Regular simplices and lower volume bounds for hyperbolic n-manifolds

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0. Introduction

A hyperbolic n-manifold M^n is a complete, connected n-dimensional Riemannian manifold of constant sectional curvature $K = -1$. The volume $\mathrm{vol}_n(M^n)$ is its most natural and dominant geometric-topological invariant; it is related to other quantities in form of (in-)equalities. For example, by the well-known theorem of Gauss-Bonnet, the volume is proportional to the Euler-Poincaré characteristic $\chi(M^{2n})$, and by inequalities, it appears in combination with Betti numbers $b_i(M^n)$ (see [X]), the diameter $diam(M^n)$, and the first eigenvalue $\lambda_1(M^n)$ of the Laplacian on M^n (see [BS]). However, the exact evaluation of a single invariant is usually very difficult.
The aim of this paper is to derive a new lower volume bound for hyperbolic n-manifolds M. Denote by $r(x, M)$ the injectivity radius of M at $x \in M$, and by

$$r(M) := \sup \{ r(x, M) \mid x \in M \}$$

the *in-radius* of M associated to the largest ball embeddable in M. Finally, let $\Omega_n := \mathrm{vol}_n(S^n)$ be the volume of the standard n-sphere.

THEOREM.

For $n \geq 2$, let M be a compact hyperbolic n-manifold with in-radius $r(M)$. Then,

$$\mathrm{vol}_n(M) \geq \frac{\Omega_{n-1}}{n+1} \cdot \frac{\mathrm{vol}_n(S_{reg}(2\alpha))}{\mathrm{vol}_{n-1}(s_{reg}(2\alpha))} \quad ,$$

where $S_{reg}(2\alpha) \subset H^n$ denotes the regular simplex with dihedral angle 2α and edge length $2r(M)$ related by

$$\frac{1}{n} < \cos(2\alpha) = \frac{\cosh(2r(M))}{1 + (n-1)\cosh(2r(M))} < \frac{1}{n-1} \quad ,$$

and where $s_{reg}(2\alpha)$ *is a spherical vertex simplex of* $S_{reg}(2\alpha)$.

The theorem is the generalization to arbitrary dimension of a result used by R. Meyerhoff in the three dimensional case (see [Me]). Since $s_{reg}(2\alpha) \subset S^{n-1}$, we obtain the following coarser but simpler estimate:

COROLLARY.

For $n \geq 2$, *let* M *be a compact hyperbolic* n*-manifold with in-radius* $r(M)$. *Then,*

$$\mathrm{vol}_n(M) \geq \frac{1}{n+1}\,\mathrm{vol}_n(S_{reg}) \quad ,$$

where $S_{reg} \subset H^n$ *is the regular simplex of edge length* $2r(M)$ *as above.*

Notice that these results appropriately modified are also valid for Euclidean and spherical space forms. Moreover, it will be obvious from the proof, which is based on K. Böröczky's upper density bound for hyperbolic sphere packings, that the theorem improves the known volume estimates referring to the volume of the hyperbolic or even of the Euclidean ball of radius $r(M)$. For example, for compact Riemannian manifolds M whose injectivity radius is bigger than or equal to ρ, say, M. Berger [Be] showed that

$$\mathrm{vol}_n(M) \geq \frac{\Omega_n}{\pi^n}\,\rho^n \quad .$$

For compact Riemannian manifolds M of sectional curvature $-1 \leq K < 0$, P. Buser and H. Karcher [BK] derived the bounds

$$r(M) \geq \frac{1}{4^{n+3}} \quad \text{and} \quad \mathrm{vol}_n(M) \geq \frac{\Omega_{n-1}}{n} \cdot \frac{1}{4^{n(n+3)}} \quad .$$

To obtain valuable estimates for the minimum volume of compact hyperbolic n-manifolds M^n necessitates, by the theorem, to find accurate bounds for the in-radius $r(M^n)$. For $n = 3$, Meyerhoff's solid tube radius bound refined by F. Gehring and G. Martin [GM] to $r(M^3) \simeq 0.05725$ gives rise to the best lower volume estimate known up to now. For $n > 3$, however, there are no satisfactory in-radius bounds available. The Buser-Karcher estimate, which is the best result at our disposal (see also [Ma] and [FH]), is rapidly decreasing with the dimension n and already for $n = 3$ much more disadvantageous than Gehring-Martin's value. In fact, the true growth behavior of

$$r_n := \inf\{\, r(M^n) \mid M^n \text{ hyperbolic } n\text{-manifold}\}$$

still lies in the dark, but there are indications coming from the study of arithmetic lattices that the volumes of hyperbolic n-manifolds are increasing with n (see [Ma, p. 258]).

This article is organized as follows. In the first chapter, we collect all necessary informations about regular simplices and their characteristic orthoschemes in spaces X of constant curvature. In chapter two, we give a short account on sphere packings of X and present K. Böröczky's basic theorem about local density bounds of sphere packings of X in terms of regular simplex volumes. With these preparations, we are ready to prove the theorem (see chapter three). In chapter four, we provide elementary upper and lower volume bounds for non-Euclidean regular simplices in terms of their dihedral angles, only. This is of considerable practical use since there are no explicit volume formulae available for non-Euclidean simplices of dimensions bigger than or equal to seven. Finally, in chapter five, we discuss and apply these results to study volume spectra of compact hyperbolic manifolds of dimensions $n \geq 2$ and their minima which, by a result of D. Kazdan and G. Margulis [KM], are always positive.

1. Non-Euclidean regular simplices

Let $n \geq 2$, and denote by X^n either the sphere S^n, the Euclidean space E^n or the hyperbolic space $\overline{H^n} = H^n \cup \partial H^n$ extended by the set ∂H^n of points at infinity. For $P, Q \in X^n$, let $l(PQ) = l_{X^n}(PQ)$ be the length of the geodesic segment from P to Q.
A simplex in X^n is *regular* if its symmetry group operates transitively on the k-dimensional faces ($0 \leq k \leq n-1$). In particular, all edge lengths and dihedral angles are of the same sizes. Moreover, all facial and vertex simplices of a regular simplex are regular (a vertex polytope of a polytope in X^n arises as $(n-1)$-dimensional intersection of a sufficiently small sphere around a vertex with the polytope).
We denote by $S_{reg}(2\alpha)$ the regular simplex of dihedral angle $0 \leq 2\alpha \leq \pi$. $S_{reg}(2\alpha)$ is realizable (see [BH, Satz 1, p. 276])

$$\begin{aligned} &\text{in } S^n && \text{for} && -1 < \cos(2\alpha) < \frac{1}{n} \quad ; \\ &\text{in } E^n && \text{for} && \cos(2\alpha) = \frac{1}{n} \quad ; \\ &\text{in } \overline{H^n} && \text{for} && \frac{1}{n} < \cos(2\alpha) \leq \frac{1}{n-1} \quad . \end{aligned} \tag{1}$$

