Almost Nonnegative Curvature and Cohomogeneity One

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Abstract

We show that any closed cohomogeneity one manifold supports metrics of almost nonnegative sectional curvature which are moreover invariant under the cohomogeneity one action, thereby establishing a conjecture of Grove and Ziller in the almost nonnegatively curved setting. Applications of our result include that there are infinitely many dimensions in which there exist almost nonnegatively curved topological spheres and homotopy lens as well as homotopy real projective spaces, all differentiably distinct from the standard ones.

1 Introduction

The quest for new constructions and examples of manifolds with certain given lower curvature bounds constitutes a central issue of global Riemannian geometry. Seminal work of K. Grove and W. Ziller has recently revealed that manifolds of cohomogeneity one play hereby a fundamental role.

Cohomogeneity one manifolds denote connected smooth manifolds which admit a smooth action by a compact Lie group with codimension one principal orbits. Grove and Ziller proved (cf. [GZ1]) that any closed cohomogeneity one manifold with two singular orbits of codimension two carries an invariant metric of nonnegative sectional curvature, where a metric is called invariant if the Lie group acts by isometries. They also showed that any cohomogeneity one manifold admits an invariant metric of nonnegative Ricci curvature and that a compact cohomogeneity one manifold admits an invariant metric of positive Ricci curvature if and only if its fundamental group is finite (cf. [GZ2]).

The above codimension two result may be considered as the first manifestation of a potentially much more general phenomenon. In [GZ1] the following conjecture was proposed:

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Conjecture (Grove–Ziller) *Any closed cohomogeneity one manifold supports an invariant metric of nonnegative sectional curvature.*

For noncompact cohomogeneity one manifolds it is easy to see that these admit nonnegatively curved invariant metrics, since they are either the product of a compact homogeneous space with the real line, or a homogeneous disc bundle over a compact homogeneous space (cf. [Mo]). The above conjecture's importance and scope are illustrated by the fact that when neglecting standard constructions as well as examples of a rather special nature (cf. [Ch], [Ri], [Yan]), till now the only closed manifolds which are known to carry metrics of nonnegative sectional curvature are homogeneous spaces, biquotients of compact Lie groups and the codimension one manifolds considered in [GZ1].

As one of the main results of this paper, we show that the Grove-Ziller conjecture is true in the almost nonnegatively curved setting:

Theorem A *Any closed cohomogeneity one manifold supports invariant metrics of almost nonnegative sectional curvature.*

Recall that a closed manifold M is said to carry almost nonnegatively curved metrics, or, in short, to be almost nonnegatively curved, if for any $\varepsilon > 0$ there exists a Riemannian metric g on M whose sectional curvatures and diameter satisfy the inequality $Sec_g \cdot (diam_g)^2 \geq -\varepsilon$. This is equivalent to say that in the Gromov-Hausdorff distance M can be collapsed to a single point under a lower curvature and upper diameter bound. In particular, in collapsing under lower curvature bounds manifolds of almost nonnegative curvature play the same basic role as almost flat manifolds do in the Cheeger-Fukaya-Gromov theory of collapse with bounded curvature (cf. [Yam], [FY], [CFG]).

Let us briefly outline the construction of the metrics needed to prove Theorem A. If a connected closed smooth manifold M admits a cohomogeneity one action, then the orbit space is either a circle or a compact interval (cf. [Mo]). In the first case, M is the total space of a homogeneous bundle over the circle, and it is easy to construct metrics of nonnegative sectional curvature. In the second case, M is obtained by glueing together two homogeneous disc bundles along their common boundary. As in [Ch], [GZ1], [GZ2], we construct metrics on such disc bundles which close to the boundary are isometric to the product of an interval and a principal orbit with a fixed (normal) homogeneous metric. Therefore, these metrics can be glued together to yield a smooth Riemannian metric on all of M. If the metrics on the disc bundles both have almost nonnegative curvature, then so does the metric on M.

We would like to point out that this method will not be suitable to prove the Grove-Ziller conjecture in full generality. Indeed, there are homogeneous disc bundles which do not admit an invariant metric of nonnegative sectional curvature which close to the boundary is isometric to the product of an interval and a normal homogeneous space. While by [GZ1] any homogeneous disc bundle with fiber dimension 2 carries such a metric, there is to our knowledge no known example of such a metric if the fiber dimension is bigger than 2. It is therefore an interesting question to characterize the disc bundles admitting such metrics.

There are classical examples of closed manifolds which admit metrics of positive Ricci curvature but cannot admit metrics of almost nonnegative curvature (cf. [Gr], [SY]). In view of this fact and the main results from [GZ2] it is worth pointing out that all metrics in our construction have nonnegative Ricci curvature. Also, if the singular orbits have codimension two, then our metrics have indeed nonnegative sectional curvature as in [GZ1].

It is well known that there exist closed manifolds of almost nonnegative curvature which do not admit nonnegatively curved metrics. In the simply connected case, however, there is not a single obstruction to deforming almost nonnegatively curved metrics into nonnegatively curved ones known today. When thinking of the potential further consequences of Theorem A for both the Grove-Ziller conjecture and the results below, one is thus urged to ask for an answer to the following

Question *Is there any closed simply connected almost nonnegatively curved manifold which does* not *admit a metric with nonnegative sectional curvature?*

We return to our discussion of Theorem A by describing some of its further consequences.

Note first that the class of closed cohomogeneity one manifolds is, like the class of manifolds with almost nonnegative curvature, very rich and enjoys nice extension and closedness properties. For instance (see the appendix), for cohomogeneity one actions of compact connected Lie groups any principal torus bundle P over a closed cohomogeneity one manifold M with finite fundamental group supports itself a cohomogeneity one structure whose singular orbits' codimensions and orbit space equals the ones of M, and Theorem A implies that P admits an almost nonnegatively curved metric which is invariant under the cohomogeneity one action on P. Moreover (cf. [FY]), if P is the total space of a fibre bundle over an almost nonnegatively curved manifold M with compact Lie structure group G and compact fibre F so that F admits a G invariant metric with nonnegative sectional curvature, then P is almost nonnegatively curved as well.

Particularly interesting implications of Theorem A arise when we apply our result to certain specific classes of cohomogeneity one manifolds. For example, since the Kervaire spheres allow descriptions as manifolds with cohomogeneity one structures (see section 6), Theorem A yields:

Corollary B *All Kervaire spheres admit metrics of almost nonnegative curvature which are invariant under cohomogeneity one group actions. In particular, there are infinitely many dimensions in which there exist almost nonnegatively curved exotic spheres.*

Until now in dimensions higher than seven not a single example of an exotic sphere with almost nonnegative curvature has been known. It is also interesting to compare Corollary B with the facts that by [BH] no exotic Kervaire sphere admits a metric of positive sectional curvature which is invariant under a cohomogeneity one group action, whereas by [Ch] (cf. [BH] and [GZ2]) all Kervaire spheres support invariant metrics of positive Ricci curvature.

Up to now no smooth homotopy lens space, differentiably distinct from a standard one, has been known to support metrics of almost nonnegative curvature. In this connection, we obtain:

Corollary C *For any integer $m \geq 3$ there are infinitely many dimensions in which there exist almost nonnegatively curved smooth homotopy lens spaces, differentiably distinct from the standard ones, with fundamental group isomorphic to the cyclic group of order m.*

In dimensions bigger than five no examples of almost nonnegatively curved homotopy real projective spaces not diffeomorphic to the standard ones have been known till now. Another consequence of Theorem A is the following:

Corollary D *For any integer $k \geq 1$ there exist at least 2^{2k} oriented diffeomorphism types of almost nonnegatively curved homotopy $\mathbb{RP}^{4k+1}$.*

We note that all manifolds arising in Corollary C and Corollary D carry by [GZ2] in addition also metrics of positive Ricci curvature.

To put our results into further perspective, let us mention some known necessary conditions which a manifold has to satisfy in order to be almost nonnegatively curved.

Let M be a closed smooth n-dimensional manifold. If M admits metrics of almost nonnegative curvature, then:

1. For any field of coefficients the total Betti number of M must be bounded above by a constant depending only on n ([Gr]).

2. A finite cover of M must fibre over a $b_1(\mathrm{M})$-dimensional torus, and if $b_1(\mathrm{M}) = n$, then M must be diffeomorphic to a torus ([Yam]). (The latter statement also holds when M supports metrics of almost nonnegative Ricci curvature (cf. [CC2]).)

3. If M has infinite fundamental group, then the Euler characteristic of M must vanish ([FY]).

4. If the fundamental group of M is finite, for some universal constant C which depends only on n the diameters of M and its universal Riemannian covering $\tilde{M}$ must satisfy the inequality $\mathrm{diam}(\tilde{M}) < C \cdot \mathrm{diam}(M)$ ([FY]).

5. The fundamental group of M must be almost nilpotent, i.e., it must contain a nilpotent subgroup Λ of finite index ([FY]). (This statement also holds under the weaker condition of almost nonnegative Ricci curvature (cf. [CC1]).) Moreover, Λ is generated by at most n elements and the degree of nilpotency of Λ is not greater than n ([FY]).

6. If M is spin, the $\hat{A}$-genus of M must be bounded by $|\hat{A}(M)| \leq 2^{\frac{n-1}{2}}$ ([Ga]).

   (This condition is already necessary for M to admit almost nonnegative Ricci curvature.)

The remaining parts of the paper are organized as follows. In sections $2-5$ the proof of Theorem A is given. In section 6 we present some constructions of cohomogeneity one manifolds and the relevant differential topological facts to deduce Corollary B, C, and D from Theorem A. As to further illustrate Theorem A, in an appendix bundle liftings of group actions are discussed.

2 Curvature of homogeneous metrics

Let $H \subset G$ be compact Lie groups, and let $\mathfrak{h} \subset \mathfrak{g}$ be their Lie algebras. Fix a bi-invariant inner product Q on $\mathfrak{g}$ for which we shall also use the notation $Q(X,Y) = \langle X, Y\rangle$ for $X, Y \in \mathfrak{g}$. Thus, we have a Q-orthogonal decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

Let $\varphi : \mathfrak{m} \to \mathfrak{m}$ be an $Ad(H)$-equivariant linear map which is symmetric and positive definite w.r.t. Q, i.e. such that the bilinear form on $\mathfrak{m}$ given by

$$g_\varphi(X,Y) := \langle X, \varphi Y\rangle$$

yields an inner product on $\mathfrak{m}$. Clearly, we can extend φ to $\mathfrak{g}$ by setting $\varphi|_{\mathfrak{h}} = Id_{\mathfrak{h}}$ and thus obtain an $Ad(H)$-equivariant inner product on $\mathfrak{g}$ which in turn induces a left invariant, $Ad(H)$-invariant Riemannian metric on G which we also denote by g_φ. Also, there is a unique G-invariant Riemannian metric on the homogeneous space G/H such that the natural projection $\pi : (G, g_\varphi) \to G/H$ becomes a Riemannian submersion. By abuse of notation, we denote this metric by g_φ as well.

