ADJOINT RINGS ARE FINITELY GENERATED

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ABSTRACT. This paper proves finite generation of the log canonical ring without Mori theory.

CONTENTS

1. INTRODUCTION

The main goal of this paper is to prove the following theorem while avoiding techniques of the Minimal Model Program.

Theorem 1.1. *Let (X,Δ) be a projective klt pair. Then the log canonical ring $R(X,K_X+\Delta)$ is finitely generated.*

Let me sketch the strategy for the proof of finite generation in this paper and present difficulties that arise on the way. The natural idea is to pick a smooth divisor S on X and to restrict the algebra to it. If we are very lucky, the restricted algebra will be finitely generated and we might hope that the generators lift to generators on X. There are several issues with this approach.

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First, to obtain something meaningful on S, we require S to be a log canonical centre of some pair (X,Δ') such that the rings $R(X,K_X+\Delta)$ and $R(X,K_X+\Delta')$ share a common truncation.

Second, even if the restricted algebra were finitely generated, the same might not be obvious for the kernel of the restriction map. So far this seems to have been the greatest conceptual issue in attempts to prove the finite generation by the plan just outlined.

Third, the natural strategy is to use the Hacon-M$^{\mathrm{c}}$Kernan extension theorem, and hence we must be able to ensure that S does not belong to the stable base locus of $K_X+\Delta'$.

The idea to resolve the kernel issue is to view $R(X,K_X+\Delta)$ as a subalgebra of a much bigger algebra containing generators of the kernel by construction. The new algebra is graded by a monoid whose rank corresponds roughly to the number of components of Δ and of an effective divisor $D\sim_{\mathbb{Q}}K_X+\Delta$. A basic example which models the general lines of the proof in §10 is presented in Lemma A.2.

It is natural to try and restrict to a component of Δ, the issue of course being that (X,Δ) does not have log canonical centres. Therefore I allow restrictions to components of some effective divisor $D\sim_{\mathbb{Q}}K_X+\Delta$, and a tie-breaking-like technique allows me to create log canonical centres. Algebras encountered this way are, in effect, plt algebras, and their restriction is handled in §7. This is technically the most involved part of the proof.

Since the algebras we consider are of higher rank, not all divisors will have the same log canonical centres. I therefore restrict to available centres, and lift generators from algebras that live on different divisors. Since the restrictions will also be algebras of higher rank, the induction process must start from them. The contents of this paper can be summarised in the following result.

Theorem 1.2. *Let X be a projective variety, and let $D_i=k_i(K_X+\Delta_i+A)\in\mathrm{Div}(X)$, where A is an ample $\mathbb{Q}$-divisor and (X,Δ_i+A) is a klt pair for $i=1,\dots,\ell$. Then the adjoint ring $R(X;D_1,\dots,D_\ell)$ is finitely generated.*

Theorem 1.1 is a corollary to the previous theorem. Techniques of the MMP were used to prove Theorem 1.1 in the seminal paper [BCHM06], and also in the recent preprint [BP09]. A proof of finite generation of the canonical ring of general type by analytic methods is announced in [Siu06].

In the appendix I give a very short history of Mori theory, and also outline a new approach which aims to turn the conventional thinking about classification on its head. Finite generation comes at the beginning of the theory and all main results of the Minimal Model Program should be derived from it. In light of this new viewpoint, it is my hope that the techniques of this paper could be adapted to handle finite generation in the case of log canonical singularities and the abundance conjecture.

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2. Notation and conventions

Unless stated otherwise, varieties in this paper are projective and normal over $\mathbb{C}$. However, all results hold when X is, instead of being projective, assumed to be projective over an affine variety Z. The group of Weil, respectively Cartier, divisors on a variety X is denoted by $\mathrm{WDiv}(X)$, respectively $\mathrm{Div}(X)$. Subscripts denote the rings in which the coefficients are taken.

We say an ample $\mathbb{Q}$-divisor A on a variety X is *general* if there is a sufficiently divisible positive integer k such that kA is very ample and kA is a general section of $|kA|$. In particular we can assume that for some $k \gg 0$, kA is a smooth divisor on X. In practice, we fix k in advance, and generality is most often needed to ensure that A does not make singularities of pairs worse.

For any two divisors $P = \sum p_i E_i$ and $Q = \sum q_i E_i$ on X set

$$P \wedge Q = \sum \min\{p_i, q_i\} E_i.$$

For the definition and basic properties of multiplier ideals used in this paper see [HM08].

The sets of non-negative (respectively non-positive) rational and real numbers are denoted by $\mathbb{Q}_+$ and $\mathbb{R}_+$ (respectively $\mathbb{Q}_-$ and $\mathbb{R}_-$), and similarly for $\mathbb{Z}_{>0}$ and $\mathbb{R}_{>0}$.

b-Divisors. I use basic properties of b-divisors, see [Cor07]. The cone of mobile b-divisors on X is denoted by $\mathbf{Mob}(X)$.

Definition 2.1. Let (X,Δ) be a log pair. For a model $f\colon Y \to X$ we can write uniquely

$$K_Y + B_Y = f^*(K_X + \Delta) + E_Y,$$

where B_Y and E_Y are effective with no common components and E_Y is f-exceptional. The *boundary* b-divisor $\mathbf{B}(X,\Delta)$ is given by $\mathbf{B}(X,\Delta)_Y = B_Y$ for every model $Y \to X$.

Lemma 2.2. *If (X,Δ) is a log pair, then the b-divisor $\mathbf{B}(X,\Delta)$ is well-defined.*

Proof. Let $g\colon Y' \to X$ be a model such that there is a proper birational morphism $h\colon Y' \to Y$. Pushing forward $K_{Y'} + B_{Y'} = g^*(K_X + \Delta) + E_{Y'}$ via h_* yields

$$K_Y + h_* B_{Y'} = f^*(K_X + \Delta) + h_* E_{Y'},$$

and thus $h_* B_{Y'} = B_Y$ since $h_* B_{Y'}$ and $h_* E_{Y'}$ have no common components. □

If $\{D\}$ denotes the fractional part of a divisor D, we have:

Lemma 2.3. *Let (X,Δ) be a log canonical pair. There exists a log resolution $Y \to X$ such that the components of $\{\mathbf{B}(X,\Delta)_Y\}$ are disjoint.*

Proof. See [KM98, 2.36] or [HM05, 6.7]. □

Convex geometry. If $\mathscr{S} = \sum \mathbb{N}e_i$ is a submonoid of $\mathbb{N}^n$, I denote $\mathscr{S}_\mathbb{Q} = \sum \mathbb{Q}_+ e_i$ and $\mathscr{S}_\mathbb{R} = \sum \mathbb{R}_+ e_i$. A monoid $\mathscr{S} \subset \mathbb{N}^n$ is *saturated* if $\mathscr{S} = \mathscr{S}_\mathbb{R} \cap \mathbb{N}^n$.

If $\mathscr{S} = \sum_{i=1}^n \mathbb{N}e_i$ and $\kappa_1,\dots,\kappa_n$ are positive integers, the submonoid $\mathscr{S}' = \sum_{i=1}^n \mathbb{N}\kappa_i e_i$ is called a *truncation* of $\mathscr{S}$. If $\kappa_1 = \cdots = \kappa_n = \kappa$, I denote $\mathscr{S}^{(\kappa)} := \sum_{i=1}^n \mathbb{N}\kappa e_i$, and this truncation does not depend on a choice of generators of $\mathscr{S}$.

A submonoid $\mathscr{S} = \sum \mathbb{N}e_i$ of $\mathbb{N}^n$ (respectively a cone $\mathscr{C} = \sum \mathbb{R}_+ e_i$ in $\mathbb{R}^n$) is called *simplicial* if its generators e_i are linearly independent in $\mathbb{R}^n$, and the e_i form a *basis* of $\mathscr{S}$ (respectively $\mathscr{C}$).

I often use without explicit mention that if $\lambda\colon \mathscr{M} \to \mathscr{S}$ is an additive surjective map between finitely generated saturated monoids, and if $\mathscr{C}$ is a rational polyhedral cone in $\mathscr{S}_\mathbb{R}$, then $\lambda^{-1}(\mathscr{S} \cap \mathscr{C}) = \mathscr{M} \cap \lambda^{-1}(\mathscr{C})$. In particular, if $\mathscr{M}$ and $\mathscr{S}$ are saturated, the inverse image of a saturated finitely generated submonoid of $\mathscr{S}$ is a saturated finitely generated submonoid of $\mathscr{M}$.

For a polytope $\mathscr{P} \subset \mathbb{R}^n$, I denote $\mathscr{P}_\mathbb{Q} = \mathscr{P} \cap \mathbb{Q}^n$. A polytope is *rational* if it is the convex hull of finitely many rational points.

If $\mathscr{B} \subset \mathbb{R}^n$ is a convex set, then $\mathbb{R}_+\mathscr{B}$ will denote the set $\{rb : r \in \mathbb{R}_+, b \in \mathscr{B}\}$. In particular, if $\mathscr{B}$ is a rational polytope, $\mathbb{R}_+\mathscr{B}$ is a rational polyhedral cone. The dimension of the rational polytope $\mathscr{P}$, denoted $\dim \mathscr{P}$, is the dimension of the smallest rational affine space containing $\mathscr{P}$.

Let $\mathscr{S} \subset \mathbb{N}^n$ be a finitely generated monoid, $\mathscr{C} \in \{\mathscr{S}, \mathscr{S}_\mathbb{Q}, \mathscr{S}_\mathbb{R}\}$ and V an $\mathbb{R}$-vector space. A function $f\colon \mathscr{C} \to V$ is: *positively homogeneous* if $f(\lambda x) = \lambda f(x)$ for $x \in \mathscr{C}, \lambda \geq 0$; *superadditive* if $f(x) + f(y) \leq f(x+y)$ for $x,y \in \mathscr{C}$; and *superlinear* if $\lambda f(x) + \mu f(y) \leq f(\lambda x + \mu y)$ for $x,y \in \mathscr{S}_\mathbb{R}, \lambda,\mu \in \mathbb{R}_+$. Similarly for *additive*, *subadditive*, *sublinear*. It is *piecewise additive* if there is a finite polyhedral decomposition $\mathscr{C} = \bigcup \mathscr{C}_i$ such that $f_{|\mathscr{C}_i \cap \mathscr{S}}$ is additive for every i; additionally, if each $\mathscr{C}_i$ is a rational cone, it is *rationally piecewise additive*. Similarly for (rationally) piecewise linear. Assume furthermore that f is linear on $\mathscr{C}$ and $\dim \mathscr{C} = n$. The *linear extension of f to $\mathbb{R}^n$* is the unique linear function $\ell\colon \mathbb{R}^n \to V$ such that $\ell_{|\mathscr{C}} = f$.

In this paper the *relative interior* of a cone $\mathscr{C} = \sum \mathbb{R}_+ e_i \subset \mathbb{R}^n$, denoted by $\operatorname{relint}\mathscr{C}$, is the topological interior of $\mathscr{C}$ in the space $\sum \mathbb{R}e_i$ union the origin. If $\dim \mathscr{C} = n$, we instead call it the *interior* of $\mathscr{C}$ and denote it by $\operatorname{int}\mathscr{C}$. The boundary of a closed set $\mathscr{D}$ is denoted by $\partial \mathscr{D}$. If a norm $\|\cdot\|$ on $\mathbb{R}^n$ is given, then for $x \in \mathbb{R}^n$ and for any $r > 0$, the closed ball of radius r with centre at x is denoted by $B(x,r)$. Unless otherwise stated, the norm considered is always the sup-norm $\|\cdot\|_\infty$, and note that then $B(x,r)$ is a hypercube in the Euclidean norm.

Asymptotic invariants. The standard references on asymptotic invariants arising from linear series are [Nak04, ELM$^+$06].

Definition 2.4. Let X be a variety and $D \in \mathrm{WDiv}(X)_\mathbb{R}$. For $k \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$, define

$$|D|_k = \{C \in \mathrm{WDiv}(X)_k : C \geq 0, C \sim_k D\}.$$

If T is a prime divisor on X such that $T \not\subset \operatorname{Fix}|D|$, then $|D|_T$ denotes the image of the linear system $|D|$ under restriction to T. The *stable base locus* of D is $\mathbf{B}(D) = \bigcap_{C \in |D|_\mathbb{R}} \operatorname{Supp} C$ if $|D|_\mathbb{R} \neq \emptyset$, otherwise we set $\mathbf{B}(D) = X$. The *diminished base locus* is $\mathbf{B}_-(D) = \bigcup_{\varepsilon > 0} \mathbf{B}(D + \varepsilon A)$ for an ample divisor A; this definition does not depend on a choice of A. In particular $\mathbf{B}_-(D) \subset \mathbf{B}(D)$.

We denote $\mathrm{WDiv}(X)^{\kappa \geq 0} = \{D \in \mathrm{WDiv}(X) : |D|_\mathbb{R} \neq \emptyset\}$, and similarly for $\mathrm{Div}(X)^{\kappa \geq 0}$ and for versions of these sets with subscripts $\mathbb{Q}$ and $\mathbb{R}$. Observe that when $D \in \mathrm{WDiv}(X)$, the condition $|D|_\mathbb{R} \neq \emptyset$ is equivalent to $\kappa(X, D) \geq 0$ by Lemma 2.8 below, where κ is the Iitaka dimension.

It is elementary that $\mathbf{B}(D_1 + D_2) \subset \mathbf{B}(D_1) \cup \mathbf{B}(D_2)$ for $D_1, D_2 \in \mathrm{WDiv}(X)_\mathbb{R}$. In other words, the set $\{D \in \mathrm{WDiv}(X)_\mathbb{R} : x \notin \mathbf{B}(D)\}$ is convex for every point $x \in X$. By [BCHM06, 3.5.3], $\mathbf{B}(D) = \bigcap_{C \in |D|_\mathbb{Q}} \operatorname{Supp} C$ when D is a $\mathbb{Q}$-divisor, which is the standard definition of the stable base locus.

Definition 2.5. Let Z be a closed subvariety of a smooth variety X and let $D \in \mathrm{Div}(X)_\mathbb{R}^{\kappa \geq 0}$. The *asymptotic order of vanishing of D along Z* is

$$\operatorname{ord}_Z \|D\| = \inf\{\operatorname{mult}_Z C : C \in |D|_\mathbb{R}\}.$$

In the case of rational divisors, the infimum above can be taken over rational divisors, see Lemma 2.8 below. More generally, one can consider any discrete valuation ν of $k(X)$ and define

$$\nu\|D\| = \inf\{\nu(C) : C \in |D|_\mathbb{Q}\}$$

for an effective $\mathbb{Q}$-divisor D. Then [ELM$^+$06] shows that $\nu\|D\| = \nu\|E\|$ if D and E are numerically equivalent big divisors, and that ν extends to a sublinear function on $\mathrm{Big}(X)_\mathbb{R}$.

Remark 2.6. When X is projective, Nakayama [Nak04] defines a function $\sigma_Z : \overline{\mathrm{Big}(X)} \to \mathbb{R}_+$ by

$$\sigma_Z(D) = \lim_{\varepsilon \downarrow 0} \operatorname{ord}_Z \|D + \varepsilon A\|$$

for any ample $\mathbb{R}$-divisor A, and shows that it agrees with $\operatorname{ord}_Z \|\cdot\|$ on big classes. Analytic properties of these invariants were studied in [Bou04].

We can define the restricted version of the invariant introduced.

Definition 2.7. Let S be a smooth divisor on a smooth variety X and let $D \in \mathrm{Div}(X)_\mathbb{R}^{\kappa \geq 0}$ be such that $S \not\subset \mathbf{B}(D)$. Let P be a closed subvariety of S. The *restricted asymptotic order of vanishing of $|D|_S$ along P* is

$$\operatorname{ord}_P \|D\|_S = \inf\{\operatorname{mult}_P C_{|S} : C \in |D|_\mathbb{R}, S \not\subset \operatorname{Supp} C\}.$$

Lemma 2.8. *Let X be a smooth variety, $D \in \mathrm{Div}(X)_{\mathbb{Q}}^{\kappa\geq 0}$ and let $D' \geq 0$ be an $\mathbb{R}$-divisor such that $D \sim_{\mathbb{R}} D'$. Then for every $\varepsilon > 0$ there is a $\mathbb{Q}$-divisor $D'' \geq 0$ such that $D \sim_{\mathbb{Q}} D''$, $\mathrm{Supp}\, D' = \mathrm{Supp}\, D''$ and $\|D' - D''\| < \varepsilon$. In particular, if $S \subset X$ is a smooth divisor such that $S \not\subset \mathbf{B}(D)$, then for every closed subvariety $P \subset S$ we have*

$$\mathrm{ord}_P \|D\|_S = \inf\{\mathrm{mult}_P C_{|S} : C \in |D|_{\mathbb{Q}}, S \not\subset \mathrm{Supp}\, C\}.$$

Proof. Let $D' = D + \sum_{i=1}^p r_i(f_i)$ for $r_i \in \mathbb{R}$ and $f_i \in k(X)$. Let $F_1, \dots, F_N$ be the components of D and of all (f_i), and assume that $\mathrm{mult}_{F_j} D' = 0$ for $j = 1, \dots, \ell$ and $\mathrm{mult}_{F_j} D' > 0$ for $j = \ell+1, \dots, N$. Let $(f_i) = \sum_{j=1}^N \varphi_{ij} F_j$ for all i and $D = \sum_{j=1}^N \delta_j F_j$. Then we have $\delta_j + \sum_{i=1}^p \varphi_{ij} r_i = 0$ for $j = 1, \dots, \ell$. Let $\mathscr{K} \subset \mathbb{R}^p$ be the space of solutions of the system $\sum_{i=1}^p \varphi_{ij} x_i = -\delta_j$ for $j = 1, \dots, \ell$. Then $\mathscr{K}$ is a rational affine subspace and $(r_1, \dots, r_p) \in \mathscr{K}$, thus for $0 < \eta \ll 1$ there is a rational point $(s_1, \dots, s_p) \in \mathscr{K}$ with $\|s_i - r_i\| < \eta$ for all i. Therefore for η sufficiently small, setting $D'' = D + \sum_{i=1}^p s_i(f_i)$ we have the desired properties. □

Remark 2.9. Similarly as in Remark 2.6, [Hac08] introduces a function $\sigma_P \| \cdot \|_S \colon \mathscr{C}_- \to \mathbb{R}_+$ by

$$\sigma_P \|D\|_S = \lim_{\varepsilon \downarrow 0} \mathrm{ord}_P \|D + \varepsilon A\|_S$$

for any ample $\mathbb{R}$-divisor A, where $\mathscr{C}_- \subset \overline{\mathrm{Big}(X)}$ is the set of classes of divisors D such that $S \not\subset \mathbf{B}_-(D)$. Then one can define a formal sum $N_\sigma \|D\|_S = \sum \sigma_P \|D\|_S \cdot P$ over all prime divisors P on S. If $S \not\subset \mathbf{B}(D)$, then for every $\varepsilon_0 > 0$ we have $\lim_{\varepsilon \downarrow \varepsilon_0} \sigma_P \|D + \varepsilon A\|_S = \mathrm{ord}_P \|D + \varepsilon_0 A\|_S$ for any ample divisor A on X similarly as in [Nak04, 2.1.1].

In this paper I need a few basic properties cf. [Hac08].

Lemma 2.10. *Let S be a smooth divisor on a smooth projective variety X and let P be a closed subvariety of S.*

(1) Let $D \in \mathrm{Div}(X)_{\mathbb{R}}^{\kappa \geq 0}$ be such that $S \not\subset \mathbf{B}(D)$. If A is an ample $\mathbb{R}$-divisor on X, then $\mathrm{ord}_P \|D + A\|_S \leq \mathrm{ord}_P \|D\|_S$, and in particular $\sigma_P \|D\|_S \leq \mathrm{ord}_P \|D\|_S$.

(2) Let D be a pseudo-effective $\mathbb{R}$-divisor on X such that $S \not\subset \mathbf{B}_-(D)$. If A_m is a sequence of ample $\mathbb{R}$-divisors on X such that $\lim_{m\to\infty} \|A_m\| = 0$, then $\lim_{m\to\infty} \mathrm{ord}_P \|D + A_m\|_S = \sigma_P \|D\|_S$.

(3) Let D be a pseudo-effective $\mathbb{Q}$-divisor on X such that $\sigma_P \|D\|_S = 0$. If A is an ample $\mathbb{Q}$-divisor on X, then there is a positive integer l such that $\mathrm{mult}_P \mathrm{Fix}\, |l(D+A)|_S = 0$.

Proof. Statement (1) is trivial. The proof of (2) is standard: fix an ample divisor A on X, and let $0 < \varepsilon \ll 1$. For $m \gg 0$ the divisor $\varepsilon A - A_m$ is ample, and so by (1) we have

$$\mathrm{ord}_P \|D + \varepsilon A\|_S = \mathrm{ord}_P \|D + A_m + (\varepsilon A - A_m)\|_S \leq \mathrm{ord}_P \|D + A_m\|_S.$$

Letting $m \to \infty$, and then $\varepsilon \downarrow 0$ we obtain

$$\sigma_P \|D\|_S \leq \lim_{m\to\infty} \mathrm{ord}_P \|D + A_m\|_S,$$

and similarly for the opposite inequality.

For (3), first we have $\operatorname{ord}_P \|D+\frac{1}{2}A\|_S = 0$. Set $n = \dim X$, let H be a very ample divisor on X and fix a positive integer l such that $H' = \frac{l}{2}A - (K_X+S) - (n+1)H$ is very ample. Let $\Delta \sim_{\mathbb{Q}} D + \frac{1}{2}A$ be a $\mathbb{Q}$-divisor such that $S \not\subset \operatorname{Supp}\Delta$ and $\operatorname{mult}_P \Delta_{|S} < 1/l$. We have

$$H^i(X, \mathscr{J}_{l\Delta_{|S}}(K_S + H'_{|S} + (n+1)H_{|S} + l\Delta_{|S} + mH_{|S})) = 0$$

for $m \geq -n$ by Nadel vanishing. Since $l(D+A) \sim_{\mathbb{Q}} K_X + S + H' + (n+1)H + l\Delta$, the sheaf $\mathscr{J}_{l\Delta_{|S}}(l(D+A))$ is globally generated by [HM08, 5.7] and its sections lift to $H^0(X, l(D+A))$ by [HM08, 4.4(3)]. Since $\operatorname{mult}_P(l\Delta_{|S}) < 1$, $\mathscr{J}_{l\Delta_{|S}}$ does not vanish along P and so $\operatorname{mult}_P \operatorname{Fix}|l(D+A)|_S = 0$. □

Remark 2.11. Analogously one can prove that if D is a pseudo-effective $\mathbb{R}$-divisor such that $\sigma_Z\|D\| = 0$ for a closed subvariety of Z of X, then $Z \not\subset \mathbf{B}_-(D)$. Further, let $f\colon Y \to X$ be a log resolution and denote $Z' = f_*^{-1}Z$. Then I claim $\sigma_{Z'}\|f^*D\| = 0$. To prove this, we have first that $Z \not\subset \mathbf{B}(D+\varepsilon A)$ for an ample divisor A and for any $\varepsilon > 0$. Therefore $Z' \not\subset \mathbf{B}(f^*D + \varepsilon f^*A)$, and thus $\sigma_{Z'}\|f^*D+\varepsilon f^*A\| = \operatorname{ord}_{Z'}\|f^*D + \varepsilon f^*A\| = 0$. But then

$$\sigma_{Z'}\|f^*D\| = \lim_{\varepsilon\downarrow 0} \sigma_{Z'}\|f^*D + \varepsilon f^*A\| = 0$$

by [Nak04, 2.1.4(2)].

Convex sets in $\operatorname{WDiv}(X)_{\mathbb{R}}$. Let X be a variety and let V be a finite dimensional affine subspace of $\operatorname{WDiv}(X)_{\mathbb{R}}$. Fix an ample $\mathbb{Q}$-divisor A and a prime divisor G on X, and define

$$\begin{aligned}
\mathscr{L}_V &= \{\Phi \in V : K_X + \Phi \text{ is log canonical}\},\\
\mathscr{E}_{V,A} &= \{\Phi \in \mathscr{L}_V : K_X + \Phi + A \text{ is pseudo-effective}\},\\
\mathscr{B}^G_{V,A} &= \{\Phi \in \mathscr{L}_V : G \not\subset \mathbf{B}(K_X + \Phi + A)\},\\
\mathscr{B}^{G=1}_{V,A} &= \{\Phi \in \mathscr{L}_V : \operatorname{mult}_G \Phi = 1, G \not\subset \mathbf{B}(K_X+\Phi+A)\}.
\end{aligned}$$

If V is a rational affine subspace, the set $\mathscr{L}_V$ is a rational polytope by [BCHM06, 3.7.2]. Similarly as in Lemma 5.8 below, one can prove that Theorem 1.2 implies that then also $\mathscr{E}_{V,A}$, $\mathscr{B}^G_{V,A}$ and $\mathscr{B}^{G=1}_{V,A}$ are rational polytopes.

3. Outline of the induction

As part of the induction, I will prove the following three theorems.