In the extremal case $\cos(2\alpha_0^n) = \frac{1}{n}$, a *non-Euclidean* regular n-simplex $S_{reg}(2\alpha_0^n)$ is degenerated in dimension and hence of zero n-volume. In $\overline{H^n}$, the condition $\cos(2\alpha_\infty^n) = \frac{1}{n-1}$ characterizes a totally asymptotic or ideal regular n-simplex $S_{reg}(2\alpha_\infty^n)$ all of whose vertices are at infinity. Notice that $\alpha_\infty^n = \alpha_0^{n-1} < \alpha_0^n < \frac{\pi}{4}$.

In Euclidean space, $S_{reg}(2\alpha)$ is determined by its dihedral angle 2α only up to homotheties while, in non-Euclidean space, there is the following relation between angle 2α and edge length $2l$ (see [BH, §6.4, p. 276–270]):

$$\text{In } S^n \quad : \quad \frac{1}{\cos(2\alpha)} = n - 1 + \frac{1}{\cos(2l)} \quad \text{or} \quad \cos(2l) = \frac{\cos(2\alpha)}{1-(n-1)\cos(2\alpha)} \quad ; \tag{2}$$

$$\text{In } H^n \quad : \quad \frac{1}{\cos(2\alpha)} = n - 1 + \frac{1}{\cosh(2l)} \quad \text{or} \quad \cosh(2l) = \frac{\cos(2\alpha)}{1-(n-1)\cos(2\alpha)} \quad . \tag{3}$$

Regular simplices, like all regular polytopes, are most conveniently described by their *characteristic orthoschemes.* An orthoscheme $R \subset X^n$, $n \geq 1$, is a simplex bounded by hyperplanes $H_0, \ldots, H_n$ such that $H_i \perp H_j$ for $|i-j| > 1$. Up to isometry, R is uniquely determined by its dihedral angles $\alpha_i = \angle(H_{i-1}, H_i) \leq \frac{\pi}{2}, 1 \leq i \leq n$. Associated to R is its (linear) scheme or weighted graph $\Sigma(R)$ whose nodes i correspond to H_i. Two nodes are disjoint if their hyperplanes are orthogonal; otherwise, $i-1, i$ are connected by an edge with weight α_i (apart from the exceptional case $\alpha_i = \frac{\pi}{3}$ when the mark is dropped according to the standard notation of Coxeter-Dynkin-Schläfli-Vinberg).
Denote by P_i the vertex of R opposite to the facet $R \cap H_i$. Then, $l(P_0P_i) \leq l(P_0P_j)$ for $1 \leq i < j \leq n$. Discarding in $\Sigma(R)$ one node i, say, together with the edge(s) emanating from it, the remaining graph describes the vertex orthoscheme R_i of R at P_i. All vertex orthoschemes are spherical apart from the hyperbolic situation where P_0, P_n may belong to ∂H^n; therefore R_0, R_n may be Euclidean orthoschemes.

The barycenter C of a regular simplex $S_{reg}(2\alpha) \subset X^n$ is the unique fixpoint under the symmetry group and center of in-sphere and circum-sphere. By drawing successively perpendiculars to lower dimensional faces starting from C, $S_{reg}(2\alpha)$ is decomposed into congruent orthoschemes $R(\alpha)$ with dihedral angles $\alpha, \frac{\pi}{3}, \ldots, \frac{\pi}{3}$. More precisely,

$$S_{reg}(2\alpha) = (n+1)!\, R(\alpha) \quad , \quad \text{where} \tag{4}$$

$$\Sigma(R(\alpha)) \quad : \quad \underset{0}{\circ} \overset{\alpha}{\text{---}} \circ - \circ - \cdots - \circ - \circ - \underset{n}{\circ} \quad .$$

Denote by $C =: P_0, \ldots, P_n$ the vertices of $R(\alpha)$. Then, the in-radius ${}_1R$ of $S_{reg}(2\alpha) \subset X^n$ equals P_0P_1. The circum-radius ${}_nR$ of $S_{reg}(2\alpha)$ is given by the longest hypotenuse P_0P_n of $R(\alpha)$.

LEMMA 1.

In-radius ${}_1R$ *and circum-radius* ${}_nR$ *of a non-Euclidean regular simplex* $S_{reg}(2\alpha)$ *of edge length* $2l$ *are given by:*

$$\text{In } S^n \quad : \quad \cos({}_1R) = \sqrt{\frac{2n}{n+1}}\,\cos(\alpha) \quad \text{and} \quad \sin({}_nR) = \sqrt{\frac{2n}{n+1}}\,\sin(l)\ ; \tag{5}$$

$$\text{In } H^n \quad : \quad \cosh({}_1R) = \sqrt{\frac{2n}{n+1}}\,\cos(\alpha) \quad \text{and} \quad \sinh({}_nR) = \sqrt{\frac{2n}{n+1}}\,\sinh(l)\ . \tag{6}$$

Proof:

Decompose $S_{reg}(2\alpha)$ into orthoschemes congruent to $R(\alpha)$ with vertices $C = P_0, \ldots, P_n$. Observe that the vertex orthoscheme R_0 of $R(\alpha)$ at P_0 is given by the scheme

$$\Sigma_0 = A_n \quad : \quad \circ - \circ - \cdots - \circ - \circ \quad ;$$

Σ_0 is the characteristic orthoscheme of $S_{reg}(\frac{2\pi}{3}) \subset S^{n-1}$ of edge length $2l_0$ such that (see (2))

$$\sin(l_0) = \sqrt{\frac{n+1}{2n}} \quad .$$

(i) The in-radius ${}_1R = P_0P_1$ of $S_{reg}(2\alpha)$ appears as cathetus in the right-angled triangle $P_0P_1P_2$ whose angle at P_2 is α; this follows from the definition of α as $\alpha = \angle(H_0, H_1)$ formed by the facets $P_1P_2\cdots P_n$ and $P_0P_2\cdots P_n$ sitting at the apex orthoscheme $P_2\cdots P_n$, and from the property $P_0P_1P_2 \perp P_2\cdots P_n$. To determine the angle $\angle P_1P_0P_2$ at P_0 in $P_0P_1P_2$, we make use of the fact that $\Sigma(R_0) = \Sigma_0$. If Q_i is the vertex of R_0 on the edge P_0P_i of R ($1 \le i \le n$), then

$$l_0 = \arcsin(\sqrt{\frac{n+1}{2n}}) = l(Q_{n-1}Q_n) = l(Q_1Q_2) \quad ;$$

the second equality is a consequence of the reflection symmetry of the graph Σ_0. Since $\angle P_1P_0P_2 = l(Q_1Q_2) = l_0$, we obtain

$$\text{in } S^n \quad \text{resp.} \quad \text{in } H^n \quad : \quad \frac{\cos(\alpha)}{\sin(l_0)} = \cos(l(P_0P_1)) \quad \text{resp.} \quad \cosh(l(P_0P_1)) \quad .$$

