}))$ then T_α and T_β have the same local structure.* $\mathbf{E}^{\perp}$ *is totally irrational if and only if $Z(\mathbf{E}) = I\!R^n$.*

For a point $\delta \in \mathbf{E}^{\perp}/_{\mathbf{p}^{\perp}(Z(\mathbf{E}))} = I\!R^n/_{Z(\mathbf{E})}$ (we regard δ also as a point of $\mathbf{E}^{\perp}$) consider the class $\mathcal{T}_{E;\delta}$ of tilings of type T_α where $\alpha \in (\delta + \mathbf{p}^{\perp}(\mathbf{Z}(\mathbf{E}))$. The closure $\overline{\mathcal{T}_{E;\delta}}$ of $\mathcal{T}_{E;\delta}$ is the set of all the tilings having the same local structure as that of a fixed one T_α of $\mathcal{T}_{E;\delta}$.

*Example.*Consider $I\!R^5$ with basis ε_i and actions of group $\mathbf{Z}_5 =< g \mid g^5 = 1 >$ by $g(\varepsilon_i = \varepsilon_{i+1})$. Then $I\!R^5$ decomposes into three invariant subspaces $\mathbf{E}$,$\bar{\mathbf{E}}$, and Δ. Here Δ is a 1-dimensional subspace on which g acts as identity, $\mathbf{E}$ is a quadratic (over $\mathbf{Q}\sqrt{5})$ 2-plane on which g acts as rotation by 72^0 and $\bar{\mathbf{E}}$ is the conjugation of $\mathbf{E}$, on $\bar{\mathbf{E}}$ g acts as rotation by 144^0. Here $Z(\mathbf{E}) = \mathbf{E} + \bar{\mathbf{E}}$, and $\mathbf{E}^{\perp}/_{\mathbf{p}^{\perp}(Z(\mathbf{E}))} = I\!R^n/_{Z(\mathbf{E})} = \Delta = I\!R$. So for each $\delta \in \Delta = I\!R$ there is a class of tilings $\overline{\mathcal{T}_{E;\delta}}$ having the same local structure. When $\delta = 0$ this class of tilings is the class of Penrose tilings , it is proved by de Bruijn that $\overline{\mathcal{T}_{E;\delta}}$ admits local rules when $\delta = 0$. By a result of Ingersent and Steindhart it follows that if $\overline{\mathcal{T}_{E;\delta}}$ admits local rule then $\delta = p + q\tau$ where p, q are integers and τ is the golden ratio, $\tau = (1+\sqrt{5})/2$.In [Le1] we prove that if $\delta = p + q\tau$ then the class $\overline{\mathcal{T}_{E;\delta}}$ *does admit local rule.* Hence a criterion for this case is found.

The structure of the closure $\overline{\mathcal{T}_{E;\delta}}$: Consider the intersection of $\mathbf{B}$ with $\delta + \mathbf{p}^{\perp}(\mathbf{Z}(\mathbf{E}))$. If there is a regular α in $\delta + \mathbf{p}^{\perp}(\mathbf{Z}(\mathbf{E}))$ then this intersection is the union of several families of parallel planes of codimension 1 in $\delta + \mathbf{p}^{\perp}(\mathbf{Z}(\mathbf{E}))$. Suppose α is an irregular point of $\delta + \mathbf{p}^{\perp}(\mathbf{Z}(\mathbf{E}))$, then there are several planes of codimension 1 from $\delta + \mathbf{p}^{\perp}(\mathbf{Z}(\mathbf{E}))$ going through α. They divide $\delta + \mathbf{p}^{\perp}(\mathbf{Z}(\mathbf{E}))$ into many parts and each part,by considering the limits, defines a unique quasiperiodic tilings. As in section §1.5. One can prove that every tiling of $\overline{\mathcal{T}_{E;\delta}}$ is defined in such a way. Note that the union of all $\overline{\mathcal{T}_{E;\delta}}$ with $\delta \in \Delta$ is not $\overline{\mathcal{T}_E}$, but only a subset of $\overline{\mathcal{T}_E}$.

Proposition 2.5.2: *If Ω is a section lying in $\delta + Z(\mathbf{E})$ and not meeting $\mathbf{B} + U_{2r}$ then the tiling T_Ω defined by this section belongs to the class $\overline{\mathcal{T}_{E;\delta}}(r)$, that is, every r-map of T_Ω is a translate of an r-map of a tiling from $\overline{\mathcal{T}_{E;\delta}}$.*

The proof is like in the previous case and we omit it.

Theorem 2.5.3: *If $\overline{\mathcal{T}_{E;\delta}}$ for some $\delta \in \mathbf{E}^{\perp}$ admits local rule then $\mathbf{E}$ satisfies the SI condition.*

Proof: In the proof of theorem 2.3.3 one should choose the line h lying in $\delta + Z(\mathbf{E})$.□

2.6 The case $\dim(Z(\mathbf{E})) = k + 1$

We recall that $Z(\mathbf{E})$ is the minimal rational subspace of $I\!R^n$ containing $\mathbf{E}$. The following proposition is a generalization of proposition 2.2.1.