Theorem A. *Let X be a smooth projective variety, and for $i = 1, \dots, \ell$ let $D_i = k_i(K_X + \Delta_i + A) \in \operatorname{Div}(X)$, where A is an ample $\mathbb{Q}$-divisor and $(X, \Delta_i + A)$ is a log smooth log canonical pair with $|D_i| \neq \emptyset$. Then the adjoint ring $R(X; D_1, \dots, D_\ell)$ is finitely generated.*

Theorem B. *Let X be a smooth projective variety, let B be a simple normal crossings divisor and let A be a general ample $\mathbb{Q}$-divisor on X. Let $V \subset \operatorname{Div}(X)_{\mathbb{R}}$ be the vector* *space spanned by the components of B. Then for any component G of B, the set* $\mathscr{B}_{V,A}^{G=1}$ *is a rational polytope, and we have*

$$\mathscr{B}_{V,A}^{G=1} = \{\Phi \in \mathscr{L}_V : \operatorname{mult}_G \Phi = 1, \sigma_G \| K_X + \Phi + A \| = 0\}.$$

Theorem C. *Let X be a smooth projective variety, let B be a simple normal crossings divisor and let A be a general ample* $\mathbb{Q}$*-divisor on X. Let* $V \subset \operatorname{Div}(X)_{\mathbb{R}}$ *be the vector space spanned by the components of B. Then the set* $\mathscr{E}_{V,A}$ *is a rational polytope, and we have*

$$\mathscr{E}_{V,A} = \{\Phi \in \mathscr{L}_V : |K_X + \Phi + A|_{\mathbb{R}} \neq \emptyset\}.$$

Let me give an outline of the paper, where e.g. "Theorem A_n" stands for "Theorem A in dimension n."

Sections 4 and 5 develop tools to deal with algebras of higher rank and to test whether functions are piecewise linear. Section 6 contains results from Diophantine approximation which will be necessary in Sections 7, 8 and 9.

In §8 I prove that Theorems A_{n-1} and C_{n-1} imply Theorem B_n, and this part of the proof uses techniques from §7.

In §9 I prove that Theorems A_{n-1}, B_n and C_{n-1} imply Theorem C_n, which is essentially done in [Hac08]. Another proof of Theorem C uses the non-vanishing result from [Pău08] whose proof is by analytic tools, and also avoids the MMP.

Finally, Sections 7 and 10 contain the proof that Theorems A_{n-1}, B_n and C_{n-1} imply Theorem A_n. Section 7 is technically the most difficult part of the proof, whereas §10 contains the main new idea on which the whole paper is based.

At the end of this section, let me sketch the proofs of Theorems A, B and C when X is a curve of genus g. Since by Riemann-Roch the condition that a divisor E on X is pseudo-effective is equivalent to $\deg E \geq 0$, and this condition is linear on the coefficients, this proves Theorem C. For Theorem A, when $g \geq 1$ we have that every divisor D_i is ample, and when $g = 0$, since $\deg D_i \geq 0$ we have that D_i is basepoint free, so the statement follows from [HK00, 2.8]. Furthermore, this shows that every divisor of the form $K_X + \Phi + A$ is semiample, so $\mathscr{B}_{V,A}^{G} = \mathscr{E}_{V,A}$ and Theorem B follows.

4. Convex geometry

Results of this section will be used in the rest of the paper to study relations between superadditive and superlinear functions, and to test their piecewise linearity. The following proposition can be found in [HUL93] and I add the proof for completeness.

Proposition 4.1. *Let* $\mathscr{C}$ *be a cone in* $\mathbb{R}^n$ *and let* $f\colon \mathscr{C} \to \mathbb{R}$ *be a concave function. Then f is locally Lipschitz continuous on the topological interior of* $\mathscr{C}$ *with respect to any norm* $\|\cdot\|$ *on* $\mathbb{R}^n$.

In particular, let $\mathscr{C}$ *be a rational polyhedral cone and assume a function* $g\colon \mathscr{C}_{\mathbb{Q}} \to \mathbb{Q}$ *is superadditive and satisfies* $g(\lambda x) = \lambda g(x)$ *for all* $x \in \mathscr{C}_{\mathbb{Q}}$ *and all* $\lambda \in \mathbb{Q}_+$*. Then g extends to a unique superlinear function on* $\mathscr{C}$.

Proof. Since f is locally Lipschitz if and only if $-f$ is locally Lipschitz, we can assume f is convex. Fix $x=(x_1,\dots,x_n)\in\operatorname{int}\mathscr{C}$, and let $\Delta=\{(y_1,\dots,y_n)\in\mathbb{R}^n_+:\sum y_i\le 1\}$. It is easy to check that translations of the domain do not affect the result, so we may assume $x\in\operatorname{int}\Delta\subset\operatorname{int}\mathscr{C}$.

First, let us prove that f is locally bounded above around x. Let $\{e_i\}$ be the standard basis in $\mathbb{R}^n$ and set $M=\max\{f(0),f(e_1),\dots,f(e_n)\}$. If $y=(y_1,\dots,y_n)\in\Delta$ and $y_0=1-\sum y_i\ge 0$, then

$$f(y)=f\Big(\sum y_ie_i+y_0\cdot 0\Big)\le\sum y_if(e_i)+y_0f(0)\le M.$$

Now choose δ such that $B(x,2\delta)\subset\operatorname{int}\Delta$. Again by translating the domain and composing f with a linear function we may assume that $x=0$ and $f(0)=0$. Then for all $y\in B(0,2\delta)$ we have

$$-f(y)=-f(y)+2f(0)\le -f(y)+f(y)+f(-y)=f(-y)\le M,$$

so $|f|\le M$ on $B(0,2\delta)$.

Set $L=2M/\delta$ and fix $u,v\in B(0,\delta)$. Set $\alpha=\frac{1}{\delta}\|v-u\|$ and $w=v+\frac{1}{\alpha}(v-u)\in B(0,2\delta)$ so that $v=\frac{\alpha}{\alpha+1}w+\frac{1}{\alpha+1}u$. Then by convexity,

$$\begin{aligned}f(v)-f(u)&\le\tfrac{\alpha}{\alpha+1}f(w)+\tfrac{1}{\alpha+1}f(u)-f(u)\\&=\tfrac{\alpha}{\alpha+1}\big(f(w)-f(u)\big)\le 2M\alpha=L\|v-u\|,\end{aligned}$$

and similarly $f(u)-f(v)\le L\|u-v\|$, which proves the first claim.

For the second one, observe that the sup-norm $\|\cdot\|_\infty$ takes values in $\mathbb{Q}$ on $\mathscr{C}_\mathbb{Q}$. The proof above applied to the interior of $\mathscr{C}$ and to the relative interiors of the faces of $\mathscr{C}$ shows that g is locally Lipschitz, and therefore extends to a unique superlinear function on the whole $\mathscr{C}$. □

The following result is classically referred to as Gordan's lemma, and I often use it without explicit mention.

Lemma 4.2. *Let $\mathscr{S}\subset\mathbb{N}^r$ be a finitely generated monoid and let $\mathscr{C}\subset\mathbb{R}^r$ be a rational polyhedral cone. Then the monoid $\mathscr{S}\cap\mathscr{C}$ is finitely generated.*

Proof. Assume first that $\dim\mathscr{C}=r$. Let $\ell_1,\dots,\ell_m$ be linear functions on $\mathbb{R}^r$ with integral coefficients such that $\mathscr{C}=\bigcap_{i=1}^m\{z\in\mathbb{R}^r:\ell_i(z)\ge 0\}$ and define $\mathscr{S}_0=\mathscr{S}$ and $\mathscr{S}_i=\mathscr{S}_{i-1}\cap\{z\in\mathbb{R}^r:\ell_i(z)\ge 0\}$ for $i=1,\dots,m$; observe that $\mathscr{S}\cap\mathscr{C}=\mathscr{S}_m$. Assuming by induction that $\mathscr{S}_{i-1}$ is finitely generated, by [Swa92, Theorem 4.4] we have that $\mathscr{S}_i$ is finitely generated.

Now assume $\dim\mathscr{C}<r$ and let $\mathscr{H}$ be a rational hyperplane containing $\mathscr{C}$. Let ℓ be a linear function with integral coefficients such that $\mathscr{H}=\ker(\ell)$. From the first part of the proof applied to the functions ℓ and $-\ell$ we have that the monoid $\mathscr{S}\cap\mathscr{H}$ is finitely generated. Now we proceed by descending induction on r. □

The next lemma will turn out to be indispensable and it shows that it is enough to check additivity of a superadditive map at one point only.

Lemma 4.3. *Let $\mathscr{S}=\sum_{i=1}^n \mathbb{N}e_i$ be a monoid and let $f\colon \mathscr{S}\to G$ be a superadditive map to a monoid G (respectively let $f\colon \mathscr{S}_\mathbb{R}\to V$ be a superlinear map to a cone V). Assume that there is a point $s_0=\sum s_ie_i\in\mathscr{S}$ with all $s_i>0$ such that $f(s_0)=\sum s_if(e_i)$ and that $f(\kappa s_0)=\kappa f(s_0)$ for every positive integer κ (respectively assume that there is a point $s_0=\sum s_ie_i\in\mathscr{S}_\mathbb{R}$ with all $s_i>0$ such that $f(s_0)=\sum s_if(e_i)$). Then the map f is additive (respectively linear).*

Proof. I will prove the lemma when f is superadditive, the other claim is proved analogously. For $p=\sum p_ie_i\in\mathscr{S}$, let κ_0 be a positive integer such that $\kappa_0s_i\geq p_i$ for all i. Then we have

$$\begin{aligned}\sum\kappa_0s_if(e_i)&=\kappa_0f(s_0)=f(\kappa_0s_0)\geq f(p)+\sum f\big((\kappa_0s_i-p_i)e_i\big)\\&\geq\sum p_if(e_i)+\sum(\kappa_0s_i-p_i)f(e_i)=\sum\kappa_0s_if(e_i).\end{aligned}$$

Therefore all inequalities are equalities and $f(p)=\sum p_if(e_i)$. □

Now we are ready to prove the main result of this section.

Lemma 4.4. *Let f be a superlinear function on a polyhedral cone $\mathscr{C}\subset\mathbb{R}^{r+1}$ with $\dim\mathscr{C}=r+1$ such that for every 2-plane $H\subset\mathbb{R}^{r+1}$ the function $f_{|H\cap\mathscr{C}}$ is piecewise linear. Then f is piecewise linear.*

Proof. I will prove the lemma by induction on r. In the proof, $\|\cdot\|$ denotes the standard Euclidean norm and $S^r\subset\mathbb{R}^{r+1}$ is the unit sphere.

Step 1. Fix a ray $R\subset\mathscr{C}$. In this step I prove that for any ray $R'\subset\mathscr{C}$ there is an $(r+1)$-dimensional cone $\mathscr{C}_{(r+1)}\subset\mathscr{C}$ containing R such that the map $f_{|\mathscr{C}_{(r+1)}}$ is linear and $\mathscr{C}_{(r+1)}\cap(R+R')\neq R$.

Let $H_r\supset(R+R')$ be any hyperplane. By induction there is an r-dimensional polyhedral cone $\mathscr{C}_{(r)}=\sum_{i=1}^r\mathbb{R}_+e_i\subset H_r\cap\mathscr{C}$ containing R such that $f_{|\mathscr{C}_{(r)}}$ is linear and $\mathscr{C}_{(r)}\cap(R+R')\neq R$. Set $e_0=e_1+\cdots+e_r$ and let P be a 2-plane such that $P\cap H_r=\mathbb{R}_+e_0$. Since $f_{|P\cap\mathscr{C}}$ is piecewise linear, there is a point $e_{r+1}\in P\cap\mathscr{C}$ such that $f|_{\mathbb{R}_+e_0+\mathbb{R}_+e_{r+1}}$ is linear. Set $\mathscr{C}_{(r+1)}=\mathbb{R}_+e_1+\cdots+\mathbb{R}_+e_{r+1}$. Then we have

$$f\Big(\sum e_i\Big)=f(e_0+e_{r+1})=f(e_0)+f(e_{r+1})=\sum f(e_i),$$

so the map $f_{|\mathscr{C}_{(r+1)}}$ is linear by Lemma 4.3. Observe that by choosing e_{r+1} appropriately we can ensure that the cone $\mathscr{C}_{(r+1)}$ is contained in either of the half-spaces into which H_r divides $\mathbb{R}^{r+1}$.

Step 2. Fix a ray $R\subset\mathscr{C}$ and let $\mathscr{C}_{(r+1)}=\mathbb{R}_+e_1+\cdots+\mathbb{R}_+e_{r+1}$ be any $(r+1)$-dimensional cone such that $f_{|\mathscr{C}_{(r+1)}}$ is linear. Let ℓ be the linear extension of $f_{|\mathscr{C}_{r+1}}$ to $\mathbb{R}^{r+1}$. Assume that for a point $h \in \mathscr{S}_\mathbb{R}$ we have $f|_{\mathbb{R}_+ h} = \ell|_{\mathbb{R}_+ h}$. Then there are real numbers λ_i such that $h = \sum \lambda_i e_i$, and setting $e := \sum(1+|\lambda_i|)e_i + h = \sum(1+|\lambda_i|+\lambda_i)e_i \in \mathscr{C}$ we have

$$\begin{aligned} f(e) &= \ell\Big(\sum(1+|\lambda_i|+\lambda_i)e_i\Big) = \sum(1+|\lambda_i|+\lambda_i)\ell(e_i) \\ &= \sum(1+|\lambda_i|)\ell(e_i) + \ell(h) = \sum(1+|\lambda_i|)f(e_i) + f(h), \end{aligned}$$

so f is linear on the cone $\mathscr{C}_{(r+1)} + \mathbb{R}_+ h$ by Lemma 4.3. Therefore the set $\widehat{\mathscr{C}} = \{z \in \mathscr{C} : f(z) = \ell(z)\}$ is a cone. Let $\mathscr{Q}$ denote its closure and let q be a point in $\widehat{\mathscr{C}}$. Then for every point $p \in \mathscr{Q}$ the function f is piecewise linear, and in particular continuous, on the cone $\mathbb{R}_+ p + \mathbb{R}_+ q$. Since f and ℓ agree on $\operatorname{int}\mathscr{Q}$, this implies that f is linear on $\mathbb{R}_+ p + \mathbb{R}_+ q$, so $\widehat{\mathscr{C}}$ is a closed cone.

Step 3. I claim $\widehat{\mathscr{C}}$ has finitely many extremal rays, and thus is polyhedral. Otherwise there exist extremal rays R_n, for $n \in \mathbb{N} \cup \{\infty\}$, such that $\lim_{n\to\infty} R_n = R_\infty$.

Let $T \supset R_\infty$ be a hyperplane tangent to $\widehat{\mathscr{C}}$. Fix an $(r-1)$-plane $H_{r-1} \subset T$ containing R_∞ and let $H_{r-1}^\perp$ be the unique 2-plane orthogonal to H_{r-1}. For each $n \in \mathbb{N}$ consider a hyperplane $H_r^{(n)}$ generated by H_{r-1} and R_n (if $R_n \subset H_{r-1}$ then we can finish by induction on the dimension). The set of points $\bigcup_{n\in\mathbb{N}} \big(S^r \cap H_{r-1}^\perp \cap H_r^{(n)}\big)$ has an accumulation point P_∞ on the circle $S^r \cap H_{r-1}^\perp$, and let $H_r^{(\infty)}$ be the hyperplane generated by H_{r-1} and P_∞; without loss of generality I can assume all R_n are on the same side of $H_r^{(\infty)}$.

Now by the construction in Step 1, there is an $(r+1)$-dimensional cone $\mathscr{C}_\infty$ such that $\mathscr{C}_\infty \cap H_r^{(\infty)}$ is a face of $\mathscr{C}_\infty$, $f_{|\mathscr{C}_\infty}$ is linear and $\mathscr{C}_\infty$ intersects hyperplanes $H_r^{(n)}$ for all $n \gg 0$. In particular $R_n \subset \mathscr{C}_\infty$ for all $n \gg 0$ and $(\operatorname{int}\mathscr{C}_\infty) \cap \widehat{\mathscr{C}} \neq \emptyset$. Let $w \in (\operatorname{int}\mathscr{C}_\infty) \cap \widehat{\mathscr{C}}$ and let $B \subset \operatorname{int}\mathscr{C}_\infty$ be a small ball centred at w. Then the set $B \cap \widehat{\mathscr{C}}$ is $(r+1)$-dimensional (otherwise the cone $\widehat{\mathscr{C}}$ would be contained in a hyperplane) and thus $\mathscr{C}_\infty \cap \widehat{\mathscr{C}}$ is an $(r+1)$-dimensional cone. Therefore the linear extension of $f_{|\mathscr{C}_\infty}$ coincides with ℓ and thus $\mathscr{C}_\infty \subset \widehat{\mathscr{C}}$. Since $R_n \not\subset \operatorname{int}\widehat{\mathscr{C}}$ we must have $R_n \subset \mathscr{C}_\infty \cap H_r^{(\infty)}$, and we finish by induction on the dimension.

Step 4. Again fix a ray $R \subset \mathscr{C}$. By Steps 1 to 3 there is a collection of $(r+1)$-dimensional polyhedral cones $\{\mathscr{C}_\alpha\}_{\alpha\in I_R}$ such that $R \subset \mathscr{C}_\alpha \subset \mathscr{C}$ for every $\alpha \in I_R$, for every ray $R' \subset \mathscr{C}$ there is $\alpha \in I_R$ such that $\mathscr{C}_\alpha \cap (R+R') \neq R$ and for every two distinct $\alpha, \beta \in I_R$ the linear extensions of $f_{|\mathscr{C}_\alpha}$ and $f_{|\mathscr{C}_\beta}$ to $\mathbb{R}^{r+1}$ are different. I will prove that I_R is a finite set.

Arguing by contradiction, assume I_R is infinite. For each $\alpha \in I_R$ pick $x_\alpha \in \operatorname{int}\mathscr{C}_\alpha$ and denote $\mathscr{H}_\alpha = (R + \mathbb{R}_+ x_\alpha) \cup (-R + \mathbb{R}_+ x_\alpha)$. Let $R_\alpha \subset \mathscr{H}_\alpha$ be the unique ray orthogonal to R. Let $R^\perp$ be the hyperplane orthogonal to R, and let $S^r \cap R^\perp \cap \mathscr{H}_\alpha = \{Q_\alpha\}$. The set $\{Q_\alpha : \alpha \in I_R\}$ has an accumulation point Q_∞. Let $\mathscr{H}_\infty = (R + \mathbb{R}_+ Q_\infty) \cup (-R + \mathbb{R}_+ Q_\infty)$, by relabelling pick a sequence $\mathscr{H}_n$ in the set $\{\mathscr{H}_\alpha\}$ such that $\lim\limits_{n\to\infty} Q_n = Q_\infty$, and let $\mathscr{C}_n$ be the corresponding cones in $\{\mathscr{C}_\alpha\}$.

By assumption there is a point $y \in \mathscr{H}_\infty \backslash R$ such that $f|_{R+\mathbb{R}_+y}$ is linear. Let $x \in R\backslash\{0\}$ and let $\mathscr{H}$ be any hyperplane such that $\mathscr{H} \cap (\mathbb{R}x + \mathbb{R}y) = \mathbb{R}(x+y)$. By induction there are r-dimensional polyhedral cones $\mathscr{Q}_1, \dots, \mathscr{Q}_k$ in $\mathscr{H} \cap \mathscr{C}$ such that $x+y \in \mathscr{Q}_i$ for all i, there is a small r-dimensional ball $B_{(r)} \subset \mathscr{H}$ centred at $x+y$ such that $B_{(r)} \cap \mathscr{C} = B_{(r)} \cap (\mathscr{Q}_1 \cup \dots \cup \mathscr{Q}_k)$ and the map $f_{|\mathscr{Q}_i}$ is linear for every i. Fix i and let g_{ij} be generators of $\mathscr{Q}_i$. Then

$$f\Big(\sum g_{ij} + x + y\Big) = \sum f(g_{ij}) + f(x+y) = \sum f(g_{ij}) + f(x) + f(y),$$

so f is linear on the cone $\widetilde{\mathscr{Q}}_i = \mathscr{Q}_i + \mathbb{R}_+x + \mathbb{R}_+y$ by Lemma 4.3. Therefore if we denote $\widetilde{\mathscr{Q}} = \widetilde{\mathscr{Q}}_1 \cup \dots \cup \widetilde{\mathscr{Q}}_k$, then $f_{|\widetilde{\mathscr{Q}}}$ is piecewise linear and there is a ball $B_{(r+1)}$ of radius $\varepsilon \ll 1$ centred at $x+y$ such that $B_{(r+1)} \cap \mathscr{C} = B_{(r+1)} \cap \widetilde{\mathscr{Q}}$ and $x \notin B_{(r+1)}$.

Since $\|Q_n - Q_\infty\| < \varepsilon$ for $n \gg 0$, then considering the subspace generated by R, Q_n and Q_∞ we obtain that $\mathscr{H}_n$ intersects $\operatorname{int} B_{(r+1)}$ for $n \gg 0$. Since $\widetilde{\mathscr{Q}} = \bigcup \widetilde{\mathscr{Q}}_i$, there is an index i_0 such that $\widetilde{\mathscr{Q}}_{i_0} \cap \operatorname{int} B_{(r+1)}$ intersects infinitely many $\mathscr{H}_n$. In particular, $\widetilde{\mathscr{Q}}_{i_0} \cap \operatorname{int} \mathscr{C}_n \neq \emptyset$ for infinitely many n and therefore $\widetilde{\mathscr{Q}}_{i_0} \cap \mathscr{C}_n$ is an $(r+1)$-dimensional cone. Thus for every such n the linear extensions of $f_{|\widetilde{\mathscr{Q}}_{i_0}}$ and $f_{|\mathscr{C}_n}$ to $\mathbb{R}^{r+1}$ are the same since they coincide with the linear extension of $f_{|\widetilde{\mathscr{Q}}_{i_0} \cap \mathscr{C}_n}$, which is a contradiction and I_R is finite.

Step 5. Finally, we have that for every ray $R \subset \mathscr{C}$ the map $f|_{\mathscr{C}_\alpha}$ is linear for $\alpha \in I_R$, and there is small ball B_R centred at $R \cap S^r$ such that $B_R \cap \mathscr{C} = B_R \cap \bigcup_{\alpha \in I_R} \mathscr{C}_\alpha$. There are finitely many open sets $\operatorname{int} B_R$ which cover the compact set $S^r \cap \mathscr{C}$ and therefore we can choose finitely many cones $\mathscr{C}_\alpha$ with $\mathscr{C} = \bigcup \mathscr{C}_\alpha$. Thus f is piecewise linear. □

5. Higher rank algebras

Definition 5.1. Let X be a variety, $\mathscr{S}$ a finitely generated submonoid of $\mathbb{N}^r$, let $\mu\colon \mathscr{S} \to \operatorname{WDiv}(X)^{\kappa \geq 0}$ be an additive map and let $\mathbf{Mob}_\mu\colon \mathscr{S} \to \mathbf{Mob}(X)$ be the subadditive map defined by $\mathbf{Mob}_\mu(s) = \mathbf{Mob}(\mu(s))$ for every $s \in \mathscr{S}$. The algebra

$$R(X, \mu(\mathscr{S})) = \bigoplus_{s \in \mathscr{S}} H^0(X, \mathscr{O}_X(\mu(s)))$$

is called the *divisorial $\mathscr{S}$-graded algebra associated to μ*. The *b-divisorial $\mathscr{S}$-graded algebra associated to μ* is

$$R(X, \mathbf{Mob}_\mu(\mathscr{S})) = \bigoplus_{s \in \mathscr{S}} H^0(X, \mathscr{O}_X(\mathbf{Mob}_\mu(s))),$$

and we obviously have $R(X, \mathbf{Mob}_\mu(\mathscr{S})) \simeq R(X, \mu(\mathscr{S}))$. If $e_1, \dots, e_\ell$ are generators of $\mathscr{S}$ and if $\mu(e_i) = k_i(K_X + \Delta_i)$, where Δ_i is an effective $\mathbb{Q}$-divisor for every i, the algebra $R(X, \mu(\mathscr{S}))$ is called the *adjoint ring associated to μ*.

Remark 5.2. When $\mathscr{S} = \bigoplus_{i=1}^{\ell} \mathbb{N}e_i$ is a simplicial cone, the algebra $R(X, \mu(\mathscr{S}))$ is denoted also by $R(X; \mu(e_1), \dots, \mu(e_\ell))$. If $\mathscr{S}'$ is a finitely generated submonoid of $\mathscr{S}$,

$R(X,\mu(\mathscr{S}'))$ is used to denote $R(X,\mu_{|\mathscr{S}'}(\mathscr{S}'))$. If $\mathscr{S}$ is a submonoid of $\mathrm{WDiv}(X)^{\kappa\geq 0}$ and $\iota\colon \mathscr{S}\to\mathscr{S}$ is the identity map, $R(X,\mathscr{S})$ is used to denote $R(X,\iota(\mathscr{S}))$.

Remark 5.3. Algebras considered in this paper are *algebras of sections* when varieties are smooth. I will occasionally, and without explicit mention, view them as algebras of rational functions, in particular to be able to write $H^0(X,D)\simeq H^0(X,\mathrm{Mob}(D))\subset k(X)$.

Assume now that X is smooth, $D\in\mathrm{Div}(X)$ and that Γ is a prime divisor on X. If σ_Γ is the global section of $\mathscr{O}_X(\Gamma)$ such that $\operatorname{div}\sigma_\Gamma=\Gamma$, from the exact sequence

$$0\to H^0(X,\mathscr{O}_X(D-\Gamma))\xrightarrow{\cdot\sigma_\Gamma} H^0(X,\mathscr{O}_X(D))\xrightarrow{\rho_{D,\Gamma}} H^0(\Gamma,\mathscr{O}_\Gamma(D))$$

we define $\mathrm{res}_\Gamma H^0(X,\mathscr{O}_X(D))=\mathrm{Im}(\rho_{D,\Gamma})$. For $\sigma\in H^0(X,\mathscr{O}_X(D))$, denote $\sigma_{|\Gamma}:=\rho_{D,\Gamma}(\sigma)$. Observe that

$$\ker(\rho_{D,\Gamma})=H^0(X,\mathscr{O}_X(D-\Gamma))\cdot\sigma_\Gamma, \tag{1}$$

and that $\mathrm{res}_\Gamma H^0(X,\mathscr{O}_X(D))=0$ if $\Gamma\subset\mathrm{Bs}\,|D|$. If $D\sim D'$ is such that the restriction $D'_{|\Gamma}$ is defined, then

$$\mathrm{res}_\Gamma H^0(X,\mathscr{O}_X(D))\simeq\mathrm{res}_\Gamma H^0(X,\mathscr{O}_X(D'))\subset H^0(\Gamma,\mathscr{O}_\Gamma(D'_{|\Gamma})).$$

The *restriction of* $R(X,\mu(\mathscr{S}))$ *to* Γ is defined as

$$\mathrm{res}_\Gamma R(X,\mu(\mathscr{S}))=\bigoplus_{s\in\mathscr{S}}\mathrm{res}_\Gamma H^0(X,\mathscr{O}_X(\mu(s))).$$

This is an $\mathscr{S}$-graded, not necessarily divisorial algebra.