It is well known that this procedure establishes a one-to-one correspondence between G-invariant Riemannian metrics on G/H and symmetric, positive definite $Ad(H)$-equivariant linear maps $\varphi : \mathfrak{m} \to \mathfrak{m}$. Moreover, if we let

$$\pi^\pm(X,Y) := \frac{1}{2}([X, \varphi Y] \pm [Y, \varphi X]),$$

then the Levi-Civita connection ∇^φ and the curvature tensor R^φ on G/H have been calculated in [Pü] to take the forms

$$\begin{aligned}
\nabla^\varphi_X Y &= -\tfrac{1}{2}[X,Y]_{\mathfrak{m}} + \varphi^{-1}\pi^+(X,Y),\\
R^\varphi(X,Y;Z,W) &= -\tfrac{1}{2}\left(\langle \pi^-(X,Y), [Z,W]\rangle + \langle \pi^-(Z,W), [X,Y]\rangle\right)\\
&- \tfrac{1}{4}\left(\langle \varphi[X,W]_{\mathfrak{m}}, [Y,Z]_{\mathfrak{m}}\rangle - \langle \varphi[X,Z]_{\mathfrak{m}}, [Y,W]_{\mathfrak{m}}\rangle - 2\,\langle \varphi[X,Y]_{\mathfrak{m}}, [Z,W]_{\mathfrak{m}}\rangle\right)\\
&- \left(\langle \pi^+(X,W), \varphi^{-1}\pi^+(Y,Z)\rangle - \langle \pi^+(X,Z), \varphi^{-1}\pi^+(Y,W)\rangle\right).
\end{aligned}$$

Here we use the convention $R^\varphi(X,Y;Z,W) := g_\varphi(R(X,Y)Z,W)$. In particular, this formula implies that

$$\begin{aligned}
R^\varphi(X,Y;Y,X) &= \langle \pi^-(X,Y), [X,Y]\rangle - \tfrac{3}{4}\,\langle \varphi[X,Y]_{\mathfrak{m}}, [X,Y]_{\mathfrak{m}}\rangle\\
&+ \langle \pi^+(X,Y), \varphi^{-1}\pi^+(X,Y)\rangle - \langle \pi^+(X,X), \varphi^{-1}\pi^+(Y,Y)\rangle\,.
\end{aligned} \tag{1}$$

Consider an $Ad(H)$-invariant Q-orthogonal decomposition

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_r. \tag{2}$$

Then any map $\varphi : \mathfrak{m} \to \mathfrak{m}$ given by

$$\varphi : \mathfrak{m} \longrightarrow \mathfrak{m}, \quad \varphi|_{\mathfrak{m}_i} = f_i^2 \; Id_{\mathfrak{m}_i} \text{ for } i = 1, \ldots, r \tag{3}$$

with $f_i > 0$ is $Ad(H)$-equivariant, and we call the tuple $(f_1, \ldots, f_r) \in \mathbb{R}^r$ the *parameters of* φ w.r.t. the decomposition (2).

Definition 2.1 *Let $H \subset G$ be compact Lie groups and choose a decomposition (2). A point $(f_1, \ldots f_r) \in \mathbb{R}^r$ with $f_i > 0$ for all i is said to be a* positive curvature parameter *if the G-invariant Riemannian metric g_φ on G/H induced by the map $\varphi : \mathfrak{m} \to \mathfrak{m}$ from (3) has positive sectional curvature.*

A path $f : [a, b] \to \mathbb{R}^r$ is called a path of positive curvature parameters *if $f(s)$ is a positive curvature parameter for all $s \in [a, b]$.*

A path $f : [a, b] \to \mathbb{R}^r$ is called a coordinate edge of the k-th coordinate *if f' is a non-zero multiple of the k-th coordinate vector.*

Also, we call f a coordinate polygon *if it is piecewise a coordinate edge.*

Evidently, the set of positive curvature parameters forms an open cone in $\mathbb{R}^r$.

Let $X, Y \in \mathfrak{m}$. Decomposing $X = X_1 + \ldots + X_r$ and $Y = Y_1 + \ldots + Y_r$ with $X_i, Y_i \in \mathfrak{m}_i$, we define

$$B^{ij} := \frac{1}{2}([X_i, Y_j] + [X_j, Y_i]), \text{ and the decomposition } B^{ij} = B_0^{ij} + B_1^{ij} + \ldots + B_r^{ij}, \tag{4}$$

where $B_0^{ij} \in \mathfrak{h}$ and $B_k^{ij} \in \mathfrak{m}_k$ for $k = 1, \ldots, r$. Then evidently, $B^{ij} = B^{ji}$ and $[X, Y] = \sum_{i,j} B^{ij}$. It is then easy to verify from the definition that

$$\begin{aligned} \pi^+(X, Y) &= \tfrac{1}{2} \textstyle\sum_{i<j} (f_j^2 - f_i^2)([X_i, Y_j] - [X_j, Y_i]), \\ \pi^-(X, Y) &= \textstyle\sum_i f_i^2 B^{ii} + \sum_{i<j} (f_i^2 + f_j^2) B^{ij}. \end{aligned} \tag{5}$$

To end this section, we shall prove the following easy

Lemma 2.2 *Let $(V, \langle\ ,\ \rangle)$ be a Euclidean vector space and let $p(x) = ax^2 + bx + c$ be a real quadratic polynomial such that $p(x) \geq 0$ for all $x \in \mathbb{R}$. Then for all $X, Y \in V$ we have*

$$a \langle X, X \rangle + b \langle X, Y \rangle + c \langle Y, Y \rangle \geq 0.$$

Moreover, if p is non-constant and $p(x) > 0$ for all $x \in \mathbb{R}$, then equality holds iff $X = Y = 0$.

Proof. Since $p(x) \geq 0$ for all $x \in \mathbb{R}$, it follows that $a \geq 0$, and $a > 0$ if p is non-constant, whence the claim follows if $Y = 0$.

If $Y \neq 0$, we let $x := \frac{\langle X,Y \rangle}{\langle Y,Y \rangle}$ and observe that by the Cauchy-Schwarz inequality, $x^2 \leq \frac{\langle X,X \rangle}{\langle Y,Y \rangle}$. Thus, since $a \geq 0$, it follows that $a\frac{\langle X,X \rangle}{\langle Y,Y \rangle} + b\frac{\langle X,Y \rangle}{\langle Y,Y \rangle} + c \geq ax^2 + bx + c = p(x) \geq 0$. ∎

3 Chains of Lie groups

So far, the description of the curvature has been completely general. We now wish to restrict to some special cases.

We call a sequence of compact Lie groups $H \subset K_1 \subset \ldots \subset K_r = G$ a *chain*. As before, we fix a bi-invariant inner product Q on $\mathfrak{g}$. For the Lie algebras we have $\mathfrak{h} \subset \mathfrak{k}_1 \subset \ldots \subset \mathfrak{k}_r = \mathfrak{g}$. Thus we obtain an $Ad(H)$-invariant Q-orthogonal decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_r, \qquad \mathfrak{k}_i = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_i, \tag{6}$$

so that this induces a decomposition $\mathfrak{m} := \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_r$ as in (2). We assert that

$$[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_j \ \text{ for } i < j. \tag{7}$$

This follows since on the one hand, $[\mathfrak{m}_i, \mathfrak{m}_j] \subset [\mathfrak{k}_j, \mathfrak{k}_j] \subset \mathfrak{k}_j$ and on the other hand by the $Ad(G)$-invariance of Q, $\langle [\mathfrak{m}_i, \mathfrak{m}_j], \mathfrak{k}_{j-1} \rangle \subset \langle [\mathfrak{k}_{j-1}, \mathfrak{m}_j], \mathfrak{k}_{j-1} \rangle = \langle \mathfrak{m}_j, [\mathfrak{k}_{j-1}, \mathfrak{k}_{j-1}] \rangle \subset \langle \mathfrak{m}_j, \mathfrak{k}_{j-1} \rangle = 0$. As a consequence, for the decomposition (4) we get

$$B^{ii}_j = B^{ij}_k = 0 \text{ if } i < j \text{ and } k \neq j. \tag{8}$$

One of the cases which shall be of much interest to us is the case where $r = 2$. Here, we obtain the following

Proposition 3.1 *Given compact Lie groups $H \subset K \subset G$ and constants $f_1, f_2 > 0$, we decompose the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ such that $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_1$ as before, and define $\varphi : \mathfrak{m} \to \mathfrak{m}$ as in (3). Let $X, Y \in \mathfrak{m}$ and decompose $X = X_1 + X_2, Y = Y_1 + Y_2$ with $X_i, Y_i \in \mathfrak{m}_i$. Then the curvature R^φ of $(G/H, g_\varphi)$ satisfies*

$$\begin{aligned} \frac{1}{f_2^2} R^\varphi(X, Y; Y, X) \;=\; & \tfrac{3}{4} q \left\langle [X,Y]_{\mathfrak{h}}, [X,Y]_{\mathfrak{h}} \right\rangle + \tfrac{1}{4} \left\langle B^{22}_2 + 2q B^{12}_2, B^{22}_2 + 2q B^{12}_2 \right\rangle \\ & + \tfrac{1}{4} q \left\langle B^{11}, B^{11} \right\rangle + \tfrac{1}{2} q (3 - 2q) \left\langle B^{11}, B^{22}_{\mathfrak{k}} \right\rangle + (1 - \tfrac{3}{4} q) \left\langle B^{22}_{\mathfrak{k}}, B^{22}_{\mathfrak{k}} \right\rangle, \end{aligned}$$

where $q = \dfrac{f_1^2}{f_2^2}$ and $B^{22}_{\mathfrak{k}} := B^{22}_0 + B^{22}_1$.

Proof. In the present case, (5) and (7) imply

$$\begin{aligned} \pi^+(X,Y) &= \tfrac{1}{2}(f_2^2 - f_1^2)([X_1, Y_2] - [X_2, Y_1]) \in \mathfrak{m}_2, \\ \pi^-(X,Y) &= f_1^2 B^{11} + (f_1^2 + f_2^2) B^{12} + f_2^2 B^{22}. \end{aligned}$$

Thus, calculating the first line in (1) yields

$$\begin{aligned} & \langle \pi^-(X,Y), [X,Y] \rangle - \tfrac{3}{4} \langle \varphi [X,Y], [X,Y]_{\mathfrak{m}} \rangle \\ & \quad = \langle f_1^2 B^{11} + (f_1^2 + f_2^2) B^{12} + f_2^2 B^{22}, B^{11} + 2B^{12} + B^{22} \rangle \\ & \qquad - \tfrac{3}{4} (f_1^2 \langle B^{11}_1 + B^{22}_1, B^{11}_1 + B^{22}_1 \rangle + f_2^2 \langle 2B^{12} + B^{22}_2, 2B^{12} + B^{22}_2 \rangle) \\ & \quad = \tfrac{3}{4} f_1^2 \langle [X,Y]_{\mathfrak{h}}, [X,Y]_{\mathfrak{h}} \rangle \\ & \qquad + \tfrac{1}{4} f_1^2 \langle B^{11}, B^{11} \rangle + (f_2^2 - \tfrac{1}{2} f_1^2) \langle B^{11}, B^{22}_{\mathfrak{k}} \rangle + (f_2^2 - \tfrac{3}{4} f_1^2) \langle B^{22}_{\mathfrak{k}}, B^{22}_{\mathfrak{k}} \rangle \\ & \qquad + \tfrac{1}{4} f_2^2 \langle B^{22}_2, B^{22}_2 \rangle + f_1^2 \langle B^{22}_2, B^{12}_2 \rangle + (2f_1^2 - f_2^2) \langle B^{12}_2, B^{12}_2 \rangle . \end{aligned}$$