Proposition 2.6.1: *If $\mathbf{E}$ is a totally irrational k-dimensional subspace of $I\!R^n$ such that $\dim(Z(\mathbf{E})) = k + 1$ then $\mathbf{E}$ does not satisfy the SI condition, hence*

$\overline{T_E}$ *does not admit local rule.*

Proof: Let $v_1, \ldots, v_{k+1}$ be integer vectors which span Z(**E**). There must be $n-k-1$ vectors from $\{\varepsilon_1, \ldots, \varepsilon_n\}$, say $\varepsilon_{k+2}, \ldots, \varepsilon_n$ which together with $v_1, \ldots, v_{k+1}$ span $\mathbb{R}^n$. Then every integer vector $\xi \in \mathbf{Z}^n$ can be expressed uniquely as a linear combination of $v_1, \ldots, v_{k+1}, \varepsilon_{k+2}, \ldots, \varepsilon_n$ with *rational* coefficients. If $\varepsilon_{k+2}, \ldots, \varepsilon_n$,**E** span a space containing an $(n-k)$-dimensional rational subspace then there is a vector $v \in \mathbf{E} \subset Z(\mathbf{E})$ such that $v, \varepsilon_{k+2}, \ldots, \varepsilon_n$ span a ratioanal space. That is, there are $\lambda_{k+2}, \ldots, \lambda_n, \lambda \in \mathbb{R}$ such that $\lambda v + \lambda_{k+2}\varepsilon_{k+2} + \cdots + \lambda_n \varepsilon_n \in \mathbf{Z}^n$. Let $\lambda v = \lambda_1 v_1 + \cdots + \lambda_{k+1} v_{k+1}$. Then we have $\lambda_1 v_1 + \cdots + \lambda_{k+1} v_{k+1} + \lambda_{k+2}\varepsilon_{k+2} + \cdots + \lambda_n \varepsilon_n \in \mathbf{Z}^n$. By the above observation we see that all λ_i are rational, that means λv is rational. Hence **E** contains a rational vector which is contradict to the total irrationality of **E**.□

3 2-dimensional tilings

In this section we will present a stronger necessary condition for 2-dimensional tilings in $\mathbb{R}^4$. There are only a countable number of **E** subject to this condition, while there are a continuum number of **E** satisfying the SI condition. In the whole section we assume that **E** is a totally irrational 2-dimensional subspace of $\mathbb{R}^4$ and $\dim Z(\mathbf{E}) = 4$.

3.1 On the Grassmanian $G_{4,2}$ and non-degeneration

Suppose **E** in $\mathbb{R}^4$ satisfies the SI condition. This means for every $i = 1,2,3,4$ the space spanned by ε_i and **E** contains a rational 2-dimensional subspace, denoted by F_i. This 2-plane F_i is defined uniquely because $Z(\mathbf{E}) = \mathbb{R}^4$. Let f_i be the intersection of F_i and **E**. This must be a 1-dimensional subspace (= a line).

The set $G_{4,2}$ of all 2-dimensional subspaces of $\mathbb{R}^4$ can be parametrized as follow: Each 2-dimensional subspace **E** is determined by two linear equations

$$a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4 = 0$$
$$b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 + b_4\lambda_4 = 0$$

where λ_i are coordinates of points in $\mathbb{R}^4$ and regarded here as variables while a_i, b_i are real numbers. Let $A_{ij} = \det \begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}$. Then

$$A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23} = 0 \qquad (1)$$

Conversely every six numbers $A_I, I \in M_2$ satisfying (1), not all zeros, define a 2-dimensional subspaces of $\mathbb{R}^4$. Two collections A_I, A'_I define the same 2-plane if and only if there exists a real number λ such that $A_I = \lambda A'_I$. Other words, the Grassmanian $G_{4,2}$ is a quadric in the projective space $\mathbb{R}P^5$ defined by equation (1) (cf. for example [GH]). Projective coordinates of F_i is denoted by $(F_i)_I$. A subspace is rational if and only if its projective coordinates, after multiplication by a same number, are rational.**E** intersects F_i by line, hence **E** and F_i are not in generic position. The set of all 2-dimensional subspaces with projective coordinates A_I having intersection of dimension greater or equal 1 with F_i is defined by the following equation

$$(F_i)_{12}A_{34} - (F_i)_{13}A_{24} + (F_i)_{14}A_{23} + (F_i)_{23}A_{14} - (F_i)_{24}A_{13} + (F_i)_{34}A_{12} = 0 \qquad (2_i)$$

Definition: *Suppose* **E** *satisfies the SI condition.* **E** *is called non-degenerate if four planes F_i, regarded as vector in $\mathbb{R}^6$ are linear independent.*

Lemma 3.1.1: *If* **E** *is degenerate then there is a continuous non-constant curve* $\mathbf{E}(t), t \in \mathbb{R}$ *in* $G_{4,2}$ *such that* $\mathbf{E}(0) = \mathbf{E}$ *and all 2-planes* $\mathbf{E}(t)$ *intersect* F_i *by lines,that is* $\dim(\mathbf{E}(t) \cap F_i) \geq 1, i = 1,2,3,4.$

Proof:(2_i) are linear equations on A_I. If $(F_i)_I$ are not linear independent then these equations define a projective space X of dimension greater or equal 2 in $\mathbb{R}P^5$. The intersection of X and the quadric defined by (1) contains **E**. If **E** is not an isolated point of this intersection then the intersection contains a curve going through **E** and we are done. But if **E** is an isolated point, then the projective subspace X have tangent point **E** with the quadric defined by (1). Because both X and the quadric have rational coefficients and degree of X is 1, it is easy to see that their tangent point must be a rational point, this contradicts the fact that **E** is totally irrational.□

3.2 Main theorem

Theorem 3.2.1: *Suppose* **E** *is a 2-dimensional subspace of* $\mathbb{R}^4$. *If* $\overline{\mathcal{T}_E}$ *admits local rule then* **E** *is quadratic and non-degenerate.*

Note that when **E** is quadratic then **E** satisfies the SI condition (see [L] or [LPS1]).We divide the proof into several cases.

Proof: *The case* $\mathbf{E}^\perp$ *intersects* F_i *by lines.* This case is essential. An example of this case is the 8-fold symmetry case considered by Burkov [B],Beenker, and De Bruijn [dB2] see also examples below. In this case F_i is a prism, $F_i = f_i + f_i^\perp$ where $f_i^\perp = F_i \cap \mathbf{E}^\perp$.