The following lemma summarises the basic properties of higher rank finite generation.

Lemma 5.4. *Let $\mathscr{S}\subset\mathbb{N}^n$ be a finitely generated monoid and let $R=\bigoplus_{s\in\mathscr{S}}R_s$ be an $\mathscr{S}$-graded algebra.*

(1) *Let $\mathscr{S}'$ be a truncation of $\mathscr{S}$. If the $\mathscr{S}'$-graded algebra $R'=\bigoplus_{s\in\mathscr{S}'}R_s$ is finitely generated over R_0, then R is finitely generated over R_0.*
(2) *Assume furthermore that $\mathscr{S}$ is saturated and let $\mathscr{S}''\subset\mathscr{S}$ be a finitely generated saturated submonoid. If R is finitely generated over R_0, then the $\mathscr{S}''$-graded algebra $R''=\bigoplus_{s\in\mathscr{S}''}R_s$ is finitely generated over R_0.*
(3) *Let X be a variety and let $\mu\colon\mathscr{S}\to\mathrm{WDiv}(X)^{\kappa\geq 0}$ be an additive map. If there exists a rational polyhedral subdivision $\mathscr{S}_\mathbb{R}=\bigcup_{i=1}^k\Delta_i$ such that, for each i, the map $\mathbf{Mob}_{\mu|\Delta_i\cap\mathscr{S}}$ is additive up to truncation, then the algebra $R(X,\mu(\mathscr{S}))$ is finitely generated.*

Proof. For (1) it is enough to observe that R is an integral extension of R': if $\mathscr{S}=\sum_{i=1}^n\mathbb{N}e_i$ and $\mathscr{S}'=\sum_{i=1}^n\mathbb{N}\kappa_ie_i$, then for any $f\in R$ we have $f^{\kappa_1\cdots\kappa_n}\in R'$.

Claim (2) is [ELM+06, 4.8].

For (3), denote $\mathbf{m}=\mathbf{Mob}_\mu$. Let $\{e_{ij}:j\in I_i\}$ be a finite set of generators of $\Delta_i\cap\mathscr{S}$, and let κ_{ij} be positive integers such that $\mathbf{m}|_{\sum_{j\in I_i}\mathbb{N}\kappa_{ij}e_{ij}}$ is additive for each i. Set $\kappa=\prod_{i,j}\kappa_{ij}$ and $\mathscr{S}^{(\kappa)}=\sum_{i,j}\mathbb{N}\kappa e_{ij}$. If $\sum_{i,j}\lambda_{ij}\kappa e_{ij}\in\Delta_i\cap\mathscr{S}^{(\kappa)}$ for some $\lambda_{ij}\in\mathbb{N}$, then

$\sum_{i,j}\lambda_{ij}e_{ij}\in\Delta_i\cap\mathscr{S}$ and thus there are $\mu_j\in\mathbb{N}$ such that $\sum_{i,j}\lambda_{ij}e_{ij}=\sum_{j\in I_i}\mu_je_{ij}$. Therefore $\Delta_i\cap\mathscr{S}^{(\kappa)}=\sum_{j\in I_i}\mathbb{N}\kappa e_{ij}$, and this is a truncation of $\sum_{j\in I_i}\mathbb{N}\kappa_{ij}e_{ij}$; in particular $\mathbf{m}_{|\Delta_i\cap\mathscr{S}^{(\kappa)}}$ is additive for each i.

I claim the algebra $R(X,\mathbf{m}(\mathscr{S}^{(\kappa)}))$ is finitely generated, and thus $R(X,\mathbf{m}(\mathscr{S}))$ is finitely generated by (1). To that end, let $Y\to X$ be a model such that $\mathbf{m}(\kappa e_{ij})$ descend to Y for all i,j, and let $s=\sum_{j\in I_i}\nu_{ij}\kappa e_{ij}\in\Delta_i\cap\mathscr{S}^{(\kappa)}$ for some i and some $\nu_{ij}\in\mathbb{N}$. Then

$$\mathbf{m}(s)=\sum\nolimits_{j\in I_i}\nu_{ij}\mathbf{m}(\kappa e_{ij})=\sum\nolimits_{j\in I_i}\nu_{ij}\overline{\mathbf{m}(\kappa e_{ij})_Y}=\overline{\sum\nolimits_{j\in I_i}\nu_{ij}\mathbf{m}(\kappa e_{ij})_Y}=\overline{\mathbf{m}(s)_Y},$$

and thus $\mathbf{m}(s)$ descends to Y and

$$R(X,\mathbf{m}(\mathscr{S}^{(\kappa)}))\simeq\bigoplus_{s\in\mathscr{S}^{(\kappa)}}H^0(Y,\mathbf{m}(s)_Y).$$

For each i consider the free monoid $\mathscr{S}_i^{(\kappa)}=\bigoplus_{j\in I_i}\mathbb{N}\kappa e_{ij}$. Since the divisorial algebra $R(Y,\mathbf{m}(\mathscr{S}_i^{(\kappa)}))$ is finitely generated by [HK00, 2.8], so is the algebra $R(X,\mathbf{m}(\Delta_i\cap\mathscr{S}^{(\kappa)}))$ by projection. Now the set of generators of all $R(X,\mathbf{m}(\Delta_i\cap\mathscr{S}^{(\kappa)}))$ spans $R(X,\mathbf{m}(\mathscr{S}^{(\kappa)}))$. □

I will need the next result in the proof of Proposition 5.7 and in §7.

Lemma 5.5. *Let X be a variety, let $\mathscr{S}\subset\mathbb{N}^r$ be a finitely generated monoid and let $f\colon\mathscr{S}\to G$ be a superadditive map to a monoid G which is a subset of* $\mathrm{WDiv}(X)$ *or* $\mathbf{Mob}(X)$, *such that for every $s\in\mathscr{S}$ there is a positive integer ι_s such that $f_{|\mathbb{N}\iota_s s}$ is an additive map.*

Then there is a unique superlinear function $f^\sharp\colon\mathscr{S}_\mathbb{R}\to G_\mathbb{R}$ such that for every $s\in\mathscr{S}$ there is a positive integer λ_s with $f(\lambda_s s)=f^\sharp(\lambda_s s)$. Furthermore, let $\mathscr{C}$ be a rational polyhedral subcone of $\mathscr{S}_\mathbb{R}$. Then $f_{|\mathscr{C}\cap\mathscr{S}}$ is additive up to truncation if and only if $f^\sharp_{|\mathscr{C}}$ is linear.

If $\mu\colon\mathscr{S}\to$ $\mathrm{Div}(X)$ *is an additive map and $\mathbf{m}=\mathbf{Mob}_\mu$ is such that for every $s\in\mathscr{S}$ there is a positive integer ι_s such that $\mathbf{m}_{|\mathbb{N}\iota_s s}$ is an additive map, then we have*

$$\mathbf{m}^\sharp(s)=\overline{\mu(s)}-\sum\big(\mathrm{ord}_E\|\mu(s)\|\big)E,\qquad(2)$$

where the sum runs over all geometric valuations E on X.

Proof. The construction will show that $f^\sharp$ is the unique function with the stated properties. To start with, fix a point $s\in\mathscr{S}_\mathbb{Q}$ and let κ be a positive integer such that $\kappa s\in\mathscr{S}$. Set

$$f^\sharp(s):=\frac{f(\iota_{\kappa s}\kappa s)}{\iota_{\kappa s}\kappa}.$$

This is well-defined: take another κ' such that $\kappa' s\in\mathscr{S}$. Then

$$f(\iota_{\kappa s}\iota_{\kappa' s}\kappa\kappa' s)=\iota_{\kappa s}\kappa f(\iota_{\kappa' s}\kappa' s)=\iota_{\kappa' s}\kappa' f(\iota_{\kappa s}\kappa s),$$

so $f(\iota_{\kappa s}\kappa s)/\iota_{\kappa s}\kappa=f(\iota_{\kappa' s}\kappa' s)/\iota_{\kappa' s}\kappa'$.

Now let $s \in \mathscr{S}_{\mathbb{Q}}$, $\xi \in \mathbb{Q}_+$ and let λ be a positive integer such that $\lambda\xi s \in \mathscr{S}$ and $\lambda\xi \in \mathbb{N}$. Then

$$f^\sharp(\xi s) = \frac{f\big((\iota_{\lambda\xi s}\lambda)\xi s\big)}{\iota_{\lambda\xi s}\lambda} = \xi\frac{f\big((\iota_{\lambda\xi s}\lambda\xi)s\big)}{\iota_{\lambda\xi s}\lambda\xi} = \xi f^\sharp(s),$$

so $f^\sharp$ is positively homogeneous (with respect to rational scalars). Further, let $s_1, s_2 \in \mathscr{S}_{\mathbb{Q}}$ and let κ be a positive integer such that $f(\kappa s_1) = f^\sharp(\kappa s_1)$, $f(\kappa s_2) = f^\sharp(\kappa s_2)$ and $f\big(\kappa(s_1+s_2)\big) = f^\sharp\big(\kappa(s_1+s_2)\big)$. By superadditivity of f we have

$$f(\kappa s_1) + f(\kappa s_2) \leq f\big(\kappa(s_1+s_2)\big),$$

so dividing the inequality by κ we obtain superadditivity of $f^\sharp$.

Let E be any divisor on X, respectively any geometric valuation E over X, when $G \subset \mathrm{WDiv}(X)$, respectively $G \subset \mathbf{Mob}(X)$. Consider the function $f_E^\sharp$ given by $f_E^\sharp(s) = \mathrm{mult}_E f^\sharp(s)$. Proposition 4.1 applied to each $f_E^\sharp$ shows that $f^\sharp$ extends to a superlinear function on $\mathscr{S}_{\mathbb{R}}$.

As for the statement on cones, necessity is clear. Now assume $f^\sharp|_{\mathscr{C}}$ is linear and let $\mathscr{C} \cap \mathscr{S} = \sum_{i=1}^n \mathbb{N}e_i$. For $s_0 = e_1 + \cdots + e_n$ we have

$$f^\sharp(s_0) = f^\sharp(e_1) + \cdots + f^\sharp(e_n). \tag{3}$$

Let μ be a positive integer such that $f(\mu s_0) = f^\sharp(\mu s_0)$ and $f(\mu e_i) = f^\sharp(\mu e_i)$ for all i. ¿From (3) we obtain

$$f(\mu s_0) = f(\mu e_1) + \cdots + f(\mu e_n),$$

and Lemma 4.3 implies that $f^\sharp|_{\sum \mathbb{N}\mu e_i}$ is additive. □

Definition 5.6. In the context of Lemma 5.5, the function $f^\sharp$ is called *the straightening of* f.

Proposition 5.7. *Let X be a variety, $\mathscr{S} \subset \mathbb{N}^r$ a finitely generated saturated monoid and $\mu\colon \mathscr{S} \to \mathrm{WDiv}(X)^{\kappa\geq 0}$ an additive map. Let $\mathscr{L}$ be a finitely generated submonoid of $\mathscr{S}$ and assume $R(X, \mu(\mathscr{S}))$ is finitely generated. Then $R(X, \mu(\mathscr{L}))$ is finitely generated. Moreover, the map $\mathbf{m} = \mathbf{Mob}_{\mu|\mathscr{L}}$ is piecewise additive up to truncation. In particular, there is a positive integer p such that $\mathbf{Mob}_\mu(ips) = i\mathbf{Mob}_\mu(ps)$ for every $i \in \mathbb{N}$ and every $s \in \mathscr{L}$.*

Proof. Denote $\mathscr{M} = \mathscr{L}_{\mathbb{R}} \cap \mathbb{N}^r$. By Lemma 5.4(2), $R(X, \mu(\mathscr{M}))$ is finitely generated, and by the proof of [ELM+06, 4.1], there is a finite rational polyhedral subdivision $\mathscr{M}_{\mathbb{R}} = \bigcup \Delta_i$ such that for every geometric valuation E on X, the map $\mathrm{ord}_E \|\cdot\|$ is linear on Δ_i for every i. Since for every saturated rank 1 submonoid $\mathscr{R} \subset \mathscr{M}$ the algebra $R(X, \mu(\mathscr{R}))$ is finitely generated by Lemma 5.4(2), the map $\mathbf{m}_{|\mathscr{R}}$ is additive up to truncation by [Cor07, 2.3.53], and thus there is the well-defined straightening $\mathbf{m}^\sharp\colon \mathscr{L}_{\mathbb{R}} \to \mathbf{Mob}(X)_{\mathbb{R}}$ since $\mathscr{M}_{\mathbb{R}} = \mathscr{L}_{\mathbb{R}}$. Then equation (2) implies that the map $\mathbf{m}^\sharp|_{\Delta_i}$ is linear for every i, hence by Lemma 5.5 the map $\mathbf{m}$ is piecewise additive up to truncation. Therefore $R(X, \mu(\mathscr{L}))$ is finitely generated by Lemma 5.4(3). □

The following lemma shows that finite generation implies certain boundedness on the convex geometry of boundaries.

Lemma 5.8. *Let X be a smooth projective variety of dimension n, let B be a simple normal crossings divisor on X and let A be a general ample $\mathbb{Q}$-divisor. Let $V \subset \mathrm{Div}(X)_{\mathbb{R}}$ be the vector space spanned by the components of B. Assume Theorems A_n and C_n. Then for each prime divisor G on X, the set $\mathscr{B}^G_{V,A}$ is a rational polytope. Furthermore, there exists a positive integer r such that:*

(1) *for every $\Phi \in (\mathscr{L}_V)_{\mathbb{Q}}$ with the property that no component of Φ is in $\mathbf{B}(K_X + \Phi + A)$, and for every positive integer k such that $k(K_X + \Phi + A)/r$ is Cartier, no component of B is in $\mathrm{Fix}\,|k(K_X + \Phi + A)|$,*
(2) *for every $\Phi \in (\mathscr{L}_V)_{\mathbb{Q}}$ with the property that $K_X + \Phi + A$ is pseudo-effective, and for every positive integer k such that $k(K_X + \Phi + A)/r$ is Cartier, we have $|k(K_X + \Phi + A)| \neq \emptyset$.*

Proof. Let K_X be a divisor such that $\mathscr{O}_X(K_X) \simeq \omega_X$ and $\mathrm{Supp}\,A \not\subset \mathrm{Supp}\,K_X$, and let $\Lambda \subset \mathrm{Div}(X)$ be the monoid spanned by components of K_X, B and A. Let G be a components of B. By Theorem C the set $\mathscr{E}_{V,A}$ is a rational polytope, and let $D_1, \dots, D_\ell$ be generators of the finitely generated monoid $\mathscr{C} = \mathbb{R}_+(K_X + A + \mathscr{E}_{V,A}) \cap \Lambda$. Since every D_i is proportional to an adjoint bundle, by Theorem A and Lemma 5.4(1) the ring $R(X; D_1, \dots, D_\ell)$ is finitely generated, and thus so is the algebra $R(X, \mathscr{C})$ by projection. By Proposition 5.7 the map $\mathbf{Mob}_{\iota|\mathscr{C}\cap\Lambda^{(r)}}$ is additive for some positive integer r, where $\iota\colon \Lambda \to \Lambda$ is the identity map. Now (1) and (2) are straightforward. Furthermore, the set $\mathscr{O} = \{\Upsilon \in \mathscr{C} : \mathrm{ord}_G \|\Upsilon\| = 0\}$ is a rational polyhedral cone by the proof of [ELM+06, 4.1], and $\mathbb{R}_+(K_X + A + \mathscr{B}^G_{V,A}) \subset \mathscr{O}$. Since for every $\Upsilon \in \mathscr{O}_{\mathbb{Q}}$ we have $G \not\subset \mathbf{B}(\Upsilon)$ by Theorem A, this implies $\mathscr{O} \subset \mathbb{R}_+(K_X + A + \mathscr{B}^G_{V,A})$ as extremal rays of $\mathscr{O}$ are rational. Therefore $\mathscr{B}^G_{V,A}$ is a rational polytope. $\square$

6. Diophantine approximation

I need a few results from Diophantine approximation theory.

Lemma 6.1. *Let $\Lambda \subset \mathbb{R}^n$ be a lattice spanned by rational vectors, and let $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. Fix a vector $v \in V$ and denote $X = \mathbb{N}v + \Lambda$. Then the closure of X is symmetric with respect to the origin. Moreover, if $\pi\colon V \to V/\Lambda$ is the quotient map, then the closure of $\pi(X)$ is a finite disjoint union of connected components. If v is not contained in any proper rational affine subspace of V, then X is dense in V.*

Proof. I am closely following the proof of [BCHM06, 3.7.6]. Let G be the closure of $\pi(X)$. Since G is infinite and V/Λ is compact, G has an accumulation point. It then follows that zero is also an accumulation point and that G is a closed subgroup. The connected component G_0 of the identity in G is a Lie subgroup of V/Λ and so by [Bum04, Theorem 15.2], G_0 is a torus. Thus $G_0 = V_0/\Lambda_0$, where $V_0 = \Lambda_0 \otimes_{\mathbb{Z}} \mathbb{R}$ is a rational subspace of V. Since G/G_0 is discrete and compact, it is finite, and it is straightforward that X is symmetric with respect to the origin. Therefore a translate of v by a rational vector is contained in V_0, and so if v is not contained in any proper rational affine subspace of V, then $V_0 = V$. □

Definition 6.2. Let $x \in \mathbb{R}^n$, let ε be a positive real number and let k be a positive integer. We say that, for $i = 1, \dots, p$, points $(w_i, k, k_i, r_i) \in \mathbb{Q}^n \times \mathbb{Z}^2_{>0} \times \mathbb{R}_{>0}$ *uniformly approximate x with error* ε if

(1) $k_i w_i / k$ is integral for every i,
(2) $\|x - w_i\| < \varepsilon / k_i$ for every i,
(3) $x = \sum r_i w_i$ and $\sum r_i = 1$.

The next result is [BCHM06, 3.7.7].

Lemma 6.3. *Let $x \in \mathbb{R}^n$ and let W be the smallest rational affine space containing x. Fix a positive integer k and a positive real number ε. Then there are finitely many $(w_i, k, k_i, r_i) \in (W \cap \mathbb{Q}^n) \times \mathbb{Z}^2_{>0} \times \mathbb{R}_{>0}$ which uniformly approximate x with error ε.*

I will need a refinement of this lemma when the approximation is not necessarily happening in the smallest rational affine space containing a point.

Lemma 6.4. *Let $x \in \mathbb{R}^n$, let $0 < \varepsilon, \eta \ll 1$ be rational numbers and let k be a positive integer. Assume that there are $w_1 \in \mathbb{Q}^n$ and $k_1 \in \mathbb{N}$ such that $\|x - w_1\| < \varepsilon / k_1$ and $k_1 w_1 / k$ is integral. Then there are $r_1 \in \mathbb{R}_{>0}$, and points $(w_i, k, k_i, r_i) \in \mathbb{Q}^n \times \mathbb{Z}^2_{>0} \times \mathbb{R}_{>0}$ for $i = 2, \dots, m$, such that (w_i, k, k_i, r_i) uniformly approximate x with error ε, for $i = 1, \dots, m$. Furthermore, we can assume that $w_3, \dots, w_m$ belong to the smallest rational affine space containing x, and we can write*

$$x = \frac{k_1}{k_1 + k_2} w_1 + \frac{k_2}{k_1 + k_2} w_2 + \xi,$$

with $\|\xi\| < \eta / (k_1 + k_2)$.

Proof. Rescaling by k, we can assume that $k = 1$. Let W be the minimal rational affine subspace containing x, let $\pi \colon \mathbb{R}^n \to \mathbb{R}^n / \mathbb{Z}^n$ be the quotient map and let G be the closure of the set $\pi(\mathbb{N}x + \mathbb{Z}^n)$. Then by Lemma 6.1 we have $\pi(-k_1 x) \in G$ and there is $k_2 \in \mathbb{N}$ such that $\pi(k_2 x)$ is in the connected component of $\pi(-k_1 x)$ in G and $\|k_2 x - y\| < \eta$ for some $y \in \mathbb{R}^n$ with $\pi(y) = \pi(-k_1 x)$. Thus there is a point $w_2 \in \mathbb{Q}^n$ such that $k_2 w_2 \in \mathbb{Z}^n$, $\|k_2 x - k_2 w_2\| < \varepsilon$ and the open segment (w_1, w_2) intersects W.

Pick $t \in (0,1)$ such that $w_t = t w_1 + (1-t) w_2 \in W$, and choose, by Lemma 6.3, rational points $w_3, \dots, w_m \in W$ and positive integers $k_3, \dots, k_m$ such that $k_i w_i \in \mathbb{Z}^n$, $\|x - w_i\| < \varepsilon / k_i$ and $x = \sum_{i=3}^m r_i w_i + r_t w_t$ with $r_t > 0$ and all $r_i > 0$, and $r_t + \sum_{i=3}^m r_i = 1$. Thus $x = \sum_{i=1}^m r_i w_i$ with $r_1 = t r_t$ and $r_2 = (1-t) r_t$.

Finally, observe that the vector $y / k_2 - w_2$ is parallel to the vector $x - w_1$ and $\|y - k_2 w_2\| = \|k_1 x - k_1 w_1\|$. Denote $z = x - y / k_2$. Then

$$\frac{x - w_1}{(w_2 + z) - x} = \frac{x - w_1}{w_2 - y / k_2} = \frac{k_2}{k_1},$$

so

$$x=\frac{k_1}{k_1+k_2}w_1+\frac{k_2}{k_1+k_2}(w_2+z)=\frac{k_1}{k_1+k_2}w_1+\frac{k_2}{k_1+k_2}w_2+\xi,$$

where $\|\xi\|=\|k_2z/(k_1+k_2)\|<\eta/(k_1+k_2)$. □

Remark 6.5. Assuming notation from the previous proof, the connected components of G are precisely the connected components of the closure of the set $\pi(\bigcup_{k>0}kW)$. Therefore $y/k_2\in W$.

Remark 6.6. Assume $\lambda\colon V\to W$ is a linear map between vector spaces such that $\lambda(V_{\mathbb{Q}})\subset W_{\mathbb{Q}}$. Let $x\in V$, and let $H\subset V$ and $H'\subset W$ be the smallest rational affine subspaces containing x and $\lambda(x)$, respectively. Then, by definition, $H'\subset\lambda(H)$ and $H\subset\lambda^{-1}(H')$, thus $H'=\lambda(H)$.

7. Restricting plt algebras

In this section I establish one of the technically most difficult steps in the scheme of the proof, that Theorems A_{n-1}, B_n and C_{n-1} imply Theorem A_n. Crucial techniques will be those developed in [HM08] and in Sections 4 and 5.

The key result is the following Hacon-M$^{\mathrm{c}}$Kernan extension theorem [HM08, 6.2], whose proof relies on deep techniques initiated by [Siu98].

Theorem 7.1. *Let $(X,\Delta=S+A+B)$ be a projective plt pair such that $S=\lfloor\Delta\rfloor$ is irreducible, $\Delta\in\mathrm{WDiv}(X)_{\mathbb{Q}}$, (X,S) is log smooth, A is a general ample $\mathbb{Q}$-divisor and $(S,\Omega+A_{|S})$ is canonical, where $\Omega=(\Delta-S)_{|S}$. Assume $S\not\subset\mathbf{B}(K_X+\Delta)$, and let*

$$F=\liminf_{m\to\infty}\tfrac{1}{m}\operatorname{Fix}|m(K_X+\Delta)|_S.$$

If $\varepsilon>0$ is any rational number such that $\varepsilon(K_X+\Delta)+A$ is ample and if Φ is any $\mathbb{Q}$-divisor on S and $k>0$ is any integer such that both $k\Delta$ and $k\Phi$ are Cartier, and $\Omega\wedge(1-\frac{\varepsilon}{k})F\leq\Phi\leq\Omega$, then

$$|k(K_S+\Omega-\Phi)|+k\Phi\subset|k(K_X+\Delta)|_S.$$

The immediate consequence is:

Corollary 7.2. *Let $(X,\Delta=S+A+B)$ be a projective plt pair such that $S=\lfloor\Delta\rfloor$ is irreducible, $\Delta\in\mathrm{WDiv}(X)_{\mathbb{Q}}$, (X,S) is log smooth, A is a general ample $\mathbb{Q}$-divisor and $(S,\Omega+A_{|S})$ is canonical, where $\Omega=(\Delta-S)_{|S}$. Assume $S\not\subset\mathbf{B}(K_X+\Delta)$, and let $\Phi_m=\Omega\wedge\frac{1}{m}\operatorname{Fix}|m(K_X+\Delta)|_S$ for every m such that $m\Delta$ is Cartier. Then*

$$|m(K_S+\Omega-\Phi_m)|+m\Phi_m=|m(K_X+\Delta)|_S.$$

The following result will be used several times to test inclusions of linear series. It is extracted and copied almost verbatim from the proof of the non-vanishing theorem in [Hac08], and Step 2 of the proof below first appeared in [Tak06]. Similar techniques in the analytic setting appeared in [Pău08].