Therefore, $\cos({}_1R)$ resp. $\cosh({}_1R)$ equals $\sqrt{\frac{2n}{n+1}}\cos(\alpha)$ as required.

(ii) To determine the circum-radius ${}_nR = P_0P_n$ of $S_{reg}(2\alpha)$, we look at the right-angled triangle $P_0P_{n-1}P_n$ with hypotenuse ${}_nR$ and cathetus $P_{n-1}P_n$ of length l opposite to the angle $l(Q_{n-1}Q_n) = \arcsin(\sqrt{\frac{n+1}{2n}})$. This yields

$$\begin{aligned} \text{in } S^n \quad &: \quad \sin({}_nR) = \frac{\sin(l)}{\sin(l(Q_{n-1}Q_n))} = \sqrt{\frac{2n}{n+1}}\sin(l) \quad ; \\ \text{in } H^n \quad &: \quad \sinh({}_nR) = \sqrt{\frac{2n}{n+1}}\sinh(l) \quad . \end{aligned}$$

Q.E.D.

2. Sphere packings

Let $\mathcal{B}$ be a packing of X^n, $n \ge 2$, with non-overlapping balls $B(r)$ of radius r (for $X^n = S^n$, assume that $r < \frac{\pi}{4}$). We are interested in bounds for the "packing density". In general, for two non-empty sets $S, T \subset X^n$, the density $d(S,T)$ of S in T is given by

$$d(S,T) := \frac{\mathrm{vol}_n(S \cap T)}{\mathrm{vol}_n(T)} < 1 \quad .$$

The density of a ball in an orthoscheme satisfies the following monotonicity property with respect to its edge lengths (see [B, Lemma 10, p. 256]):

LEMMA 2.

In X^n, let $R = P_0P_1\cdots P_n$ and $R' = P_0P_1'\cdots P_n'$ be two orthoschemes with $l(P_0P_i) \geq l(P_0P_i')$ for $i = 1,\ldots,n$. Denote by B a ball centered at P_0 such that B is disjoint to the facets $P_1\cdots P_n$ and $P_1'\cdots P_n'$. Then, $d(B,R) \leq d(B,R')$, and equality holds precisely for $l(P_0P_i) = l(P_0P_i')$ for $i = 1,\ldots,n$.

Associate to each ball $B = B(r)$ of the packing $\mathcal{B}$ its *Dirichlet-Voronoi-cell*

$$D = D(B,\mathcal{B}) = \{\, p \in X^n \mid \mathrm{dist}(p,B) \leq \mathrm{dist}(p,B')\,,\ \forall B' \subset \mathcal{B}\,\} \quad .$$

Then, the *local density* of B with respect to $\mathcal{B}$ is defined by

$$ld(B,\mathcal{B}) := d(B,D) = \frac{\mathrm{vol}_n(B)}{\mathrm{vol}_n(D)} \quad . \tag{7}$$

Finally, denote by $d_n(r)$ the density of $n+1$ balls $B(r)$ of radius r mutually touching one another with respect to the regular simplex S_{reg} of edge length $2r$ spanned by their centers, that is:

$$d_n(r) = (n+1)\,\frac{\mathrm{vol}_n(B(r)\cap S_{reg})}{\mathrm{vol}_n(S_{reg})} < 1 \quad .$$

Then, K. Böröczky sen. proved the following local density bound formerly conjectured by H.S.M. Coxeter and L. Fejes Tóth (see [B, Theorem 1, p. 243]):

THEOREM 1 (K. Böröczky).

Let $\mathcal{B}$ be a packing of X^n with balls $B = B(r)$ of radius r (for $X^n = S^n$, suppose that $r < \frac{\pi}{4}$). Then, $ld(B,\mathcal{B}) \leq d_n(r)$.

Remark:

For $n = 3$, this result was settled by K. Böröczky sen. already in 1964 (see [BF]); in addition, A. Florian showed that $d_3(r)$ is monotonely increasing with respect to r. The limiting value

$$d := \lim_{r\to\infty} d_3(r) = \left(1 + \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} - + \cdots\right)^{-1} \simeq 0.85328$$

is achieved by the local density of the in-balls of the regular honeycomb $\{6,3,3\}$ forming a horoball packing of $\overline{H^3}$.

Of interest is the behavior of the upper bound $d_n(r)$ with increasing dimension.

LEMMA 3.

For $r > 0$, $\quad d_n(r) < d_{n-1}(r)$.

Proof:

We follow an idea of K. Böröczky sen. in order to compare densities of a ball with respect to simplices of different dimensions. Let $R = P_0 \cdots P_k$ denote a k-orthoscheme. R can be considered as degeneration of an n-simplex $\overline{R} = P_0 \cdots P_k P_{k+1} \cdots P_n$ for $P_{k+i} \to P_k$ for $i = 1, \ldots, n-k$.

The *limiting density* $d(B, P_k, R)$ of an n-ball B with respect to R is then defined by

$$d(B, P_k, R) := \lim_{\substack{P_{k+i} \to P_k \\ 1 \le i \le n-k}} d(B, \overline{R}) \quad .$$

Now, dissect the regular n-simplex S_{reg} of edge length $2r$ and with center $C =: P_0$ appearing in the definition of $d_n(r)$ into orthoschemes congruent to $R = P_0 \cdots P_n$. It is clear from the definition that

$$d_n(r) = d(B^n_{P_n}(r), R) = \frac{\mathrm{vol}_n(B^n_{P_n}(r) \cap R)}{\mathrm{vol}_n(R)} \quad , \tag{8}$$

where $B^n_{P_n}(r)$ is the n-ball of radius r centered at P_n.