Likewise, calculating the second line in (1) yields

$$\begin{aligned}
&\langle \pi^+(X,Y), \varphi^{-1}\pi^+(X,Y)\rangle - \langle \pi^+(X,X), \varphi^{-1}\pi^+(Y,Y)\rangle \\
&\quad = \tfrac{(f_2^2-f_1^2)^2}{4f_2^2} \langle [X_1,Y_2]-[X_2,Y_1],[X_1,Y_2]-[X_2,Y_1]\rangle - \tfrac{(f_2^2-f_1^2)^2}{f_2^2} \langle [X_1,X_2],[Y_1,Y_2]\rangle \\
&\quad = \tfrac{(f_2^2-f_1^2)^2}{4f_2^2} \left(\begin{array}{l} \langle [X_1,Y_2]-[X_2,Y_1],[X_1,Y_2]-[X_2,Y_1]\rangle \\ \qquad\qquad + 4(\langle [X_1,Y_1],[Y_2,X_2]\rangle + \langle [X_1,Y_2],[X_2,Y_1]\rangle) \end{array} \right) \\
&\quad = \tfrac{(f_2^2-f_1^2)^2}{4f_2^2} (\langle [X_1,Y_2]+[X_2,Y_1],[X_1,Y_2]+[X_2,Y_1]\rangle - 4\langle [X_1,Y_1],[X_2,Y_2]\rangle) \\
&\quad = \tfrac{(f_2^2-f_1^2)^2}{f_2^2} (\langle B^{12},B^{12}\rangle - \langle B^{11},B^{22}\rangle).
\end{aligned}$$

Here, going from the second to the third line, we used the $Ad(G)$-invariance of Q and the Jacobi identity, i.e.

$$\begin{aligned}
\langle [X,Y],[Z,W]\rangle &= \langle X,[Y,[Z,W]]\rangle = -(\langle X,[Z,[W,Y]]\rangle + \langle X,[W,[Y,Z]]\rangle) \\
&= -(\langle [X,Z],[W,Y]\rangle + \langle [X,W],[Y,Z]\rangle).
\end{aligned}$$

Adding these expressions yields the asserted formula. ∎

We shall now cite a lemma which is needed for the proof of Proposition 3.3.

Lemma 3.2 (cf. [Ch]) *Let $\pi : P \to B$ be a principal K-bundle where K is a compact Lie group, and let g be a K-invariant Riemannian metric on P whose restriction to each fibre is isomorphic to a fixed bi-invariant metric on K. Define the vertical and horizontal distributions on P as $\mathcal{V} := \ker(d\pi)$ and $\mathcal{H} := \mathcal{V}^\perp$. For $c > 0$, let g^c be the Riemannian metric defined by*

$$g^c|_{\mathcal{V}} = cg|_{\mathcal{V}}, \quad g^c(\mathcal{V},\mathcal{H}) = 0, \quad g^c|_{\mathcal{H}} = g|_{\mathcal{H}}.$$

Then g^c has positive sectional curvature for all $c \in (0,1]$, provided g does.

This is proven by realizing that for $c \in (0,1)$ and $\lambda := \frac{c}{1-c} > 0$ there is a Riemannian submersion $(P \times K, g + \lambda g|_K) \to (P, g^c)$ induced from the identification $P \cong P \times_K K$. Then one uses O'Neill's formula.

Proposition 3.3 *Let $K \subset O(n+1)$ be a Lie subgroup which acts transitively on $S^n \subset \mathbb{R}^{n+1}$, i.e. $S^n = K/H$ for some subgroup $H \subset K$, and let $\mathfrak{h} \subset \mathfrak{k}$ be their Lie algebras. Let Q be an Ad_K-invariant bilinear form on $\mathfrak{k}$, and denote the induced normal homogeneous metric on $S^n = K/H$ by g_Q.*

Then there is a chain $H \subset K_1 \subset \ldots \subset K_r = K$ and a coordinate polygon of positive curvature parameters $f : [0,1] \to \mathbb{R}^r$ such that $g_{\varphi(0)}$ is a multiple of the standard metric, while $g_{\varphi(1)} = g_Q$.

Proof. We recall the classification of these actions (cf. [MS], [Bes, p.179]). Indeed, if $K \subset O(n+1)$ acts transitively on $S^n \subset \mathbb{R}^{n+1}$ then so does its identity component $K_0 \subset SO(n+1)$, i.e. we have $S^n = K/H = K_0/H_0$. Then the isotropy decomposition of the Lie algebra $\mathfrak{k}$ in each case is listed in Table 1.

Table 1 LIST OF TRANSITIVE ACTIONS OF CONNECTED LIE GROUPS ON SPHERES

	K_0	S^n	H_0	isotropy representation
1	$SO(n+1)$	n	$SO(n)$	$\mathfrak{m} = \mathfrak{m}_1 = \mathbb{R}^n$
2	$U(m+1)$	$n = 2m+1$	$U(m)$	$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, $\mathfrak{m}_1 = \mathbb{R}, \mathfrak{m}_2 = \mathbb{C}^m$
3	$SU(m+1)$	$n = 2m+1$	$SU(m)$	$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, $\mathfrak{m}_1 = \mathbb{R}, \quad \mathfrak{m}_2 = \mathbb{C}^m$
4	$Sp(1) \cdot Sp(m+1)$	$n = 4m+3$	$Sp(1) \cdot Sp(m)$	$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ $\mathfrak{m}_1 = Im\mathbb{H}, \quad \mathfrak{m}_2 = \mathbb{H}^m$
5	$U(1) \cdot Sp(m+1)$	$n = 4m+3$	$U(1) \cdot Sp(m)$	$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ $\mathfrak{m}_1 = \mathbb{R}, \quad \mathfrak{m}_2 = \mathbb{C}, \quad \mathfrak{m}_3 = \mathbb{H}^m$
6	$Sp(m+1)$	$n = 4m+3$	$Sp(m)$	$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ $\mathfrak{m}_1 = \mathbb{R}^3_{triv}, \quad \mathfrak{m}_2 = \mathbb{H}^m$
7	G_2	$n = 6$	$SU(3)$	$\mathfrak{m} = \mathfrak{m}_1 = \mathbb{C}^3$
8	$Spin(7)$	$n = 7$	G_2	$\mathfrak{m} = \mathfrak{m}_1 = \mathbb{C}a$
9	$Spin(9)$	$n = 15$	$Spin(7)$	$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ $\mathfrak{m}_1 = \mathbb{R}^7, \quad \mathfrak{m}_2 = \Delta_7$

It follows that in each case we get a chain $H \subset K_1 \subset \ldots \subset K_r = K$ with $r = 3$ in case 5, $r = 2$ in cases 2,3,4,6,9 and $r = 1$ in the remaining cases 1,7,8.

We assert that the standard metric on S^n is induced by a map $\varphi : \mathfrak{m} \to \mathfrak{m}$ of the form (3), and moreover that any map φ of this form is Ad_H-invariant. For all cases but 6, this follows since $\mathfrak{m}_i$ is irreducible and $\mathfrak{m}_i \not\cong \mathfrak{m}_j$ for $i \neq j$. In case 6, note that for the standard metric, the normalizer $Norm_{O(n)} H_0 = Sp(1) \cdot Sp(m)$ operates by isometries on the tangent space at eH_0, whence $\varphi|_{\mathfrak{m}_1}$ must be a multiple of the identity. Likewise, since $H \subset Norm_{O(n)} H_0$, any metric of this form is Ad_H-invariant.

If $r = 1$ then there is – up to homothety – only one K-invariant metric on S^n, hence the normal homogeneous metric must be a multiple of the standard metric.

If $r = 2$, i.e. if $H \subset K_1 \subset K$, we get the Hopf fibration $K_1/H \subset K/H \to K/K_1$. In cases 2,3,4,6, K_1/H is one of the Lie groups $U(1)$ or $Sp(1)$, so that the Hopf fibration is principal. If we denote the curvature parameters of the standard and the normal homogeneous metric on S^n by (f_1, f_2) and $(\tilde{f}_1, \tilde{f}_2)$, respectively, then after rescaling the standard metric, we may assume that $f_2 = \tilde{f}_2$. Note that both of these metrics have positive sectional curvature (for the normal homogeneous metric, see [Ber]), thus by Lemma 3.2 it follows that the coordinate edge joining (f_1, f_2) and $(\tilde{f}_1, f_2)$ is a positive parameter curve as claimed.

In case 9, it is known [GZ2] that the curvature parameters (f_1, f_2) which correspond to the standard metric satisfy $f_1 > f_2$ (in fact, $f_1 = 2f_2$), whence one obtains the normal homogeneous metric from the standard metric by shrinking the Hopf fibers. It is known that this procedure maintains positive sectional curvature [BK].

Finally, a straightforward calculation shows that in case 5, if Q on $\mathfrak{u}(1) \cdot \mathfrak{sp}(n+1)$ is given by $Q((t\sqrt{-1}, A), (s\sqrt{-1}, B)) = \lambda st - trace(AB)$ for some constant $\lambda > 0$, then the standard metric on S^{4n+3} is determined by the triple $(f_1^2, f_2^2, f_3^2) = \left(\frac{\lambda+1}{\lambda}, 1, \frac{1}{2}\right)$.

We consider the principal Hopf fibration $S^3 \hookrightarrow S^{4n+3} \to \mathbb{HP}^n$ and observe that at eH, $\mathcal{V} \cong \mathfrak{m}_1 \oplus \mathfrak{m}_2$ while $\mathcal{H} \cong \mathfrak{m}_3$. Thus, by Lemma 3.2, it follows that the curve $c \mapsto (cf_1, cf_2, f_3)$, $c \in (0, 1]$ is a path of positive curvature parameters, whence so is $c \mapsto \left(f_1, f_2, \frac{1}{c} f_3\right)$, $c \in (0, 1]$, and since $f_2 > f_3$, this is a coordinate edge joining (f_1, f_2, f_3) and (f_1, f_2, f_2).

Next, we consider the principal Hopf fibration $S^1 \hookrightarrow S^{4n+3} \to \mathbb{CP}^{2n+1}$. In this case we have at eH that $\mathcal{V} \cong \mathfrak{m}_1$ while $\mathcal{H} \cong \mathfrak{m}_2 \oplus \mathfrak{m}_3$, thus by Lemma 3.2 the curve $c \mapsto (cf_1, f_2, f_2)$, $c \in (0, 1]$ is a coordinate edge of positive curvature parameters joining (f_1, f_2, f_2) and (f_2, f_2, f_2) which corresponds to a multiple of the normal homogeneous metric. ∎

4 One-parameter families of homogeneous spaces

Let $I \subset \mathbb{R}$ be an interval, and consider a G-invariant metric g on the product $M := I \times G/H$. Assuming that the curves $t \mapsto (t, p)$ are unit speed geodesics for all $p \in G/H$, g takes the form

$$g = dt^2 + g_{\varphi(t)},$$

where $\varphi(t) : \mathfrak{m} \to \mathfrak{m}$ is as above, depending smoothly on $t \in I$. One can easily verify that the Levi-Civita connection ∇ on (M, g) is given by

$$\nabla_X Y = \nabla^{\varphi}_X Y - \langle S_t X, \varphi Y \rangle \, \partial_t, \quad \nabla_{\partial_t} X = \nabla_X \partial_t = S_t X, \quad \nabla_{\partial_t} \partial_t = 0,$$

where

$$S_t := \frac{1}{2} \varphi^{-1} \dot{\varphi}$$

is the shape operator and where ∇^{φ} is the Levi-Civita connection of $(G/H, g_{\varphi(t)})$.