Proposition 7.3. *Let $(X,\Delta = S+A+B)$ be a projective plt pair such that $S = \lfloor \Delta \rfloor$ is irreducible, $\Delta \in \mathrm{WDiv}(X)_{\mathbb{Q}}$, (X,S) is log smooth, A is a general ample $\mathbb{Q}$-divisor and $(S,\Omega+A_{|S})$ is canonical, where $\Omega = (\Delta - S)_{|S}$. Let $0 \leq \Theta \leq \Omega$ be a $\mathbb{Q}$-divisor on S, let k be a positive integer such that $k\Delta$ and $k\Theta$ are integral, and denote $A' = A/k$. Assume that $S \not\subset \mathbf{B}(K_X+\Delta+A')$ and that for any $l > 0$ sufficiently divisible we have*

$$\Omega \wedge \tfrac{1}{l}\operatorname{Fix}|l(K_X+\Delta+A')|_S \leq \Omega - \Theta. \tag{4}$$

Then

$$|k(K_S+\Theta)| + k(\Omega-\Theta) \subset |k(K_X+\Delta)|_S.$$

Proof. Step 1. We first prove that there exists an effective divisor H on X not containing S such that for all sufficiently divisible positive integers m we have

$$|m(K_S+\Theta)| + m(\Omega-\Theta) + (mA'+H)_{|S} \subset |m(K_X+\Delta)+mA'+H|_S. \tag{5}$$

Taking l as in (4) sufficiently divisible, we can assume $S \not\subset \operatorname{Bs}|l(K_X+\Delta+A')|$. Let $f\colon Y \to X$ be a log resolution of $(X,\Delta+A')$ and of $|l(K_X+\Delta+A')|$. Let $\Gamma = \mathbf{B}(X,\Delta+A')_Y$ and $E = K_Y + \Gamma - f^*(K_X+\Delta+A')$, and define

$$\Xi = \Gamma - \Gamma \wedge \tfrac{1}{l}\operatorname{Fix}|l(K_Y+\Gamma)|.$$

We have that $l(K_Y+\Xi)$ is Cartier, $\operatorname{Fix}|l(K_Y+\Xi)| \wedge \Xi = 0$ and $\operatorname{Mob}(l(K_Y+\Xi))$ is free. Since $\operatorname{Fix}|l(K_Y+\Xi)|+\Xi$ has simple normal crossings support, it follows that $\mathbf{B}(K_Y+\Xi)$ contains no log canonical centres of $(Y,\lceil \Xi \rceil)$. Let $T = f_*^{-1}S, \Gamma_T = (\Gamma - T)_{|T}$ and $\Xi_T = (\Xi - T)_{|T}$, let m be any positive integer divisible by l and consider a section

$$\sigma \in H^0(T,\mathscr{O}_T(m(K_T+\Xi_T))) = H^0(T,\mathscr{J}_{\|m(K_T+\Xi_T)\|}(m(K_T+\Xi_T))).$$

By [HM08, 5.3], there is an ample divisor H' on Y such that if $\tau \in H^0(T,\mathscr{O}_T(H'))$, then $\sigma\cdot\tau$ is in the image of the homomorphism

$$H^0(Y,\mathscr{O}_Y(m(K_Y+\Xi)+H')) \to H^0(T,\mathscr{O}_T(m(K_Y+\Xi)+H')).$$

Therefore

$$|m(K_T+\Xi_T)| + m(\Gamma_T - \Xi_T) + H'_{|T} \subset |m(K_Y+\Gamma)+H'|_T. \tag{6}$$

We claim that

$$\Omega + A'_{|S} \geq (f_{|T})_*\Xi_T \geq \Theta + A'_{|S} \tag{7}$$

and so, as $(S,\Omega+A'_{|S})$ is canonical, we have

$$|m(K_S+\Theta)| + m((f_{|T})_*\Xi_T - \Theta) \subset |m(K_S+(f_{|T})_*\Xi_T)| = (f_{|T})_*|m(K_T+\Xi_T)|.$$

Pushing forward the inclusion (6), we obtain (5) for $H = f_*H'$.

We will now prove the inequality (7) claimed above. We have $\Xi_T \leq \Gamma_T$ and $(f_{|T})_*\Gamma_T = \Omega + A'_{|S}$ and so the first inequality follows.

In order to prove the second inequality, let P be any prime divisor on S and let $P' = (f_{|T})_*^{-1}P$. Assume that $P \subset \operatorname{Supp}\Omega$, and thus $P' \subset \operatorname{Supp}\Gamma_T$. Then there is a component Q of the support of Γ such that

$$\operatorname{mult}_{P'}\operatorname{Fix}|l(K_Y+\Gamma)|_T = \operatorname{mult}_Q \operatorname{Fix}|l(K_Y+\Gamma)|$$

and $\operatorname{mult}_{P'}\Gamma_T = \operatorname{mult}_Q\Gamma$. Therefore

$$\operatorname{mult}_{P'}\Xi_T = \operatorname{mult}_{P'}\Gamma_T - \min\{\operatorname{mult}_{P'}\Gamma_T, \operatorname{mult}_{P'}\tfrac{1}{l}\operatorname{Fix}|l(K_Y+\Gamma)|_T\}.$$

Notice that $\operatorname{mult}_{P'}\Gamma_T = \operatorname{mult}_P(\Omega + A'_{|S})$ and since $E_{|T}$ is exceptional, we have that

$$\operatorname{mult}_{P'}\operatorname{Fix}|l(K_Y+\Gamma)|_T = \operatorname{mult}_P\operatorname{Fix}|l(K_X+\Delta+A')|_S.$$

Therefore $(f_{|T})_*\Xi_T = \Omega + A'_{|S} - \Omega \wedge \frac{1}{l}\operatorname{Fix}|l(K_X+\Delta+A')|_S$. The inequality now follows from (4).

Step 2. Therefore, for any $\Sigma \in |k(K_S+\Theta)|$ and any $m>0$ sufficiently divisible, we may choose a divisor $G \in |m(K_X+\Delta)+mA'+H|$ such that $G_{|S} = \frac{m}{k}\Sigma + m(\Omega-\Theta) + (mA'+H)_{|S}$. If we define $\Lambda = \frac{k-1}{m}G + \Delta - S - A$, then

$$k(K_X+\Delta) \sim_{\mathbb{Q}} K_X + S + \Lambda + A' - \tfrac{k-1}{m}H,$$

where $A' - \frac{k-1}{m}H$ is ample as $m \gg 0$. By [HM08, 4.4(3)], we have a surjective homomorphism

$$H^0(X, \mathscr{J}_{S,\Lambda}(k(K_X+\Delta))) \to H^0(S, \mathscr{J}_{\Lambda_{|S}}(k(K_X+\Delta))).$$

Since (S,Ω) is canonical, $(S, \Omega + \frac{k-1}{m}H_{|S})$ is klt as $m \gg 0$, and therefore $\mathscr{J}_{\Omega+\frac{k-1}{m}H_{|S}} = \mathscr{O}_S$. Since

$$\Lambda_{|S} - (\Sigma + k(\Omega-\Theta)) = \tfrac{k-1}{m}G_{|S} + \Omega - A_{|S} - (\Sigma + k(\Omega-\Theta)) \le \Omega + \tfrac{k-1}{m}H_{|S},$$

then by [HM08, 4.3(3)] we have $\mathscr{I}_{\Sigma+k(\Omega-\Theta)} \subset \mathscr{J}_{\Lambda_{|S}}$, and so

$$\Sigma + k(\Omega-\Theta) \in |k(K_X+\Delta)|_S,$$

which finishes the proof. □

The main result of this section is the following.

Theorem 7.4. *Let X be a smooth variety of dimension n, S a smooth prime divisor and A a general ample $\mathbb{Q}$-divisor on X. For $i = 1,\dots,\ell$ let $D_i = k_i(K_X+\Delta_i)$, where $(X, \Delta_i = S + B_i + A)$ is a log smooth plt pair with $\lfloor\Delta_i\rfloor = S$ and $|D_i| \neq \emptyset$. Assume Theorems A_{n-1}, B_n and C_{n-1}. Then the algebra $\operatorname{res}_S R(X; D_1,\dots,D_\ell)$ is finitely generated.*

Proof. Step 1. I first show that we can assume $S \notin \operatorname{Fix}|D_i|$ for all i.

To prove this, let K_X be a divisor with $\mathscr{O}_X(K_X) \simeq \omega_X$ and $\operatorname{Supp} A \not\subset \operatorname{Supp} K_X$, and let Λ be the monoid in $\operatorname{Div}(X)$ generated by the components of K_X and $\sum \Delta_i$. Denote $\mathscr{C}_S = \{P \in \Lambda_{\mathbb{R}} : S \notin \mathbf{B}(P)\}$. By Theorem B, the set $\mathscr{A} = \sum_i \mathbb{R}_+ D_i \cap \mathscr{C}_S$ is a rational polyhedral cone.

The monoid $\sum_{i=1}^{\ell}\mathbb{R}_+D_i\cap\Lambda$ is finitely generated and let $P_1,\dots,P_q$ be its generators with $P_i=D_i$ for $i=1,\dots,\ell$. Let $\mu\colon\bigoplus_{i=1}^{q}\mathbb{N}e_i\to\operatorname{Div}(X)$ be an additive map from a simplicial monoid such that $\mu(e_i)=P_i$. Therefore $\mathscr{S}=\mu^{-1}(\mathscr{A}\cap\Lambda)\cap\bigoplus_{i=1}^{\ell}\mathbb{N}e_i$ is a finitely generated monoid. Let $h_1,\dots,h_m$ be generators of $\mathscr{S}$, and observe that $\mu(h_i)$ is a multiple of an adjoint bundle for every i.

Since $\operatorname{res}_S H^0(X,\mu(s))=0$ for every $s\in\big(\bigoplus_{i=1}^{\ell}\mathbb{N}e_i\big)\backslash\mathscr{S}$, we have that the restricted algebra $\operatorname{res}_S R(X,\mu(\bigoplus_{i=1}^{\ell}\mathbb{N}e_i))=\operatorname{res}_S R(X;D_1,\dots,D_\ell)$ is finitely generated if and only if $\operatorname{res}_S R(X,\mu(\mathscr{S}))$ is. Since we have the diagram

$$\begin{array}{ccc} R(X;\mu(h_1),\dots,\mu(h_m)) & \longrightarrow & R(X,\mu(\mathscr{S})) \\ \downarrow & & \downarrow \\ \operatorname{res}_S R(X;\mu(h_1),\dots,\mu(h_m)) & \longrightarrow & \operatorname{res}_S R(X,\mu(\mathscr{S})) \end{array}$$

where the horizontal maps are natural projections and the vertical maps are restrictions to S, it is enough to prove that the restricted algebra $\operatorname{res}_S R(X;\mu(h_1),\dots,\mu(h_m))$ is finitely generated. By passing to a truncation, I can assume further that $S\notin\operatorname{Fix}|\mu(h_i)|$ for $i=1,\dots,m$.

Step 2. Therefore I can assume $\mathscr{S}=\bigoplus_{i=1}^{\ell}\mathbb{N}e_i$ and $\mu(e_i)=D_i$ for every i. For $s=\sum_{i=1}^{\ell}t_ie_i\in\mathscr{S}_{\mathbb{Q}}$ and $t_s=\sum_{i=1}^{\ell}t_ik_i$, denote $\Delta_s=\sum_{i=1}^{\ell}t_ik_i\Delta_i/t_s$ and $\Omega_s=(\Delta_s-S)_{|S}$. Observe that

$$R(X;D_1,\dots,D_\ell)=\bigoplus_{s\in\mathscr{S}}H^0(X,t_s(K_X+\Delta_s)).$$

In this step I show that we can assume that the pair $(S,\Omega_s+A_{|S})$ is terminal for every $s\in\mathscr{S}_{\mathbb{Q}}$.

Let $\sum F_k=\bigcup_i\operatorname{Supp}B_i$, and denote $\mathbf{B}_i=\mathbf{B}(X,\Delta_i)$ and $\mathbf{B}=\mathbf{B}(X,S+\nu\sum_k F_k+A)$, where $\nu=\max_{i,k}\{\operatorname{mult}_{F_k}B_i\}$. By Lemma 2.3 there is a log resolution $f\colon Y\to X$ such that the components of $\{\mathbf{B}_Y\}$ do not intersect, and denote $D_i'=k_i(K_Y+\mathbf{B}_{iY})$. Observe that

$$R(X;D_1,\dots,D_\ell)\simeq R(Y;D_1',\dots,D_\ell'). \tag{8}$$

Since $B_i\leq\nu\sum_k F_k$, by comparing discrepancies we see that the components of $\{\mathbf{B}_{iY}\}$ do not intersect for every i, and notice that $f^*A=f_*^{-1}A\leq\mathbf{B}_{iY}$ since A is general. Denote $\Delta_s'=\sum_{i=1}^{\ell}t_ik_i\mathbf{B}_{iY}/t_s$. Let H be a small effective f-exceptional $\mathbb{Q}$-divisor such that $A'\sim_{\mathbb{Q}}f^*A-H$ is a general ample $\mathbb{Q}$-divisor, and let $T=f_*^{-1}S$. Then, setting $\Psi_s=\Delta_s'-f^*A-T+H\geq 0$ and $\Omega_s'=\Psi_{s|T}+A'_{|T}$, the pair $(T,\Omega_s'+A'_{|T})$ is terminal and $K_Y+T+\Psi_s+A'\sim_{\mathbb{Q}}K_Y+\Delta_s'$. Now replace X by Y, S by T, Δ_s by $T+\Psi_s+A'$ and Ω_s by Ω_s'.

Step 3. For every $s\in\mathscr{S}$, denote $F_s=\frac{1}{t_s}\operatorname{Fix}|t_s(K_X+\Delta_s)|_S$ and $F_s^\sharp=\liminf\limits_{m\to\infty}F_{ms}$. Define the maps $\Theta\colon\mathscr{S}\to\operatorname{Div}(S)_{\mathbb{Q}}$ and $\Theta^\sharp\colon\mathscr{S}\to\operatorname{Div}(S)_{\mathbb{Q}}$ by

$$\Theta(s)=\Omega_s-\Omega_s\wedge F_s,\qquad\Theta^\sharp(s)=\Omega_s-\Omega_s\wedge F_s^\sharp.$$

Then, denoting $\Theta_s = \Theta(s)$ and $\Theta_s^\sharp = \Theta^\sharp(s)$, we have

$$\operatorname{res}_S R(X; D_1, \dots, D_\ell) \simeq \bigoplus_{s \in \mathscr{S}} H^0(S, t_s(K_S + \Theta_s)) \tag{9}$$

by Corollary 7.2. Furthermore, for $s \in \mathscr{S}$ let $\varepsilon > 0$ be a rational number such that $\varepsilon(K_X + \Delta_s) + A$ is ample. Then by Theorem 7.1 we have

$$|k_s(K_S + \Omega_s - \Phi_s)| + k_s\Phi_s \subset |k_s(K_X + \Delta_s)|_S$$

for any Φ_s and k_s such that $k_s\Delta_s, k_s\Phi_s \in \operatorname{Div}(X)$ and $\Omega_s \wedge (1 - \frac{\varepsilon}{k_s})F_s \le \Phi_s \le \Omega_s$. Then similarly as in the proof of [HM08, 7.1], by Lemma 5.8 we have that $\Omega_s \wedge F_s^\sharp$ is rational and

$$\operatorname{res}_S R(X, k_s^\sharp(K_X + \Delta_s)) \simeq R(S, k_s^\sharp(K_S + \Theta_s^\sharp)), \tag{10}$$

where $k_s^\sharp\Theta_s^\sharp$ and $k_s^\sharp\Delta_s$ are both Cartier. Note also, by the same proof, that $G \not\subset \mathbf{B}(K_S + \Theta_s^\sharp)$ for every component G of $\Theta_s^\sharp$. In particular, $\Theta_{k_s^\sharp ps} = \Theta_{k_s^\sharp s} = \Theta_s^\sharp$ for every $p \in \mathbb{N}$.

Define maps $\lambda\colon \mathscr{S} \to \operatorname{Div}(S)_\mathbb{Q}$ and $\lambda^\sharp\colon \mathscr{S} \to \operatorname{Div}(S)_\mathbb{Q}$ by

$$\lambda(s) = t_s(K_S + \Theta_s), \qquad \lambda^\sharp(s) = t_s(K_S + \Theta_s^\sharp).$$

Then by Lemma 5.5, $\lambda^\sharp$ extends to $\mathscr{S}_\mathbb{R}$. By Theorem 7.6 below, there is a finite rational polyhedral subdivision $\mathscr{S}_\mathbb{R} = \bigcup \mathscr{C}_i$ such that $\lambda^\sharp$ is linear on each $\mathscr{C}_i$. In particular, there is a sufficiently divisible positive integer κ such that $\kappa\lambda^\sharp(s)$ is Cartier for every $s \in \mathscr{S}$, and thus $\kappa\lambda^\sharp(s) = \lambda(\kappa s)$ for every $s \in \mathscr{S}$. Therefore the restriction of λ to $\mathscr{S}_i^{(\kappa)}$ is additive, where $\mathscr{S}_i = \mathscr{S} \cap \mathscr{C}_i$. If $s_1^i, \dots, s_z^i$ are generators of $\mathscr{S}_i^{(\kappa)}$, then the ring $R(S; \lambda(s_1^i), \dots, \lambda(s_z^i))$ is finitely generated by Theorem A and Lemma 5.4(1), and so is the algebra $R(S, \lambda(\mathscr{S}_i^{(\kappa)}))$ by projection. Hence the algebra $\bigoplus_{s \in \mathscr{S}} H^0(S, \lambda(s))$ is finitely generated, and this together with (9) finishes the proof. □

It remains to prove that the map $\lambda^\sharp$ is rationally piecewise linear, and this is done in Theorem 7.6 below. The first step towards this goal is the following result.

Theorem 7.5. *For any $s, t \in \mathscr{S}_\mathbb{R}$ we have*

$$\lim_{\varepsilon \downarrow 0} \Theta^\sharp_{s+\varepsilon(t-s)} = \Theta_s^\sharp.$$

Proof. Step 1. First I prove that $\Theta_s^\sigma = \Theta_s^\sharp$, where

$$\Theta_s^\sigma = \Omega_s - \Omega_s \wedge N_\sigma\|K_X + \Delta_s\|_S,$$

cf. Remark 2.9. I am closely following the argument from the proof of the non-vanishing theorem in [Hac08]. Let r be a positive integer as in Lemma 5.8, let $\phi < 1$ be the smallest positive coefficient of $\Omega_s - \Theta_s^\sigma$ if it exists, and set $\phi = 1$ otherwise. Let $V \subset \operatorname{Div}(X)_\mathbb{R}$ and $W \subset \operatorname{Div}(S)_\mathbb{R}$ be the smallest rational affine spaces containing Δ_s and Θ_s^σ respectively. Let $0 < \eta \ll 1$ be a rational number such that $\eta(K_X + \Delta_s) + \frac{1}{2}A$ is ample, and such that if $\Delta' \in V$ with $\|\Delta' - \Delta_s\| < \eta$, then $\Delta' - \Delta_s + \frac{1}{2}A$ is ample. Then by Lemma 6.3 there are rational points $(\Psi_i, \Theta_i) \in V \times W$, integers $p_i \gg 0$ and $r_i \in \mathbb{R}_{>0}$ such that $(\Psi_i, \Theta_i, r, p_i, r_i)$ uniformly approximate $(\Delta_s, \Theta_s^\sigma) \in V \times W$ with error $\phi\eta/2$. Observe that then $\Theta_i \le \Omega_i = (\Psi_i - S)_{|S}$.

Step 2. Set $A_i = A/p_i$. In this step I prove

$$|p_i(K_S + \Theta_i)| + p_i(\Omega_i - \Theta_i) \subset |p_i(K_X + \Psi_i)|_S. \tag{11}$$

First observe that since $S \not\subset \mathbf{B}(K_X + \Delta_s)$ and $\Psi_i - \Delta_s + A_i$ is ample, we have $S \not\subset \mathbf{B}(K_X + \Psi_i + A_i)$, and thus by Proposition 7.3 it is enough to show that for any component $P \subset \operatorname{Supp}\Omega_s$, and for any $l > 0$ sufficiently divisible, we have

$$\operatorname{mult}_P(\Omega_i \wedge \tfrac{1}{l}\operatorname{Fix}|l(K_X + \Psi_i + A_i)|_S) \le \operatorname{mult}_P(\Omega_i - \Theta_i). \tag{12}$$

If $\phi = 1$, (12) follows immediately from Lemma 2.10. Now assume $0 < \phi < 1$. Since $\|\Omega_s - \Omega_i\| < \phi\eta/2p_i$ and $\|\Theta_s^\sigma - \Theta_i\| < \phi\eta/2p_i$, it suffices to show that

$$\operatorname{mult}_P(\Omega_i \wedge \tfrac{1}{l}\operatorname{Fix}|l(K_X + \Psi_i + A_i)|_S) \le (1 - \tfrac{\eta}{p_i})\operatorname{mult}_P(\Omega_s - \Theta_s^\sigma).$$

Let $\delta > \eta/p_i$ be a rational number such that $\delta(K_X + \Psi_i) + \frac{1}{2}A_i$ is ample. Since

$$K_X + \Psi_i + A_i = (1-\delta)(K_X + \Psi_i + \tfrac{1}{2}A_i) + \big(\delta(K_X + \Psi_i) + \tfrac{1+\delta}{2}A_i\big),$$

we have

$$\operatorname{ord}_P \|K_X + \Psi_i + A_i\|_S \le (1-\delta)\operatorname{ord}_P \|K_X + \Psi_i + \tfrac{1}{2}A_i\|_S,$$

and thus

$$\operatorname{mult}_P \tfrac{1}{l}\operatorname{Fix}|l(K_X + \Psi_i + A_i)|_S \le (1 - \tfrac{\eta}{p_i})\sigma_P\|K_X + \Psi_i\|_S$$

for l sufficiently divisible, cf. Lemma 2.10.

Step 3. Let A_m be ample divisors with $\operatorname{Supp}A_m \subset \operatorname{Supp}(\Delta_s - S)$ such that $\Delta_s + A_m$ are $\mathbb{Q}$-divisors and $\lim\limits_{m\to\infty}\|A_m\| = 0$. Denote $\Delta_m = \Delta_s + A_m$, $\Omega_m = (\Delta_m - S)_{|S}$ and

$$\Theta_m^\sigma = \Omega_m - \Omega_m \wedge N_\sigma\|K_X + \Delta_m\|_S$$

for $m \gg 0$. Observe that $\Theta_s^\sigma = \lim\limits_{m\to\infty}\Theta_m^\sigma$ by Lemma 2.10(2), and note that

$$N_\sigma\|K_X + \Delta_m\|_S = \sum \operatorname{ord}_P \|K_X + \Delta_m\|_S \cdot P$$

for all prime divisors P on S for all m, cf. Remark 2.9. But then as in Step 3 of the proof of Theorem 7.4, no component of Θ_m^σ is in $\mathbf{B}(K_S + \Theta_m^\sigma)$, and thus, by Lemma 5.8 and since $\Theta_m^\sigma \ge \Theta_s^\sigma$ for every m, no component of Θ_s^σ is in $\mathbf{B}(K_S + \Theta_s^\sigma)$. Since $p_i\Theta_i/r$ is Cartier and $\Theta_i \in W$, by (11) we have

$$\Omega_i - \Theta_i \ge \Omega_i \wedge \tfrac{1}{p_i}\operatorname{Fix}|p_i(K_X + \Psi_i)|_S \ge \Omega_i - \Theta_i^\sharp,$$

and so $\Theta_i^\sharp \ge \Theta_i$, where

$$\Theta_i^\sharp = \Omega_i - \Omega_i \wedge \liminf_{m\to\infty}\tfrac{1}{m}\operatorname{Fix}|m(K_X + \Psi_i)|_S.$$

Let P be a prime divisor on S. If $\operatorname{mult}_P\Theta_s^\sigma=0$, then $\operatorname{mult}_P\Theta_s^\sharp=0$ since $\Theta_s^\sigma\geq\Theta_s^\sharp$ by Lemma 2.10. Otherwise $\operatorname{mult}_P\Theta_i>0$ for all i and thus $\operatorname{mult}_P\Theta_i^\sharp>0$. Therefore by concavity we have

$$\operatorname{mult}_P\Theta_s^\sharp\geq\sum r_i\operatorname{mult}_P\Theta_i^\sharp\geq\sum r_i\operatorname{mult}_P\Theta_i=\operatorname{mult}_P\Theta_s^\sigma,$$

proving the claim from Step 1.

Step 4. Now let C be an ample $\mathbb{Q}$-divisor such that $\Delta_t-\Delta_s+C$ is ample. Then by the claim from Step 1 and by Lemma 2.10,

$$\begin{aligned}\Omega_s-\Theta_s^\sharp&=\Omega_s\wedge\lim_{\varepsilon\downarrow 0}\big(\sum\operatorname{ord}_P\|K_X+\Delta_s+\varepsilon(\Delta_t-\Delta_s+C)\|_S\cdot P\big)\\&\leq\Omega_s\wedge\lim_{\varepsilon\downarrow 0}\big(\sum\operatorname{ord}_P\|K_X+\Delta_s+\varepsilon(\Delta_t-\Delta_s)\|_S\cdot P\big)\leq\Omega_s-\Theta_s^\sharp,\end{aligned}$$

where the last inequality follows from convexity. Therefore all inequalities are equalities, and this completes the proof. □

Recall that $\mathscr{S}=\bigoplus_{i=1}^\ell\mathbb{N}e_i$. Let Z be a prime divisor on S and let $\mathscr{L}_Z$ be the closure in $\mathscr{S}_\mathbb{R}$ of the set $\{s\in\mathscr{S}_\mathbb{R}:\operatorname{mult}_Z\Theta_s^\sharp>0\}$. Then $\mathscr{L}_Z$ is a closed cone. Let $\lambda_Z^\sharp\colon\mathscr{S}_\mathbb{R}\to\mathbb{R}$ be the function given by $\lambda_Z^\sharp(s)=\operatorname{mult}_Z\lambda^\sharp(s)$, and similarly for $\Theta_Z^\sharp$.