Together with S_{reg} all its facets S'_{reg} of edge length $2r$ are subdivided into $(n-1)$-orthoschemes congruent to $R' = P_1 \cdots P_n$. Extend each $R' = P_1 \cdots P_n$ in $R = P_0 P_1 \cdots P_n$ to an n-orthoscheme $\overline{R} = \overline{P}_0 P_1 \cdots P_n$ with $\overline{P}_0$ on $P_0 P_1$ in between P_0 and P_1. Since $l(P_0 P_n) \ge l(\overline{P}_0 P_n) > l(P_1 P_n)$, Lemma 2.1 implies that the density of the ball $B = B^n_{P_n}(r)$ in $\overline{R}$ dominates the one with respect to R, that is,

$$d(B, R) \le d(B, \overline{R}) \quad .$$

On the other hand side,

$$d_{n-1}(r) = d(B, P_1, R') = \lim_{\overline{P}_0 \to P_1} d(B, \overline{R}) \quad .$$

Therefore,

$$d_n(r) = d(B, R) \le d(B, \overline{R}) \le d(B^{n-1}_{P_n}(r), R') = d_{n-1}(r) \quad .$$

Q.E.D.

Remark:

It is still an unsolved problem whether also for $n > 3$ the density $d_n(r)$ is strictly monotonely increasing with r. Florian's proof for $n = 3$ made use of the *explicit* form of $d_3(r)$ with respect to r. We hope to come back to this question elsewhere.

3. A lower volume bound for hyperbolic manifolds

Denote by

$$\Omega_n = \mathrm{vol}_n(S^n) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$$

the volume of the sphere S^n with its canonical metric of curvature $+1$. Let M be a hyperbolic n-manifold, that is, a complete, connected Riemannian n-manifold of constant curvature -1 and hence of the form $M = H^n/\Gamma$, where Γ is a discrete, torsion-free Möbius group. Denote by

$$r(x, M) = \sup\{\, r > 0 \,|\, \exp_x : T_xM \to M \text{ injective on the ball of radius } r \text{ around zero} \,\}$$

the injectivity radius of M at the point x, and let

$$r(M) = \sup\{\, r(x, M) \mid x \in M \,\}$$

be the *in-radius* of M, that is, the radius of the largest ball embedded in M^n. With the preparations of chapters one and two, we are ready to prove the lower volume bound for M announced in the Introduction, that is:

THEOREM 2.

For $n \geq 2$, let M be a compact hyperbolic n-manifold with in-radius $r(M)$. Then,

$$\mathrm{vol}_n(M) \geq \frac{\Omega_{n-1}}{n+1} \cdot \frac{\mathrm{vol}_n(S_{reg}(2\alpha))}{\mathrm{vol}_{n-1}(s_{reg}(2\alpha))} \quad , \tag{9}$$

where $S_{reg}(2\alpha) \subset H^n$ denotes the regular simplex with dihedral angle 2α and edge length $2r(M)$ related by

$$\frac{1}{n} < \cos(2\alpha) = \frac{\cosh(2r(M))}{1 + (n-1)\cosh(2r(M))} < \frac{1}{n-1} \quad , \tag{10}$$

and where $s_{reg}(2\alpha)$ is a spherical vertex simplex of $S_{reg}(2\alpha)$.

Remark:

For $n = 3$, the result of Theorem 2 is due to R. Meyerhoff (see [Me, p. 277] and chapter 5).

By decomposing $S_{reg}(2\alpha)$ and $s_{reg}(2\alpha)$ into their characteristic orthoschemes $R(\alpha)$ and $r(\alpha)$ (see (4)), that is,

$$S_{reg}(2\alpha) = (n+1)!\, R(\alpha) \quad , \quad s_{reg}(2\alpha) = n!\, r(\alpha) \quad ,$$

one obtains the following equivalent of Theorem 2:

COROLLARY 1.

For $n \geq 2$, let M be a compact hyperbolic n-manifold with in-radius $r(M)$. Then,

$$\mathrm{vol}_n(M) \geq \Omega_{n-1} \frac{\mathrm{vol}_n(R(\alpha))}{\mathrm{vol}_{n-1}(r(\alpha))} \quad , \quad \text{where} \quad \Sigma(R(\alpha)) \,:\, \underset{0}{\circ} \overset{\alpha}{\text{---}} \circ - \circ - \cdots - \circ - \circ - \underset{n}{\circ} \tag{11}$$

denotes the characteristic orthoscheme of $S_{reg}(2\alpha)$ with $l(P_0P_1) = r(M)$ and vertex figure $r(\alpha)$.

Since $n!\,r(\alpha) = s_{reg}(2\alpha) \subset S^{n-1}$, we finally obtain

COROLLARY 2.

For $n \geq 2$, let M be a compact hyperbolic n-manifold with in-radius $r(M)$. Then,

$$\mathrm{vol}_n(M) \geq \frac{1}{n+1}\,\mathrm{vol}_n(S_{reg}(2\alpha)) = n!\,\mathrm{vol}_n(R(\alpha)) \quad , \tag{12}$$

where $S_{reg}(2\alpha)$ and $R(\alpha)$ are as above.

Proof of Theorem 2:
By assumption, $M = H^n/\Gamma$ contains a ball isometric to a ball of radius $r := r(M)$ in H^n. The lifts of this ball to the universal covering space H^n yield a packing $\mathcal{B}$ with balls of radius r. Each Dirichlet-Voronoi-cell $D = D(B, \mathcal{B})$ is a fundamental domain for the action of Γ on H^n. Therefore, $\mathrm{vol}_n(D) = \mathrm{vol}_n(M)$. Consider the local density (see (7))

$$ld(B, \mathcal{B}) = d(B, D) = \frac{\mathrm{vol}_n(B)}{\mathrm{vol}_n(D)}$$

of a ball B with respect to $\mathcal{B}$. By Böröczky's theorem (see Theorem 1, **2.**),

$$ld(B, \mathcal{B}) \leq d_n(r) = (n+1)\,\frac{\mathrm{vol}_n(B \cap S_{reg}(2\alpha))}{\mathrm{vol}_n(S_{reg}(2\alpha))} \quad ,$$

where

$$\cos(2\alpha) = \frac{\cosh(2r)}{1 + (n-1)\cosh(2r)} \quad .$$

This implies that

$$\frac{\mathrm{vol}_n(B)}{\mathrm{vol}_n(D)} \leq (n+1) \cdot \frac{\mathrm{vol}_{n-1}(s_{reg}(2\alpha))}{\Omega_{n-1}}\,\mathrm{vol}_n(B) \cdot \frac{1}{\mathrm{vol}_n(S_{reg}(2\alpha))} \quad ,$$

where $s_{reg}(2\alpha)$ denotes a spherical vertex figure of $S_{reg}(2\alpha)$, and where

$$\frac{\mathrm{vol}_{n-1}(s_{reg}(2\alpha))}{\Omega_{n-1}}\,\mathrm{vol}_n(B)$$

equals the volume of the ball sector of B cut out by $S_{reg}(2\alpha)$. Finally, we deduce

$$\mathrm{vol}_n(M) \geq \frac{\Omega_{n-1}}{n+1} \cdot \frac{\mathrm{vol}_n(S_{reg}(2\alpha))}{\mathrm{vol}_{n-1}(s_{reg}(2\alpha))} \quad ,$$

as required.