Let $\mathcal{F} = \cup_{i=1}^{i=4}(F_i + \mathbf{Z}^4)$, this is a set of 2-planes in $\mathbb{R}^4$. Then $\mathcal{W}(r) = \mathcal{F} + U_r$. For every $r > 0$ we have to construct a section Ω not meeting $\mathbf{B} + U_r$ and not reduced to planar section. In fact here we will construct Ω not meeting $\mathbf{B} + U_r$ such that the projection $\mathbf{p}^\perp(\Omega)$ is not a bounded set in $\mathbf{E}^\perp$. This is of course a stronger assertion. The idea is as follow. Let $U_r(t)$ be the image of U_r under the projector on $\mathbf{E}(t)$ along $\mathbf{E}^\perp$. That is $U_r(t) = (U_r + \mathbf{E}^\perp) \cap \mathbf{E}(t)$. We will construct a continuous map $\phi : \mathbb{R}^4 \to \mathbb{R}^4$ satisfying :

a)preserving fibers, that is, $\phi(x + \mathbf{E}^\perp) = x + \mathbf{E}^\perp$ for every $x \in \mathbf{E}$.

b)not far from the identity, that is, there is a constant such that $|\phi(x) - x| <$ constant for every $x \in \mathbb{R}^4$.

c)$\phi(\mathcal{F} + U_r) = \mathcal{F} + U_r(t)$ and $\phi^{-1}(\mathcal{F} + U_r(t)) = \mathcal{F} + U_r$ for a number $t > 0$ such that $\mathbf{E}(t) \neq \mathbf{E}$.

If such ϕ exists. Choose a point $\alpha \in \mathbf{E}^{\perp}$ such that $\alpha + \mathbf{E}(t)$ does not meet $\mathcal{F}$, and hence does not meet $\mathcal{F} + U_r(t)$. Consider $\Omega = \phi(\alpha + \mathbf{E}(t)$. Then by a) Ω is a section, by b) the set $\mathbf{p}^{\perp}(\Omega)$ is not bounded and by c) section Ω does not meet $\mathcal{F} + U_r$. That is Ω is the section to find.

Step 1.Fix $r > 0$. We call a *big wall* a set of type $F_i + \xi + U_r$ where $\xi \in \mathbf{Z}^4$. Each big wall is contained in a unique 3-plane. There are 6 prisms C_I which together form a fundamental domain of group $\mathbf{Z}^4$. Each C_I intersects with a finite number of big walls. The 3-planes going through these big walls divide C_I into smaller prisms, each smaller prism has the same projection on $\mathbf{E}$ as C_I has. Other words, $P_I^{\perp}$ is divided into smaller polygons and the division of C_I is just the collection of the sums of each smaller polygon with P_I. By this way we divide all six prisms C_I into smaller prisms and spread out this division to all the other prisms of $\mathcal{O}$ just by translation. Denote $\mathcal{O}_r$ the new family of prisms, it is called a refinement of $\mathcal{O}$. Let $\mathbf{B}_r$ be the parallel boundary of $\mathcal{O}_r$. From the construction one sees that $(\mathcal{F} + U_r) \subset \mathbf{B}_r$, but $\mathbf{B}_r \subset \mathcal{F} + U_{r'}$ for a large r'. If $x \in \mathbf{E}$ then the 2-plane $x + \mathbf{E}^{\perp}$ is divided into polygons by the intersections with $\mathbf{B}_r$. There are only a finite number, up to translations, of different polygons in the intersection of $x + \mathbf{E}^{\perp}$ with $\mathbf{B}_r$.

Step 2.For a fixed r we can parametrize the curve $\mathbf{E}(t)$ such that the distance between y and $\mathbf{p}(y)$ is less than t for every $y \in U_{r+1}(t)$. Consider the intersection of a fiber $x + \mathbf{E}^{\perp}$ with the set of all big walls $\mathcal{W} = \mathcal{F} + U_r$. First consider the intersection with a big wall $(x + \mathbf{E}^{\perp}) \cap (F_i + \xi + U_r)$. Both are prisms, and $F_i + \xi + U_r = (f_i^{\perp}) + (f_i + U_r) + \xi$. It is easy to chech that $x + \mathbf{E}^{\perp}$ and $F_i + \xi + U_r$ have non-empty intersection if and only if x is lying in the ball $\mathbf{p}(\xi) + U_r$ and in this case the intersection is the line $x + f_i^{\perp} + \mathbf{p}^{\perp}(\xi)$ which is parallel to $f_i^{\perp}$. Hence the intersection of $x + \mathbf{E}^{\perp}$ with $\mathcal{F} + U_r$ is the union of 4 families of parallel lines. These 4 families divide $x + \mathbf{E}^{\perp}$ into polygons. There are only a finite number of polygons in this division, up to translation.

Step 3. **Lemma 3.2.2:** *The 2-plane $x + \mathbf{E}^{\perp}$ has non-empty intersection with $F_i + \xi + U_{r'}$ if and only if it has non-empty intersection with $F_i + \xi + U_{r'}(t)$. In the case the intersections are not empty both are lines and the distance between them is less than t.Here r' is any number between r and $r + 1$.*

The proof is quite easy and we omit it.This lemma means that the two system of lines are very close to each other when t is small.

We call the line $(x + \mathbf{E}^{\perp}) \cap (F_i + \xi + U_r(t))$ the corresponding line of $(x + \mathbf{E}^{\perp}) \cap (F_i + \xi + U_r)$, if both are not empty.