Lemma 4.1 *Let $M = I \times G/H$ be as above, and let $g = dt^2 + g_{\varphi(t)}$. Then the curvature operator R of (M, g) is given by*

$$\begin{aligned}
R(X, Y; Z, W) &= R^{\varphi(t)}(X, Y; Z, W) + \tfrac{1}{4}(\langle \dot{\varphi} X, Z \rangle \langle \dot{\varphi} Y, W \rangle - \langle \dot{\varphi} Y, Z \rangle \langle \dot{\varphi} X, W \rangle) \\
R(X, Y; Z, \partial_t) &= -\tfrac{1}{2} \langle \dot{\pi}^-(X, Y), Z \rangle + \tfrac{1}{2} \langle \dot{\varphi}[X, Y], Z \rangle + \langle Y, S_t \pi^+(X, Z) \rangle - \langle X, S_t \pi^+(Y, Z) \rangle \\
R(\partial_t, Y; Z, \partial_t) &= \left\langle (-\tfrac{1}{2} \ddot{\varphi} + \tfrac{1}{4} \dot{\varphi} \varphi^{-1} \dot{\varphi}) Y, Z \right\rangle.
\end{aligned}$$

where $X, Y, Z, W \in \mathfrak{m} \cong T_{eH} G/H$ and where $R^{\varphi(t)}$ is the curvature operator of $(G/H, g_{\varphi(t)})$.

Proof. Since $\{t\} \times G/H \subset M$ is a hypersurface and S_t is the shape operator which determines the second fundamental form, the first equation follows from the Gauß equation

$$R(X, Y; Z, W) = R^{\varphi(t)}(X, Y; Z, W) + II_t(X, Z) II_t(Y, W) - II_t(X, W) II_t(Y, Z),$$

where $II_t(A, B) = g(S_t A, B) = \langle \varphi S_t A, B \rangle = \frac{1}{2} \langle \dot{\varphi} A, B \rangle$ is the second fundamental form.

For the second, note that

$$\begin{aligned}
g(\nabla_X\nabla_Y Z,\partial_t) &= -\langle S_tX,\varphi\nabla^{\varphi}_Y Z\rangle - g(\nabla_X \langle S_tY,\varphi Z\rangle \partial_t,\partial_t)\\
&= -\tfrac{1}{2}\langle\dot\varphi X,\nabla^{\varphi}_Y Z\rangle - \langle S_tY,\varphi Z\rangle g(\nabla_X\partial_t,\partial_t)\\
&= -\tfrac{1}{2}\left\langle\dot\varphi X,\tfrac{1}{2}[Y,Z]+\varphi^{-1}\pi^+(Y,Z)\right\rangle - 0\\
&= \tfrac{1}{4}\left\langle[Y,\dot\varphi X],Z\right\rangle - \langle X,S_t\pi^+(Y,Z)\rangle\,.
\end{aligned}$$

Now interchange X and Y, and note that $(\nabla_{[X,Y]}Z,\partial_t) = -\langle S_t[X,Y],\varphi Z\rangle = -\frac{1}{2}\langle\dot\varphi[X,Y],Z\rangle$.

Finally, $g(\nabla_{\partial_t}\nabla_Y Z,\partial_t) = \partial_t g(\nabla_Y Z,\partial_t) = -\partial_t\langle S_tY,\varphi Z\rangle = -\frac{1}{2}\langle\ddot\varphi Y,Z\rangle$, whereas $g(\nabla_Y\nabla_{\partial_t}Z,\partial_t) = -g(\nabla_{\partial_t}Z,\nabla_Y\partial_t) = -g(S_tZ,S_tY) = -\frac{1}{4}\langle\dot\varphi\varphi^{-1}\dot\varphi Y,Z\rangle$. The third identity then follows since $[\partial_t,Y]=0$. ∎

Corollary 4.2 *Let $M = I\times G/H$ and $g = dt^2+g_{\varphi(t)}$ be as above, and let $c\in\mathbb{R}$, $X,Y\in T_{eH}G/H\cong\mathfrak{m}$. Then*

$$\begin{aligned}
R(c\partial_t+X,Y;\quad Y,c\partial_t+X) &= R^{\varphi(t)}(X,Y;Y,X)+\tfrac{1}{4}\left(\langle\dot\varphi X,Y\rangle^2-\langle\dot\varphi X,X\rangle\langle\dot\varphi Y,Y\rangle\right)\\
&+\ \tfrac{1}{2}c\ \left(3\langle\dot\varphi[X,Y],Y\rangle+4\left(\langle Y,S_t\pi^+(X,Y)\rangle-\langle X,S_t\pi^+(Y,Y)\rangle\right)\right)\\
&-\ \tfrac{1}{4}c^2\left\langle(2\ddot\varphi-\dot\varphi\varphi^{-1}\dot\varphi)Y,Y\right\rangle.
\end{aligned}$$

Proof. All we need to do is to use the lemma and to observe the identity $2\langle\dot\pi^-(X,Y),Y\rangle = \langle[X,\dot\varphi Y],Y\rangle-\langle[Y,\dot\varphi X],Y\rangle = -\langle\dot\varphi Y,[X,Y]\rangle+0$. ∎

Proposition 4.3 *Given a chain of compact Lie groups $H\subset K_1\subset\ldots\subset K_r=G$ and smooth functions $f_1,\ldots,f_r: I\to\mathbb{R}^+$ where $I\subset\mathbb{R}$ is an interval, we define a metric $g=dt^2+g_{\varphi(t)}$ on $M:=I\times G/H$ such that $\varphi(t)$ is determined by $f_i(t)$ as in (3) and(6).*

Let $c\in\mathbb{R}$ and $X,Y\in T_{eH}G/H\cong\mathfrak{m}$. Then when defining the vectors B_k^{ij} by (4), the curvature at $(t,eH)\in M$ satisfies

$$\begin{aligned}
&R(c\partial_t+X,Y;Y,c\partial_t+X) =\\
&\quad R^{\varphi(t)}(X,Y;Y,X)-\textstyle\sum_i f_i^2(f_i')^2||X_i\wedge Y_i||^2-\sum_{i<j}f_if_jf_i'f_j'||X_i\wedge Y_j+X_j\wedge Y_i||^2\\
&\quad +\ 3c\textstyle\sum_{i<j}f_if_j\left(\frac{f_i}{f_j}\right)'\left\langle B_i^{jj},Y_i\right\rangle-c^2\sum_i f_if_i''\langle Y_i,Y_i\rangle\,.
\end{aligned}$$

Proof. We calculate the terms of the formula for the curvature in Corollary 4.2. First, observe that $\dot\varphi|_{\mathfrak{m}_i}=2f_if_i'Id_{\mathfrak{m}_i}$, whence

$$\begin{aligned}
\tfrac{1}{4}(\langle\dot\varphi X,Y\rangle^2 &- \langle\dot\varphi X,X\rangle\langle\dot\varphi Y,Y\rangle)\\
&= \left(\textstyle\sum_i f_if_i'\langle X_i,Y_i\rangle\right)^2-\left(\sum_i f_if_i'\langle X_i,X_i\rangle\right)\left(\sum_j f_jf_j'\langle Y_j,Y_j\rangle\right)\\
&= \textstyle\sum_{i,j}f_if_jf_i'f_j'(\langle X_i,Y_i\rangle\langle X_j,Y_j\rangle-\langle X_i,X_i\rangle\langle Y_j,Y_j\rangle)\\
&= -\textstyle\sum_i f_i^2(f_i')^2||X_i\wedge Y_i||^2-\sum_{i<j}f_if_jf_i'f_j'||X_i\wedge Y_j+X_j\wedge Y_i||^2.
\end{aligned}$$

For the second line, we calculate $\langle\dot\varphi[X,Y],Y\rangle = 2\sum_{i,j,k} f_k f_k' \left\langle B_k^{ij}, Y_k\right\rangle$. Now $\langle B_i^{ii}, Y_i\rangle = \langle [X_i,Y_i],Y_i\rangle = 0$ and $2\langle B^{ij},Y_j\rangle = \langle [X_i,Y_j],Y_j\rangle + \langle [X_j,Y_i],Y_j\rangle = 0 - \langle Y_i,[X_j,Y_j]\rangle = -\left\langle B_i^{jj},Y_i\right\rangle$. This together with (8) yields

$$\begin{aligned}
\tfrac12\langle\dot\varphi[X,Y],Y\rangle &= \textstyle\sum_{k<i} f_k f_k' \langle B_k^{ii},Y_k\rangle + 2\sum_{i<j} f_j f_j' \left\langle B_j^{ij},Y_j\right\rangle\\
&= \textstyle\sum_{i<j} f_i f_i' \left\langle B_i^{jj},Y_i\right\rangle - \sum_{i<j} f_j f_j' \left\langle B_i^{jj},Y_i\right\rangle\\
&= \textstyle\sum_{i<j} (f_i f_i' - f_j f_j') \left\langle B_i^{jj},Y_i\right\rangle.
\end{aligned}$$

If $i<j$ then $\pi^+(\mathfrak{m}_i,\mathfrak{m}_j)\subset\mathfrak{m}_j$ by (7), and $\langle [X_i,Y_j]-[X_j,Y_i],Y_j\rangle = \langle X_i,[Y_j,Y_j]\rangle + \langle Y_i,[X_j,Y_j]\rangle = \left\langle B_i^{jj},Y_i\right\rangle$, as well as $\langle [Y_i,Y_j],X_j\rangle = -\langle Y_i,[X_j,Y_j]\rangle = -\left\langle B_i^{jj},Y_i\right\rangle$. Hence, by (5)

$$\begin{aligned}
\langle Y, S_t\pi^+(X,Y)\rangle &= \tfrac12 \textstyle\sum_{i<j} \frac{f_j'}{f_j}(f_j^2-f_i^2)\left\langle B_i^{jj},Y_i\right\rangle\\
\langle X, S_t\pi^+(Y,Y)\rangle &= \textstyle\sum_{i<j} \frac{f_j'}{f_j}(f_j^2-f_i^2)\langle [Y_i,Y_j],X_j\rangle = -\sum_{i<j}\frac{f_j'}{f_j}(f_j^2-f_i^2)\left\langle B_i^{jj},Y_i\right\rangle.
\end{aligned}$$

Finally, $(2\ddot\varphi - \dot\varphi\varphi^{-1}\dot\varphi)|_{\mathfrak{m}_i} = 4 f_i f_i'' Id_{\mathfrak{m}_i}$, and hence, putting all of this together yields the asserted formula. ∎

The main result of this section will be given by the

Theorem 4.4 *Consider a chain of compact Lie groups $H\subset K_1\subset\ldots\subset K_r = G$, let $f:[0,1]\to\mathbb{R}^r$ be a coordinate polygon of positive curvature parameters, and let $\varphi(s):\mathfrak{m}\to\mathfrak{m}$ be the map with parameters $(f_1(s),\ldots,f_r(s))$ (cf. (3), (6)).*

Then there exists a Riemannian metric g on $M := I\times G/H$ where $I=(a,b)\subset\mathbb{R}$ is an open interval, satisfying the following properties:

1. *(M,g) is G-invariant,*
2. *(M,g) has positive sectional curvature,*
3. *there is an $\varepsilon>0$ such that on $(a,a+\varepsilon)\times G/H$ we can write $g = dt^2 + f_a(t)^2 g_{\varphi(0)}$ for some smooth function $f_a:(a,a+\varepsilon)\to\mathbb{R}^+$ with $f_a'>0$,*
4. *there is an $\varepsilon>0$ such that on $(b-\varepsilon,b)\times G/H$ we can write $g = dt^2 + f_b(t)^2 g_{\varphi(1)}$ for some smooth function $f_b:(b-\varepsilon,b)\to\mathbb{R}^+$ with $f_b'>0$.*

Moreover, there is a constant $a_0>0$ such that we can choose $I=(a,b)$ for any $a>a_0$, and such that we may assume that $f_a=\sqrt{t}|_{(a,a+\varepsilon)}$ and $f_b=\sqrt{t}|_{(b-\varepsilon,b)}$.

