On continuity of rotation numbers associated to geodesic laminations

H. K. Farahat
Department of Mathematics
2500 University Drive
Calgary T2N 1N4 Canada
hfarahat@math.ucalgary.ca

I. V. Nikolaev
Department of Mathematics
2500 University Drive
Calgary T2N 1N4 Canada
nikolaev@math.ucalgary.ca

July 30, 2003

Abstract

Geodesic lamination on the Riemann surface is an analogy of a family of parallel lines in the plane. There exists a notion of "slope" of this family on any Riemann surface ([6]). In this note we prove that the "slope function" is in fact a continuous function of the lamination. It is remarkable, that no geometric argument is required to establish this fact. We use exclusively postulates of continued fraction theory and arithmetic group theory.

Key words and phrases: Geodesic lamination, arithmetic group, continued fraction

AMS (MOS) Subj. Class.: 11F06, 20H10, 57M50

Introduction

Geodesic lines on the Riemann surfaces are critically important in the understanding of algebra and geometry of such surfaces. They have been extensively studied in the past by Hadamard ([3]), Artin ([1]), Morse ([4]) and others. Topologically, geodesic lines on the Riemann surface S can be subdivided into the three types: (i) immersed lines, i.e. lines with self-intersections on S; (ii) embedded periodic lines and (iii) embedded non-periodic lines on S. In this article we deal with the geodesic lines of types (ii) and (iii) and their topological closures on S, known as *geodesic lamination.* Let us remind the reader that geodesic laminations constitute a core of Thurston's theory of 3-dimensional manifolds ([7]).

In ([6]) we attached to given geodesic lamination Λ certain real quantity α, called *rotation number*. Number α is a numerical invariant of lamination Λ which measures the average "slope" of geodesics relatively a fixed generator of the fundamental group of S. To this end, the rotation numbers can be viewed as a generalization of the Poincaré rotation numbers to the case $S \neq T^2$.

For given geodesic lamination Λ its rotation number is a complete topological invariant of Λ. Namely, two geodesic laminations Λ, Λ' on surface S are topologically equivalent if and only if their rotation numbers α, α' are modular equivalent ([6]). This fact has various important applications in the theory of C^*-algebras (classification of the AF C^*-algebras), algebraic geometry (moduli problem of Riemann surfaces) and geometric topology (classification of 3-dimensional manifolds).

If S is an arithmetic Riemann surface [1] , then the rotation number, α, has especially nice presentation. Namely, α unfolds into the regular continued fraction

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}} = [a_0, a_1, a_2, \dots], \tag{1}$$

where $a_i = a_i(\Gamma_n)$ are positive integers defined by the principal congruence group Γ_n. In fact, this is exactly the way rotation numbers were introduced in ([6]).

The aim of present note is to study the range of the "rotation function" defined by equation (1). Namely, for fixed principal congruence group Γ_n, what values of α will appear in formula (1)? The answer is contained in the following statement, which to the best of our knowledge has not been studied in the literature.

[1]We call S arithmetic if $S = \mathbb{H}/\Gamma_n$, where $\mathbb{H}$ is the Lobachevsky plane and Γ_n the principal congruence group of order n, see Section 1 for the definition.

Theorem 1 (Surjectivity of rotation function) *Let $n \geq 1$ be an integer number. Suppose that S is an arithmetic Riemann surface $S = \mathbb{H}/\Gamma_n$, where Γ_n is the principal congruence (sub-)group of step n of the modular group $SL_2(\mathbb{Z})$. Then for any irrational number $\alpha \in \mathbb{R}$, there exists a geodesic lamination Λ on S whose rotation number is equal to α. In other words, geodesic laminations on S can have any (irrational) slope on Riemann surface S.*

1 Notation

In this section we briefly review notation and definitions to be used in this note. We refer the reader to ([4]), ([5]) and ([6]) for the details.

1.1 Geodesics

We identify Riemann surfaces of genus $g \geq 2$ with the compact surfaces of constant negative curvature -1. Any such a surface will be denoted by S. By a *geodesic* on S we understand the maximal arc on S consisting of the locally shortest sub-arcs (in the given metric on S). Topologically, each geodesic is the image of the open real interval I under a continuous mapping $I \to S$.

Using symbolic dynamics Artin, Morse and Myrberg established existence of the following types of geodesic lines on the surfaces of constant negative curvature.