Theorem 7.6. *For every prime divisor Z on S, the map $\lambda_Z^\sharp$ is rationally piecewise linear. Therefore, $\lambda^\sharp$ is rationally piecewise linear.*

Proof. Step 1. Let $G_1,\dots,G_w$ be prime divisors on X different from S and $\operatorname{Supp}A$ such that $\operatorname{Supp}(\Delta_s-S-A)\subset\sum G_i$ for every $s\in\mathscr{S}$. Let $\nu=\max\{\operatorname{mult}_{G_i}\Delta_s:s\in\mathscr{S},i=1,\dots,w\}<1$, and let $0<\eta\ll 1-\nu$ be a rational number such that $A-\eta\sum G_i$ is ample. Let $A'\sim_\mathbb{Q}A-\eta\sum G_i$ be a general ample $\mathbb{Q}$-divisor. Define $\Delta_s'=\Delta_s-A+\eta\sum G_i+A'\geq 0$, and observe that $\Delta_s'\sim_\mathbb{Q}\Delta_s$, $\lfloor\Delta_s'\rfloor=S$ and $(S,(\Delta_s'-S)_{|S})$ is terminal.

Define the map $\chi\colon\mathscr{S}\to\operatorname{Div}(X)$ by $\chi(s)=\kappa t_s(K_X+\Delta_s')$, for κ sufficiently divisible. Then as before, we can construct maps $\tilde\Theta^\sharp\colon\mathscr{S}_\mathbb{R}\to\operatorname{Div}(S)_\mathbb{R}$, $\tilde\lambda^\sharp\colon\mathscr{S}_\mathbb{R}\to\operatorname{Div}(S)_\mathbb{R}$ and $\tilde\lambda_Z^\sharp\colon\mathscr{S}_\mathbb{R}\to\mathbb{R}$ associated to χ. By construction, $\operatorname{ord}_Z\|\tilde\lambda_s^\sharp/\kappa t_s\|_S=\operatorname{ord}_Z\|\lambda_s^\sharp/t_s\|_S$, and thus $\operatorname{mult}_Z\tilde\Theta_s^\sharp=\operatorname{mult}_Z\Theta_s^\sharp+\eta$ for every $s\in\mathscr{L}_Z$. Let $\tilde{\mathscr{L}}_Z$ be the closure in $\mathscr{S}_\mathbb{R}$ of the set $\{s\in\mathscr{S}_\mathbb{R}:\operatorname{mult}_Z\tilde\Theta_s^\sharp>0\}$, and observe that $\mathscr{L}_Z$ is the closure in $\mathscr{S}_\mathbb{R}$ of the set $\{s\in\mathscr{S}_\mathbb{R}:\operatorname{mult}_Z\tilde\Theta_s^\sharp>\eta\}$. Note that $\operatorname{mult}_Z\tilde\Theta_s^\sharp\geq\eta$ for every $s\in\mathscr{L}_Z$ by Theorem 7.5 applied to 2-planes that intersect $\mathscr{L}_Z$.

Step 2. In this step I prove that there is a rational polyhedral cone $\mathscr{M}_Z$ such that $\mathscr{L}_Z\subset\mathscr{M}_Z\subset\tilde{\mathscr{L}}_Z$, and so the map $\tilde\lambda_Z^\sharp|_{\mathscr{M}_Z}$ is *superlinear.*

To that end, I will show that for every point $x\in\mathscr{L}_Z$ there is a neighbourhood $\mathscr{U}$ of x such that $\mathscr{U}\cap\mathscr{S}_\mathbb{R}\subset\tilde{\mathscr{L}}_Z$, in the sup-norm. Namely, let $x_1,\dots,x_m\in\mathscr{S}_\mathbb{R}$ be points different from x such that $x\in\sum\mathbb{R}_+x_i$ and $B(x,\varepsilon)\cap\mathscr{S}_\mathbb{R}=B(x,\varepsilon)\cap\sum\mathbb{R}_+x_i$ for some $\varepsilon>0$. Then by Theorem 7.5, for each i there exists a point y_i in the segment (x,x_i) such that $\operatorname{mult}_Z\tilde\Theta_{y_i}^\sharp>0$. Therefore it is sufficient to take any neighbourhood $\mathscr{U}$ of x such that $\mathscr{U}\cap\mathscr{S}_\mathbb{R}\subset\sum\mathbb{R}_+y_i$.

By compactness, there is a rational number $0 < \xi \ll 1$ and finitely many rational points $z_1, \dots, z_p \in \mathscr{L}_Z$ such that $\mathscr{L}_Z \subset \bigcup B(z_i, \xi) \cap \mathscr{S}_\mathbb{R} \subset \tilde{\mathscr{L}}_Z$. The convex hull $\mathscr{B}$ of $\bigcup B(z_i, \xi)$ is a rational polytope, and define $\mathscr{M}_Z = \mathscr{B} \cap \mathscr{S}_\mathbb{R}$.

Step 3. By Theorem 7.9 below, for any 2-plane $H \subset \mathbb{R}^\ell$ the map $\tilde{\lambda}_Z^\sharp|_{\mathscr{M}_Z \cap H}$ is piecewise linear, and thus $\tilde{\lambda}_Z^\sharp|_{\mathscr{M}_Z}$ is piecewise linear by Lemma 4.4.

To prove that $\tilde{\lambda}_Z^\sharp|_{\mathscr{M}_Z}$ is rationally piecewise linear, we can assume that $\mathscr{M}_Z \subset \mathbb{R}^k$ and $\dim \mathscr{M}_Z = k$. Let $\mathscr{M}_Z = \bigcup \mathscr{C}_m$ be a finite polyhedral decomposition such that the maps $\tilde{\lambda}_Z^\sharp|_{\mathscr{C}_m}$ are linear and their linear extensions to $\mathbb{R}^k$ are pairwise different. Let $\mathscr{H}$ be the $(k-1)$-plane which contains a common $(k-1)$-dimensional face of cones $\mathscr{C}_i$ and $\mathscr{C}_j$ and assume $\mathscr{H}$ is not rational. One can easily prove by induction on the dimension that a real vector space cannot be a union of countably many codimension 1 affine subspaces, and in particular there is a point $s \in \mathscr{C}_i \cap \mathscr{C}_j$ such that the minimal affine rational space containing s has dimension k. Then as in Step 1 of the proof of Theorem 7.9 below there is a k-dimensional cone $\widetilde{\mathscr{C}}$ such that $s \in \operatorname{int} \widetilde{\mathscr{C}}$ and the map $\tilde{\lambda}_Z^\sharp|_{\widetilde{\mathscr{C}}}$ is linear. But then the cones $\widetilde{\mathscr{C}} \cap \mathscr{C}_i$ and $\widetilde{\mathscr{C}} \cap \mathscr{C}_j$ are k-dimensional and linear extensions of $\tilde{\lambda}_Z^\sharp|_{\mathscr{C}_i}$ and $\tilde{\lambda}_Z^\sharp|_{\mathscr{C}_j}$ coincide since they are equal to the linear extension of $\tilde{\lambda}_Z^\sharp|_{\widetilde{\mathscr{C}}}$, a contradiction. Therefore all $(k-1)$-dimensional faces of the cones $\mathscr{C}_m$ belong to rational $(k-1)$-planes and thus $\mathscr{C}_m$ are rational cones.

Therefore the map $\tilde{\lambda}_Z^\sharp|_{\mathscr{M}_Z}$ is rationally piecewise linear, and since $\mathscr{L}_Z$ is the closure of the set $\{s \in \mathscr{S}_\mathbb{R} : \operatorname{mult}_Z \tilde{\Theta}_s^\sharp > \eta\}$, we have that $\mathscr{L}_Z$ is a rational polyhedral cone, the map $\tilde{\lambda}_Z^\sharp|_{\mathscr{L}_Z}$ is rationally piecewise linear, and therefore so is $\lambda_Z^\sharp$. Now it is trivial that $\lambda^\sharp$ is a rationally piecewise linear map. □

Thus it remains to prove that $\lambda_Z^\sharp|_{\mathscr{M}_Z \cap H}$ is piecewise linear for every 2-plane $H \subset \mathbb{R}^\ell$. Replacing $\mathscr{S}$ by a free monoid spanned by generators of $\mathscr{M}_Z \cap \mathscr{S}$, it is enough to assume, and I will until the end of the section, that $\lambda_Z^\sharp$ is a superlinear function on $\mathscr{S}_\mathbb{R}$.

I will need the following result in the proof of Theorem 7.9.

Lemma 7.7. *If $\Theta_Z^\sharp$ is not identically zero, then for every $s \in \mathscr{S}_\mathbb{R}$ we have $Z \not\subset \mathbf{B}(K_S + \Theta_s^\sharp)$.*

Proof. Fix $s \in \mathscr{S}_\mathbb{R}$, let t be any element of $\mathscr{S}_\mathbb{R}$ such that $\Theta_Z^\sharp(t) \neq 0$ and denote $s_\varepsilon = \varepsilon t + (1-\varepsilon)s$ for $\varepsilon \in [0,1]$. By concavity we have

$$\Theta_Z^\sharp(s_\varepsilon) \geq \varepsilon \Theta_Z^\sharp(t) + (1-\varepsilon)\Theta_Z^\sharp(s) \geq \varepsilon \Theta_Z^\sharp(t),$$

and thus $Z \not\subset \mathbf{B}(K_S + \Theta_{s_\varepsilon}^\sharp)$ for $\varepsilon > 0$ by Step 3 of the proof of Theorem 7.5. But then $Z \not\subset \mathbf{B}(K_S + \Theta_s^\sharp)$ by Lemma 5.8 since $\lim\limits_{\varepsilon \to 0} s_\varepsilon = s$. □

Let C_s be a local Lipschitz constant of $\Theta^\sharp$ around $s \in \mathscr{S}_\mathbb{R}$ in the smallest rational affine space containing s. For every $s \in \mathscr{S}$, let ϕ_s be the smallest positive coefficient of $\Omega_s - \Theta_s^\sharp$, or set $\phi_s = 0$ if $\Omega_s = \Theta_s^\sharp$. Observe that ϕ_s is a continuous function around every point in a neighbourhood contained in the smallest rational affine space containing that point. Also, ϕ_S is continuous on segments by Theorem 7.5.

Theorem 7.8. *Fix $s \in \mathscr{S}_{\mathbb{R}}$ and let $U \subset \mathbb{R}^\ell$ be the smallest rational affine subspace containing s. If $\phi_s > 0$, let $0 < \delta \ll 1$ be a rational number such that $\phi_u > 0$ for $u \in U$ with $\|u - s\| \leq \delta$, set $\phi = \min\{\phi_u : u \in U, \|u - s\| \leq \delta\}$ and let $0 < \varepsilon \ll \delta$ be a rational number such that $(C_s/\phi + 1)\varepsilon(K_X + \Delta_s) + A$ is ample. If $\phi_s = 0$ and $\operatorname{Supp}\Delta_s = \sum F_i$, let $0 < \varepsilon \ll 1$ be a rational number such that $\sum f_i F_i + A$ is ample for any $f_i \in (-\varepsilon, \varepsilon)$, and set $\phi = 1$. Let $t \in U \cap \mathscr{S}_{\mathbb{Q}}$ and $p_t \gg 0$ be an integer such that $\|t - s\| < \varepsilon/p_t$, $p_t\Delta_t/r$ is Cartier for r as in Lemma 5.8 and $S \not\subset \mathbf{B}(K_X + \Delta_t)$. Then for any divisor Θ on S such that $0 \leq \Theta \leq \Omega_t$, $\|\Theta - \Theta_s^\sharp\| < \phi\varepsilon/p_t$ and $p_t\Theta/r$ is Cartier we have*

$$|p_t(K_S + \Theta)| + p_t(\Omega_t - \Theta) \subset |p_t(K_X + \Delta_t)|_S.$$

Proof. Set $A_t = A/p_t$. By Proposition 7.3 it is enough to prove that for any component $P \subset \operatorname{Supp}\Omega_s$, and for any $l > 0$ sufficiently divisible, we have

$$\operatorname{mult}_P(\Omega_t \wedge \tfrac{1}{l}\operatorname{Fix}|l(K_X + \Delta_t + A_t)|_S) \leq \operatorname{mult}_P(\Omega_t - \Theta). \tag{13}$$

Assume first that $\phi_s = 0$. Then in particular $\operatorname{ord}_P \|K_X + \Delta_s\|_S = 0$ and $\Delta_t - \Delta_s + A_t$ is ample since $\|\Delta_t - \Delta_s\| < \varepsilon/p_t$, so

$$\operatorname{ord}_P \|K_X + \Delta_t + A_t\|_S = \operatorname{ord}_P \|K_X + \Delta_s + (\Delta_t - \Delta_s + A_t)\|_S \leq \operatorname{ord}_P \|K_X + \Delta_s\|_S = 0.$$

Since for l sufficiently divisible we have

$$\operatorname{mult}_P \tfrac{1}{l}\operatorname{Fix}|l(K_X + \Delta_t + A_t)|_S = \operatorname{ord}_P \|K_X + \Delta_t + A_t\|_S \tag{14}$$

as in Step 3 of the proof of Theorem 7.4, we obtain (13).

Now assume that $\phi_s \neq 0$ and set $C = C_s/\phi$. By Lipschitz continuity we have $\|\Theta_t^\sharp - \Theta_s^\sharp\| < C\phi\varepsilon/p_t$, so $\|\Theta_t^\sharp - \Theta\| < (C+1)\phi\varepsilon/p_t$. Therefore it suffices to show that

$$\operatorname{mult}_P(\Omega_t \wedge \tfrac{1}{l}\operatorname{Fix}|l(K_X + \Delta_t + A_t)|_S) \leq (1 - \tfrac{C+1}{p_t}\varepsilon)\operatorname{mult}_P(\Omega_t - \Theta_t^\sharp).$$

Since $p_t \gg 0$, we can choose a rational number $\eta > (C+1)\varepsilon/p_t$ such that $\eta(K_X + \Delta_t) + A_t$ is ample. From

$$K_X + \Delta_t + A_t = (1 - \eta)(K_X + \Delta_t) + (\eta(K_X + \Delta_t) + A_t)$$

we have

$$\operatorname{ord}_P \|K_X + \Delta_t + A_t\|_S \leq (1 - \eta)\operatorname{ord}_P \|K_X + \Delta_t\|_S,$$

and thus by (14),

$$\operatorname{mult}_P \tfrac{1}{l}\operatorname{Fix}|l(K_X + \Delta_t + A_t)|_S \leq (1 - \tfrac{C+1}{p_t}\varepsilon)\operatorname{ord}_P \|K_X + \Delta_t\|_S$$

for l sufficiently divisible. □

Finally, we have

Theorem 7.9. *Fix $s \in \mathscr{S}_{\mathbb{R}}$ and let R be a ray in $\mathscr{S}_{\mathbb{R}}$ not containing s. Then there exists a ray $R' \subset \mathbb{R}_+ s + R$ not containing s such that the map $\lambda^{\sharp}_Z|_{\mathbb{R}_+ s + R'}$ is linear. In particular, for every 2-plane $H \subset \mathbb{R}^{\ell}$, the map $\lambda^{\sharp}_Z|_{\mathscr{S}_{\mathbb{R}} \cap H}$ is piecewise linear.*

Proof. Step 1. Let $U \subset \mathbb{R}^{\ell}$ be the smallest rational affine space containing s. In this step I prove that the map $\Theta^{\sharp}$ is linear in a neighbourhood of s contained in U.

Let ε and ϕ be as in Theorem 7.8. Let $W \subset \mathbb{R}^{\ell}$ and $V \subset \mathrm{Div}(S)_{\mathbb{R}}$ be the smallest rational affine spaces containing s and $\Theta^{\sharp}_s$ respectively, and let r be as in Lemma 5.8. By Lemma 6.3, there exist rational points $(t_i, \Theta'_{t_i}) \in W \times V$, integers $p_{t_i} \gg 0$ and $r_{t_i} \in \mathbb{R}_{>0}$ such that $(t_i, \Theta'_{t_i}, r, p_{t_i}, r_{t_i})$ uniformly approximate $(\Delta_s, \Theta^{\sharp}_s) \in W \times V$ with error $\phi\varepsilon$. Note that then $\Theta'_{t_i} \leq \Omega_{t_i}$.

Observe that $S \not\subset \mathbf{B}(K_X + \Delta_{t_i})$ since $t_i \in W$ for every i, $\varepsilon \ll 1$ and $\mathscr{S}_{\mathbb{R}}$ is a rational polyhedral cone. By local Lipschitz continuity and by Theorem 7.8 we have that

$$|p_{t_i}(K_S + \Theta'_{t_i})| + p_{t_i}(\Omega_{t_i} - \Theta'_{t_i}) \subset |p_{t_i}(K_X + \Delta_{t_i})|_S.$$

Since $\Theta'_{t_i} \in V$ and $p_{t_i}\Theta'_{t_i}/r$ is Cartier, $Z \not\subset \mathrm{Fix}\,|p_{t_i}(K_S + \Theta'_{t_i})|$ for every i by Lemmas 5.8 and 7.7. In particular,

$$\mathrm{mult}_Z(\Omega_{t_i} - \Theta'_{t_i}) \geq \mathrm{mult}_Z\left(\Omega_{t_i} \wedge \tfrac{1}{p_{t_i}} \mathrm{Fix}\,|p_{t_i}(K_X + \Delta_{t_i})|_S\right) \geq \mathrm{mult}_Z(\Omega_{t_i} - \Theta^{\sharp}_{t_i}),$$

and so $\Theta^{\sharp}_Z(t_i) \geq \mathrm{mult}_Z \Theta'_{t_i}$. But then by uniform approximation and since the map $\Theta^{\sharp}_Z$ is concave, we have

$$\Theta^{\sharp}_Z(s) \geq \sum r_{t_i} \Theta^{\sharp}_Z(t_i) \geq \sum r_{t_i} \mathrm{mult}_Z \Theta'_{t_i} = \Theta^{\sharp}_Z(s),$$

which proves the statement by Lemma 4.3.

Step 2. Now assume $s \in \mathscr{S}_{\mathbb{Q}}$, $\phi_s = 0$ and fix $u \in R$ such that s and u belong to a rational affine subspace $\mathscr{P}$ of $\mathbb{R}^{\ell}$. Let $\Delta\colon \mathbb{R}\mathscr{P} \to \mathrm{Div}(X)_{\mathbb{R}}$ be a linear map given by $\Delta(q_i) = \Delta_{q_i}$ for linearly independent points $q_1, \dots, q_{\ell} \in \mathscr{P} \cap \mathscr{S}_{\mathbb{Q}}$, and then extended linearly. Observe that $\Delta(q) = \Delta_q$ for every $q \in \mathscr{P} \cap \mathscr{S}_{\mathbb{R}}$.

Let W be the smallest rational affine subspace containing s and u. If there is a sequence $s_m \in (s, u]$ such that $\lim\limits_{m\to\infty} s_m = s$ and $\phi_{s_m} = 0$, then $\lambda^{\sharp}$ is linear on the cone $\mathbb{R}_+ s + \mathbb{R}_+ s_1$ by Lemma 4.3.

Therefore we can assume that there are rational numbers $0 < \varepsilon, \eta \ll 1$ such that for all $v \in [s, u]$ with $0 < \|v - s\| < 2\varepsilon$ we have $\phi_v > 0$, that for every prime divisor P on S, we have either $\mathrm{mult}_P \Omega_v > \mathrm{mult}_P \Theta^{\sharp}_v$ or $\mathrm{mult}_P \Omega_v = \mathrm{mult}_P \Theta^{\sharp}_v$ and either $\mathrm{mult}_P \Theta^{\sharp}_v = 0$ or $\mathrm{mult}_P \Theta^{\sharp}_v > 0$ for all such v, and that $\Delta_v - \Delta_s + \Xi + A$ is ample for all such v and for any divisor Ξ such that $\mathrm{Supp}\,\Xi \subset \mathrm{Supp}\,\Delta_s \cup \mathrm{Supp}\,\Delta_u$ and $\|\Xi\| < \eta$.

Let p_s be a positive integer such that $p_s\Delta_s/r$ and $p_s\Theta^{\sharp}_s/r$ are integral, where r is as in Lemma 5.8. Pick $t \in (s, u]$ such that $\|s - t\| < \varepsilon/p_s$, $\|\Theta^{\sharp}_s - \Theta^{\sharp}_t\| < \varepsilon/p_s$ which is possible by Theorem 7.5, and the smallest rational affine subspace containing t is precisely W. Let $0 < \delta \ll 1$ be a rational number such that $\phi_v > 0$ for $v \in W$ with $\|v - t\| \leq \delta$, set

$\phi = \min\{\phi_v : v \in W, \|v-t\| \leq \delta\}$ and let $0 < \xi \ll \min\{\delta, \varepsilon\}$ be a rational number such that $(C_t/\phi + 1)\xi(K_X + \Delta_t) + A$ is ample. Denote by $V \subset \mathrm{Div}(S)_{\mathbb{R}}$ the smallest rational affine space containing $\Theta_s^\sharp = \Omega_s$ and $\Theta_t^\sharp$. Then by Lemma 6.4 there exist rational points $(t_i, \Theta'_{t_i}) \in W \times V$, integers $p_{t_i} \gg 0$ and $r_{t_i} \in \mathbb{R}_{>0}$ for $i = 1, \dots, w$ such that:

(1) $(t_i, \Theta'_{t_i}, r, p_{t_i}, r_{t_i})$ uniformly approximate $(\Delta_t, \Theta_t^\sharp) \in W \times V$ with error ε, where $t_1 = s$, $\Theta'_{t_1} = \Theta^\sharp_{t_1} = \Omega_{t_1}$, $p_{t_1} = p_s$,
(2) $\|t - t_i\| < \xi/p_{t_i}$, $\|\Theta_t^\sharp - \Theta'_{t_i}\| < \phi\xi/p_{t_i}$ and (t_i, Θ'_{t_i}) belong to the smallest rational affine space containing $(t, \Theta_t^\sharp)$ for $i = 2, \dots, w-1$,
(3) $\Delta_t = \frac{p_{t_1}}{p_{t_1}+p_{t_w}}\Delta_{t_1} + \frac{p_{t_w}}{p_{t_1}+p_{t_w}}\Delta_{t_w} + \Psi$, where $\|\Psi\| < \eta/(p_{t_1}+p_{t_w})$,
(4) $\Theta_t^\sharp = \frac{p_{t_1}}{p_{t_1}+p_{t_w}}\Theta'_{t_1} + \frac{p_{t_w}}{p_{t_1}+p_{t_w}}\Theta'_{t_w} + \Phi$, where $\|\Phi\| < \eta/(p_{t_1}+p_{t_w})$.

Note that then $\Theta'_{t_i} \leq \Omega_{t_i}$ for all i, and that $\mathrm{Supp}\,\Psi \subset \mathrm{Supp}\,\Delta_t$ and $\mathrm{Supp}\,\Phi \subset \mathrm{Supp}\,\Theta_t^\sharp$ by Remarks 6.5 and 6.6 applied to the linear map Δ defined at the beginning of Step 2. Then by Theorem 7.8,

$$|p_{t_i}(K_S + \Theta'_{t_i})| + p_{t_i}(\Omega_{t_i} - \Theta'_{t_i}) \subset |p_{t_i}(K_X + \Delta_{t_i})|_S$$

for $i = 2, \dots, w-1$. Let P be a component in $\mathrm{Supp}\,\Omega_t$ and denote $A_{t_w} = A/p_{t_w}$. I claim that for $l > 0$ sufficiently divisible we have

$$\mathrm{mult}_P(\Omega_{t_w} \wedge \tfrac{1}{l}\,\mathrm{Fix}\,|l(K_X + \Delta_{t_w} + A_{t_w})|_S) \leq \mathrm{mult}_P(\Omega_{t_w} - \Theta'_{t_w}). \tag{15}$$

To that end, assume first that $\mathrm{mult}_P\,\Theta_t^\sharp = 0$. Then $\mathrm{mult}_P\,\Theta_s^\sharp = 0$ by the choice of ε, and thus $\mathrm{mult}_P\,\Theta'_{t_w} = 0$ since $\Theta'_{t_w} \in V$. Therefore

$$\mathrm{mult}_P(\Omega_{t_w} \wedge \tfrac{1}{l}\,\mathrm{Fix}\,|l(K_X + \Delta_{t_w} + A_{t_w})|_S) \leq \mathrm{mult}_P\,\Omega_{t_w} = \mathrm{mult}_P(\Omega_{t_w} - \Theta'_{t_w}).$$

Now assume that $\mathrm{mult}_P\,\Theta_t^\sharp > 0$. For l sufficiently divisible we have

$$\mathrm{mult}_P\,\tfrac{1}{l}\,\mathrm{Fix}\,|l(K_X + \Delta_{t_w} + A_{t_w})|_S = \mathrm{ord}_P\,\|K_X + \Delta_{t_w} + A_{t_w}\|_S$$

as in Step 3 of the proof of Theorem 7.4, and since $\Delta_t - \Delta_{t_1} - \frac{p_{t_1}+p_{t_w}}{p_{t_1}}\Psi + \frac{1}{p_{t_1}}A$ is ample by the choice of η,

$$\begin{aligned}\mathrm{mult}_P(\Omega_{t_w} \wedge \tfrac{1}{l}\,\mathrm{Fix}\,|l(K_X + \Delta_{t_w} + A_{t_w})|_S) &\leq \mathrm{ord}_P\,\|K_X + \Delta_{t_w} + A_{t_w}\|_S \\ &= \mathrm{ord}_P\,\big\|K_X + \Delta_t + \tfrac{p_{t_1}}{p_{t_w}}\big(\Delta_t - \Delta_{t_1} - \tfrac{p_{t_1}+p_{t_w}}{p_{t_1}}\Psi + \tfrac{1}{p_{t_1}}A\big)\big\|_S \\ &\leq \mathrm{ord}_P\,\|K_X + \Delta_t\|_S = \mathrm{mult}_P(\Omega_t - \Theta_t^\sharp).\end{aligned}$$

Combining assumptions (3) and (4) above we have

$$\Omega_t - \Theta_t^\sharp \leq \Omega_t - \Theta_t^\sharp + \tfrac{p_{t_1}}{p_{t_w}}\big(\Omega_t - \Theta_t^\sharp - \tfrac{p_{t_1}+p_{t_w}}{p_{t_1}}(\Psi_{|S} - \Phi)\big) = \Omega_{t_w} - \Theta'_{t_w},$$

and (15) is proved. Furthermore, we can choose $\varepsilon \ll 1$ and $p_{t_w} \gg 0$ such that $S \not\subset \mathbf{B}(K_X + \Delta_{t_w})$ since $\mathscr{S}_{\mathbb{R}}$ is a rational polyhedral cone. Therefore by Proposition 7.3 we have

$$|p_{t_w}(K_S + \Theta'_{t_w})| + p_{t_w}(\Omega_{t_w} - \Theta'_{t_w}) \subset |p_{t_w}(K_X + \Delta_{t_w})|_S.$$

Let $V_S \subset \mathrm{Div}(S)_{\mathbb{R}}$ be the vector space spanned by the components of $\bigcup_{s \in \mathscr{S}_{\mathbb{R}}} \mathrm{Supp}(\Omega_s - A_{|S})$. Then by Lemma 5.8, $\mathscr{B}^Z_{V_S, A_{|S}}$ is a rational polytope and $\Theta^\sharp_p \in \mathscr{B}^Z_{V_S, A_{|S}}$ for every $p \in \mathscr{S}_{\mathbb{R}}$ by Lemma 7.7. Therefore when $\varepsilon \ll 1$, as in Step 1 we have that $\lambda^\sharp_Z$ is linear on the cone $\sum_{i=1}^w \mathbb{R}_+ t_i$, and in particular on the cone $\mathbb{R}_+ s + \mathbb{R}_+ t$.