Proof. Since s ranges over a compact interval, it follows that under the hypotheses of the theorem, there are constants $C_{min}, C_{max}>0$ such that

$$\begin{aligned}
&0 < C_{min} \le Sec(G/H, g_{\varphi(s)}) \qquad \text{for all } s\in[0,1],\\
&||[X,Y]_{\mathfrak{m}}||^2_{\varphi(0)} \le C_{max}||X\wedge Y||^2_{\varphi(0)}.
\end{aligned} \tag{9}$$

Observe that it suffices to show the theorem for the case where the polygon consists of one coordinate edge. Indeed, if the theorem holds in this case, then for an arbitrary polygon we can choose the metrics corresponding to each edge such that f_a and f_b equal $\sqrt{t}$, and the intervals can be chosen such that these metrics can be glued together for consecutive edges.

The proof of the theorem for the case of one edge will be split into two lemmas.

Lemma 4.5 *Suppose that all the hypotheses of Theorem 4.4 are satisfied where f is a coordinate edge of the k-th coordinate with vertices $(c_1, \dots, c_r)$ and $(c_1, \dots, \tilde{c}_k, \dots, c_r)$. Let*

$$\rho_0 := min\left\{1, \frac{\tilde{c}_k}{c_k}\right\} \qquad and \qquad \rho_1 := max\left\{1, \frac{\tilde{c}_k}{c_k}\right\},$$

and suppose that there are constants $C_1, C_2, C_3 > 0$ and smooth functions $\mu, \rho : [a,b) \to \mathbb{R}$, such that

1. $C_{min} - 4C_1^2 - \dfrac{C_2 C_{max}}{m\rho_0^4 \mu(a)^2} > 0$, *where* $m = \begin{cases} \frac{4}{9}\ min\left\{\rho_0^2, \frac{C_3}{(r-k)\rho_1}\right\} & if\ \ k < r \\ \frac{4}{9}\ \rho_0^2 & if\ \ k = r \end{cases}$,
2. $0 < \dot{\mu} \le C_1 \mu$ *and* $\ddot{\mu} < 0$,
3. ρ *is monotone,* $\rho|_{(a,a+\varepsilon)} \equiv 1$ *and* $\rho|_{(b-\varepsilon,b)} \equiv \dfrac{\tilde{c}_k}{c_k}$ *for some* $\varepsilon > 0$*; whence* $\rho_0 \le \rho \le \rho_1$,
4. $(\mu\rho)'' \le C_3 \ddot{\mu} < 0$,
5. $\dot{\rho} \le C_1 \rho$ *and* $(\dot{\rho})^2 \le -C_2 \dfrac{\ddot{\mu}}{\mu}$.

Then the metric $g = dt^2 + g_{\varphi(t)}$ on $M = (a,b) \times G/H$ with $\varphi(t) : \mathfrak{m} \to \mathfrak{m}$ defined as in (3) and

$$f_i := c_i \mu \text{ for } i \neq k, \text{ and } f_k := c_k \mu \rho$$

fulfills all the asserted properties of Theorem 4.4, and $f_a = \mu|_{(a,a+\varepsilon)}$ while $f_b = \mu|_{(b-\varepsilon,b)}$.

Proof. The G-invariance of g is automatic, while the last two conditions of the proposition are obviously implied by the third property of the Lemma. Thus, it remains to show that g has positive sectional curvature. For this, we first observe that

$$\frac{f_i'}{f_i} = \begin{cases} \dfrac{\dot{\mu}}{\mu} & \le \quad C_1 \quad \text{if } i \neq k \\[2ex] \dfrac{\dot{\mu}}{\mu} + \dfrac{\dot{\rho}}{\rho} & \le \quad 2C_1 \quad \text{if } i = k \end{cases},$$

whence $f_i' \le 2C_1 f_i$ for all i, and therefore

$$\begin{aligned} \textstyle\sum_i f_i^2 (f_i')^2 ||X_i \wedge Y_i||^2 \ &+ \ \textstyle\sum_{i<j} f_i f_j f_i' f_j' ||X_i \wedge Y_j + X_j \wedge Y_i||^2 \\ &= \ \textstyle\sum_i \left(\frac{f_i'}{f_i}\right)^2 ||X_i \wedge Y_i||_g^2 + \sum_{i<j} \frac{f_i' f_j'}{f_i f_j} ||X_i \wedge Y_j + X_j \wedge Y_i||_g^2 \\ &\le \ \textstyle\sum_i 4C_1^2 ||X_i \wedge Y_i||_g^2 + \sum_{i<j} 4C_1^2 ||X_i \wedge Y_j + X_j \wedge Y_i||_g^2 \\ &= \ 4C_1^2 \left|\left|\textstyle\sum_i X_i \wedge Y_i + \sum_{i<j} X_i \wedge Y_j + X_j \wedge Y_i\right|\right|_g^2 \\ &= \ 4C_1^2 ||X \wedge Y||_g^2. \end{aligned} \tag{10}$$

Now we define the following quadratic polynomials for $i \neq k$:

$$\begin{array}{rcrcl} p_i(x) & = & -f_i f_i'' x^2 + 3 f_i f_k \left(\frac{f_i}{f_k}\right)' x + \frac{C_2}{m} f_i^2 & = & -c_i^2 \mu\ddot{\mu}\, x^2 - 3c_i^2 \mu^2 \frac{\dot\rho}{\rho}\, x + \frac{C_2}{m} c_i^2 \mu^2, \text{ and} \\ q_i(x) & = & -\frac{1}{r-k} f_k f_k'' x^2 + 3 f_i f_k \left(\frac{f_k}{f_i}\right)' x + \frac{C_2}{m} \frac{f_k^2}{\rho^2} & = & -\frac{1}{r-k} c_k^2 \mu\rho(\mu\rho)''\, x^2 + 3c_k^2 \mu^2 \rho\dot\rho\, x + \frac{C_2}{m} c_k^2 \mu^2. \end{array}$$

Evidently, q_i is not defined if $k = r$. The discriminant of p_i satisfies

$$-\frac{C_2}{m} c_i^4 \mu^3 \ddot{\mu} - \frac{9}{4} c_i^4 \frac{\mu^4}{\rho^2} (\dot\rho)^2 \geq -\frac{C_2}{m} c_i^4 \mu^3 \ddot{\mu} + \frac{9}{4} C_2 c_i^4 \frac{\mu^3}{\rho^2} \ddot{\mu} = -\frac{C_2}{m} c_i^4 \frac{\mu^3}{\rho^2} \ddot{\mu} \left(\rho^2 - \frac{9}{4} m\right) \geq 0,$$

where the last inequality follows from $\rho \geq \rho_0$, whereas the discriminant of q_i satisfies

$$\begin{array}{rcl} -\frac{C_2}{(r-k)m} c_k^4 \mu^3 \rho(\mu\rho)'' - \frac{9}{4} c_k^4 \mu^4 \rho^2 (\dot\rho)^2 & \geq & -\frac{C_2 C_3}{(r-k)m} c_k^4 \mu^3 \ddot{\mu}\rho + \frac{9}{4} C_2 c_k^4 \mu^3 \ddot{\mu} \rho^2 \\ & = & -\frac{C_2 C_3}{(r-k)m} c_k^4 \mu^3 \ddot{\mu} \rho^2 \left(\rho^{-1} - \frac{9}{4} \frac{r-k}{C_3} m\right) \geq 0, \end{array}$$

where the last inequality follows since $\rho^{-1} \geq \rho_1^{-1}$. Since evidently $p_i(0) > 0$ and $q_i(0) > 0$, this implies that

$$p_i(x), q_i(x) \geq 0 \text{ for all } x \in \mathbb{R} \text{ and all } i \neq k,$$

and therefore, Lemma 2.2 implies that for all $c \in \mathbb{R}$ and $X, Y \in \mathfrak{m}$ and $i \neq k$,

$$\begin{array}{rcrcl} -c^2 f_i f_i'' \langle Y_i, Y_i \rangle + 3c f_i f_k \left(\frac{f_i}{f_k}\right)' \left\langle B_i^{kk}, Y_i \right\rangle & \geq & -\frac{C_2}{m} f_i^2 \left\langle B_i^{kk}, B_i^{kk} \right\rangle & = & -\frac{C_2}{m} \mu^2 ||B_i^{kk}||^2_{\varphi(0)}, \\ -\frac{c^2}{r-k} f_k f_k'' \langle Y_k, Y_k \rangle + 3c f_i f_k \left(\frac{f_k}{f_i}\right)' \langle B_k^{ii}, Y_k \rangle & \geq & -\frac{C_2}{m} \frac{f_k^2}{\rho^2} \langle B_k^{ii}, B_k^{ii} \rangle & = & -\frac{C_2}{m} \mu^2 ||B_k^{ii}||^2_{\varphi(0)}. \end{array}$$

Summation then yields

$$\begin{array}{rcl} -c^2 \sum_{i<k} f_i f_i'' \langle Y_i, Y_i \rangle + 3c \sum_{i<k} f_i f_k \left(\frac{f_i}{f_k}\right)' \left\langle B_i^{kk}, Y_i \right\rangle & \geq & -\frac{C_2}{m} \mu^2 \sum_{i<k} ||B_i^{kk}||^2_{\varphi(0)}, \\ -c^2 f_k f_k'' \langle Y_k, Y_k \rangle + 3c \sum_{i>k} f_i f_k \left(\frac{f_k}{f_i}\right)' \langle B_k^{ii}, Y_k \rangle & \geq & -\frac{C_2}{m} \mu^2 \sum_{i>k} ||B_k^{ii}||^2_{\varphi(0)}. \end{array} \tag{11}$$

Now,

$$\begin{array}{rcccccl} \sum_{i<k} ||B_i^{kk}||^2_{\varphi(0)} & \leq & ||B_{\mathfrak{m}}^{kk}||^2_{\varphi(0)} & \leq & C_{max} ||X_k \wedge Y_k||^2_{\varphi(0)} & \leq & \frac{C_{max}}{\rho^4 \mu^4} ||X_k \wedge Y_k||^2_g, \\ ||B_k^{ii}||^2_{\varphi(0)} & \leq & ||B_{\mathfrak{m}}^{ii}||^2_{\varphi(0)} & \leq & C_{max} ||X_i \wedge Y_i||^2_{\varphi(0)} & \leq & \frac{C_{max}}{\mu^4} ||X_i \wedge Y_i||^2_g, \end{array}$$

where in the second line we assume that $i > k$. Note that $\frac{f_i}{f_j} \equiv \frac{c_i}{c_j}$ and hence $\left(\frac{f_i}{f_j}\right)' = 0$ if $i, j \neq k$. Thus, using the preceding estimates, we can add the lines in (11) to

$$\begin{array}{rcl} -c^2 \sum_{i \leq k} f_i f_i'' \langle Y_i, Y_i \rangle & + & 3c \sum_{i<j} f_i f_j \left(\frac{f_i}{f_j}\right)' \left\langle B_i^{jj}, Y_i \right\rangle \\ & \geq & -\frac{C_2}{m} \mu^2 \left(\sum_{i<k} ||B_i^{kk}||^2_{\varphi(0)} + \sum_{i>k} ||B_k^{ii}||^2_{\varphi(0)} \right) \\ & \geq & -\frac{C_2}{m} \left(\frac{C_{max}}{\rho^4 \mu^2} ||X_k \wedge Y_k||^2_g + \sum_{i>k} \frac{C_{max}}{\mu^2} ||X_i \wedge Y_i||^2_g \right) \\ & \geq & -\frac{C_2 C_{max}}{m \rho_0^4 \mu(a)^2} \sum_{i \geq k} ||X_i \wedge Y_i||^2_g \\ & \geq & -\frac{C_2 C_{max}}{m \rho_0^4 \mu(a)^2} ||X \wedge Y||^2_g, \end{array} \tag{12}$$

where we used for the second to last estimate that $\rho \geq \rho_0$, $\rho_0 \leq 1$ and $\mu \geq \mu(a)$.