Step 4.**Lemma 3.2.3:** *If three lines from the intersection of $x + \mathbf{E}^{\perp}$ with $\mathcal{F} + U_r$ intersect at a point then their corresponding lines also intersect at a point.*

Proof: If three lines $(x + \mathbf{E}^{\perp}) \cap (F_i + \xi_i + U_r), i = 1, 2, 3$ intersect at a point then $\mathbf{p}^{\perp}(\xi_i) + f_i^{\perp}$ also intersect at a point. But in this case, due to the fact that $\mathbf{E}$ is totally irrational and F_i are rational, ξ_i are interger points, one easily prove that the three 2-planes $F_i + \xi_i, i = 1, 2, 3$ intersect at a point. From this it follows that three corresponding lines $(x + \mathbf{E}^{\perp}) \cap (F_i + \xi_i + U_r(t)), i = 1, 2, 3$

intersect at a point.□

If two lines from $(x + \mathbf{E}^{\perp}) \cap (\mathcal{F} + U_r)$ intersect at a point v then the intersection point of the two corresponding lines is called the corresponding vertex of v. The lemma guarantees that the definition is correct.

Step 5. The system of line $(x + \mathbf{E}^{\perp}) \cap (\mathcal{F} + U_r)$ divides $x + \mathbf{E}^{\perp}$ into polygons. If we choose t very small then the system of corresponding lines divides $x + \mathbf{E}^{\perp}$ into polygons in a similar manner. This means that if we take the set of vertices of a polygon, then the convex hull of the corresponding vertices is a polygons of $x + \mathbf{E}^{\perp}$ divided by the corresponding system of lines. Now we define a map $\psi : (x + \mathbf{E}^{\perp}) \to (x + \mathbf{E}^{\perp})$ as follow. If x is a vertex of a polygon of $x + \mathbf{E}^{\perp}$ divided by the system of lines $(x + \mathbf{E}^{\perp}) \cap (\mathcal{F} + U_r)$ then let $\psi(x) =$ the corresponding vertex. For a convex polygon with vertices $v_1, \ldots, v_m$ define its center as the unique point v such that $\overrightarrow{vv_1} + \cdots + \overrightarrow{vv_m} = \vec{0}$. The center is unique and lying inside the convex polygon. We have defined ψ for vetices of polygons and now can define ψ for centers of polygons, just take the center of the corresponding polygon. By connecting the center with each vertex of a polygon, we get a linear simplicial structure of $x + \mathbf{E}^{\perp}$ and spread the map ψ on $x + \mathbf{E}^{\perp}$ by linearity.

The map ψ is defined on $\mathbb{R}^4$. It satisfies all three properties a), b), c) listed above , but unfortunately it is not continuous. However the set of continuous points of ψ is a "big" one.

Step 6.The boundary of the projection of a big wall of width r is two parallel lines. Denote V_r the union of all such boundaries of the projections of all the big walls of width r. This is the union of 4 families of lines, each family consists of a countable number of parallel lines.

Lemma 3.2.4: *Suppose $y \in \mathbb{R}^4$ such that $\mathrm{p}(y) = x$ does not lie in V_r, then there is a neighborhood of y in $\mathbb{R}^4$ such that ψ is continuous in this neighborhood.*

Proof: Suppose $P_1, \ldots, P_m$ are polygons of $(x + \mathbf{E}^{\perp}) \cap (\mathcal{F} + U_r)$ containing y (including the case when y lies on the boundary of some polygon). Each side of a polygon is a segment of the intersection of $x + \mathbf{E}^{\perp}$ with a big wall. Let $W_1, \ldots, W_p$ are those big walls whose intersection with $x + \mathbf{E}^{\perp}$ containing a side of one of $P_1, \ldots, P_m$. Then x is lying inside in the intersection X of the projections of these big walls on $\mathbf{E}$, but x is not lying on the boundary of X due to the condition of the lemma. The union of all the sets $X + P_i, i = 1, \ldots, m$ is a neighborhood of y and obviously ψ is continuous in this neighborhood.□

Step 7. For a fixed number r we can construct a map ψ. If we choose another, say r', $r < r' < r + 1$ then we can construct in a similar way another map ψ', with properties:

i)preserving fibres

ii)$|\psi'(x) - x| < t$

iii)$\psi'(\mathcal{F} + U_r) = \mathcal{F} + U_r(t), \psi'^{-1}(\mathcal{F} + U_r(t)) = \mathcal{F} + U_r$ and if y is an interior point of a polygon of $x + \mathbf{E}^{\perp}$ divided by the system of lines $(x + \mathbf{E}^{\perp}) \cap (\mathcal{F} + U_r)$

then $\psi'(y)$ is an interior point of the corresponding polygon. This follows from the fact that ψ' is bijective, and when $r' > r$ the division of $x + \mathbf{E}^{\perp}$ into polygons by the system of lines $(x + \mathbf{E}^{\perp}) \cap (\mathcal{F} + U_{r'})$ is finer than the division by the system of lines $(x + \mathbf{E}^{\perp}) \cap (\mathcal{F} + U_r)$.

Now for each $y \in \mathbb{R}^4$ we can choose an $r', r < r' < r + 1$ such that the map ψ' is continuous in a neighborhood of y. The space $\mathbb{R}^4$ is covered by such neighborhoods. Choose a subfamily of neighborhhoods which is locally finite and by using the partition of unity with respect to this locally finite family we can glue all the continuous maps ψ' in these neighborhoods and get a continuous map ϕ. It is easy to check that this map ϕ satisfies all three properties a),b),c). The theorem for the case when $\mathbf{E}^{\perp}$ intersects F_i by lines is proved.

The general case. Choose a 2-dimensional subspace $\mathbf{E}'$ in the curve $\mathbf{E}(t)$, it intersects F_i by lines.Suppose $\mathbf{E} \cap \mathbf{E}' = \{0\}$. Denote π and π' resp. the projection on $\mathbf{E}$ and $\mathbf{E}'$ corresponding to the decomposition $\mathbb{R}^4 = \mathbf{E} + \mathbf{E}'$. A 2-dimensional surface Ω is called a π-section if the restriction of π on Ω is a homeomorphism between Ω and $\mathbf{E}$. A π-fiber is a set of type $x + \mathbf{E}'$. Then the proof of the previous case yields the following: for every $r > 0$ there is a π-section Ω which does not meet $\mathcal{F} + U_r$ and the projection $\pi'(\Omega)$ is not bounded, or equivalently, the projection $\mathbf{p}^{\perp}(\Omega)$ is not bounded.