Step 3. Assume now that $s \in \mathscr{S}_{\mathbb{Q}}$, $\phi_s > 0$ and fix $u \in R$. Let again W be the smallest rational affine space containing s and u. Since $\Theta^\sharp$ is continuous on $[s,u]$ by Theorem 7.5, let $0 < \xi \ll 1$ be a rational number such that $\phi_v > 0$ for $v \in [s,u]$ with $\|v - s\| \leq 2\xi$, that for every prime divisor P on S we have either $\mathrm{mult}_P \Omega_v > \mathrm{mult}_P \Theta^\sharp_v$ or $\mathrm{mult}_P \Omega_v = \mathrm{mult}_P \Theta^\sharp_v$ for all such v, and let $\phi = \min\{\phi_v : v \in [s,u], \|v - s\| \leq 2\xi\}$.

Let p_s be a positive integer such that $p_s \Delta_s / r$ and $p_s \Theta^\sharp_s / r$ are integral, where r is as in Lemma 5.8. Let me first show that there exist a real number $0 < \varepsilon \ll \xi$ and $t \in (s,u]$ such that $\|t - s\| = \varepsilon / p_s$ and $(C_t/\phi + 1)\varepsilon(K_X + \Delta_v) + A$ is ample for all $v \in \mathscr{S}_{\mathbb{R}}$ with$\|v - s\| < 2\xi$. If $\Theta^\sharp$ is locally Lipschitz around s this is straightforward. Otherwise, assume $\Theta^\sharp$ is not locally Lipschitz around s and assume we cannot find such ε. But that means that $(C_t/\phi + 1)\|s - t\|$ is bounded from below as $t \to s$, thus there is a sequence $s_m \in (s,u]$ such that $\lim\limits_{m \to \infty} s_m = s$ and $C_{s_m}\|s_m - s\| \geq M$, where M is a constant and $C_{s_m} \to \infty$. Since a local Lipschitz constant can be taken as the maximum of local slopes of the concave function $\Theta^\sharp|_{[s,u]}$, we have that

$$\frac{\Theta^\sharp_{s_m} - \Theta^\sharp_s}{\|s_m - s\|} > C_{s_m}.$$

Therefore

$$\Theta^\sharp_{s_m} - \Theta^\sharp_s > C_{s_m}\|s_m - s\| \geq M$$

for all $m \in \mathbb{N}$, which contradicts Theorem 7.5.

Increase ε a bit, and pick $t \in (s,u]$ such that $\|s - t\| < \varepsilon / p_s$, the smallest rational subspace containing t is precisely W and $(C_t/\phi + 1)\varepsilon(K_X + \Delta_v) + A$ is ample for all $v \in \mathscr{S}_{\mathbb{R}}$ such that $\|v - s\| < 2\varepsilon$. In particular, $\Theta^\sharp$ is locally Lipschitz in a neighbourhood of t contained in W. Furthermore, by changing ϕ slightly I can assume that $\phi \leq \min\{\phi_v : v \in W, \|v - t\| \ll 1\}$. Denote by V the smallest rational affine space containing $\Theta^\sharp_s$ and $\Theta^\sharp_t$. Then by Lemma 6.4 there exist rational points $(t_i, \Theta'_{t_i}) \in W \times V$, integers $p_{t_i} \gg 0$ and $r_{t_i} \in \mathbb{R}_{>0}$ such that $(t_i, \Theta'_{t_i}, r, p_{t_i}, r_{t_i})$ uniformly approximate $(\Delta_t, \Theta^\sharp_t) \in W \times V$ with error ε, where $t_1 = s$, $\Theta'_{t_1} = \Theta^\sharp_{t_1}$, $p_{t_1} = p_s$. Note that then $\Theta'_{t_i} \leq \Omega_{t_i}$, and similarly as in Step 2 we have $S \not\subset \mathbf{B}(K_X + \Delta_{t_i})$ for all i. Therefore by Theorem 7.8,

$$|p_{t_i}(K_S + \Theta'_{t_i})| + p_{t_i}(\Omega_{t_i} - \Theta'_{t_i}) \subset |p_{t_i}(K_X + \Delta_{t_i})|_S$$

for all i. Then we finish as in Steps 1 and 2.

Step 4. Assume in this step that $s \in \mathscr{S}_{\mathbb{R}}$ is a non-rational point and fix $u \in R$. By Step 1 there is a rational cone $\mathscr{C} = \sum_{i=1}^{k} \mathbb{R}_+ g_i$ with $g_i \in \mathscr{S}_{\mathbb{Q}}$ and $k > 1$ such that $\lambda_Z^\sharp$ is linear on $\mathscr{C}$ and $s = \sum \alpha_i g_i$ with all $\alpha_i > 0$. Consider the rational point $g = \sum_{i=1}^{k} g_i$. Then by Steps 2 and 3 there is a point $s' = \alpha g + \beta u$ with $\alpha, \beta > 0$ such that the map $\lambda_Z^\sharp$ is linear on the cone $R_+ g + \mathbb{R}_+ s'$. Now we have

$$\lambda_Z^\sharp\Big(\sum g_i + s'\Big) = \lambda_Z^\sharp(g + s') = \lambda_Z^\sharp(g) + \lambda_Z^\sharp(s') = \sum \lambda_Z^\sharp(g_i) + \lambda_Z^\sharp(s'),$$

so the map $\lambda_Z^\sharp|_{\mathscr{C}+\mathbb{R}_+ s'}$ is linear by Lemma 4.3. Taking $\mu = \max_i\{\frac{\alpha}{\alpha_i \beta}\}$ and taking a point $\hat{u} = \mu s + u$ in the relative interior of $\mathbb{R}_+ s + R$, it is easy to check that

$$\hat{u} = \sum(\mu\alpha_i - \tfrac{\alpha}{\beta})t_i + \tfrac{1}{\beta}s' \in \mathscr{C} + \mathbb{R}_+ s',$$

so the map $\lambda_Z^\sharp|_{\mathbb{R}_+ s + \mathbb{R}_+ \hat{u}}$ is linear.

Step 5. Finally, let H be any 2-plane in $\mathbb{R}^\ell$. Then by the previous steps, for every ray $R \subset \mathscr{S}_{\mathbb{R}} \cap H$ there is a polyhedral cone $\mathscr{C}_R$ with $R \subset \mathscr{C}_R \subset \mathscr{S}_{\mathbb{R}} \cap H$ such that there is a polyhedral decomposition $\mathscr{C}_R = \mathscr{C}_{R,1} \cup \mathscr{C}_{R,2}$ with $\lambda_Z^\sharp|_{\mathscr{C}_{R,1}}$ and $\lambda_Z^\sharp|_{\mathscr{C}_{R,2}}$ being linear maps, and if $R \subset \operatorname{relint}(\mathscr{S}_{\mathbb{R}} \cap H)$, then $R \subset \operatorname{relint} \mathscr{C}_R$.

Let $S^{\ell-1}$ be the unit sphere. Restricting to the compact set $S^{\ell-1} \cap \mathscr{S}_{\mathbb{R}} \cap H$ we see that $\lambda_Z^\sharp|_{\mathscr{S}_{\mathbb{R}} \cap H}$ is piecewise linear. □

8. Stable base loci

Theorem 8.1. *Theorems A_{n-1} and C_{n-1} imply Theorem B_n.*

Proof. Step 1. Let K_X be a divisor such that $\mathscr{O}_X(K_X) \simeq \omega_X$ and $A \not\subset \operatorname{Supp} K_X$. It is enough to prove that the cone $\mathscr{C} = \mathbb{R}_+(K_X + A + \mathscr{B}_{V,A}^{G=1})$ is rational polyhedral.

In Steps 1 and 2, I first show that

$$\mathscr{B}_{V,A}^{G=1} = \{\Phi \in \mathscr{L}_V : \operatorname{mult}_G \Phi = 1, \sigma_G\|K_X + \Phi + A\| = 0\}.$$

To that end, let $\Delta \in \mathscr{L}_V + A$ be such that $K_X + \Delta$ is pseudo-effective, $\operatorname{mult}_G \Delta = 1$ and $\sigma_G\|K_X + \Delta\| = 0$, and denote $\Omega = (\Delta - G)_{|G}$. Replacing A by a general ample $\mathbb{Q}$-divisor $\mathbb{Q}$-linearly equivalent to $A - \Xi$ for some $\Xi \in V$ with $\|\Xi\| \ll 1$, I can assume that (X, Δ) is plt and $\lfloor \Delta \rfloor = G$. Furthermore, let $f\colon Y \to X$ be a log resolution such that the components of $\{\mathbf{B}(X,\Delta)_Y\}$ are disjoint as in Lemma 2.3. Then since $f_*^{-1}G \not\subset \mathbf{B}(D)$ implies $G \not\subset \mathbf{B}(f_*D)$ for $D \in \operatorname{Div}(Y)_{\mathbb{R}}$, since $K_Y + B(X,\Delta)_Y$ is pseudo-effective and $\sigma_{f_*^{-1}G}\|K_Y + B(X,\Delta)_Y\| = 0$ by Remark 2.11, I can replace X by Y, G by $f_*^{-1}G$, Δ by $\mathbf{B}(X,\Delta)_Y$, A by $f^*A - H$ for some small effective f-exceptional divisor H on Y, and V by the vector space spanned by proper transforms of elements of V and by exceptional divisors.

Since $\sigma_G\|K_X+\Delta\|=0$, G is not contained in $\mathbf{B}_-(K_X+\Delta)$ by Remark 2.11, and so $N_\sigma\|K_X+\Delta\|_G$ is defined. Let

$$\Phi=\sum_{P\subset\operatorname{Supp}\Omega}\sigma_P\|K_X+\Delta\|_G\cdot P\leq N_\sigma\|K_X+\Delta\|_G$$

and $\Theta=\Omega-\Omega\wedge\Phi$. Observe that $K_G+\Theta=\lim_{\delta\downarrow 0}(K_G+\Omega_\delta-\Omega_\delta\wedge\Phi_\delta)$, where $\Omega_\delta=(\Delta+\delta A-G)_{|G}$ and

$$\Phi_\delta=\sum_{P\subset\operatorname{Supp}\Omega_\delta}\operatorname{ord}_P\|K_X+\Delta+\delta A\|_G\cdot P.$$

Thus $K_G+\Theta$ is a limit of effective divisors, and so is pseudo-effective. Let $\phi<1$ be the smallest positive coefficient of Φ, or set $\phi=0$ if $\Phi=0$.

Let $W\subset\operatorname{Div}(X)_\mathbb{R}$ be the smallest rational affine subspace containing Δ and let $U,Z\subset\operatorname{Div}(G)_\mathbb{R}$ be the smallest rational affine subspaces containing Φ,Θ respectively. There exists a number $\varepsilon>0$ such that $\varepsilon(K_X+\Delta)+\frac{1}{2}A$ is ample and if $\Delta'\in W$ with $\|\Delta-\Delta'\|<\varepsilon$, then $\Delta-\Delta'+\frac{1}{2}A$ is ample. Let r be a positive integer as in Lemma 5.8. Then by Lemma 6.3 there exist rational pairs $(\Delta_i,\Phi_i)\in W\times U$, integers $k_i\gg 0$ and $r_i\in\mathbb{R}_{>0}$ such that $(\Delta_i,\Phi_i,r,k_i,r_i)$ uniformly approximate $(\Delta,\Phi)\in W\times U$ with error $\phi\varepsilon$. Note that then (X,Δ_i) is plt and $(G,\Omega_i+A_{|G})$ is terminal, where $\Omega_i=(\Delta_i-G)_{|G}$.

Step 2. Since $\sigma_G\|K_X+\Delta\|=0$ we have $G\not\subset\mathbf{B}(K_X+\Delta+\frac{1}{2}A_i)$ by Remark 2.11, and since $\Delta-\Delta_i+\frac{1}{2}A_i$ is ample, it follows that $G\not\subset\mathbf{B}(K_X+\Delta_i+A_i)$, so similarly as in Step 2 of the proof of Theorem 7.5 we have

$$|k_i(K_G+\Theta_i)|+k_i(\Omega_i-\Theta_i)\subset|k_i(K_X+\Delta_i)|_G, \tag{16}$$

where $\Theta_i=\Omega_i-\Omega_i\wedge\Phi_i$. One easily shows that $\Theta_i\in Z$. In particular, by Theorem C and Lemma 5.8, for $\varepsilon\ll 1$ we have that $|k_i(K_G+\Theta_i)|\neq\emptyset$, and therefore (16) implies that there is an effective divisor D_i with $G\not\subset\operatorname{Supp}D_i$ such that $k_i(K_X+\Delta_i)\sim D_i$. But then $K_X+\Delta\sim_\mathbb{R}\sum\frac{r_i}{k_i}D_i$ and $G\not\subset\mathbf{B}(K_X+\Delta)$, as desired.

Step 3. To prove the cone $\mathscr{C}$ is closed, let $D_m\in\mathscr{C}$ be a sequence such that $\lim_{m\to\infty}D_n=D$, and assume $D\notin\mathscr{C}$. Therefore by Step 1 we have $\sigma_G\|D\|>0$. Then for any ample divisor B on X there exists a positive number $\delta\ll 1$ such that $\operatorname{ord}_G\|D+\delta B\|>0$. Pick $m\gg 0$ such that $\delta B+D-D_m$ is ample. Then

$$0<\operatorname{ord}_G\|D+\delta B\|=\operatorname{ord}_G\|D_m+(\delta B+D-D_m)\|\leq\operatorname{ord}_G\|D_m\|=0,$$

a contradiction.

Step 4. Steps 1 and 2 also show that the condition $\sigma_G\|K_X+\Phi+A\|=0$ implies $\operatorname{ord}_G\|K_X+\Phi+A\|=0$, and that the cone $\mathscr{C}$ is rational, i.e. its extremal rays are rational. It remains to prove that $\mathscr{C}$ is polyhedral. To that end, I will prove it has only finitely many extremal rays.

Assume there are divisors Δ_m in $\mathscr{B}_{V,A}^{G=1}+A$ for all $m\in\mathbb{N}\cup\{\infty\}$ such that the rays $\mathbb{R}_+(K_X+\Delta_m)$ are extremal in $\mathscr{C}$ and $\lim_{m\to\infty}\Delta_m=\Delta_\infty$. By the previous steps, Δ_m are rational divisors. I will achieve contradiction by showing that for some $m\gg 0$ there is a ray $R\subset\mathscr{C}$ such that $K_X+\Delta_m\subset\operatorname{int}(\mathbb{R}_+(K_X+\Delta_\infty)+R)$, so that the ray $\mathbb{R}_+(K_X+\Delta_m)$ cannot be extremal.

Since the problem is local around Δ_∞, by taking a log resolution as in Step 1, I can assume that (X,Δ_m) is plt, $\lfloor\Delta_m\rfloor=G$, and each pair $(G,\Omega_m+A_{|G})$ is canonical, where $\Omega_m=(\Delta_m-G)_{|G}$.

Let

$$\Phi_m=\sum_{P\subset\operatorname{Supp}\Omega_m}\operatorname{ord}_P\|K_X+\Delta_m\|_G\cdot P$$

and set $\Theta_m^\sharp=\Omega_m-\Omega_m\wedge\Phi_m$. By Step 3 of the proof of Theorem 7.4 each $\Theta_m^\sharp$ is a rational divisor, and as in the proof of [Nak04, 2.1.4] we have $\Theta_\infty^\sharp\geq\limsup_{m\to\infty}\Theta_m^\sharp$. By passing to a subsequence, we can assume that there is a divisor Θ_∞^0 such that $\Theta_\infty^0=\lim_{m\to\infty}\Theta_m^\sharp$. Let ϕ be the smallest positive coefficient of $\Omega_\infty-\Theta_\infty^0$, or set $\phi=0$ if $\Omega_\infty=\Theta_\infty^0$.

Step 5. Assume first that $\phi>0$. Let $0<\varepsilon\ll 1$ be a rational number such that $\varepsilon(K_X+\Delta_m)+\frac{1}{2}A$ is ample for every $m\gg 0$, and $\tilde{\Delta}+\frac{1}{2}A$ is ample for every $\tilde{\Delta}\in V$ with $\|\tilde{\Delta}\|<\varepsilon$. By Diophantine approximation there is a $\mathbb{Q}$-divisor Ψ in the minimal rational affine space containing Θ_∞^0 and a positive integer k such that $\Psi\leq\Theta_\infty^\sharp$, $\|\Psi-\Theta_\infty^0\|<\phi\varepsilon/2k$, and $k\Psi/r$ and $k\Delta_\infty/r$ are integral. Pick m such that $\|\Delta_\infty-\Delta_m\|<\varepsilon/2k$, $\|\Psi-\Theta_m^\sharp\|<\phi\varepsilon/2k$, and such that for every prime divisor $P\subset V$, $\operatorname{mult}_P\Omega_m=\operatorname{mult}_P\Theta_m^\sharp$ if and only if $\operatorname{mult}_P\Omega_\infty=\operatorname{mult}_P\Theta_\infty^\sharp$. Then by Lemma 6.4 there is a point $(\Delta',\Psi')\in\operatorname{Div}(X)_\mathbb{Q}\times\operatorname{Div}(G)_\mathbb{Q}$ and a positive integer $k'\gg 0$ such that:

(1) $\Delta_m=\frac{k}{k+k'}\Delta_\infty+\frac{k'}{k+k'}\Delta'$ and $\Theta_m^\sharp=\frac{k}{k+k'}\Psi+\frac{k'}{k+k'}\Psi'$,
(2) $k'\Delta'/r$ is integral and $\|\Delta_m-\Delta'\|<\varepsilon/2k'$,
(3) $\Psi'\leq\Omega'$, where $\Omega'=(\Delta'-G)_{|G}$, $k'\Psi'/r$ is integral and $\|\Theta_m^\sharp-\Psi'\|<\phi\varepsilon/2k'$.

Since $G\not\subset\mathbf{B}(K_X+\Delta_m)$ and $\Delta'-\Delta_m+A/k'$ is ample, we have $G\not\subset\mathbf{B}(K_X+\Delta'+A/k')$, so as in Step 2 of the proof of Theorem 7.5 we have that

$$|k'(K_G+\Psi')|+k'(\Omega'-\Psi')\subset|k'(K_X+\Delta')|_G. \tag{17}$$

Let me prove that $K_G+\Psi'$ is pseudo-effective. Let $V_S\subset\operatorname{Div}(S)_\mathbb{R}$ be the vector space spanned by the components of divisors $E_{|S}$ with $E\in V$ and $S\not\subset\operatorname{Supp}E$. Then by Theorem C the cone $\mathscr{C}_S=\mathbb{R}_+(K_X+A_{|S}+\mathscr{E}_{V_S,A_{|S}})$ is rational polyhedral. Since k' can be taken arbitrarily large, $\Psi'\notin\mathscr{C}_S$ implies $\Theta_m^\sharp\in\partial\mathscr{C}_S$. If this stands for every $m\gg 0$, we have that $\Theta_\infty^0\in\partial\mathscr{C}_S$. Therefore, possibly passing to a subsequence, there is a face $\mathscr{F}$ of $\mathscr{C}_S$ such that $\Theta_m^\sharp$, Θ_∞^0 and Ψ belong to $\mathscr{F}$ for $m\gg 0$, and we finish by descending induction on $\dim\mathscr{F}$.

Therefore $|k'(K_G+\Psi')|\neq\emptyset$ by Lemma 5.8, and thus $G\not\subset \mathbf{B}(K_X+\Delta')$ by (17). But then by the condition (1) above, the ray $\mathbb{R}_+(K_X+\Delta_m)$ is not extremal, a contradiction.

Step 6. Now assume that $\phi=0$, and in particular $\Psi=\Theta^\sharp_\infty=\Omega_\infty$. Let $0<\varepsilon\ll 1$ be a rational number such that $\varepsilon(K_X+\Delta_m)+\frac{1}{2}A$ is ample for every $m\gg 0$, and $\tilde\Delta+\frac{1}{2}A$ is ample for every $\tilde\Delta\in V$ with $\|\tilde\Delta\|<\varepsilon$. Let k be a positive integer such that $k\Theta^\sharp_\infty/r$ and $k\Delta_\infty/r$ are integral. Pick m such that $\|\Delta_\infty-\Delta_m\|<\varepsilon/2k$, $\|\Theta^\sharp_\infty-\Theta^\sharp_m\|<\varepsilon/2k$, and such that for every prime divisor $P\subset V$, $\operatorname{mult}_P\Omega_m=\operatorname{mult}_P\Theta^\sharp_m$ if and only if $\operatorname{mult}_P\Omega_\infty=\operatorname{mult}_P\Theta^\sharp_\infty$, and $\operatorname{mult}_P\Theta^\sharp_m=0$ if and only if $\operatorname{mult}_P\Theta^\sharp_\infty=0$. Then by Lemma 6.4 there is a point $(\Delta',\Psi')\in\operatorname{Div}(X)_\mathbb{Q}\times\operatorname{Div}(G)_\mathbb{Q}$ and a positive integer $k'\gg 0$ such that conditions (1)–(3) from Step 5 are satisfied for $\Psi=\Theta^\sharp_\infty$, and denote $A'=A/k'$.

Let me prove that

$$\operatorname{mult}_P(\Omega'\wedge\tfrac{1}{l}\operatorname{Fix}|l(K_X+\Delta'+A')|_S)\leq\operatorname{mult}_P(\Omega'-\Psi') \tag{18}$$

for every prime divisor P on G and for all $l\gg 0$ sufficiently divisible. To that end, assume first that $\operatorname{mult}_P\Theta^\sharp_m=0$. Then $\operatorname{mult}_P\Theta^\sharp_\infty=0$ by the choice of ε and m, and thus $\operatorname{mult}_P\Psi'=0$ by the condition (1) above. Therefore

$$\operatorname{mult}_P(\Omega'\wedge\tfrac{1}{l}\operatorname{Fix}|l(K_X+\Delta'+A')|_S)\leq\operatorname{mult}_P\Omega'=\operatorname{mult}_P(\Omega'-\Psi').$$

Now assume that $\operatorname{mult}_P\Theta^\sharp_m>0$. For l sufficiently divisible we have

$$\operatorname{mult}_P\tfrac{1}{l}\operatorname{Fix}|l(K_X+\Delta'+A')|_S=\operatorname{ord}_P\|K_X+\Delta'+A'\|_S$$

as in Step 3 of the proof of Theorem 7.4, and since $\Delta_m-\Delta_\infty+\frac{1}{k}A$ is ample by the choice of ε,

$$\begin{aligned}\operatorname{mult}_P(\Omega'\wedge\tfrac{1}{l}\operatorname{Fix}|l(K_X+\Delta'+A')|_S)&\leq\operatorname{ord}_P\|K_X+\Delta'+A'\|_S\\&=\operatorname{ord}_P\big\|K_X+\Delta_m+\tfrac{k}{k'}\big(\Delta_m-\Delta_\infty+\tfrac{1}{k}A\big)\big\|_S\\&\leq\operatorname{ord}_P\|K_X+\Delta_m\|_S=\operatorname{mult}_P(\Omega_m-\Theta^\sharp_m).\end{aligned}$$

¿From the condition (1) above we have

$$\Omega_m-\Theta^\sharp_m\leq\Omega_m-\Theta^\sharp_m+\tfrac{k}{k'}\big(\Omega_m-\Theta^\sharp_m\big)=\Omega'-\Psi',$$

and (18) is proved. Since $G\not\subset\mathbf{B}(K_X+\Delta'+A')$ as in Step 5, by Proposition 7.3 we have that

$$|k'(K_G+\Psi')|+k'(\Omega'-\Psi')\subset|k'(K_X+\Delta')|_G.$$

The contradiction now follows as in Step 5. □

9. PSEUDO-EFFECTIVITY AND NON-VANISHING

The starting point for this section is the following result.