Now from Proposition 4.3, (9), (10) and (12), we get for all $c \in \mathbb{R}$ and $X, Y \in \mathfrak{m}$

$$R(c\partial_t + X, Y; Y, c\partial_t + X) \geq \left(C_{min} - 4C_1^2 - \frac{C_2 C_{max}}{m\rho_0^4 \mu(a)^2}\right) ||X \wedge Y||_g^2 \geq 0,$$

with equality only if $X \wedge Y = 0$. In this case, assuming that $Y \neq 0$, we may assume that $X = 0$, so that $0 = c^2 R(\partial_t, Y; Y, \partial_t) = -c^2 \sum_i f_i f_i'' \langle Y_i, Y_i \rangle$, and since $f_i'' < 0$ for all i this implies that $c = 0$. Therefore, $R(c\partial_t + X, Y; Y, c\partial_t + X) > 0$ whenever $(c\partial_t + X) \wedge Y \neq 0$. ∎

Lemma 4.6 *Let $C_{max}, C_{min}, c_k, \tilde{c}_k > 0$ and $k, r \in \mathbb{Z}^+$ with $k \leq r$ and $a > 0$ with $a^2 > C_{min}^{-1}$ be arbitrary constants. Let $C_1 := \frac{1}{2a}$, and define ρ_0, ρ_1 as in Lemma 4.5. Then there are constants $C_2, C_3 > 0$ and smooth functions $\mu, \rho : [a, b) \to \mathbb{R}$ for some $b > a$ satisfying the hypotheses of Lemma 4.5. Indeed, we may choose $\mu(t) = \sqrt{t}$.*

Proof. Let $\mu : [a, \infty) \to \mathbb{R}$ be given by $\mu(t) = \sqrt{t}$. Note that $4C_1^2 = \frac{1}{a^2} < C_{min}$, whence for any choice of $C_3 \in (0, \rho_0)$ we can choose $C_2 > 0$ such that the first condition is satisfied. Moreover, one also verifies that μ satisfies the second condition.

Given any $t_0 > a+2$, we choose a smooth function $\sigma : [a, \infty)$ with the following properties:

1. $0 \leq \sigma \leq 1$, $\sigma|_{[a,a+1]} \equiv 0$, $\sigma|_{[a+2,t_0]} \equiv 1$ and $\sigma|_{[2t_0,\infty)} \equiv 0$,
2. On $(a+1, a+2)$ we have $0 \leq \dot{\sigma} \leq 2$,
3. On $(t_0, 2t_0)$ we have $-2 \leq t\dot{\sigma} \leq 0$.

In particular, this implies that $|t\dot{\sigma}| \leq 2(a+2)$. Moreover, we may assume that t_0 is sufficiently large such that

$$\int_a^\infty \frac{\sigma(s)}{s}\, ds \geq \frac{1}{m_0} \left| \frac{\tilde{c}_k}{c_k} - 1 \right|,$$

where $m_0 := min \left\{ \frac{1}{2}, \frac{1}{2}\sqrt{C_2}, \frac{\rho_0 - C_3}{8(a+2)} \right\} > 0$.

Now, let $\kappa \in \mathbb{R}$ be the number defined by

$$\kappa \int_a^\infty \frac{\sigma(s)}{s}\, ds = \frac{\tilde{c}_k}{c_k} - 1.$$

Evidently, $|\kappa| \leq m_0$, and we claim that

$$\rho(t) := 1 + \kappa \int_a^t \frac{\sigma(s)}{s}\, ds$$

satisfies all the asserted properties. Namely, $\rho|_{(a,a+1)} \equiv 1$ and $\rho|_{(2t_0,\infty)} \equiv \frac{\tilde{c}_k}{c_k}$ is obvious, and $\dot{\rho} = \kappa \frac{\sigma}{t}$ has the same sign as κ, hence ρ is monotone.

Next, $\frac{(\mu\rho)''}{\ddot{\mu}} = \rho - 4t\dot{\rho} - 4t^2\ddot{\rho} = \rho - 4\kappa t\dot{\sigma} \geq C_3$ where the last inequality follows since $\rho \geq \rho_0$ and $\kappa t\dot{\sigma} \leq |\kappa||t\dot{\sigma}| \leq 2m_0(a+2)$.

For the last property, $\dot{\rho} \leq 0 \leq C_1\rho$ if $\tilde{c}_k \leq c_k$, whereas $\dot{\rho} = \kappa\frac{\sigma}{t} \leq \frac{\kappa}{a} \leq 2m_0C_1 \leq C_1 \leq C_1\rho$ if $\tilde{c}_k > c_k$.

Finally, $(\dot{\rho})^2 = \kappa^2\frac{\sigma^2}{t^2} \leq m_0^2\frac{1}{t^2} \leq C_2\frac{1}{4t^2} = -C_2\frac{\ddot{\mu}}{\mu}$. ▮

Theorem 4.4 now follows immediately from Lemmas 4.5 and 4.6. For the last assertion, one chooses $a_0 := (C_{min})^{-\frac{1}{2}}$. ▮

5 Metrics on homogeneous disc bundles

We now wish to prove two important consequences of the results of the preceding section.

Theorem 5.1 *Let $K \subset O(n+1)$ be a Lie subgroup which acts transitively on $S^n \subset \mathbb{R}^{n+1}$, and let g_Q be a normal homogeneous metric on S^n induced by some Ad_K-invariant inner product Q on $\mathfrak{k}$. Let $r(x) := ||x||$ be the radius function on $\mathbb{R}^{n+1}$.*

Then there exists a K-invariant metric g on the unit ball $B_1(0) \subset \mathbb{R}^{n+1}$ with positive sectional curvature, and an $\varepsilon > 0$, such that on $r^{-1}(1-\varepsilon, 1)$ we have $g = dr^2 + f(r)^2 g_Q$ where $f : (1-\varepsilon, 1) \to \mathbb{R}$ satisfies $f > 0, f' > 0$.

Proof. By Theorem 4.4 and Proposition 3.3 it follows that there is a K-invariant metric g_1 on $(a,b) \times S^n = \{x \in \mathbb{R}^{n+1} \mid a < r(x) < b\}$ of positive sectional curvature such that on $(a, a+\varepsilon) \times S^n$ and on $(b-\varepsilon, b) \times S^n$, the metric takes the form $dr^2 + rc_0^2 g_0$ and $dr^2 + rg_Q$, respectively, where g_0 denotes the standard metric on S^n. Moreover, we may choose a arbitrarily large.

Let $r_0 \in (a, a+\varepsilon)$. It is easy to see that for a sufficiently large choice of a there is a smooth function $f : (0, a+\varepsilon) \to \mathbb{R}$ such that $f(r) = \sin r$ for $r < \varepsilon$, $f(r) = c_0\sqrt{r}$ for $r > r_0$ and $f'' < 0$.

On $B_{a+\varepsilon}(0) \subset \mathbb{R}^{n+1}$, define the $O(n+1)$-invariant metric $g := dr^2 + f(r)^2 g_0$. It is known that the standard metric on S^{n+1} w.r.t. the normal coordinate chart has the form $dr^2 + \sin^2 r g_0$, whence the germ of g at $r = 0$ is a smooth metric of constant sectional curvature 1. Also, it is known that the curvature of g at some point with $r > 0$ is given by $R^g(c\partial_r + X, Y; Y, c\partial_r + X) = \frac{1-(f')^2}{f^2}||X \wedge Y||_g^2 - \frac{f''}{f}||cY||_g^2$ where X, Y are tangent vectors of the radial sphere. (This can be seen e.g. using Proposition 4.3 with $G = O(n+1), H = O(n)$ and $r = 1$.)

Since $f'' < 0$, it follows that $f'(r) < f'(0) = 1$ for all $r > 0$, whence g has positive sectional curvature and is K-invariant. Since the germ of g at $r = a+\varepsilon$ coincides with the germ of g_1, we can glue these two metrics together to obtain a K-invariant metric of positive sectional curvature on $B_b(0) \subset \mathbb{R}^{n+1}$ such that for $r \in (b-\varepsilon, b)$ we have $g = dr^2 + rg_Q$. Finally, one rescales the metric to replace b by 1. ▮

We shall now recall some generalities and set up some notation. Let $K \subset G$ be compact Lie groups and fix a bi-invariant metric Q on $\mathfrak{g}$. Suppose there is a representation $\imath : K \to O(n+1)$ under which K acts transitively on $S^n \subset \mathbb{R}^{n+1}$, and let $H \subset K$ be the isotropy subgroup of this action, i.e. $K/H = S^n$. We let $V := \mathbb{R}^{n+1}$ and consider the homogeneous vector bundle $D := G \times_K V$; for $R > 0$ we also let

$$D_R := G \times_K B_R(0) \subset D, \tag{13}$$

where $B_R(0) \subset \mathbb{R}^{n+1}$ denotes the open ball of radius R. Since $r : V \to \mathbb{R},\ x \mapsto ||x||$ parametrizes the K-orbits in V, it follows that the induced map $r : D \to \mathbb{R}$ parametrizes the G-orbits of D. In fact, for $t > 0$ we have $r^{-1}(t) \cong G \times_K K/H \cong G/H$, whence $D_R \backslash r^{-1}(0) \cong (0, R) \times G/H$.