Now consider the cut method of pair ($\mathbf{E}$,$\mathbf{E}'$). That is, in the construction of family $\mathcal{O}$, instead of $\mathbf{E}^{\perp}$ we use $\mathbf{E}'$. We get 6 new π-prisms C_I^{π}. These six π-prisms and their translates by integer vectors cover the whole $\mathbb{R}^4$, but may be with overlaps. When $\mathbf{E}' = \mathbf{E}^{\perp}$ or when $\mathbf{E}'$ is near to $\mathbf{E}^{\perp}$ there are no overlaps. But we can always get a tiling of $\mathbb{R}^4$: the superpositions of all the π-prisms of type $C_I^{\pi} + \xi, \xi \in \mathbf{Z}^4$ divide $\mathbb{R}^4$ into covex polyhedra, each is a π-prism. The collection of all these polyhedra is a tiling of $\mathbb{R}^4$, denoted by $\mathcal{O}'$. This tiling is invariant under translations from $\mathbf{Z}^4$. The parallel boundary of this family is denoted by B'. As in the case $\mathbf{E}' = \mathbf{E}^{\perp}$ the set $\mathbf{B}' + U_r$ is a subset of $\mathcal{F} + U_{r_1}$ for some r_1. We can also refine the family $\mathcal{O}'$ as in the case of $\mathcal{O}$ by the set $\mathcal{F} + U_r$ to get a family $\mathcal{O}'_r$ with parallel boundary $\mathbf{B}'_r$. Note that when r tends to infinity, the maximal diameter of the projection $\pi'(C), C \in \mathcal{O}'(r)$ tends to zero, while the refinement does not affect the projections $\pi(C)$: these projections are always the same, and up to translations there are a finite number of them.

For every $r > 0$ choose r_1 such that $\mathbf{B}'_r$ is contained in $\mathcal{F} + U_{r_1}$. By the previous construstion there is a π-section Ω not meeting $\mathcal{F} + U_{r_1}$ and hence not meeting $\mathbf{B}'_r$. Let $\mathcal{S}$ be the set of all π-prisms from $\mathcal{O}'_r$ meeting Ω. By projecting (along $\mathbf{E}'$) on $\mathbf{E}$ the collection $\mathcal{S}$ we get a tiling T, whose tiles are convex polygons. We divide each tile of T into triangles by putting some diagonals arbitrarily. The result is a simplicial structure of the 2-dimensional plane $\mathbf{E}$. For every vertex v of T define $\varphi(v)$ be the point of Ω lying upon v, that is $\varphi(v) = \Omega \cap \pi^{-1}(v)$. Then we spread the map φ on the whole $\mathbf{E}$ by linearity using the simplicial structure. The surface $\varphi(\mathbf{E})$ is of course lying inside the union of all prisms from $\mathcal{S}$, it consists of triangles and defines the same tiling as Ω. Each triangle is contained in a prism of $\mathcal{S}$. Because the size of the projection on $\mathbf{E}'$ of prisms from $\mathcal{O}'_r$ tends to zero when $r \to \infty$, the 2-plane containing the triangle tends (uniformly on the set of all triangles) to a plane parallel to $\mathbf{E}$. Hence when r is sufficiently large, the surface $\varphi(\mathbf{E})$ is a section with respect to $\mathbf{p}$. Is is a section to find.

Now suppose $\mathbf{E}$ intersects $\mathbf{E}'$ by line,$\mathbf{E} \cap \mathbf{E}' = h$. Note that among F_1, F_2, F_3, F_4 there may be coincident 2-planes. If F_i and F_j intersect by a subspace of dimension ≥ 1 then they are coincident. In fact if $F_i \cap F_j = l$ is a line then $l + \mathbf{E}$ contains both F_i and F_j , hence $\dim Z(\mathbf{E}) = 3$.

Case a). There are 3 different 2-planes from F_1, F_2, F_3, F_4, say F_1, F_2, F_3. Since the intersection of two rational space is a rational space, two of the three 2-planes F_1, F_2, F_3, say F_1, F_2 do not go through h. In this case because F_1, F_2 intersect $\mathbf{E}$,$\mathbf{E}'$ by lines we see that both F_1, F_2 are contained in $\mathbf{E} + \mathbf{E}'$ which is a 3-dimensional space while F_1, F_2 span a 4-dimensional space. So this case is impossible.

Case b). There are only two different 2-planes from F_1, F_2, F_3, F_4, say F_1, F_2. We can choose $v_1 \in F_1; v_2 \in F_2$ such that $v_1, v_2, \mathbf{E}$ span $F_1 + F_2 = \mathbb{R}^4$. Then E" spanned by v_1, v_2 is a 2-dimensional space intersects all F_i by lines and $\mathbf{E} \cap$ E" $= \{0\}$.

The theorem is completely proved.□

Example. As an application of the theorem consider the following case. The group $\mathbf{Z}_8 =< g \mid g^8 = 1 >$ acts in $\mathbb{R}^4$ by $g(\varepsilon_1) = \varepsilon_2, g(\varepsilon_2) = \varepsilon_3, g(\varepsilon_3) = \varepsilon_4, g(\varepsilon_4) = -\varepsilon_1$. The space $\mathbb{R}^4$ decomposes into two invariant 2-dimensional subspaces $\mathbf{E}$ and $\mathbf{E}^{\perp}$, on $\mathbf{E}$ g acts as rotation by 45^0 and on $\mathbf{E}^{\perp}$ by 135^0. A tiling of $\overline{\mathcal{T}_E}$ is called a quasiperiodic tiling having 8-fold symmetry. This class of tilings has been investigated by Beenker and Burkov. In this case $\mathbf{E}$ satisfies the SI condition so theorem 2.3.3 does not say any thing about this class. Burkov [B] and De Bruijn [dB2] proved that this class does not admit local rule. Here this can be obtained directly from the theorem , because $\mathbf{E}$ in this case is degenerate.