Q.E.D.

Remark:

The proof of Theorem 2 follows from the inequality

$$\mathrm{vol}_n(M) \geq \frac{\mathrm{vol}_n(B(r(M))}{d_n(r(M))} \tag{13}$$

based on Böröczky's result that the local sphere packing density $ld(B(r),\mathcal{B})$ of a ball $B(r)$ with respect to $\mathcal{B}$ is bounded from above by $d_n(r) < 1$. Hence, (13) is an improvement of the often used coarse estimate

$$\mathrm{vol}_n(M) \geq \mathrm{vol}_n(B(r(M))) = \Omega_{n-1} \int_0^{r(M)} \sinh^{n-1}(x)\,dx > \frac{\Omega_{n-1}}{n}\,(r(M))^n \quad .$$

4. Bounds for the regular simplex volume

By Theorem 2, we can estimate the volume of a compact hyperbolic n-manifold M with in-radius $r(M)$ in terms of the volume of a hyperbolic regular simplex $S_{reg}(2\alpha)$ in H^n of edge length $2r(M)$ and dihedral angle 2α. However, explicit volume formulae for non-Euclidean n-simplices are known only up to dimension $n = 6$ (see [K1] for the even dimensional volume problem and [K2], [K3] for the cases $n = 3, 5$); for $3 \leq n \leq 6$, these formulae are complicated expressions involving polylogarithms of orders less or equal to $\left[\frac{n+1}{2}\right]$.
The aim is therefore to *approximate* $\mathrm{vol}_n(S_{reg}(2\alpha))$ in terms of n and α. For this, we make use of Schläfli's fundamental formula for the volume differential (see [K1, **2.2**, p. 199]):

THEOREM 3 (L. Schläfli).

Let $n \geq 2$, and denote by $S \subset S^n$ a family of simplices with dihedral angles $\alpha_{jk} = \angle(S_j, S_k)$ formed by the facets S_j, S_k at $S_{jk} = S_j \cap S_k$ $(1 \leq j < k \leq n)$. Then,

$$d\,\mathrm{vol}_n(S) = \frac{1}{n-1} \sum_{1\leq j<k\leq n} \mathrm{vol}_{n-2}(S_{jk})\,d\alpha_{jk} \quad , \quad \mathrm{vol}_0 := 1 \quad . \tag{14}$$

Formula (14) is, up to the sign, also valid for *hyperbolic* n-simplices. An elegant proof for both curvature cases was given by H. Kneser (see [K1, **2.2**, p. 199]).

Consider $S_{reg}(2\alpha) \subset X^n$, $X^n \neq E^n$, of edge length $2l$ according to (2) and (3), together with its $(n+1)!$ characteristic orthoschemes $R(\alpha)$ given by

$$\Sigma(R(\alpha)) \;:\; \underset{0}{\circ} \overset{\alpha}{\text{---}} \circ \text{---} \circ \text{---} \cdots \text{---} \circ \text{---} \circ \text{---} \underset{n}{\circ} \quad .$$

By Schläfli's formula (14), we deduce for $\mathrm{vol}_n(R(\alpha) = \frac{1}{(n+1)!}\,\mathrm{vol}_n(S_{reg}(2\alpha))$, $n \geq 2$, that

(A1) in S^n : $f_n(\alpha) := \mathrm{vol}_n(R(\alpha))$ is strictly monotonely *increasing* with α;

(A2) in H^n : $F_n(\alpha) := \mathrm{vol}_n(R(\alpha))$ is strictly monotonely *decreasing* with α.

Theorem 3 allows to write the regular simplex volume as *single* integral. Denote by $0 \leq \alpha_0^n < \frac{\pi}{2}$ the angle with $\cos^2(\alpha_0^n) = \frac{n+1}{2n}$, which is such that $\mathrm{vol}_n(S_{reg}(2\alpha_0^n)) = 0$ (see (1)).

Then, for $n = 2$, the integration of (14) starting from α_0^2 gives immediately the well-known excess- and defect-formulae for the area of $S_{reg}(2\alpha)$ and $R(\alpha)$ in S^2 and H^2.

From now on, let $n \geq 3$. The face of $R(\alpha) \subset X^n$ associated to $\alpha = \angle(H_0, H_1)$ is the orthoscheme $R(\beta(\alpha)) = R(\alpha) \cap H_0 \cap H_1 \subset X^{n-2}$ with graph

$$\Sigma(R(\beta(\alpha))) \;:\; \underset{2}{\circ} \overset{\beta(\alpha)}{\text{——}} \circ - \circ - \cdots - \circ - \circ - \underset{n}{\circ} \quad .$$

Here, the *apex angle* $\beta(\alpha)$ of $R(\alpha)$ can be seen in $R(\alpha) = P_0P_1P_2P_3 \subset X^3$ as length $l(P_2P_3)$, and for $R(\alpha) \subset X^n$, $n \geq 4$, as angle $\angle P_2P_4P_3$. Therefore, it is analytically expressible by (see [BH, Hilfssatz 1, p. 276])

$$0 \leq \beta(\alpha) < \infty \quad \text{with} \quad \cosh(\beta(\alpha)) = \frac{\sin(\alpha)}{\sqrt{4\sin^2(\alpha) - 1}} \quad \text{in} \quad H^3\,; \tag{15}$$

$$0 \leq \beta(\alpha) \leq \frac{\pi}{2} \quad \text{with} \quad \cos(\beta(\alpha)) = \frac{\sin(\alpha)}{\sqrt{4\sin^2(\alpha) - 1}} \quad \text{in} \quad X^n \neq H^3\,,\, n \geq 3\,. \tag{16}$$