As we now have the chain $H \subset K \subset G$, we define the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ such that $\mathfrak{h} \oplus \mathfrak{m}_1 = \mathfrak{k}$. Consider a K-invariant metric g_{φ_1} on $S^n \cong K/H$ which is induced by an Ad_K-invariant symmetric positive definite map $\varphi_1 : \mathfrak{m}_1 \to \mathfrak{m}_1$. On $G \times K/H$, we define the product metric where the metric on G is induced by Q. Then an easy calculation (cf. [GZ1, Lemma 2.1]) shows that the induced submersion metric on $G \times_K K/H \cong G/H$ is induced by the map

$$\varphi : \mathfrak{m}_1 \oplus \mathfrak{m}_2 \longrightarrow \mathfrak{m}_1 \oplus \mathfrak{m}_2, \quad \varphi|_{\mathfrak{m}_1} = \varphi_1 \left(\varphi_1 + Id_{\mathfrak{m}_1}\right)^{-1}, \quad \varphi|_{\mathfrak{m}_2} = Id_{\mathfrak{m}_2}. \tag{14}$$

In particular, if we have a K-invariant metric on $B_R(0) \subset V$ of nonnegative curvature of the form $dr^2 + g_{\varphi(t)}$, then the induced metric on $D_R = G \times_K B_R(0)$ has also nonnegative curvature and the form

$$g = dr^2 + g_{\varphi(r)}, \quad \text{with} \quad \varphi(r) : \mathfrak{m}_1 \oplus \mathfrak{m}_2 \longrightarrow \mathfrak{m}_1 \oplus \mathfrak{m}_2 \text{ as in (14).} \tag{15}$$

Theorem 5.2 *Let $H \subset K \subset G$ be compact Lie groups and let Q be a bi-invariant inner product on $\mathfrak{g}$. Suppose that $K/H = S^n$ where the action of K is induced by a representation $\imath : K \to O(n+1)$.*

Then for every $\delta > 0$ there exists a G-invariant Riemannian metric g_δ on D_R (cf. (13)) for some $R > 0$ such that

1. *$Sec(D_R, g_\delta) \geq -\delta$,*
2. *$diam(D_R, g_\delta) \leq O(\delta^{-1/6})$,*
3. *There is an $\varepsilon > 0$ such that $(r^{-1}(R-\varepsilon, R), g_\delta)$ is isometric to $(R-\varepsilon, R) \times (G/H, g_Q)$.*

Here, $O(\delta^p)$ denotes any function of δ such that $\limsup_{\delta \to 0} |\delta^{-p} O(\delta^p)| < \infty$.

Proof. We decompose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ w.r.t. the chain $H \subset K \subset G$ (cf. 6).

First, we note that one can extend the metric on $B_1(0) \subset \mathbb{R}^{n+1}$ given in Theorem 5.1 to a nonnegatively curved K-invariant metric on $\mathbb{R}^{n+1}$ which outside of $B_1(0)$ takes the form $dr^2 + c^2 r^2 g_Q$ with some constant $c > 0$, where g_Q denotes the normal homogeneous metric on $S^n = K/H$. This induces a G-invariant metric g_0 of nonnegative sectional curvature on $D = G \times_K \mathbb{R}^{n+1}$ which, according to (15), is of the form $g_0 = dt^2 + g_{\varphi(t)}$ outside of D_1, where $\varphi(t)$ is parametrized by $(f_1(t), f_2(t)) = \left(\frac{ct}{\sqrt{1+c^2t^2}}, 1\right)$. Note that $f_1 < 1$ for all t, hence $g_{\varphi(t)} \neq g_Q$. Therefore, we need to extend g_0 to the collar in a different way.

In general, for $0 < R_0 < R$ we have $D_R \backslash D_{R_0} = (R_0, R) \times G/H$. Thus, in order to extend the submersion metric on D_{R_0} from the previous paragraph to a metric on D_R which satisfies the third condition of the theorem, we need to find a function $(f_1, f_2) : (R_0, R) \to \mathbb{R}^2$ whose germs at R_0 and R are given by

$$(f_1, f_2)|_{(R_0, R_0+\varepsilon)} = \left(\frac{ct}{\sqrt{1+c^2t^2}}, 1 \right) \quad \text{and} \quad (f_1, f_2)|_{(R-\varepsilon, R)} = (1, 1), \tag{16}$$

and define the metric on $D_R \backslash D_1$ by

$$g = dt^2 + g_{\varphi(t)} \tag{17}$$

with $\varphi(t)$ parametrized by $(f_1(t), f_2(t))$.

We choose $f_2 \equiv 1$, and need to estimate the curvature of g in (17). It is well-known [Ber] that for $K/H = S^n$ and Q any Ad_K-invariant inner product on $\mathfrak{k}$ the normal homogeneous metric (S^n, g_Q) has positive sectional curvature. Thus, we can find constants $C_1 \geq C_2 > 0$ such that $\frac{1}{C_2} \geq Sec(K/H, g_Q) \geq \frac{1}{C_1}$. Now $R^{g_Q}(X_1, Y_1; Y_1, X_1) = \langle B_0^{11}, B_0^{11} \rangle + \frac{1}{4} \langle B_1^{11}, B_1^{11} \rangle$, whence it follows that

$$C_2 \left(\left\langle B_0^{11}, B_0^{11} \right\rangle + \frac{1}{4} \left\langle B_1^{11}, B_1^{11} \right\rangle \right) \leq ||X_1 \wedge Y_1||^2 \leq C_1 \left(\left\langle B_0^{11}, B_0^{11} \right\rangle + \frac{1}{4} \left\langle B_1^{11}, B_1^{11} \right\rangle \right). \tag{18}$$

Observe that $f_1^4 ||X_1 \wedge Y_1||^2 = ||X_1 \wedge Y_1||_g^2 \leq ||(c\partial_t + X) \wedge Y||_g^2$, whence in order to guarantee that $Sec(D_R \backslash D_{R_0}, g) \geq -\delta$, it suffices to show that

$$R(c\partial_t + X, Y; Y, c\partial_t + X) + \delta f_1^4 ||X_1 \wedge Y_1||^2 \geq 0. \tag{19}$$

From the curvature formulas in Propositions 3.1 and 4.3, we get for the curvature of g:

$$\begin{array}{rl} R(c\partial_t + X, Y; Y, c\partial_t + X) = & \\ & \frac{3}{4} f_1^2 \left\langle [X,Y]_{\mathfrak{h}}, [X,Y]_{\mathfrak{h}} \right\rangle + \frac{1}{4} \left\langle B_2^{22} + 2f_1^2 B_2^{12}, B_2^{22} + 2f_1^2 B_2^{12} \right\rangle \\ + & \frac{1}{4} f_1^2 \left\langle B^{11}, B^{11} \right\rangle + \frac{1}{2} f_1^2 (3 - 2f_1^2) \left\langle B^{11}, B_{\mathfrak{k}}^{22} \right\rangle + (1 - \frac{3}{4} f_1^2) \left\langle B_{\mathfrak{k}}^{22}, B_{\mathfrak{k}}^{22} \right\rangle \\ - & f_1^2 (f_1')^2 ||X_1 \wedge Y_1||^2 + 3cf_1 f_1' \left\langle B_1^{22}, Y_1 \right\rangle - c^2 f_1 f_1'' \left\langle Y_1, Y_1 \right\rangle. \end{array} \tag{20}$$

We decompose $B_1^{22} = V_1 + W_1$ with $V_1, W_1 \in \mathfrak{m}_1$ such that $\langle V_1, Y_1 \rangle = 0$ and W_1 is a multiple of Y_1; if $Y_1 = 0$ then we set $W_1 = 0$. Moreover, we let $V_{\mathfrak{k}} := B_0^{22} + V_1$. Thus, $B_{\mathfrak{k}}^{22} = V_{\mathfrak{k}} + W_1$. Since $\langle B^{11}, Y_1 \rangle = \langle [X_1, Y_1], Y_1 \rangle = 0$, it follows that

$$\left\langle B^{11}, B_{\mathfrak{k}}^{22} \right\rangle = \left\langle B^{11}, V_{\mathfrak{k}} \right\rangle, \quad \left\langle B_1^{22}, Y_1 \right\rangle = \langle W_1, Y_1 \rangle \quad \text{and} \quad \left\langle B_{\mathfrak{k}}^{22}, B_{\mathfrak{k}}^{22} \right\rangle = \langle V_{\mathfrak{k}}, V_{\mathfrak{k}} \rangle + \langle W_1, W_1 \rangle. \tag{21}$$

Substituting (18) and (21) into (20) yields

$$\begin{array}{l} R(c\partial_t + X, Y; Y, c\partial_t + X) + \delta f_1^4 ||X_1 \wedge Y_1||^2 \geq \\ \qquad \begin{array}{ccccc} \frac{1}{4} f_1^2 (1-u) \left\langle B^{11}, B^{11} \right\rangle & + & \frac{1}{2} f_1^2 (3 - 2f_1^2) \left\langle B^{11}, V_{\mathfrak{k}} \right\rangle & + & (1 - \frac{3}{4} f_1^2) \left\langle V_{\mathfrak{k}}, V_{\mathfrak{k}} \right\rangle \\ -c^2 f_1 f_1'' \left\langle Y_1, Y_1 \right\rangle & + & 3cf_1 f_1' \left\langle W_1, Y_1 \right\rangle & + & (1 - \frac{3}{4} f_1^2) \left\langle W_1, W_1 \right\rangle, \end{array} \end{array} \tag{22}$$

where $u = 4C_1 (f_1')^2 - C_2 \delta f_1^2$.

Clearly, (19) will be satisfied if both rows on the right of (22) are nonnegative which happens – according to Lemma 2.2 – if the quadratic polynomials

$$\begin{array}{rcl} p_1(x) & = & \frac{1}{4}f_1^2(1-u)x^2 + \frac{1}{2}f_1^2(3-2f_1^2)x + (1-\frac{3}{4}f_1^2) \\ p_2(x) & = & -f_1 f_1'' x^2 + 3 f_1 f_1' x + (1-\frac{3}{4}f_1^2) \end{array}$$

are nonnegative for all $x \in \mathbb{R}$. Since $f_1 \in (0,1]$, we have $p_i(0) > 0$ for all i, thus it suffices to verify that the discriminants d_i of p_i are nonnegative. These discriminants are

$$\begin{array}{rcl} d_1 & = & \frac{1}{4}f_1^2(1-u)(1-\frac{3}{4}f_1^2) - \frac{1}{16}f_1^4(3-2f_1^2)^2 \;\; = \;\; \frac{1}{16}f_1^2(4(1-f_1^2)^3 - (4-3f_1^2)u) \\ d_2 & = & -f_1 f_1''(1-\frac{3}{4}f_1^2) - \frac{9}{4}f_1^2(f_1')^2. \end{array}$$

Now we let

$$\mu := \frac{f_1}{\sqrt{4-3f_1^2}}, \qquad \text{whence} \qquad f_1 = \frac{2\mu}{\sqrt{1+3\mu^2}}. \tag{23}$$

The boundary conditions (16) for f_1 translate to

$$\mu|_{(R_0,R_0+\varepsilon)} = \frac{ct}{\sqrt{4+c^2t^2}} \qquad \text{and} \qquad \mu|_{(R-\varepsilon,R)} \equiv 1. \tag{24}$$

One calculates that $d_2 = -\frac{4\mu\ddot{\mu}}{(1+3\mu^2)^3}$, whence $d_2 \geq 0$ iff $\ddot{\mu} \leq 0$. Also, $d_1 \geq 0$ if $u \leq 0$, and since $u = \frac{16}{(1+3\mu^2)^3}C_1\dot{\mu}^2 - C_2\delta\frac{4\mu^2}{1+3\mu^2}$, we conclude that (19) is satisfied if f_1 is given as in (23) where

$$4C_1\dot{\mu}^2 \leq C_2\delta\mu^2, \qquad \text{and} \qquad \ddot{\mu} \leq 0. \tag{25}$$

If μ satisfies (24) and $\ddot{\mu} \leq 0$, then $\dot{\mu} \geq 0$, whence $(4C_1\dot{\mu}^2 - C_2\delta\mu^2)' = 2\dot{\mu}(4C_1\ddot{\mu} - C_2\delta\mu) \leq 0$. Thus, it suffices to verify that the first inequality of (25) holds close to R_0, and by (24), this happens if $R_0^2(4+c^2R_0^2)^2 = \frac{64C_1}{C_2}\delta^{-1}$, so we choose R_0 according to this expression. Note that then $R_0 = O(\delta^{-1/6})$.