3.3 On the non-degeneration

Suppose $\mathbb{R}^4 = \mathbf{E} \oplus \mathbf{E}'$, $Z(\mathbf{E}) = \mathbb{R}^4$ and $H_1, \ldots, H_m$are *rational* 2-dimensional subspaces of $\mathbb{R}^4$ such that $\dim(\mathbf{E} \cap H_i) = \dim(\mathbf{E}' \cap H_i) = 1, i = 1, 2, \ldots, m$.

Proposition 3.3.1: *The following are equivalent:*

a) There is a non-constant curve $\mathbf{E}(t), t \in \mathbb{R}$ in $G_{4,2}$ such that $\mathbf{E}(0) = \mathbf{E}$ and all $\mathbf{E}(t)$ intersects $H_i, i = 1, \ldots, m$ by lines.

b) m vectors $(H_i)_I, i = 1, 2, \ldots, m$ form a subspace of dimension ≤ 3 in $\mathbb{R}^6$.

c) There is a linear transformation $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\varphi(\mathbf{E}) = \mathbf{E}'$ and $\varphi(H_i) = H_i, i = 1, 2, \ldots, m$.

Proof: $a) \Leftrightarrow b)$.See lemma 3.1.1

$b) \Leftrightarrow c)$ Note that up to a multiple there is a unique linear transformation

φ such that $\varphi(\mathbf{E}) = \mathbf{E}'$ and $\varphi(H_i) = H_i$ for $i = 1,2,3$. Choose coordinate system x_1, x_2 on $\mathbf{E}$ such that $H_1 \cap \mathbf{E}$ is given by $x_1 = 0$,$H_2 \cap \mathbf{E}$ is given by $x_2 = 0$, $H_3 \cap \mathbf{E}$ is given by $x_1 + x_2 = 0$.Then choose coordinate system y_1, y_2 on $\mathbf{E}'$ such that $H_1 \cap \mathbf{E}'$ is given by $y_1 = 0$,$H_2 \cap \mathbf{E}'$ is given by $y_2 = 0$, $H_3 \cap \mathbf{E}'$ is given by $y_1 + y_2 = 0$. Suppose H is a rational 2-dimensional of $\mathbb{R}^4$ intersecting $\mathbf{E}$,$\mathbf{E}'$ by lines. Then $H \cap \mathbf{E}$ is given by $x_1 + ax_2 = 0$ and $H \cap \mathbf{E}'$ by $y_1 + by_2 = 0$ where a, b are real numbers. Consider the coordinate system of $\mathbb{R}^4$ given by x_1, x_2, y_1, y_2. In this coordinate system the projective coordinate of H_1, H_2, H_3, H are as below (here we write six numbers $H_{12}, H_{13}, H_{14}, H_{23}, H_{24}, H_{34}$):

H_1 : (0,1,0,0,0,0)

H_2 : (0,0,0,0,1,0)

H_3 : (0,1,1,1,1,0)

H : (0,1,b,a,ab,0)

It is easy to see that vector H is a linear combination of H_1, H_2, H_3 if and only if $a = b$, that is , if and only if $\varphi(H) = H$.□

The proof of theorem 3.2.1 gives the following.

Proposition 3.3.2: *Suppose* $\mathbb{R}^4 = \mathbf{E} \oplus \mathbf{E}'$, $Z(\mathbf{E}) = \mathbb{R}^4$, $\mathbf{E}$ *is totally irrational and* $H_1, \ldots, H_m$ *are 2-dimensional subspaces of* $\mathbb{R}^4$ *satisfying one of the three equivalent conditions of proposition 3.3.1. Let* $\mathcal{H} = \bigcup_{i=1,\ldots,m}(H_i + \mathbf{Z}^4)$. *Suppose* $r > 0$ *is fixed . Thenfor sufficiently small* $t > 0$ *there is a continuous map* $\phi : \mathbb{R}^4 \to \mathbb{R}^4$ *satisfying:*

a)preserving fibers, that is, $\phi(x + \mathbf{E}') = x + \mathbf{E}'$ *for every* $x \in \mathbf{E}$.

b)not far from the identity, that is, there is a constant such that $|\phi(x) - x| <$ constant *for every* $x \in \mathbb{R}^4$.

c)$\phi(\mathcal{H} + U_r) = \mathcal{H} + U_r(t)$ *and* $\phi^{-1}(\mathcal{H} + U_r(t)) = \mathcal{H} + U_r$.

Here $U_r(t)$ is the ball in $\mathbf{E}(t)$ lying upon U_r, i.e. $U_r(t) = \mathbf{E}(t) \cap (U_r + \mathbf{E}')$.

4 Tilings having 12-fold symmetry

4.1 Description of the tilings

Let's consider $\mathbb{R}^6$ with basis $\varepsilon_i, i = 1, \ldots, 6$ and action of group $\mathbf{Z}_{12} =< g \mid g^{12} = 1 >$ in $\mathbb{R}^6$ by $g(\varepsilon_i) = \varepsilon_{i+1}, i = 1, \ldots, 5, g(\varepsilon_6) = -\varepsilon_1$. Then $\mathbb{R}^6$ falls into three invariant 2-dimensional subspaces $\mathbf{E}$,$\bar{\mathbf{E}}$ and Δ where Δ is a rational subspace on which g acts as rotation by 90^0, $\mathbf{E}$ is a quadratic (over $\mathbf{Q}\sqrt{3}$) on which g acts as rotation by 30^0, $\bar{\mathbf{E}}$ is the conjugation of $\mathbf{E}$, on $\bar{\mathbf{E}}$ g acts as rotation by 150^0. A tiling belongs to $\overline{\mathcal{T}_E}$ is called *a quasiperiodic tiling having 12-fold symmetry.*The prototiles, up to rotations, are listed in fig.2.