Theorem 9.1. *Let (X,Δ) be a projective klt pair such that Δ is big and $K_X+\Delta$ is pseudo-effective. Then there exists an effective divisor $D\in\mathrm{Div}(X)_{\mathbb{R}}$ such that $K_X+\Delta\equiv D$. Moreover, if $\Delta\in\mathrm{WDiv}(X)_{\mathbb{Q}}$, then $|K_X+\Delta|_{\mathbb{Q}}\neq\emptyset$.*

This theorem was proved in [Pău08, §1] by using analytic techniques. However, it can be proved purely algebraically as a part of the induction given in this paper, which is sketched in Theorem 9.3 below. Note that the last claim in Theorem C is a refinement of this result.

We are now ready to prove the following.

Theorem 9.2. *Assume Theorem 9.1 in dimension n. Then Theorem B_n implies Theorem C_n.*

Proof. Step 1. Fix a divisor K_X such that $\mathscr{O}_X(K_X)\simeq\omega_X$ and $A\not\subset\operatorname{Supp}K_X$, and denote $\mathscr{C}:=\mathbb{R}_+(K_X+A+\mathscr{E}_{V,A})\subset\mathrm{Div}(X)_{\mathbb{R}}$. Fix $\Delta=\sum_{i=1}^N\delta_iF_i\in\mathscr{E}_{V,A}$. Then $K_X+\Delta+A\equiv\sum_{i=1}^N f_iF_i\geq 0$ by Theorem 9.1, where $F_i\neq A$ for all i and $\delta_i=0$ for some i. Let $W\subset\mathrm{Div}(X)_{\mathbb{R}}$ be the vector space spanned by all F_i and by the components of K_X and A. Let $\phi\colon W\to N^1(X)$ be the linear map sending a divisor to its numerical class. Let $0<\varepsilon\ll 1$ be a rational number such that $A+\Phi$ is ample for any divisor $\Phi\in W$ with $\|\Phi\|\leq\varepsilon$.

Let $0\leq f_i'\leq f_i$ be rational numbers such that $f_i-f_i'<\varepsilon$. Then

$$K_X+\Delta'+A\equiv\sum f_i'F_i,$$

where $\Delta'=\Delta-\sum(f_i-f_i')F_i$. Since $\mathscr{P}=\phi^{-1}\big(\sum f_i'[F_i]\big)$ is a rational affine subspace, there are rational divisors $\Delta_j\in W$ such that $\|\Delta'-\Delta_j\|\ll 1$, $K_X+\Delta_j+A\in\mathscr{P}$ and $K_X+\Delta'+A=\sum\rho_j(K_X+\Delta_j+A)$ for some positive numbers ρ_j with $\sum\rho_j=1$. Observe that then $\Delta_j+A=\Delta_j^0+A'$, where $\Delta_j^0=\sum\max\{0,\operatorname{mult}_{F_j}\Delta_j\}F_j$, and $A'=A+\Delta_j-\Delta_j^0$ is ample since $\|\Delta_j-\Delta_j^0\|\leq\varepsilon$. Therefore each $K_X+\Delta_j+A\sim_{\mathbb{Q}}K_X+\Delta_j^0+A'$ is a rational pseudo-effective divisor, and thus it is $\mathbb{Q}$-linearly equivalent to an effective divisor by Theorem 9.1. For each j, denote $\mathscr{B}_j=\sum[\operatorname{mult}_{F_i}\Delta_j,1]F_i$, and let $\mathscr{B}$ be the convex hull of $\bigcup\mathscr{B}_j$; observe that $\mathscr{B}$ is a rational polytope. Then $K_X+\Delta+A\in(K_X+A+\mathscr{B})\cap(K_X+A+\mathscr{L}_V)$, and so $K_X+\Delta+A$ is $\mathbb{R}$-linearly equivalent to an effective divisor, and $\mathscr{C}$ is a closed cone which is locally rational around every $K_X+\Delta+A$, and thus rational globally.

Step 2. Let $G_1,\dots,G_N$ be prime divisors on X such that $\operatorname{Supp}K_X\cup\operatorname{Supp}B\subset\sum G_i$. It remains to prove that the cone $\mathscr{C}$ is polyhedral.

Assume that $\mathscr{C}$ has infinitely many extremal rays. Thus there are distinct divisors $\Delta_m=\sum\delta_j^mG_j$ in $\mathscr{E}_{V,A}$ for all $m\in\mathbb{N}\cup\{\infty\}$ such that, denoting $\Upsilon_m=K_X+\Delta_m+A$, the rays $\mathbb{R}_+\Upsilon_m$ are extremal in $\mathscr{C}$ and $\lim\limits_{m\to\infty}\Delta_m=\Delta_\infty$. I will achieve contradiction by showing that for some $m\gg 0$ there is a ray $R\subset\mathscr{C}$ such that $\Upsilon_m\subset\operatorname{int}(\mathbb{R}_+\Upsilon_\infty+R)$, so that the ray $\mathbb{R}_+\Upsilon_m$ cannot be extremal. To that end, I am allowed and will without explicit mention, increase V, translate points by fixed divisors and re-scale by positive integers, since this does not affect the outcome of the aforementioned procedure. By Step 1, Δ_m are rational divisors.

Write $\Upsilon_\infty \sim_{\mathbb{Q}} D_\infty \geq 0$; by possibly adding components I can assume that $\operatorname{Supp} D_\infty \subset \sum_{j=1}^N G_j$ and that $V = \sum_{j=1}^N \mathbb{R}G_j$. Since the problem is local around Δ_∞, replacing A by a general ample $\mathbb{Q}$-divisor $\mathbb{Q}$-linearly equivalent to $A - \Xi$ for some $\Xi \in V$ with $\|\Xi\| \ll 1$, I can assume that (X, Δ_m) is klt and $\operatorname{Supp} \Delta_m = \sum_{j=1}^N G_j$ for all $m \gg 0$. Furthermore, let $f\colon Y \to X$ be a log resolution of $(X, \sum_{j=1}^N G_j)$. Then since D being pseudo-effective implies that f_*D is pseudo-effective for $D \in \operatorname{Div}(Y)_{\mathbb{R}}$, I can replace X by Y, A by $f^*A - H$, Δ_m by $\mathbf{B}(X, \Delta_m + A)_Y - f^*A + H$ for some small effective f-exceptional divisor H on Y, and V by the vector space spanned by proper transforms of elements of V and by exceptional divisors.

If $D_\infty = 0$, for $m \gg 0$ choose any $\Psi \in V$ such that for some $0 < t < 1$ we have $\Delta_m = (1-t)\Delta_\infty + t\Psi$. Then $K_X + \Psi + A \sim_{\mathbb{Q}} \Upsilon_m / t$, and thus $K_X + \Psi + A$ is pseudo-effective and the ray $\mathbb{R}_+ \Upsilon_m$ is not extremal.

Now assume $D_\infty \neq 0$ and write $D_\infty = \sum d_j G_j$. Then $K_X \sim_{\mathbb{Q}} -A + \sum f_j G_j$ with $f_j = d_j - \delta_j^\infty > -1$. Setting

$$r_\infty = \max_{j=1}^N \left\{ \frac{f_j + \delta_j^\infty}{f_j + 1} \right\} \quad \text{and} \quad b_j^\infty = -f_j + \frac{f_j + \delta_j^\infty}{r_\infty},$$

we have

$$\sum\nolimits_j (f_j + \delta_j^\infty) G_j = r_\infty \sum\nolimits_j (f_j + b_j^\infty) G_j.$$

Observe that $r_\infty \in (0,1]$, $b_j^\infty \in [\delta_j^\infty, 1]$ and there exists j_0 such that $b_{j_0}^\infty = 1$. Now for $m \gg 0$, setting

$$r_m = \frac{f_{j_0} + \delta_{j_0}^m}{f_{j_0} + 1} \quad \text{and} \quad b_j^m = -f_j + \frac{f_j + \delta_j^m}{r_m},$$

we have

$$\sum\nolimits_j (f_j + \delta_j^m) G_j = r_m \sum\nolimits_j (f_j + b_j^m) G_j,$$

$b_{j_0}^m = 1$ for all m, and $\lim\limits_{m \to \infty} b_j^m = b_j^\infty$ for all j. Let $0 < \eta \ll 1$ be a rational number such that $-\eta < b_j^m < 1 + \eta$ for all m and j, and such that $A - \Xi$ is ample for all $\Xi \in V$ with $\|\Xi\| \leq \eta$. By passing to a sub-sequence and by re-indexing, I can assume that $b_j^m \leq 1 - 2\eta$ for all $m \gg 0$ and all $j < j_0$, and that $b_j^m \geq 1 - 2\eta$ for all $m \gg 0$ and all $j \geq j_0$.

By replacing A by $A' \sim_{\mathbb{Q}} A - \eta \sum_{j<j_0} G_j + \eta \sum_{j>j_0} G_j$, δ_j^m by $b_j^m + \eta$ for $j < j_0$, δ_j^m by $b_j^m - \eta$ for $j > j_0$, and $\delta_{j_0}^m$ by 1, finally I can assume that all Δ_m are effective divisors such that $(X, \Delta_m + A)$ is plt and $G_{j_0} = \lfloor \Delta_m \rfloor$ for every m.

Step 3. In this step I prove that $N_\sigma \|\Upsilon_\infty\| = 0$, and therefore Theorem B implies $G_{j_0} \not\subset \mathbf{B}(\Upsilon_\infty)$.

By way of contradiction, assume that $\sigma_\Sigma \|\Upsilon_\infty\| > 0$ for a prime divisor Σ, and let $\mathscr{E} \subset N^1(X)$ denote the pseudo-effective cone. By the proof of [Bou04, 3.19], $\mathscr{E}$ is generated by $[\Sigma]$ and by the closed cone $\mathscr{E}_\Sigma = \{\Xi \in \mathscr{E} : \sigma_\Sigma \|\Xi\| = 0\}$. We have $[\Upsilon_\infty] \in \partial\mathscr{E} \setminus \mathscr{E}_\Sigma$, and let $\varphi\colon V \to N^1(X)$ be the linear map sending a divisor to its numerical class. We can assume that $[\Upsilon_n] \neq [\Upsilon_\infty]$ for all $n \gg 0$, since otherwise we obtain a contradiction as $\varphi^{-1}([\Upsilon_\infty])$ is an affine subspace of V.

If $[\Upsilon_\infty] \in \mathbb{R}_+[\Sigma]$, then for any $n \gg 0$ the set

$$\{\Upsilon_\infty + t(\Upsilon_n - \Upsilon_\infty) : t \in \mathbb{R}_+\} \cap \varphi^{-1}(\mathscr{E})$$

is strictly larger than the segment $[\Upsilon_\infty, \Upsilon_n]$, so $\mathbb{R}_+\Upsilon_n$ is not an extremal ray for $n \gg 0$.

Therefore I can assume $[\Upsilon_\infty] \notin \mathbb{R}_+[\Sigma]$ and, similarly as above, that $[\Upsilon_n] \in \partial\mathscr{E}$ for $n \gg 0$. In particular, if we consider the cone spanned by $\mathbb{R}_+[\Sigma]$ and $\mathbb{R}_+[\Upsilon_n]$ for $n \gg 0$, the dimension of this cone must be strictly smaller than $\dim\mathscr{E}$, since otherwise it would contain a point of $\operatorname{int}\mathscr{E}$. Let $\mathscr{H}$ be the smallest affine subspace of $N^1(X)$ containing that cone, and let $\mathscr{E}_\Sigma' = \mathscr{E}_\Sigma \cap \mathscr{H}$. Then it is easy to see that $\mathscr{E} \cap \mathscr{H}$ is spanned by $\mathscr{E}_\Sigma'$ and $\mathbb{R}_+[\Sigma]$, so by replacing $\mathscr{E}$ by $\mathscr{E} \cap \mathscr{H}$, we can finish by descending induction on $\dim\mathscr{E}$.

Step 4. Denote $G := G_{j_0}$. Let $0 < \eta \ll 1$ be a rational number such that $A - \Xi$ is ample for all $\Xi \in V$ with $\|\Xi\| \leq \eta$, let $\mathscr{L}_{V,\eta}$ be the η-neighbourhood of $\mathscr{L}_V$ in V in the sup-norm, and set

$$\mathscr{B}^{G=1}_{V,A,\eta} = \{\Phi \in \mathscr{L}_{V,\eta} : \operatorname{mult}_G \Phi = 1,\, G \not\subset \mathbf{B}(K_X + \Phi + A)\}$$

and $\mathscr{D} = \mathbb{R}_+(\sum f_j G_j + \mathscr{B}^{G=1}_{V,A,\eta}) \subset V$. Note that, for some $0 < \xi \ll 1$, we have

$$\mathbb{R}_+\{\Upsilon_\infty + \Xi : 0 \leq \Xi \in V, \|\Xi\| < \xi, \operatorname{mult}_G \Xi = 0\} \subset \mathscr{D},$$

so $\dim\mathscr{D} = \dim V$ and $\Upsilon_\infty \in \operatorname{int}\mathscr{D}$. Fix $\Phi \in \mathscr{B}^{G=1}_{V,A,\eta}$. Then as in Step 2 there is a $\mathbb{Q}$-divisor $\Theta \in V$ such that $\|\Theta\| = \eta$ and $A' \sim_{\mathbb{Q}} A - \Theta$ is ample, for every $\Phi' \in \mathscr{B}^{G=1}_{V,A,\eta}$ with $\|\Phi - \Phi'\| < \eta$ the divisor $\Phi' + \Theta$ is effective and $\lceil \Phi' + \Theta \rceil$ is reduced. Therefore, since $\mathscr{B}^{G=1}_{V,A'}$ is a rational polytope by Theorem B, this means that $\mathscr{B}^{G=1}_{V,A,\eta}$ is locally a rational polytope around Φ, and therefore $\mathscr{D}$ is a rational polyhedral cone. Since $\mathscr{C}$ is not polyhedral, I can assume that $\Upsilon_m \notin \mathscr{D}$ for all $m \gg 0$.

Step 5. For each $m \in \mathbb{N}$ let k_m be a positive integer such that $k_m\Upsilon_m$ is Cartier and denote

$$\Gamma_m = \Upsilon_m - \frac{\operatorname{mult}_G \operatorname{Fix}|k_m\Upsilon_m|}{k_m} G.$$

Then $G \not\subset \mathbf{B}(\Gamma_m)$. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be codimension 1 faces of the cone $\mathscr{D}$ such that $\Upsilon_\infty \in \mathscr{F}_1 \cap \mathscr{F}_2$, and let $\mathscr{H}_1$ and $\mathscr{H}_2$ be their supporting hyperplanes; if Υ_∞ belongs only to one codimension 1 face, assume that $\mathscr{F}_1 = \mathscr{F}_2$. Let $\mathscr{Q}$ be exactly one of the convex cones into which $\mathscr{H}_1$ and $\mathscr{H}_2$ subdivide V which contains $\mathscr{D}$. For every $\Psi \in \mathbb{R}_+\Gamma_m + \mathbb{R}_+\Upsilon_\infty$ we have $G \not\subset \mathbf{B}(\Psi)$, and therefore $(\mathbb{R}_+\Gamma_m + \mathbb{R}_+\Upsilon_\infty) \cap (\sum f_j G_j + \mathscr{L}_V) \subset \mathscr{D}$. This implies that $\Gamma_m \in \mathscr{Q}$, and since $\Upsilon_m \notin \mathscr{Q}$ for $m \gg 0$, the segment $[\Gamma_m, \Upsilon_m]$ intersects $\mathscr{Q}$.

Let P_m be the point of intersection of the half-ray $\Upsilon_m + \mathbb{R}_- G$ with $\partial\mathscr{Q}$ closest to Υ_m, and observe that $\lim\limits_{m\to\infty} P_m = \Upsilon_\infty$. This means that P_m belongs to $\mathscr{F}_1$ or $\mathscr{F}_2$ for $m \gg 0$, and by passing to a subsequence I can assume that all P_m belong to the same codimension 1 face of $\mathscr{D}$. Since $\mathscr{D}$ is polyhedral, for $m \gg 0$ there are points $Q_m \in \mathscr{D}$ such that $P_m \in (Q_m, \Upsilon_\infty)$ and $\|P_m - Q_m\| \ll 1$. Let Z_m be the intersection point of the half-ray $P_m + \mathbb{R}_+ G$ with the hyperplane $(G = \mathrm{mult}_G \Upsilon_\infty)$. Then it is easy to see that $\Upsilon_m \in (Z_m, \Upsilon_\infty)$, Z_m is pseudo-effective and belongs to $\sum f_j G_j + \mathscr{L}_V$ since Υ_m and Υ_∞ do. Thus $\mathbb{R}_+\Upsilon_m$ is not an extremal ray of the cone $\mathscr{C}$, a contradiction which finishes the proof. □

The following result, together with Theorem 9.2, yields that Theorems A_{n-1}, B_n and C_{n-1} imply Theorem C_n.

Theorem 9.3. *Theorems A_{n-1} and C_{n-1} imply Theorem 9.1 in dimension n.*

Proof. This was done essentially in [Hac08], and I will sketch the proof here for completeness.

By passing to a log resolution, we can assume that the pair (X,Δ) is log smooth. If $\Delta \sim_{\mathbb{Q}} A+B$, where A is an ample $\mathbb{Q}$-divisor and $B \geq 0$, then replacing Δ by $(1-\varepsilon)\Delta+\varepsilon(A+B)$ for a rational number $0 < \varepsilon \ll 1$, we can assume that $\Delta = A+B$. If $\nu(X,D) = 0$, cf. Definition A.4, then the result follows from [BCHM06, 3.3.2].

If $\nu(X,D) > 0$, then by [BCHM06, 6.2] we can assume that (X,Δ) is plt, A is a general ample $\mathbb{Q}$-divisor, $\lfloor\Delta\rfloor = S$, $(S,\Omega+A_{|S})$ is canonical, where $\Omega = (\Delta - S)_{|S}$, and $\sigma_S\|K_X + \Delta\| = 0$. But now the result follows as in Steps 1 and 2 of the proof of Theorem 8.1.

The second statement in Theorem 9.1 follows by using Shokurov's trick from his proof of the classical Non-vanishing theorem, and I will present an algebraic proof following the analytic version from [Pău08].

Assume that $\Delta = A+B \in \mathrm{Div}(X)_{\mathbb{Q}}$ and let $\Upsilon := K_X + \Delta \equiv D$ for some effective $\mathbb{R}$-divisor D. We can again assume (X,Δ) is log smooth, and that D is a $\mathbb{Q}$-divisor similarly as in Step 1 of the proof of Theorem 9.2. Let m be a positive integer such that $m\Delta$ and mD are integral. By Nadel vanishing

$$H^i\big(X, \mathscr{J}_{(m-1)D+B}(m\Upsilon)\big) = 0 \quad \text{and} \quad H^i\big(X, \mathscr{J}_{(m-1)D+B}(mD)\big) = 0$$

for $i > 0$, and since the Euler characteristic is a numerical invariant,

$$h^0\big(X, \mathscr{J}_{(m-1)D+B}(m\Upsilon)\big) = h^0\big(X, \mathscr{J}_{(m-1)D+B}(mD)\big). \tag{19}$$

Let $\sigma \in H^0(X,mD)$ be the section with $\mathrm{div}\,\sigma = mD$. Since

$$((m-1)D+B) - mD \leq B,$$

by [HM08, 4.3(3)] we have $\mathscr{I}_{mD} \subset \mathscr{J}_{(m-1)D+B}$, and thus

$$\sigma \in H^0\big(X, \mathscr{J}_{(m-1)D+B}(mD)\big).$$

Therefore (19) implies $h^0(X,m\Upsilon) > 0$. □

10. Finite generation

Theorem 10.1. *Theorems A_{n-1}, B_n and C_{n-1} imply Theorem A_n.*

Proof. Step 1. I first show that it is enough to prove the theorem in the case when A is a general ample $\mathbb{Q}$-divisor and $(X,\Delta_i + A)$ is a log smooth klt pair for every i.

Let p and k be sufficiently divisible positive integers such that all divisors $k(\Delta_i + pA)$ and $(p+1)kA$ are very ample. Let $(p+1)kA_i$ be a general section of $|k(\Delta_i + pA)|$ and let $(p+1)kA'$ be a general section of $|(p+1)kA|$. Set $\Delta_i' = \frac{p}{p+1}\Delta_i + A_i$. Then the pairs $(X, \Delta_i' + A')$ are klt and

$$(p+1)k(K_X + \Delta_i + A) \sim (p+1)k(K_X + \Delta_i' + A') =: D_i'.$$

Thus the ring $R(X; D_1, \dots, D_\ell)$ has a truncation which is isomorphic to $R(X; D_1', \dots, D_\ell')$, so it is enough to prove the latter algebra is finitely generated.

Step 2. Therefore I can assume that $\Delta_i = \sum_{j=1}^N \delta_{ij} F_j$ with $\delta_{ij} \in [0,1)$. Write $K_X + \Delta_i + A \sim_{\mathbb{Q}} \sum_{j=1}^N f_{ij}' F_j \geq 0$, where $F_j \neq A$ since A is general. By blowing up, and by possibly replacing the pair $(X, \Delta_i + A)$ by $(Y, \Delta_i' + A')$ for some model $Y \to X$ as in Step 2 of the proof of Theorem 7.4, I can assume that the divisor $\sum_{j=1}^N F_j$ has simple normal crossings. Thus for every i,

$$K_X \sim_{\mathbb{Q}} -A + \sum_{j=1}^N f_{ij} F_j,$$

where $f_{ij} = f_{ij}' - \delta_{ij} > -1$.

Denote $\Lambda := \bigoplus_{j=1}^N \mathbb{N} F_j \subset \operatorname{Div}(X)$ and $\mathscr{T} := \{(t_1, \dots, t_\ell) : t_i \geq 0, \sum t_i = 1\} \subset \mathbb{R}^\ell$. For each $\tau = (t_1, \dots, t_\ell) \in \mathscr{T}$, denote $\delta_{\tau j} = \sum_i t_i \delta_{ij}$ and $f_{\tau j} = \sum_i t_i f_{ij}$, and observe that $K_X \sim_{\mathbb{R}} -A + \sum_j f_{\tau j} F_j$. Denote $\mathscr{B}_\tau = \sum_{j=1}^N [\delta_{\tau j} + f_{\tau j}, 1 + f_{\tau j}] F_j \subset \Lambda_{\mathbb{R}}$ and let $\mathscr{B} = \bigcup_{\tau \in \mathscr{T}} \mathscr{B}_\tau$. It is easy to see that $\mathscr{B}$ is a rational polytope: every point in $\mathscr{B}$ is a barycentric combination of the vertices of $\mathscr{B}_{\tau_1}, \dots, \mathscr{B}_{\tau_\ell}$, where τ_i are the standard basis vectors of $\mathbb{R}^\ell$. Thus $\mathscr{C} = \mathbb{R}_+ \mathscr{B}$ is a rational polyhedral cone.

For each $j = 1, \dots, N$ fix a section $\sigma_j \in H^0(X, F_j)$ such that $\operatorname{div} \sigma_j = F_j$. Consider the Λ-graded algebra $\mathfrak{R} = \bigoplus_{s \in \Lambda} \mathfrak{R}_s \subset R(X; F_1, \dots, F_N)$ generated by the elements of $R(X, \mathscr{C} \cap \Lambda)$ and by all σ_j; observe that $\mathfrak{R}_s = H^0(X, s)$ for every $s \in \mathscr{C} \cap \Lambda$. I claim that it is enough to show that $\mathfrak{R}$ is finitely generated.

To see this, assume $\mathfrak{R}$ is finitely generated and denote

$$\omega_i = r k_i \sum_j (\delta_{ij} + f_{ij}) F_j \in \Lambda$$

for r sufficiently divisible and $i = 1, \dots, \ell$. Set $\mathscr{G} = \sum_i \mathbb{R}_+ \omega_i \cap \Lambda$ and observe that $\omega_i \sim r D_i$ and $\mathscr{G}_{\mathbb{R}} \subset \mathscr{C}$. Then by Lemma 5.4(2) the algebra $R(X, \mathscr{C} \cap \Lambda)$ is finitely generated, and therefore by Proposition 5.7 there is a finite rational polyhedral subdivision $\mathscr{G}_{\mathbb{R}} = \bigcup_k \mathscr{G}_k$ such that the map $\mathbf{Mob}_{\iota|\mathscr{G}_k \cap \Lambda}$ is additive up to truncation for every k, where $\iota \colon \Lambda \to \Lambda$ is the identity map.

Let $\omega_1', \dots, \omega_q'$ be generators of $\mathscr{G}$ such that $\omega_i' = \omega_i$ for $i = 1, \dots, \ell$, and denote by $\pi \colon \bigoplus_{i=1}^q \mathbb{N} \omega_i' \to \mathscr{G}$ the natural projection. Then the map $\mathbf{Mob}_{\pi|\pi^{-1}(\mathscr{G}_k \cap \Lambda)}$ is additive up to truncation for every k, and thus $R(X, \pi(\bigoplus_{i=1}^q \mathbb{N} \omega_i'))$ is finitely generated by Lemma 5.4(3). Therefore $R(X, \pi(\bigoplus_{i=1}^\ell \mathbb{N} \omega_i)) \simeq R(X; rD_1, \dots, rD_\ell)$ is finitely generated by Lemma 5.4(2), and $R(X; D_1, \dots, D_\ell)$ is finitely generated by Lemma 5.4(1).