Lemma 1 (Topological classification of geodesics) *Let $p \in S$ be a point with an attached unit vector $t \in S^1$ on the surface S. Then :*

(a) for almost all points $t \in S^1$ the geodesic line through p in direction t is an immersion $I \to S$, i.e. finite or infinite curve with self-intersections;

(b) the complementary set $\mathfrak{K} \subset S^1$ has the cardinality of continuum and geodesic lines through p in direction $t \in \mathfrak{K}$ are embedded curves of one of the three types:

(i) periodic

(ii) spiral

(iii) non-periodic filling-up the entire S (Cantor-type curve).

Proof. See Artin ([1]) and Myrberg ([5]) for the original proof. □

In the sequel, we mainly work with the geodesics of type (iiib) and following Birkhoff ([2]) and Morse ([4]) call such lines *recurrent geodesics*. As the reader will see, recurrent geodesics are a replacement for the "straight lines" on the Riemann surfaces $S \neq T^2$. The topological closure of a recurrent geodesic on S contains continuum of distinct recurrent geodesics on S (having the same closure), and such a collection of geodesics is called *(minimal) geodesic lamination*. We shall discuss laminations in Section 1.4.

1.2 Arithmetic Riemann surfaces

Denote by Γ the modular group $PSL_2(\mathbb{Z})$. For every $n = 1, 2, 3, \ldots$, consider its normal subgroup $\Gamma_n \subset \Gamma$ (of finite index) consisting of matrices of the form:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; Mod \; n = \begin{pmatrix} 1+np & nq \\ nr & 1+ns \end{pmatrix}, \tag{2}$$

where $p, q, r, s \in \mathbb{Z}$. The subgroup Γ_n is called a *principal congruence group* of step n.

Let $\mathbb{H} = \{z = x+iy \in \mathbb{C} | y > 0\}$ be the Lobachevsky (half-) plane endowed with the hyperbolic metric $ds = |dz|/y$. The principal congruence group Γ_n acts discretely on $\mathbb{H}$ by the linear-fractional transformations $z \mapsto \frac{az+b}{cz+d}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_n$. The orbit space $S = \mathbb{H}/\Gamma_n$ is an orbifold with a finite number of cusps corresponding to the (finite) stabilizers of group Γ_n. It follows from the construction that S is a Riemann surface, which we call an *arithmetic* Riemann surface to underline its connection with the arithmetic subgroups of the modular group. The genus of S can be evaluated according to the classical formula:

$$g = 1 + \frac{1}{24}\tau n^2(n-6)\prod_{p|n}\left(1 - \frac{1}{p^2}\right), \qquad \tau = \begin{cases} 2, & \text{if } n = 2, \\ 1, & \text{if } n > 2 \end{cases}. \tag{3}$$

1.3 Rotation numbers

Assume that S is an arithmetic Riemann surface of genus $g \geq 2$. We wish to associate a real number α to every recurrent geodesic γ on S. The construction is based on the continued fraction, which arises from an infinite sequence $\{g_i \mid g_i \in \Gamma_n, i = 1, \ldots, \infty\}$ and is attached "almost uniquely" to γ.

Namely, take an arbitrary fundamental region (orbit space) Ω of fixed principal congruence group Γ_n (Fig. 1). Since γ is recurrent, it intersects Ω infinitely often, and let us write down the corresponding arcs of intersection as $\{I_0, I_1, \ldots\}$ in the order they cut Ω. Since each I_k is the arc of the geodesic half-circle on $\mathbb{H}$, they can be moved to each other by transformations from Γ_n. Put then $g_k \in \Gamma_n$ be such that $I_k = g_k(I_{k-1})$. It can be shown [2] that $g_k \in \Gamma_n^+$, where Γ_n^+ is the principal congruence semi-group consisting of 2×2 matrices with positive integer entries. Finally, suppose that $g_k = \begin{pmatrix} a_0^{(k)} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n^{(k)} & 1 \\ 1 & 0 \end{pmatrix}$ is the Minkowski (prime) decomposition of matrix $g_k \in \Gamma_n^+$. Then the regular continued fraction

$$a_0^{(0)} + \cfrac{1}{a_1^{(0)} + \cfrac{1}{a_2^{(0)} + \ldots}}, \tag{4}$$

converges to an irrational number α called a *rotation number* associated to the recurrent geodesic γ.

1.4 Topological equivalence of geodesic laminations

The closure of any recurrent geodesic on S contains infinitely many (a continuum) of other recurrent geodesics. The disjoint union of such geodesics gives us a geodesic lamination, Λ, on surface S. Λ can be thought of as collection of "parallel" geodesic lines, i.e lines with the same asymptotic behaviour. It can be easily seen that the rotation number associated with *any* geodesic in Λ will be the same.