Here are vectors that span $\mathbf{E}$,$\bar{\mathbf{E}}$ and Δ.

$\mathbf{E}$ is spanned by $(2, \sqrt{3}, 1, 0, -1, -\sqrt{3})$ and $(0, 1, \sqrt{3}, 2, \sqrt{3}, 1)$.

$\bar{\mathbf{E}}$ is spanned by $(2,-\sqrt{3},1,0,-1,\sqrt{3})$ and $(0,1,-\sqrt{3},2,-\sqrt{3},1)$.

Δ is spanned by $(1,0,-1,0,1,0)$ and $(0,1,0,-1,0,1)$.

Let e_i(resp. $\bar{e}_i$,$\tilde{e}_i$) be the projection of ε_i on $\mathbf{E}$(resp. on $\bar{\mathbf{E}}$,Δ). On the planes $\mathbf{E}$,$\bar{\mathbf{E}}$,Δ these vectors looks like in fig.3.

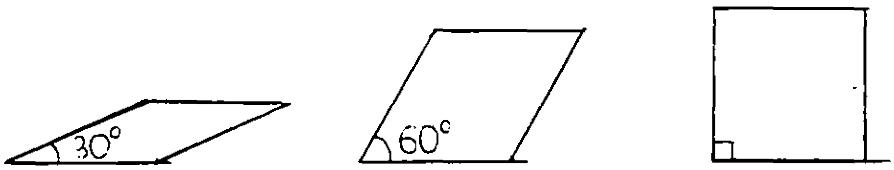

Figure 2

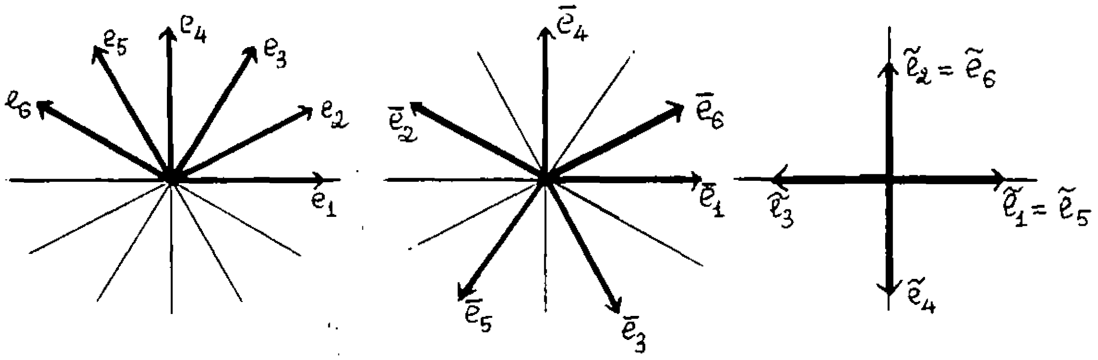

Figure 3

One can check easily that $\mathbf{E}$ satisfies the SI condition. Here the set M_3 has 20 elements. Note that $\mathrm{p}^{\perp}(Z(\mathbf{E}))$ is contained in $\mathbf{Ir}$.The subspace $h_J+\mathbf{E}$ contains $Z(\mathbf{E})$ for $J=(1,3,5)$ and $J=(2,4,6)$. Let M_3' be the subset of M_3 not containing $(1,3,5)$ and $(2,4,6)$. Then M_3' has 18 elements and for every $J \in M_3'$ the space $h_J+\mathbf{E}$ contains a unique rational 4-dimensional subspace F_J. For example when $J=(1,2,3)$ then F_J is spanned by $\varepsilon_1,\varepsilon_2,\varepsilon_3$ and e_2, when $J=(1,3,4)$ then F_J is spanned by $\varepsilon_1,\varepsilon_3,\varepsilon_4$ and e_2,when $J=(1,3,6)$ then F_J is spanned by $\varepsilon_1,\varepsilon_3,\varepsilon_6$ and e_2. These three 4-planes have the same intersection with $Z(\mathbf{E})=\mathbf{E}\oplus\bar{\mathbf{E}}$, the intersection is a rational 2-dimensional space H spanned by e_2 and $\bar{e}_2$. Other F_J where $J \in M_3'$ can be obtained from these three by actions of group $\mathbf{Z}_{12}$. Note that $g^6(F_J)=F_J, J \in M_3'$. The 2-plane H intersects $\mathbf{E}$,$\mathbf{E}'$ by lines. By actions of $\mathbf{Z}_{12}$ from H we can get six 2-planes $H_1,\ldots,H_6$. We can choose the notation so that H_i is spanned by e_i and $\bar{e}_i$.

Let Λ be the projection of $\mathbf{Z}^6$ on $Z(\mathbf{E})$, it is a lattice in the 4-dimensional space $Z(\mathbf{E})=\mathbf{E}+\mathbf{E}'$, it is generated by 4 rational vectors. Denote $\Lambda/2$ be the set of all points ξ such that 2ξ belongs to Λ.Let $\mathcal{H}=\bigcup_{i=1}^{6}(H_i+\Lambda/2)$.

4.2 Absence of local rules

Let $\delta=(\tilde{e}_1+\tilde{e}_2)/2$, it is a point of Δ.