Thus, it remains to find $\mu : (R_0, R) \to \mathbb{R}$ satisfying (24) and $\ddot{\mu} \leq 0$. This can be done on an interval of length $R - R_0 \leq \frac{1-\mu(R_0)}{\dot{\mu}(R_0)} + 1$, and a straightforward calculation shows that the right hand side of this inequality is of the form $O(\delta^{-1/6})$, whence $R \leq O(\delta^{-1/6})$.

Finally, since on $D_R \backslash D_1$ this metric is of the form (17), it follows that the curves $t \mapsto (t,p)$ are unit speed geodesics, whence $diam(D_R, g) \leq diam(D_1, g) + 2(R-1) \leq O(\delta^{-1/6})$ which completes the proof. ∎

We are now ready to prove our main result:

Theorem A *Any closed cohomogeneity one manifold supports metrics of almost nonnegative sectional curvature which are invariant under the cohomogeneity one action.*

Proof. As we mentioned in the introduction, we may assume that the orbit space M/G is a compact interval and hence M is obtained by gluing together two homogeneous disc bundles of the type considered in Theorem 5.2. The boundary condition assures that these metrics extend smoothly to metrics g_δ on M satisfying $Sec(M, g_\delta) \geq -\delta$ and $diam(M, g_\delta) \leq O(\delta^{-1/6})$. Thus, $-Inf(Sec(M, g_\delta))diam(M, g_\delta)^2 \leq \delta(O(\delta^{-1/6}))^2 = O(\delta^{2/3})$, and therefore, $\lim_{\delta\to 0} -Inf(Sec(M, g_\delta))diam(M, g_\delta)^2 = 0$. ∎

Remark: In view of the results from [GZ1] and [GZ2] it is worth to point out the following. By (19) it follows that the metrics constructed above have nonnegative sectional curvature if $\dim \mathfrak{m}_1 \leq 1$. Also, a direct calculation shows that all these metrics have nonnegative Ricci curvature. Moreover, points of positive Ricci curvature exist iff $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{m}_2 = 0$ and $\dim \mathfrak{m}_1 > 0$ which are the same algebraic criteria obtained in [GZ2] for the existence of such points.

6 Actions on Brieskorn manifolds and exotic spheres

In this section we exhibit several examples of closed cohomogeneity one manifolds and provide the differential topological facts which allow to deduce Corollaries B, C, and D from these examples and Theorem A.

Particularly interesting examples of closed cohomogeneity one manifolds are given by the odd-dimensional Brieskorn manifolds (see [Bri], [Bro2], [HM], [Mi]). Given an integer $d \geq 1$, the Brieskorn manifolds $W^{2n-1}(d)$ are the $2n-1$ dimensional real algebraic submanifolds of $\mathbb{C}^{n+1}$ defined by the equations

$$z_0^d + z_1^2 + \cdots + z_n^2 = 0 \quad \text{and} \quad |z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1.$$

The manifolds $W^{2n-1}(d)$ are invariant under the standard linear action of $O(n)$ on the $(z_1, \ldots, z_n)$ coordinates, and (compare [HH]) the circle subgroup $S^1(d) \cong SO(2) \subset U(n+1)$ consisting of diagonal matrices of the form $\mathrm{diag}(e^{2i\theta}, e^{di\theta}, \ldots, e^{di\theta})$ also acts on $W^{2n-1}(d)$. In particular, by Theorem A all Brieskorn manifolds $W^{2n-1}(d)$, $2 \leq d \in \mathbb{Z}$, admit $S^1 \times O(n)$ invariant metrics of almost nonnegative sectional curvature.

Suppose now that $n \geq 3$ is odd, and let M^{2n} denote the closed manifold with boundary obtained by plumbing two copies of the tangent disc bundle of S^n (see [Bro2]). The boundary of M^{2n} is called the $2n-1$ dimensional *Kervaire sphere* K^{2n-1}. The Kervaire sphere K^{2n-1} is a smooth homotopy sphere homeomorphic to S^{2n-1}, and if θ_{2n-1} denotes the set of equivalence classes of smooth oriented topological $2n-1$ spheres with respect to orientation preserving diffeomorphisms which is a group under taking connected sums, then K^{2n-1} is a generator of the subgroup $bP_{2n} \subset \theta_{2n-1}$ which consists of the subset of $2n-1$ spheres which bound a closed parallelizable $2n$ manifold. It was shown by Browder (cf. [Bro1]) that the Kervaire sphere K^{2n-1} is an exotic sphere, i.e., homeomorphic but not diffeomorphic to the standard sphere, if $n+1$ is not a power of 2. In particular, K^{2n-1} is exotic if $n \equiv 1 \bmod 4$.

The Brieskorn manifolds are related to the Kervaire spheres and standard spheres as follows (cf. [HM]): If $n \geq 3$ and $d \geq 3$ are odd, $W^{2n-1}(d)$ is homeomorphic to a sphere from bP_{2n}. If now in addition $d \equiv \pm 3 \bmod 8$, then $W^{2n-1}(d)$ is diffeomorphic to the Kervaire sphere K^{2n-1}, and for $d \equiv \pm 1 \mod 8$ the manifolds $W^{2n-1}(d)$ are diffeomorphic to the standard $2n-1$ sphere. Thus, if $n \geq 3$ is odd and $n+1$ is not a power of 2 and if $d > 0$ and $d \equiv \pm 3 \mod 8$, then the Brieskorn manifolds $W^{2n-1}(d)$ are diffeomorphic to Kervaire spheres which are exotic. Notice also that for each (odd) d one obtains a different cohomogeneity one action by $S^1 \times SO(n)$ action on the Brieskorn spheres $W^{2n-1}(d)$.

Recall that the orbit space of a free action of a nontrivial finite cyclic group on a homotopy sphere is called a homotopy real projective space if this group has order two, and else said to be a homotopy lens space. Notice that homotopy real projective spaces are always homotopy equivalent to standard real projective spaces (cf. [Wa]), whereas a corresponding statement for homotopy lens spaces does in general not hold.

Free actions of finite cyclic groups on the Brieskorn spheres $W^{2n-1}(d)$ and the differential topology of the resulting orbit spaces have been extensively studied (cf. [AB], [Bro3], [Gi1], [Gi2], [Gi3], [Or], [HM], [Lo]).

Suppose that $n \geq 3$ and $d \geq 1$ are odd, and consider the involution $I = I_d : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ which is defined by $I(z_0, z_1, \ldots, z_n) := (z_0, -z_1, \ldots, -z_n)$. This involution leaves the Brieskorn spheres $W^{2n-1}(d)$ invariant and descends to a fixed point free involution $I_d : W^{2n-1}(d) \to W^{2n-1}(d)$ whose quotient space $P^{2n-1}(d) := W^{2n-1}(d)/I_d$ is a smooth closed manifold homotopy equivalent to $\mathbb{RP}^{2n-1}$. Notice also that the $S^1 \times SO(n)$ action on $W^{2n-1}(d)$ commutes with I_d and therefore descends to an action on $P^{2n-1}(d)$. Now, by attaching generalized Arf-Kervaire invariants to the involutions I_d, it has been shown in [Gi2], [Gi3] (compare [AB], [Bro3], [HM]) that $P^{2n-1}(d)$ and $P^{2n-1}(d')$ are orientation preserving smoothly distinct if $0 \leq r < s \leq 2^{n-1}$, $d = 2r+1$, $d' = 2s+1$. If $n \equiv 1 \mod 4$, then 2^{n-2} of the $P^{2n-1}(d)$ are universally covered by the exotic $2n-1$ dimensional Kervaire sphere.

Suppose now that $n \geq 3$ and $m \geq 3$. Define an action of $\mathbb{Z}_m$ on $\mathbb{C}^{n+1}$ by $\alpha(z_0, z_1, \ldots, z_n) := (\alpha^2 z_0, \alpha^d z_1, \ldots \alpha^d z_n)$, where α is a primitive m-th root of unity generating $\mathbb{Z}_m \subset S^1$. One verifies that if m and d are relatively prime, this action induces a free action on $W^{2n-1}(d)$ and if $W^{2n-1}(d)$ is a sphere, the quotient $Q_m^{2n-1}(d) := W^{2n-1}(d)/\mathbb{Z}_m$ is a smooth homotopy lens space. Of course, if $W^{2n-1}(d)$ is an exotic sphere, all $Q_m^{2n-1}(d)$ will be differentiably distinct from the standard lens spaces. Whether for fixed n and $m \geq 3$ one could obtain different homotopy types among the $Q_m^{2n-1}(d)$ is presently unclear.

Appendix: Bundle liftings of group actions

The following proposition shows that the class of closed smooth manifolds with actions of a given cohomogeneity and given orbit space is in fact very rich and enjoys nice extension and closedness properties. It follows from a general lifting result of Hattori-Yoshida (cf. [HY]).

Proposition *Let M be a closed smooth manifold on which a compact connected Lie group G acts smoothly by cohomogeneity $j \geq 0$. Let $\pi : P \to M$ be a principal T^k bundle over M, where k is some natural number. Then, if $H^1(M, \mathbb{Z})$ is trivial or if G is semisimple, P admits a cohomogeneity j action by $G' \times T^k$, where T^k acts via the principal action and where G' is a finite covering group of G whose action on P is a bundle lifting of the action of G' on M which is induced by the action of G on M.*

Remark Since the T^k action on P commutes with the action of G', the orbit space of the cohomogeneity j action on P is homeomorphic to M/G. Moreover, if the G action on M has singular orbits, then the codimensions of the singular orbits of the $G' \times T^k$ action on P are equal to the codimensions of the corresponding singular orbits of the G action on M.

Remark For principal bundles with non-Abelian structure group it is in general difficult to decide whether a given action on the base admits a bundle lifting to the total space. In [GZ1] a special construction of cohomogeneity one principal bundles over cohomogeneity one manifolds is given.

Example Consider the series of cohomogeneity one G manifolds $M = M_n$ which are (cf. [GZ1]) determined by the triples $H \subset \{K_-, K_+\} \subset G$, where $G = U(n)$ and

$$H = Sp(1)^2 \times U(1)^{n-4}, \quad K_- = Sp(2) \times U(1)^{n-4}, \quad K_+ = Sp(1)^2 \times U(1)^{n-6} \times U(2).$$

The manifolds M_n are simply connected and one can show that their second Betti number equals $b_2(M_n) = n - 6$. Since equivalence classes of T^k principal bundles over M_n are classified by $H^2(M_n, \mathbb{Z}^k)$, the above proposition implies that for large n and suitable values of k the manifolds M_n will give rise to infinitely many further examples of cohomogeneity one manifolds which are total spaces of principal T^k bundles over M_n. In general, all these total spaces will be nondiffeomorphic and can moreover be chosen to be simply connected, too. For the manifolds M_n the homogeneous spheres $K_-/H = \mathbb{HP}^1 = S^4$ and $K_+/H = \mathbb{CP}^1 = S^2$ have different dimension and it seems that the only means to show that all M_n (and thus all principal torus bundles over them) carry almost nonnegatively curved metrics is given by Theorem A.

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