On the other hand, there exists infinitely many topologically distinct geodesic laminations on given surface S. By the topological equivalence

[2] This fact is an implication of Selberg's theory of the length spectra of Riemann surfaces. Indeed, the iteration process of recurrent geodesic γ by segments I_k would give us an infinite sequence of periodic geodesics $\{p_k \mid k = 1, \ldots, \infty\}$, which converges to γ. By Selberg's formula, the length of geodesic p_k is $2\ cosh(\frac{1}{2}l(p_k)) = |tr(G_k)|$, where $G_k = \prod_{j=1}^{k} g_j$. Since the lengths of periodic geodesics p_k grow, so do traces $tr\ (G_k)$. This can happen if and only if all the entries of matrices g_k are non-negative integers. In other words, "prime decomposition" of the recurrent geodesic into a continued fraction is morally equivalent to decomposition of Riemann surfaces into their "length spectra".

of two laminations Λ, Λ' one means a homeomorphism $h : S \to S$ which preserves the leaves of the laminations and sends leaves of Λ to the leaves of Λ'. It turns out that the topological equivalence of laminations is "controlled" be their rotation numbers in the following sense.

Lemma 2 (Topological equivalence of geodesic laminations) *Two geodesic laminations Λ, Λ' are topologically equivalent if and only if their associated continued fractions coincide, except possibly in the finite number of terms. In other words, the associated rotation numbers α, α' are modular equivalent:*

$$\alpha' = \frac{a\alpha + b}{c\alpha + d}, \qquad a, b, c, d \in \mathbb{Z}, \qquad ad - bc = \pm 1. \tag{5}$$

Proof. See ([6]) for the proof. $\square$

2 Proof of Theorem 1

Note that geometric interpretation of Theorem 1 is intuitively obvious and requires no formal proof. (Indeed, the slope of a geodesic lamination should be a continuous quantity, in the sense that "close" geodesic laminations must have infinitely close slopes.) We wish to give here an *algebraic* proof of this fact.

Our proof is based on the fact that Γ_n is the subgroup of finite index in the modular group $\Gamma = PSL_2(\mathbb{Z})$. Or, in geometric terms, the arithmetic Riemann surface $S_n = \mathbb{H}/\Gamma_n$ finitely covers the modular surface $S = \mathbb{H}/\Gamma$.

Let us consider the following semi-group:

$$\Gamma^+ = \left\{ \prod_{i \text{ even}} \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix} \mid a_i \in \mathbb{N} \right\}, \tag{6}$$

where the product is taken over *finite even* number of factors. The following lemma (Lemma 3) shows that every non-negative integer matrix of determinant $+1$ belongs to Γ^+ and *vice versa.*

Lemma 3 (Minkowski's lemma) *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an integral matrix with the non-negative entries and det $A = 1$. Then there exists a unique* *decomposition of A into the even number of "prime factors" $\begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}$, where $a_i \geq 0$. In other words,*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \dots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}}_{even} \tag{7}$$

Proof. See Wall ([8]) □

Now we are able to establish the range of the "rotation function" for the modular surface. Namely, the following lemma is true.

Lemma 4 (Surjectivity of rotation function for modular surface) *Let $S = \mathbb{H}/\Gamma$ be the modular surface. Then for any irrational number $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, there exists a geodesic lamination $\Lambda \subset S$ whose rotation number is equal to α.*

Proof. Let $z \mapsto Tz$ be linear fractional transformation T of the complex plane $\mathbb{C}$. It is known that under the action of infinite product $\prod_{i=1}^{\infty} T_i$, point $z = \infty$ goes to the point $z' \in \mathbb{C}$ whose coordinates unfold into an infinite continued fraction, convergent or not, see ([8]), p. 13. In the important case $T_i \in \Gamma^+$, the action has especially nice form:

$$\prod_{i=1}^{\infty} T_i(\infty) = \prod_{i=1}^{\infty} T_i(0) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}, \tag{8}$$

where a_i are entries of the prime factors of T_i (see Lemma 3). Continued fraction at the right of equation (8) always converges to a real (irrational) number, and therefore points $z = 0$ and $z = \infty$ go to an irrational point on the real line $\mathbb{R} \subset \mathbb{C}$. On the other hand, by the Lagrange theorem any irrational number unfolds uniquely into regular continued fraction (8). Lemma 4 follows. □

Remark 1 *Geometrically, Theorem 1 follows from Lemma 4. Indeed, since given arithmetic Riemann surface S_n finitely covers modular surface S, so does any geodesic lamination Λ' on S_n. Namely, Λ' covers a geodesic lamination $\Lambda \subset S$, for which the surjectivity of rotation numbers is established by Lemma 4. It remains to "lift" the obtained family of rotation numbers to the surface S_n. However, we wish to give purely algebraic proof of Theorem 1.*

To finish the proof of Theorem 1 we would need the following key lemma.