From this, it follows that $\beta(\alpha_0^n) = \alpha_0^{n-2}$, and that therefore the orthoscheme $R(\beta(\alpha_0^n)) \subset X^{n-2}$ also degenerates in dimension. Moreover, by putting for the angle α of $R(\alpha) \subset X^n$ and its apex angle $\beta(\alpha)$

$$\beta^0(\alpha) := \alpha \quad , \quad \beta^1(\alpha) := \beta(\alpha) \quad ,$$

we find for the k-th iterative $\beta(\beta^{k-1}(\alpha)) \in [0, \frac{\pi}{2}]$ that

$$\beta^k(\alpha) = \arccos\sqrt{\frac{k-1-(2k-3)b(\alpha)}{2k-1-4(k-1)b(\alpha)}} \quad , \quad \text{for } k = 2, \ldots, [\frac{n}{2}] - 1 \quad , \tag{17}$$

and, for $k = [\frac{n}{2}]$,

$$\beta^{[\frac{n}{2}]}(\alpha) = \begin{cases} \arccos\sqrt{\dfrac{[\frac{n}{2}]-1-(2[\frac{n}{2}]-3)b(\alpha)}{2[\frac{n}{2}]-1-4([\frac{n}{2}]-1)b(\alpha)}} \in [0, \dfrac{\pi}{2}] & \text{for } R(\alpha) \subset S^n \quad ; \\ \operatorname{arcosh}\sqrt{\dfrac{[\frac{n}{2}]-1-(2[\frac{n}{2}]-3)b(\alpha)}{2[\frac{n}{2}]-1-4([\frac{n}{2}]-1)b(\alpha)}} \in [0, \infty) & \text{for } R(\alpha) \subset H^n \quad , \end{cases} \tag{18}$$

whereby

$$b(\alpha) := \frac{\sin^2(\alpha)}{4\sin^2(\alpha)-1} \quad .$$

Finally, by (14), we are allowed to write

(B1) for $R(\alpha) \subset S^n$: $f_n(\alpha) = \frac{1}{n-1} \int\limits_{\alpha_0^n}^{\alpha} f_{n-2}(\beta(x))\, dx$;

(B2) for $R(\alpha) \subset H^n$: $F_n(\alpha) = \frac{1}{1-n} \int\limits_{\alpha_0^n}^{\alpha} F_{n-2}(\beta(x))\, dx = \frac{1}{n-1} \int\limits_{\alpha}^{\alpha_0^n} F_{n-2}(\beta(x))\, dx$,

where $\cos(2\alpha_0^n) = \frac{1}{n}$ and $\beta(x)$ is defined according to (15) and (16). For the sake of completeness, we add:

(C) The volume of a non-Euclidean regular simplex $S_{reg}(2\alpha)$ is a strictly convex function in α; more precisely,

$$F_n''(\alpha) > 0 \quad \text{for } n > 3 \quad ; \quad f_n''(\alpha) > 0 \quad \text{for } n \geq 3 \quad .$$

This follows from (B1) and (B2) using the fact that the apex angle $\beta(\alpha)$ satisfying (16) has $\beta'(\alpha) > 0$.

By Lemma 1, we can estimate regular simplex volume in terms of the volumes of in-sphere and circum-sphere; for example, for a regular simplex of edge length $2l$, we obtain

$$\text{in } S^n \quad : \quad \mathrm{vol}_n(S_{reg}(2\alpha)) \leq \Omega_{n-1} \int\limits_0^{{}_nR} \sin^{n-1}(x)\, dx \quad , \tag{19}$$

$$\text{where} \qquad {}_nR = \arcsin\left(\sqrt{\frac{2n}{n+1}}\, \sin(l)\right) \quad ;$$

$$\text{in } H^n \quad : \quad \mathrm{vol}_n(S_{reg}(2\alpha)) \geq \Omega_{n-1} \int\limits_0^{{}_1R} \sinh^{n-1}(x)\, dx \quad , \tag{20}$$

$$\text{where} \qquad {}_1R = \operatorname{arcosh}\left(\sqrt{\frac{2n}{n+1}}\, \cos(\alpha)\right) \quad .$$

The next two lemmata contain *elementary* upper and lower bounds for the volumes of a regular simplex and its characteristic orthoscheme which are very useful for computations in higher dimensions. We present only the inequalities needed to simplify the volume bound for hyperbolic manifolds given by Theorem 2, while the remaining cases follow after obvious sign modifications.

LEMMA 4.

Let $n \geq 3$ *and* $\alpha_0^k = \arccos\sqrt{\frac{k+1}{2k}} \in [0, \frac{\pi}{2}]$ *for* $2 \leq k \leq n$. *Denote by* $\beta^k(x)$, $k = 0, \ldots, [\frac{n}{2}]$, *the angle defined according to (17) and (18). Then, the volume* $f_n(\alpha)$ *of the orthoscheme* $R(\alpha) \subset S^n$ *with graph* $\Sigma(R(\alpha))$: $\underset{0}{\circ} \overset{\alpha}{—} \circ — \circ — \cdots — \circ — \circ — \underset{n}{\circ}$ *is bounded from above by*

$$f_n(\alpha) \leq \prod_{k=0}^{[\frac{n}{2}]-1} \frac{\beta^k(\alpha) - \alpha_0^{n-2k}}{n-(2k+1)} \cdot \varphi_n(\alpha) \ , \ \text{where } \varphi_n(\alpha) = \begin{cases} \beta^{[\frac{n}{2}]}(\alpha) \ , & \text{for } n \text{ odd ;} \\ 1 \ , & \text{for } n \text{ even .} \end{cases} \tag{21}$$

Proof:
By (B1), we can write

$$f_n(\alpha) = \frac{1}{n-1} \int_{\alpha_0^n}^{\alpha} f_{n-2}(\beta(x))\, dx \quad .$$

Since $\beta(y)$ and $f_n(y)$ are monotonely increasing by (16) and (A1), it follows that

$$f_n(\alpha) \leq \frac{1}{n-1}(\alpha - \alpha_0^n) \cdot \max_{x \in [\alpha_0^n, \alpha]} f_{n-2}(\beta(x)) = \frac{1}{n-1}(\alpha - \alpha_0^n)\, f_{n-2}(\beta(\alpha)) \quad .$$

By iteration, we obtain (21).

Q.E.D.