Lemma 4.2.1: *If $J=(1,3,5)$ or $J=(2,4,6)$ then the space $h_J+\mathbf{E}+\xi$ does not meet $\delta+Z(\mathbf{E})$ for any $\xi \in \mathbb{R}^6$.*

Proof: Note that the projection of $\mathbf{Z}^6$ on Δ is the discrete lattice generated by $\tilde{e_1}, \tilde{e_2}$, and the projections of h_J for the above J on Δ are two lines going through 1-facets of this lattice. Hence the projections of all the sets of type $h_J + \mathbf{E} + \xi$ is contained in the union of all the 1-facets of this lattice. Since δ does not lie on any 1-facet of the lattice,by considering the projections on Δ one sees easily that the set $\delta + Z(\mathbf{E})$ does not meets $h_J + \mathbf{E} + \xi$.□

We want to find a section in $\delta + Z(\mathbf{E}) = \delta + \mathbf{E} + \mathbf{E}'$ not meeting $\mathbf{B} + U_r$ and not reduced to planar section. For this reason at first we study the intersection $(\mathbf{B} + U_r) \cap (\delta + \mathbf{E} + \mathbf{E}')$.

Lemma 4.2.2:*For every $J \in M_3'$ the intersection of F_J with $\delta + \mathbf{E} + \mathbf{E}'$ is contained in $\delta + \mathcal{H}$.*

Proof: For each $J \in M_3'$ we prove that $F_J \cap (\delta + \mathbf{E} + \mathbf{E}')$ is contained in $\delta + H_i + \xi/2$ for some $i = 1, 2, \ldots, 6$ and $\xi \in \Lambda$. This can be checked easily. For example when $J = (1,2,3)$ the intersection $F_J \cap (\delta + \mathbf{E} + \mathbf{E}')$ is contained in $\delta + H_2 + \xi/2 + U_1$ where ξ is the projection of $\varepsilon_1 + \varepsilon_2$ on $Z(\mathbf{E})$. □

From this lemma one sees that the intersection of $\mathbf{B} + U_r$ with $\delta + \mathbf{E} + \mathbf{E}'$ is contained in $\delta + U_r + \mathcal{H}$.

Now in the 4-dimensional plane $X = (\delta + \mathbf{E}) + \mathbf{E}'$ we have a lattice $\delta + \Lambda/2$ which plays the role $\mathbf{Z}^4$ as in §3. The 2-planes $\delta + H_i$ intersects $\delta + \mathbf{E}$ and $(\mathbf{E}' + \delta)$ by lines, just like the 2-planes H_i intersects $\mathbf{E}$,$\mathbf{E}'$ by lines.

Lemma 4.2.3:*Six 2-planes H_i satisfy all the conditions listed in proposition 3.3.1.*

Proof: The proof consists of straight verification. The linear map $\varphi : (\mathbf{E} + \mathbf{E}') \to (\mathbf{E} + \mathbf{E}')$ defined by $\varphi(e_1) = \bar{e_1}, \varphi(e_2) = -\bar{e_2}$ sends $\mathbf{E}$ onto $\mathbf{E}'$ and preserves all the six 2-planes H_i.□

From this lemma and proposition 3.3.2 one easily constructs a section Ω lying in $\delta + \mathbf{E} + \mathbf{E}'$ not meeting $\mathbf{B} + U_r$ and not reduced to planar section. Morever, the projection $\mathrm{p}^{\perp}(\Omega)$ is not bounded. Hence we get

Theorem 4.2.4:*The class of quasiperiodic tilings having 12-fold symmetry does not admit local rules.*

5 Concluding remarks

1)The proof of theorem 4.2.4 can be applied to the case $n > 4, k = 2$ and $\dim Z(\mathbf{E}) = 4$.

2) For all quadratic $\mathbf{E}$ one can always *color* all the tilings of $\overline{\mathcal{T}_E}$ such that the resulted class admit local rules. This is true even in the case when $\mathbf{E}$ is degenerate. For the exact definition of coloring and the proof we refer to [LPS].For example, after coloring, the classes of quasiperiodic tilings having 8-fold symmetry and 12-fold symmtry admit local rule. While the class quasiperiodic tilings having 5-fold symmetry admits local rule even without any coloring.

3) In the case of 8-fold symmetry, by Main theorem (theorem 3.2.1) for a fixed

r the set $\overline{\mathcal{T}_E}(r)$ is never coincident with $\overline{\mathcal{T}_E}$. Nevertheless for large r this set consists of only quasiperiodic tilings, including some periodic tilings. More precisely for large r there is $a \in \mathbb{R}, a > 0$ such that $\overline{\mathcal{T}_E}(r) = \cup_{-a<t<a}\overline{\mathcal{T}_{E(t)}}$. This is proved in [Le2].

4)Levitov [L] introduce the definition of weak local rules, and the proof of the Main theorem also asserts that if **E** is degenerate then even weak local rule does not exists. Combining a result of [LPS] (see also [L]) we get the following:

Proposition:*Suppose* **E** *is quadratic subspace of* $\mathbb{R}^4$. *The following are equivalent:*

*1)***E** *is non-degenerate.*

2)$\overline{\mathcal{T}_E}$ *admits weak local rule.*

5)Loosely speaking when **E** is quadratic and non-degenerate, the class $\overline{\mathcal{T}_E}$ is "not far" from having local rule. Only modulo a "bootstrapped condition", and in many cases one can prove the existence of local rule for $\overline{\mathcal{T}_E}$.

6) For the case dim(**E**) greater than 2 we can prove an analog of theorem 3.2.1 which states that if **E** is a quadratic,totally irrational subspace of $\mathbb{R}^n$ with $n = 2k$ and the class of tilings $\overline{\mathcal{T}_E}$ admits local rule then **E** must be *non-degenerate*. Here non-degeneration is defined only for the case **E** is *quadratic*. In this case 2-plane F_i going through ε_i and $\pi(\varepsilon_i)$ is rational where π is the projection on **E** along the algebraic conjugation $\bar{\mathbf{E}}$ of **E**. There are 2k such 2-planes, each intersects **E**,$\bar{\mathbf{E}}$ by lines. **E** is called *degenerate* if there is a continuous family **E**(t) of k-dimensional subspaces of $\mathbb{R}^n$ such that $\mathbf{E}(0) = E$ and $\mathbf{E}(t)$ intersects F_i by lines.

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