Lemma 5 *Let G be a finite group. Then every infinite word on the elements of G is equivalent to the infinite word $1_G 1_G \dots$ consisting of the repeating identity 1_G of group G.*

Proof. Let $w = g_1, g_2, \dots$ be an infinite word on the elements g_i of group G. Consider the "partial products" $p_1 = g_1, p_2 = g_1 g_2, \quad \dots \quad p_n = g_1 g_2 \dots g_n, \quad \dots$. We proceed by induction.

Step 1. If $p_1 = p_2$, then $g_2 = 1_G$. In this case $w = g_1 1_G g_3 \dots = 1_G g_1 g_3 \dots$ by commutativity of the identity element 1_G. We resume the procedure for the word $w' = g_1, g_3, \dots$.

Step 2. If $p_1 \neq p_2$, then suppose $p_1 = p_3$. In this case $g_2 g_3 = 1_G$ and we get the word $w = g_1 1_G g_4 \dots = 1_G g_1 g_4 \dots$ and repeat the procedure for the word $w' = g_1 g_4 \dots$.

Step n. Since G is a finite group, there exists a number $n \leq |G|$ such that $p_1 = p_n$. Then the algorithm eventually stops, and as it was shown earlier, one obtains the word $w = 1_G g_1 g_{n+1} \dots$.

The induction is repeated, so that any infinite word w reduces to the form $1_G 1_G \dots$ Lemma 5 follows. □

Let $G = \Gamma / \Gamma_n$ be the factor group of the modular group by the principal congruence group of step n. It is known that G is finite group. Thus Γ splits into a finite union of cosets $g\Gamma_n$, with the well-defined group operation between the cosets. Note that Γ_n is a coset itself corresponding to the identity 1_G of group G.

Finally, take an irrational number $\alpha \in \mathbb{R} \backslash \mathbb{Q}$. By Lemma 4,

$$\alpha = \prod_{i=1}^{\infty} T_i(\infty) = \prod_{i=1}^{\infty} T_i(0), \tag{9}$$

and one gets a correctly defined infinite sequence $\{T_i \mid i = 1, 2, \dots \infty\}$ of elements $T_i \in \Gamma$. If we replace each T_i by the corresponding coset, we get a word $\{g_i \mid i = 1, 2, \dots \infty\}$, where $g_i \in G$ are elements of finite group G. By Lemma 5 the latter word is equivalent to the infinite word $1_G 1_G \dots$. In other words, the product (9) can be rearranged to the product:

$$\alpha = \prod_{i=1}^{\infty} \mathfrak{T}_i(\infty), \tag{10}$$

where all $\mathfrak{T}_i$ are elements of the principal congruence group Γ_n. Theorem 1 is proven. $\square$

References

[1] E. Artin, Ein mechanishes System mit quasiergodischen Bahnen, Abh. Math. Sem. Hamburg (1924), 170-175; in: Collected papers, 499-504, Springer Verlag.

[2] G. D. Birkhoff, Quelques théorèmes sur le mouvement des systèmes dynamiques, Bull. de la Soc. Math. de France 40 (1912), 305-323.

[3] J. Hadamard, Les surfaces à courbures opposées et leur lignes geodesiques, Journal de Mathématique pures et appliquées 4 (1898), 27-73.

[4] M. Morse, Recurrent geodesics on a surface of negative curvature, Trans. Amer. Math. Soc. 22 (1921), 84-100.

[5] P. J. Myrberg, Ein Approximationssatz für die Fuchsschen Gruppen, Acta Math. 57 (1931), 389-409.

[6] I. Nikolaev, Geometry of a crossed product, Proc. Amer. Math. Soc. 128 (2000), 1177-1183.

[7] W. P. Thurston, The Geometry and Topology of Three-Manifolds, MSRI 1997, electronic edition of 1980 Princeton Univ. notes, available at http://www.msri.org/gt3m/

[8] H. S. Wall, Analytic theory of continued fractions, Chelsea, N.Y., 1973