The quality of the volume estimate (21) can be read off by comparison, for example, with Schläfli's *exact* volume formulae for the spherical n-orthoschemes $R(\frac{\pi}{3})$ and $R(\frac{\pi}{4})$ with graphs A_{n+1} and B_{n+1}; he showed that (see [K1, (10) and (11), p. 199])

$$f_n(\frac{\pi}{3}) = \frac{\Omega_n}{(n+2)!} \quad \text{and} \quad f_n(\frac{\pi}{4}) = \frac{\Omega_n}{2^{n+1}(n+1)!} \quad .$$

Let $m(x,y) := \frac{x+y}{2}$, and define inductively the functions

$$\begin{aligned} &\mu^0(\alpha) := \alpha \quad , \quad \mu^1(\alpha) := m(\alpha_0^n, \alpha) = \frac{\alpha_0^n + \alpha}{2} \quad \text{and} \\ &\mu^k(\alpha) := m(\beta^{k-1}(\alpha_0^n), \beta(\mu^{k-1}(\alpha))) \quad \text{for} \quad k \geq 2 \quad . \end{aligned} \tag{22}$$

This means,

$$\mu^2(\alpha) = \frac{\beta(\alpha_0^n) + \beta(\frac{\alpha_0^n + \alpha}{2})}{2} \quad , \quad \mu^3(\alpha) = \frac{\beta^2(\alpha_0^n) + \beta\left(\frac{\beta(\alpha_0^n) + \beta(\frac{\alpha_0^n + \alpha}{2})}{2}\right)}{2} \quad , \quad \ldots \quad .$$

LEMMA 5.

Let $n \geq 3$ *and* $\alpha_0^k = \arccos\sqrt{\frac{k+1}{2k}} \in [0, \frac{\pi}{2}]$ *for* $2 \leq k \leq n$. *Denote by* $\beta^k(x)$, $k = 0, \ldots, [\frac{n}{2}]$, *the angle defined according to (17) and (18). Then, the volume* $F_n(\alpha)$ *of the orthoscheme* $R(\alpha) \subset H^n$ *with graph* $\Sigma(R(\alpha))$: $\underset{0}{\circ} \overset{\alpha}{—} \circ - \circ - \cdots - \circ - \circ - \underset{n}{\circ}$ *is bounded from below by*

$$F_n(\alpha) > \prod_{k=0}^{[\frac{n}{2}]-1} \frac{\alpha_0^{n-2k} - \mu^{k+1}(\alpha)}{n-(2k+1)} \cdot \psi_n(\alpha) \quad , \quad \text{where} \tag{23}$$

$$\psi_n(\alpha) = \begin{cases} \beta^{[\frac{n}{2}]}(\mu^{[\frac{n}{2}]}(\alpha)) \,, & \text{for } n \text{ odd} \,; \\ 1 \,, & \text{for } n \text{ even} \,. \end{cases}$$

Proof:
By (B2), we have

$$F_n(\alpha) = \frac{1}{n-1} \int_{\alpha}^{\alpha_0^n} F_{n-2}(\beta(x))\, dx \quad ,$$

where the integrand is positive on $[\alpha, \alpha_0^n)$, that is,

$$F_n(\alpha) > \frac{1}{n-1} \int_{\alpha}^{\frac{\alpha_0^n+\alpha}{2}} F_{n-2}(\beta(x))\, dx \quad .$$

For $n = 3$ and by (15), $\beta(y) = F_1(\beta(y))$ is strictly monotonely decreasing, while for $n > 3$ and by (16) and (A2), $\beta(y)$ (resp. $F_n(y)$) is strictly monotonely increasing (resp. decreasing). This implies that

$$F_n(\alpha) > \frac{1}{n-1} \frac{\alpha_0^n - \alpha}{2} \cdot \min_{x \in [\alpha, \frac{\alpha_0^n+\alpha}{2}]} F_{n-2}(\beta(x)) = \frac{\alpha_0^n - \mu^1(\alpha)}{n-1} F_{n-2}(\beta(\frac{\alpha_0^n+\alpha}{2})) \quad .$$

Again, by iteration, the assertion follows.

Q.E.D.

For example, the volume of the hyperbolic 4-orthoscheme $R(\frac{\pi}{5})$ is estimated by

$$F_4(\frac{\pi}{5}) > 0.00021 \quad ,$$

while its *exact* value is (see [K1, p. 206])

$$F_4(\frac{\pi}{5}) = \frac{\pi^2}{10,800} \simeq 0.00091 \quad .$$

5. Volume estimates for hyperbolic manifolds

The aim is to find a *universal* lower volume bound for compact hyperbolic manifolds M^n of dimension $n \geq 2$ and therefore an estimate for the minimum value in the *n-th volume spectrum*

$$\mathbf{Vol}_n := \{\, \mathrm{vol}_n(M^n) \mid M^n \text{ compact hyperbolic n-manifold} \,\} \subset \mathbf{R}_+ \quad .$$

For even dimensions n, the well-known theorem of Gauss-Bonnet says that

$$\mathrm{vol}_n(M^n) = (-1)^{\frac{n}{2}} \, \frac{\Omega_n}{2} \, \chi(M^n) \quad .$$

In fact, for compact Riemann surfaces M_g^2 of genus $g > 1$ the volume spectrum equals

$$\mathbf{Vol}_2 = 2\pi \, \mathbf{N} \quad . \tag{24}$$

For $n = 4$, we obtain

$$\mathrm{vol}_4(M^4) \geq \frac{4\pi^2}{3} \quad . \tag{25}$$

Moreover, J. Ratcliffe and S. Tschantz proved the existence of a *non-compact* hyperbolic 4-manifold with Euler characteristic one and with positive first Betti number (see [RT, p. 2]). Therefore,

$$\{\, \mathrm{vol}_4(M^4) \mid M^4 \text{ hyperbolic 4-manifold} \,\} \;=\; \frac{4\pi^2}{3} \, \mathbf{N} \quad . \tag{26}$$

Let $n = 3$. R. Meyerhoff found the bound (see [Me, p. 277])

$$\mathrm{vol}_3(M^3) > 0.00082 \quad . \tag{27}$$

He obtained (27) by constructing embedded tubular neighborhoods around short geodesics of lengths l less than $l_0 \simeq 0.10695$, whose volumes increase with decreasing l. Since M^3 must have either an embedded ball of radius r or a geodesic of length less than $2r$, one obtains for the in-radius of M^3 that

$$r(M^3) \geq \frac{l_0}{2} \simeq 0.05347 \quad .$$

Finally, Theorem 2 for the in-radius $\frac{l_0}{2}$ gives the volume bound 0.00082 which is *bigger* than both, the volume of the hyperbolic ball of radius $\frac{l_0}{2}$ and the volume of the solid tube around a geodesic of length l_0. Meyerhoff's tube construction is based on Jørgensen's trace inequality for a non-elementary discrete two generator group G of Möbius-transformations of $P_1(\mathbf{C})$. By a very detailed analysis of Jørgensen's trace inequality in terms of three different norms on G, F. Gehring and G. Martin improved Meyerhoff's method and results, and they obtained (see [GM, Corollaries 7.20 and 7.21, p. 73–74])

$$r(M^3) \geq r_0(M^3) := 0.05725 \quad \text{and} \quad \mathrm{vol}_3(M^3) \geq 0.00115 \quad . \tag{28}$$