Step 3. Thus it suffices to prove that $\mathfrak{R}$ is finitely generated. Take a point $\sum_j (f_{\tau j} + b_{\tau j}) F_j \in$

$\mathscr{B}_\tau \backslash \{0\}$; in particular $b_{\tau j} \in [\delta_{\tau j}, 1]$. Setting

$$r_\tau = \max_{j=1}^{N} \left\{ \frac{f_{\tau j} + b_{\tau j}}{f_{\tau j} + 1} \right\} \quad \text{and} \quad b'_{\tau j} = -f_{\tau j} + \frac{f_{\tau j} + b_{\tau j}}{r_\tau},$$

we have

$$\textstyle\sum_j (f_{\tau j} + b_{\tau j}) F_j = r_\tau \sum_j (f_{\tau j} + b'_{\tau j}) F_j. \tag{20}$$

Observe that $r_\tau \in (0,1]$, $b'_{\tau j} \in [b_{\tau j}, 1]$ and there exists j_0 such that $b'_{\tau j_0} = 1$. For every $j = 1, \dots, N$, let

$$\mathscr{F}_{\tau j} = (1 + f_{\tau j}) F_j + \textstyle\sum_{k \neq j} [\delta_{\tau k} + f_{\tau k}, 1 + f_{\tau k}] F_k,$$

and set $\mathscr{F}_j = \bigcup_{\tau \in \mathscr{T}} \mathscr{F}_{\tau j}$, which is a rational polytope. Then $\mathscr{C}_j = \mathbb{R}_+ \mathscr{F}_j$ is a rational polyhedral cone, and (20) shows that $\mathscr{C} = \bigcup_j \mathscr{C}_j$. Furthermore, since $\sum_j (f_{\tau j} + b'_{\tau j}) F_j \sim_{\mathbb{R}} K_X + \sum_j b'_{\tau j} F_j + A$ for $\tau \in \mathscr{T}$, for every j and for every $s \in \mathscr{C}_j \cap \Lambda$ there is $r_s \in \mathbb{Q}_+$ such that $s \sim_{\mathbb{Q}} r_s (K_X + F_j + \Delta_s + A)$, where $\operatorname{Supp} \Delta_s \subset \sum_{k \neq j} F_k$ and the pair $(X, F_j + \Delta_s + A)$ is log canonical.

Step 4. Assume that the restricted algebra $\operatorname{res}_{F_j} R(X, \mathscr{C}_j \cap \Lambda)$ is finitely generated for every j. I will show that then $\mathfrak{R}$ is finitely generated.

Let $V = \sum_{j=1}^N \mathbb{R} F_j \simeq \mathbb{R}^N$, and let $\|\cdot\|$ be the Euclidean norm on V. By compactness there is a constant C such that every $\mathscr{F}_j \subset V$ is contained in the closed ball centred at the origin with radius C. Let deg denote the total degree function on Λ, i.e. $\deg(\sum_{j=1}^N \alpha_j F_j) = \sum_{j=1}^N \alpha_j$; it induces the degree function on elements of $\mathfrak{R}$. Let M be a positive integer such that, for each j, $\operatorname{res}_{F_j} R(X, \mathscr{C}_j \cap \Lambda)$ is generated by $\{\sigma_{|F_j} : \sigma \in R(X, \mathscr{C}_j \cap \Lambda), \deg \sigma \leq M\}$, and such that $M \geq C N^{1/2} \max_{i,j} \{\frac{1}{1-\delta_{ij}}\}$. By Hölder's inequality we have $\|s\| \geq N^{-1/2} \deg s$ for all $s \in \mathscr{C} \cap \Lambda$, and thus

$$\|s\|/C \geq \max_{i,j} \left\{ \frac{1}{1 - \delta_{ij}} \right\} \tag{21}$$

for all $s \in \mathscr{C} \cap \Lambda$ with $\deg s \geq M$. Let $\mathscr{H}$ be a finite set of generators of the finite dimensional vector space

$$\bigoplus_{s \in \mathscr{C} \cap \Lambda, \deg s \leq M} H^0(X, s)$$

such that for every j, the set $\{\sigma_{|F_j} : \sigma \in \mathscr{H}\}$ generates $\operatorname{res}_{F_j} R(X, \mathscr{C}_j \cap \Lambda)$. I claim that $\mathfrak{R}$ is generated by $\{\sigma_1, \dots, \sigma_N\} \cup \mathscr{H}$, with σ_j as in Step 2.

To that end, take any section $\sigma \in \mathfrak{R}$. By definition, possibly by considering monomial parts of σ and dividing σ by a suitable product of sections σ_j, I can assume that $\sigma \in R(X, \mathscr{C} \cap \Lambda)$. If $\deg \sigma \leq M$, then it is generated by elements of $\mathscr{H}$ and we are done. If $\deg \sigma > M$, by Step 3 there exists $w \in \{1, \dots, N\}$ such that $\sigma \in R(X, \mathscr{C}_w \cap \Lambda)$, and thus there is $\tau \in \mathscr{T} \cap \mathbb{Q}^\ell$ such that $\sigma \in H^0(X, r_\sigma \sum_j (f_{\tau j} + b_{\tau j}) F_j)$ with $b_{\tau w} = 1$. Observe that $r_\sigma \geq \max_{i,j} \{\frac{1}{1-\delta_{ij}}\}$ by (21) since $\|\sum_j (f_{\tau j} + b_{\tau j}) F_j\| \leq C$, and in particular $\frac{r_\sigma - 1}{r_\sigma} \geq \delta_{\tau w}$.

Therefore by assumption there are elements $\theta_1,\dots,\theta_z\in\mathscr{H}$ and a polynomial $\varphi\in\mathbb{C}[X_1,\dots,X_z]$ such that $\sigma_{|F_w}=\varphi(\theta_{1|F_w},\dots,\theta_{z|F_w})$. Therefore $\sigma-\varphi(\theta_1,\dots,\theta_z)=\sigma_w\cdot\hat\sigma$ by (1) in Remark 5.3, where

$$\hat\sigma\in H^0\big(X,r_\sigma\textstyle\sum_j(f_{\tau j}+b_{\tau j})F_j-F_w\big).$$

Since

$$r_\sigma\sum_j(f_{\tau j}+b_{\tau j})F_j-F_w=r_\sigma\Big((f_{\tau w}+\tfrac{r_\sigma-1}{r_\sigma})F_w+\sum_{j\neq w}(f_{\tau j}+b_{\tau j})F_j\Big),$$

we have that $r_\sigma\sum_j(f_{\tau j}+b_{\tau j})F_j-F_w$ belongs to $\mathbb{R}_+\mathscr{B}_\tau$, and in particular it belongs to $\mathscr{C}\cap\Lambda$. Replacing σ by $\hat\sigma$, we finish by descending induction on $\deg\sigma$.

Step 5. Therefore it remains to show that for each j, the restricted algebra $\operatorname{res}_{F_j}R(X,\mathscr{C}_j\cap\Lambda)$ is finitely generated.

To that end, denote $\mathscr{N}=\{1,\dots,N\}$, choose a rational $0<\varepsilon\ll1$ such that $\varepsilon\sum_{k\in I}F_k+A$ is ample for every $I\subset\mathscr{N}$, and let $A_I\sim_{\mathbb{Q}}\varepsilon\sum_{k\in I}F_k+A$ be a general ample $\mathbb{Q}$-divisor. Fix j, and for $I\subset\mathscr{N}\setminus\{j\}$ let

$$\mathscr{F}^I_{\tau j}=(1+f_{\tau j})F_j+\sum_{k\in I}[1-\varepsilon+f_{\tau k},1+f_{\tau k}]F_k+\sum_{k\notin I\cup\{j\}}[\delta_{\tau k}+f_{\tau k},1-\varepsilon+f_{\tau k}]F_k.$$

Set $\mathscr{F}^I_j=\bigcup_{\tau\in\mathscr{T}}\mathscr{F}^I_{\tau j}$; these are rational polytopes such that $\mathscr{F}_j=\bigcup_{I\subset\mathscr{N}\setminus\{j\}}\mathscr{F}^I_j$, and therefore $\mathscr{C}^I_j=\mathbb{R}_+\mathscr{F}^I_j$ are rational polyhedral cones such that $\mathscr{C}_j=\bigcup_{I\subset\mathscr{N}\setminus\{j\}}\mathscr{C}^I_j$. Furthermore, for every $s\in\mathscr{C}^I_j\cap\Lambda$ we have $s\sim_{\mathbb{Q}}r_s(K_X+F_j+\Delta_s+A)\sim_{\mathbb{Q}}r_s(K_X+F_j+\Delta'_s+A_I)$, where $\Delta'_s=\Delta_s-\varepsilon\sum_{k\in I}F_k\geq0$ and $\lfloor F_j+\Delta'_s+A_I\rfloor=F_j$.

Therefore it is enough to prove that $\operatorname{res}_{F_j}R(X,\mathscr{C}^I_j\cap\Lambda)$ is finitely generated for every I. Fix I and let $h_1,\dots,h_m$ be generators of $\mathscr{C}^I_j\cap\Lambda$. Similarly as in Step 1 of the proof of Theorem 7.4, it is enough to prove that the restricted algebra $\operatorname{res}_{F_j}R(X;h_1,\dots,h_m)$ is finitely generated. For p sufficiently divisible, by the argument above we have $ph_v\sim\rho_v(K_X+F_j+B_v+A_I)=:H_v$, where $\lceil B_v\rceil\subset\sum_{k\neq j}F_k$, $\lfloor B_v\rfloor=0$, $\rho_v\in\mathbb{N}$ and A_I is a general ample $\mathbb{Q}$-divisor. Therefore it is enough to show that $\operatorname{res}_{F_j}R(X;H_1,\dots,H_m)$ is finitely generated by Lemma 5.4(1), and this follows from Theorem 7.4. □

Finally, we have:

Proof of Theorem 1.2. Similarly as in Step 1 of the proof of Theorem 10.1, I can assume that A is a general ample $\mathbb{Q}$-divisor. Let $f\colon Y\to X$ be a log resolution of $(X,\sum\Delta_i)$, let H be a small effective f-exceptional divisor such that $A'\sim_{\mathbb{Q}}f^*A-H$ is ample, and denote $\Gamma_i=\mathbf{B}(X,\Delta_i+A)_Y-f^*A+H$. Since $K_Y+\mathbf{B}(X,\Delta_i+A)_Y\sim_{\mathbb{Q}}K_Y+\Gamma_i+A'=:D'_i$, and since $R(Y;D'_1,\dots,D'_\ell)$ and $R(X;D_1,\dots,D_\ell)$ have isomorphic truncations, replacing X by Y, Δ_i by Γ_i and A by A' we may assume that (X,Δ_i+A) is log smooth for every i.

Let K_X be a divisor with $\mathscr{O}_X(K_X)\simeq\omega_X$ and $\operatorname{Supp}A\not\subset\operatorname{Supp}K_X$, let $V\subset\operatorname{Div}(X)_{\mathbb{R}}$ be the vector space spanned by the components of $\sum\Delta_i$ and let $\Lambda\subset\operatorname{Div}(X)$ be the monoid spanned by the components of K_X, $\sum\Delta_i$ and A. The set $\mathscr{C}=\sum\mathbb{R}_+D_i\subset\Lambda_{\mathbb{R}}$ is a rational polyhedral cone. Similarly as in Step 2 of the proof of Theorem 10.1 it is enough to prove that the algebra $R(X,\mathscr{C}\cap\Lambda)$ is finitely generated. By Theorem C the set $\mathscr{E}_{V,A}$ is a rational polytope, and denote $\mathscr{D}=\mathbb{R}_+(K_X+A+\mathscr{E}_{V,A})\subset\Lambda_\mathbb{R}$. Then the algebra $R(X,\mathscr{C}\cap\Lambda)$ is finitely generated if and only if the algebra $R(X,\mathscr{D}\cap\Lambda)$ is finitely generated. Let $H_1,\dots,H_m$ be generators of the monoid $\mathscr{D}\cap\Lambda$. Then it is enough to prove that the ring $R(X;H_1,\dots,H_m)$ is finitely generated, and this follows from Theorem A. □

Proof of Theorem 1.1. By [FM00, 5.2] and by induction on $\dim X$, we may assume $K_X+\Delta$ is big. Write $K_X+\Delta\sim_\mathbb{Q} B+C$ with B effective and C ample. Let ε be a small positive rational number and set $\Delta'=(\Delta+\varepsilon B)+\varepsilon C$. Then $K_X+\Delta'\sim_\mathbb{Q}(\varepsilon+1)(K_X+\Delta)$, and $R(X,K_X+\Delta)$ and $R(X,K_X+\Delta')$ have isomorphic truncations, so the result follows from Theorem 1.2. □

APPENDIX A. HISTORY AND THE ALTERNATIVE

In this appendix I briefly survey the development of the Minimal Model Program, and then present an alternative approach to the classification of varieties. There are many works describing Mori theory, and I do not spend much time on that. My principal goal is to outline a different strategy, whose philosophy is greatly influenced and advocated by A. Corti. I do not intend to be exhaustive, but rather to put together results and ideas that I particularly find important, some of which are scattered throughout the literature or cannot be found in written form.

For many years the guiding philosophy of the Minimal Model Program was to prove finite generation of the canonical ring as a standard consequence of the theory, namely as a corollary to the existence of minimal models and of the Abundance conjecture. Efforts in this direction culminated in [BCHM06], which derived the finite generation in the case of klt singularities from the existence of minimal models for varieties of log general type. However, passing to the case of log canonical singularities, as well as trying to prove the Abundance conjecture, although seemingly slight generalisations, seem to be substantially harder problems where different techniques and methods are welcome, if not needed. The aim of the new approach is to invert the conventional logic of the theory, where finite generation is not at the end, but at the beginning of the process, and the standard theorems and conjectures of Mori theory are derived as consequences. I hope the results of this paper give substantial ground to such claims.

There are many contributors to the initial development of Mori theory, Mori, Reid, Kawamata, Shokurov, Koll´ar, Corti to name a few. In the MMP one starts with a $\mathbb{Q}$-factorial log canonical pair (X,Δ), and then constructs a birational map $\varphi\colon X\dashrightarrow Y$ such that the pair $(Y,\varphi_*\Delta)$ has exceptionally nice properties. Namely we expect that in the case of log canonical singularities, there is the following dichotomy:

1. if $\kappa(X,K_X+\Delta)\geq 0$, then $K_Y+\varphi_*\Delta$ is nef (Y is a *minimal model*),
2. if $\kappa(X,K_X+\Delta)=-\infty$, then there is a contraction $Y\to Z$ such that $\dim Z<\dim Y$ and $-(K_Y+\varphi_*\Delta)$ is ample over Z (Y is a *Mori fibre space*).

If Y is a Mori fibre space, then it is known that $\kappa(X,K_X+\Delta)=-\infty$ and X is uniruled. The reverse implication is much harder to prove. The greatest contributions in that direction are [BDPP04], which proves that if X is smooth and K_X is not pseudo-effective, then X is uniruled, and [BCHM06], which proves that if $K_X+\Delta$ is klt and not pseudo-effective, then there is a map to Y as in (2) above.

The classical strategy is as follows: if $K_X+\Delta$ is not nef, then by the Cone theorem (known for log canonical pairs by the work of Ambro and Fujino, see [Amb03]) there is a $(K_X+\Delta)$-negative extremal ray R in $\overline{\mathrm{NE}}(X)$, and by the contraction theorem there is a morphism $\pi\colon X\to W$ which contracts curves whose classes belong to R, and only them. Then if $\dim W<\dim X$ we are done. Otherwise π is birational, and there are two cases. If $\operatorname{codim}_X\operatorname{Exc}\pi=1$, then π is a *divisorial contraction*, W is $\mathbb{Q}$-factorial, $\rho(X/W)=1$ and we continue the process starting from the pair $(W,\pi_*\Delta)$. If $\operatorname{codim}_X\operatorname{Exc}\pi\geq 2$, then π is a *flipping contraction*, $\rho(X/W)=1$, but $K_W+\pi_*\Delta$ is no longer $\mathbb{Q}$-Cartier. In order to proceed, one needs to construct a *flip* of π, namely a birational map $\pi^+\colon X^+\to W$ such that X^+ is $\mathbb{Q}$-factorial, $\rho(X^+/W)=1$ and $K_{X^+}+\phi_*\Delta$ is ample over W, where $\phi\colon X\dashrightarrow X^+$ is the birational map which completes the diagram. Continuing the procedure, one hopes that it ends in finitely many steps.

Therefore there are two conjectures that immediately arise in the theory: existence and termination of flips. The existence of the flip of a flipping contraction $\pi\colon X\to W$ is known to be equivalent to the finite generation of the *relative canonical algebra*

$$R(X/W,K_X+\Delta)=\bigoplus_{m\in\mathbb{N}}\pi_*\mathscr{O}_X(\lfloor m(K_X+\Delta)\rfloor),$$

and then the flip is given by $X^+=\operatorname{Proj}_W R(X/W,K_X+\Delta)$. The termination of flips is related to conjectures about the behaviour of the coefficients in the divisor Δ, but I do not discuss it here.

Since the paper [Zar62], one of the central questions in higher dimensional birational geometry is the following:

Conjecture A.1. *Let (X,Δ) be a projective log canonical pair. Then the canonical ring $R(X,K_X+\Delta)$ is finitely generated.*

Finite generation implies existence of flips [Fuj09, 3.9]; moreover, one only needs to assume finite generation for pairs (X,Δ) with $K_X+\Delta$ big.

The proof of the finite generation in the case of klt singularities along the lines of the classical philosophy in [BCHM06] is as follows: by [FM00, 5.2] one can assume that $K_X+\Delta$ is big. Then by applying carefully chosen flipping contractions, prove that the corresponding flips exist and terminate (*termination with scaling*), and since the process preserves the canonical ring, deduce finite generation from the basepoint free theorem.

Now assume we have a flipping contraction $\pi\colon(X,\Delta)\to W$ with additional properties that (X,Δ) is a plt pair such that $S=\lfloor\Delta\rfloor$ is an irreducible divisor which is negative over Z. This contraction is called *pl flipping*, and the corresponding flip is the pl flip. Following the work of Shokurov, one of the steps in the proof in [BCHM06] is showing that pl flips exist, and the starting point is Lemma A.2 below. Note that in the context of pl flips, the issues which occur in the problem of global finite generation outlined in the introduction to this paper do not exist. I give a slightly modified proof than the one present elsewhere in the literature in order to stress the following point: I do not *calculate* the kernel of the restriction map, but rather *chase* the generators. This reflects the basic principle: if our algebra is large enough so that it contains the equation of the divisor we are restricting to, then it is automatically finitely generated assuming the restriction to the divisor is. This is one of the main ideas guiding the proof in §10.

Lemma A.2. *Let (X,Δ) be a plt pair of dimension n, where $S = \lfloor \Delta \rfloor$ is a prime divisor, and let $f\colon X \to Z$ be a pl flipping contraction with Z affine. Then $R(X/Z, K_X+\Delta)$ is finitely generated if and only if $\operatorname{res}_S R(X/Z, K_X+\Delta)$ is finitely generated.*

Proof. We will concentrate on sufficiency, since necessity is obvious.

Numerical and linear equivalence over Z coincide by the basepoint free theorem. Since $\rho(X/Z) = 1$, and both S and $K_X+\Delta$ are f-negative, there exists a positive rational number r such that $S \sim_{\mathbb{Q},f} r(K_X+\Delta)$. By considering open subvarieties of Z we can assume that $S - r(K_X+\Delta)$ is $\mathbb{Q}$-linearly equivalent to a pullback of a principal divisor.

Therefore $S \sim_{\mathbb{Q}} r(K_X+\Delta)$, and since then $R(X,S)$ and $R(X,K_X+\Delta)$ have isomorphic truncations, it is enough to prove that $R(X,S)$ is finitely generated. Since a truncation of $\operatorname{res}_S R(X,S)$ is isomorphic to a truncation of $\operatorname{res}_S R(X,K_X+\Delta)$, we have that $\operatorname{res}_S R(X,S)$ is finitely generated. If $\sigma_S \in H^0(X,S)$ is a section such that $\operatorname{div}\sigma_S = S$ and $\mathscr{H}$ is a finite set of generators of the finite dimensional vector space $\bigoplus_{i=1}^d \operatorname{res}_S H^0(X,iS)$, for some d, such that the set $\{s_{|S} : s \in \mathscr{H}\}$ generates $\operatorname{res}_S R(X,S)$, it is easy to see that $\mathscr{H} \cup \{\sigma_S\}$ is a set of generators of $R(X,S)$, since $\ker(\rho_{kS,S}) = H^0(X,(k-1)S)\cdot\sigma_S$ for all k, in the notation of Remark 5.3. □

One of the crucial unsolved problems in higher dimensional geometry is the following Abundance conjecture.

Conjecture A.3. *Let (X,Δ) be a projective log canonical pair such that $K_X+\Delta$ is nef. Then $K_X+\Delta$ is semiample.*

Until the end of the appendix I discuss this conjecture more thoroughly. There are, to my knowledge, two different ways to approach this problem.

The first approach is close to the classical strategy, and goes back to [Kaw85b]. First let us recall the following definition from [Nak04]; the corresponding analytic version can be found in [Pău08].

Definition A.4. Let X be a projective variety. If D is a pseudo-effective divisor, denote

$$\sigma(D,A) = \sup\big\{k \in \mathbb{N} : \liminf_{m\to\infty} h^0(X,\lfloor mD\rfloor + A)/m^k > 0\big\}.$$

Then the *numerical dimension* of D is

$$\nu(X,D) = \sup\{\sigma(D,A) : A \text{ is ample}\}.$$

We know that $\nu(X,D) = 0$ if and only if $D \equiv N_\sigma\|D\|$, and that $\nu(X,D)$ is the standard numerical dimension when D is nef by [Nak04, 6.2.8]. It is well known that abundance holds when $\nu(X, K_X + \Delta)$ is equal to 0 or $\dim X$ by [Kaw85a, 8.2], and when $\nu(X, K_X + \Delta) = \kappa(X, K_X + \Delta)$ by [Kaw85b, 6.1]. Further, we have the following statement.

Theorem A.5. *Let (X, Δ) be a projective klt pair of dimension n such that $K_X + \Delta$ is nef. Assume that $\nu(Y, K_Y + \Delta_Y) > 0$ implies $\kappa(Y, K_Y + \Delta_Y) > 0$ for any klt pair (Y, Δ_Y) of dimension at most n. Then $K_X + \Delta$ is semiample.*

Proof. Let (S, Δ_S) be a $\mathbb{Q}$-factorial $(n-1)$-dimensional klt pair with $\kappa(S, K_S + \Delta_S) = 0$. Then $\nu(S, K_S + \Delta_S) = 0$ by the assumption in dimension $n-1$, and thus $K_S + \Delta_S \equiv N_\sigma \| K_S + \Delta_S \|$. By [Dru09, 3.4] there exists a minimal model of (S, Δ_S). Now the result follows along the lines of [Kaw85b, 7.3]. □

The assumption in the theorem can be seen as a stronger version of non-vanishing.

Now I present a different approach, where one derives abundance from the finite generation. It is a result of J. M$^{\text{c}}$Kernan and C. Hacon, and I am grateful to them for allowing me to include it here.

Theorem A.6. *Assume that for every $(n+1)$-dimensional projective log canonical pair (X, Δ) with $K_X + \Delta$ nef and big, the canonical ring $R(X, K_X + \Delta)$ is finitely generated. Then abundance holds for klt pairs in dimension n.*

Proof. Let (Y, Φ) be an n-dimensional projective klt pair such that $K_Y + \Phi$ is nef, and let $Y \subset \mathbb{P}^N$ be some projectively normal embedding. Let X_0 be the cone over it, let $X = \mathbb{P}(\mathscr{O}_Y \oplus \mathscr{O}_Y(1))$ be the blowup of X_0 at the origin, and let $H' \subset \mathbb{P}^N$ be a sufficiently ample divisor which does not contain the origin. Let Δ be the proper transform of Φ in X, let $E \subset X$ be the exceptional divisor, and let H be the proper transform of H' in X.

Then by inversion of adjunction the pair $(X, \Upsilon = E + \Delta + H)$ is log canonical, and of log general type since H' is ample enough. We have $Y \simeq E$, and this isomorphism maps $K_Y + \Phi$ to $K_E + \Delta_{|E}$. The divisor $K_X + \Upsilon$ is also nef: since $(K_X + E + \Delta)_{|E}$ is identified with $K_Y + \Phi$, this deals with curves lying in E by nefness, and for those curves which are not in E, the ampleness of H away from E ensures that the intersection product with $K_X + \Upsilon$ is positive. Then since the algebra $R(X, K_X + \Upsilon)$ is finitely generated by assumption, we have that $K_X + \Upsilon$ is semiample by [Laz04, 2.3.15], and then so is $K_E + \Delta_{|E} = (K_X + \Upsilon)_{|E}$. □

Finally a note about the general alternative philosophy. Since [HK00] it has become clear that adjoint rings encode many important geometric information about the variety. In particular, Theorem 1.2 in the case of a Fano variety X implies that X is a Mori dream space [HK00, 2.9], and therefore all the main theorems and conjectures of Mori theory hold on X, such as the Cone and Contraction theorems, existence and termination of flips, abundance [HK00, 1.11]. In particular, the following conjecture applied to Mori dream regions [HK00, 2.12, 2.13] seems to encode the whole Mori theory.

Conjecture A.7. *Let X be a projective variety, and let $D_i = k_i(K_X + \Delta_i) \in \mathrm{Div}(X)$, where (X, Δ_i) is a log canonical pair for $i = 1, \dots, \ell$. Then the adjoint ring $R(X; D_1, \dots, D_\ell)$ is finitely generated.*

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