For $n \geq 3$ arbitrary, G. Martin generalized the inequality of Jørgensen to *all* dimensions n and derived the lower bounds for in-radius and volume of a hyperbolic n-manifold M^n (see [M, Corollaries 3.6 and 3.7, p. 263])

$$r(M^n) \geq \frac{1}{2 \cdot 9^{2+[\frac{n}{2}]}} \quad \text{and} \quad \mathrm{vol}_n(M^n) > \Omega_{n-1}\left(\frac{1}{5 \cdot 9^{1+[\frac{n}{2}]}}\right)^n \quad . \tag{29}$$

Unfortunately, S. Friedland and S. Hersonsky discovered an error in [Ma, Corollary 3.3, p. 261] and corrected Martin's in-radius estimate as follows (see [FH, Theorem 4.6, p. 608]):

$$r(M^n) \geq \frac{0.0025}{17^{[\frac{n}{2}]}} \quad . \tag{30}$$

While Martin's estimate (29) improved the earlier result of P. Buser and H. Karcher (see [BK, Proposition 2.5.3]) saying that

$$r(M^n) \geq \frac{1}{4^{n+3}} \tag{31}$$

for compact Riemannian manifolds M^n with sectional curvature $-1 \leq K < 0$, the corrected bound (30) of Friedland-Hersonsky is worse than (31) for all n.

By Theorem 2 and its Corollaries 1 and 2, the volume of a compact hyperbolic n-manifold M with in-radius $r := r(M)$ is bounded from below by expressions involving the volume of the hyperbolic regular n-simplex of edge length $2r$. For example,

$$\mathrm{vol}_n(M) \geq \Omega_{n-1}\,\frac{\mathrm{vol}_n(R(\alpha))}{\mathrm{vol}_{n-1}(r(\alpha))} \quad , \quad \text{where} \quad \Sigma(R(\alpha)) \ : \ \underset{0}{\circ}\overset{\alpha}{—}\circ—\circ—\cdots—\circ—\circ—\underset{n}{\circ} \tag{32}$$

is the characteristic orthoscheme of the hyperbolic regular n-simplex $S_{reg}(2\alpha)$ of dihedral angle 2α and edge length $2r$ related by

$$\frac{1}{n} < \cos(2\alpha) = \frac{\cosh(2r)}{1 + (n-1)\cosh(2r)} < \frac{1}{n-1} \quad .$$

Here, $r(\alpha)$ denotes the vertex orthoscheme of $R(\alpha)$ at P_n. Recall that (32) is equivalent to (see (13))

$$\mathrm{vol}_n(M) \geq \frac{\mathrm{vol}_n(B(r))}{d_n(r)} \quad ,$$

where $d_n(r) < 1$ is the density of $n+1$ balls of radius r centered at the vertices of $S_{reg}(2\alpha)$ of edge length $2r$. The estimate (32) becomes meaningful away from infinitesimally small r; that is, it gains importance when compared to the Euclidean ball volume bound

$$\mathrm{vol}_n(M^n) > \frac{\Omega_{n-1}}{n}\, r^n$$

the bigger the in-radius $r = r(M^n)$ becomes. On the other hand side, with increasing dimension n, the lower bounds (30) and (31) for $r(M^n)$ of Friedland-Hersonsky and Buser-Karcher tend rapidly to zero. However, it is not clear at all whether this is the true growth behavior of

$$r_n := \inf \{ r(M^n) \mid M^n \text{ hyperbolic } n\text{-manifold} \} \quad ,$$

in particular, because there are indications that the minimum volume of all hyperbolic n-manifolds *increases* with the dimension n (see [Ma, p. 258]). One indication in this direction is also provided by $d_n(r) < d_{n-1}(r)$ for $n \geq 3$ and $r > 0$ (see Lemma 3).

We finish with an illustrative example for $n = 5$. By Buser-Karcher's estimate (31), the in-radius for a compact hyperbolic 5-manifold M^5 satisfies

$$r(M^5) \geq r_0 := \frac{1}{4^8} \simeq 1.52588 \cdot 10^{-5} \quad .$$

The hyperbolic regular 5-simplex with edge length $2r_0$ has the dihedral angle $2\alpha = 1.36944...$ in radians (see (3)) which is very close to the degeneration angle $2\alpha_0^5$ (see 4., (B2)). Therefore, for Buser-Karcher's in-radius estimate, the regular simplex volume bound for hyperbolic 5-manifolds is well approximated by the Euclidean ball volume $4.35410 \cdot 10^{-24}$. To get better volume estimates, it is therefore necessary to improve exisiting bounds for the in-radius $r(M^n)$. For example, if indeed the in-radius $r(M^n)$ would increase with respect to n, that is, if

$$r(M^n) \geq r_0(M^3) = \frac{l_0}{2} \simeq 0.05725 \quad ,$$

the regular simplex estimate would give

$$\mathrm{vol}_5(M^5) \geq 6.16490 \cdot 10^{-6} \quad ,$$

and the Euclidean ball volume is around $3.23728 \cdot 10^{-6}$. The only known volume of a hyperbolic 5-manifold is due to Ratcliffe and Tschantz (see [RT, p. 5]); they constructed an open, non-orientable, arithmetic hyperbolic 5-manifold M with positive first Betti number and with

$$\mathrm{vol}_5(M) = 28\zeta(3) \simeq 33.65759 \quad . \tag{33}$$

Actually, the fundamental polytope $P \subset \overline{H^5}$ of M can be subdivided into 184,320 copies of the (simply) asymptotic 5-orthoscheme R with graph

$$\Sigma(R) \quad : \quad \circ - \circ \overset{\frac{\pi}{4}}{-\!-} \circ - \circ - \circ - \circ \quad ,$$

whose volume $\frac{7\zeta(3)}{46,080}$ was computed in [K2, (39), p. 662].

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