Mixing with staircase multiplicity function ⋆

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Abstract

Every subgroup of the symmetric group defines a natural factor of the Cartesian power of a transformation. We calculate the set of values of the spectral multiplicity function of such factors (under certain conditions on a transformation) in terms of numbers of orbits of diagonal actions of these subgroups. The analogous statement is also valid for factors of tensor products of a unitary operator preserving $\mathbf{1}$. As an application we prove, in particular, that for any n there exists a mixing (of all orders) transformation having a staircase multiplicity function of length n, i.e. essential values of the spectral multiplicity function are $\{1, 2, \ldots, n\}$.

Key words: Spectral invariants; Ergodic theory; Mixing; The spectral multiplicity function

1 Introduction

By dynamical system T we mean an invertible measure preserving map (transformation) acting on a non-atomic Lebesgue space (X, μ). Iterations (powers) of this map define an action of $\mathbf{Z}$, thus forming a subgroup of a group of all measure preserving transformations.

Spectral invariants of T, i.e. the spectrum of a unitary (Koopman) operator induced on $L_2 : \widehat{T} f(x) = f(Tx)$ have become classical in the theory of dy-

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namical systems. In this paper we study (in a special interesting subclass) the following general problem:

to determine precise conditions on the spectrum of a unitary operator under which it can be realized by a dynamical system (see [6, p.36]).

Let us recall that the spectrum of any unitary operator U is completely determined by the spectral multiplicity function $M(U)$ and the measure σ_U of the maximal spectral type defined on the one dimensional torus. Below we will concentrate our attention at $M(\widehat{T})$, and its values denoted by the same symbol. It is said that a transformation has a *homogeneous* spectrum of multiplicity n if $M(\widehat{T}) \equiv n$, where the Koopman operator $\widehat{T}$ is considered on the main invariant subspace $L_2^0 = \{f \in L_2 : \int f d\mu = 0\}$. We refer the reader to surveys [8],[10] having sufficiently complete information concerning the history of spectral invariants in ergodic theory. Let us only stress the recent solution of the homogeneous spectrum problem (see the case $n = 2$ in [2],[13], and [4] for the case of the complete generality). In spite of the fact that in the set of all ergodic transformations we have almost full information concerning possible values of $M(\widehat{T})$, the case of mixing transformations is still weakly studied, because many useful methods applicable for typical transformations do not work in the subset of mixing transformations.

Next we study $M(\widehat{T})$ for mixing T. Staircase constructions of rank 1 transformations are used in search of series of mixing transformations with various multiplicity functions. In particular, we prove that there exist dynamical systems with staircase multiplicity functions. More precisely,

Theorem 1. *For any n there exists a mixing transformation T such that*

$$M(\widehat{T}) = \{2, 3, \dots, n\} \text{ on } L_2^0,$$

this gives

$$M(\widehat{T}) = \{1, 2, 3, \dots, n\} \text{ on } L_2.$$

Remark 1. For $n = 2$ it was proved by Ryzhikov (see [14]).

Remark 2. Theorem 1 also gives an answer to a question of Robinson (see [12], and [8, Subsect. 5.3]) about possible values of $m(T) = \max M(\widehat{T})$, who showed that for any n there exists a mixing T with $n < m(T) < \infty$.

Remark 3. All mixing transformations constructed in this paper are mixing of all orders, or, in other terms, they are k-fold mixing for any $k > 1$.

Let $\mathfrak{S}_n$ be the *symmetric* group acting on $\{1, \dots, n\}$ by permutations, and

G be any subgroup of $\mathfrak{S}_n$. For any $1 \leq k \leq n$ denote by the same symbol G the *diagonal* action of G on $I_k = \{1, \dots, n\}^k$. Consider a restriction of the G-action to a G invariant subset

$$I'_k = \{i_k = (i_k(1), \dots, i_k(k)) \in I_k : i_k(l) = i_k(m) \text{ iff } l = m\}.$$

Suppose $\sim$ is an orbital equivalence relation naturally defined by G on I'_k. Let us define

$$D_k = \sharp I'_k / \sim (1 \leq k \leq n), \text{ and } D_G(n) = (D_1, \dots, D_n).$$

In Sects. 3-5 we will show that $D_G(n)$ has a direct connection (see Theorems 2-4) to spectral multiplicity functions of natural factors of the Cartesian powers. Roughly speaking, in the framework of our construction to have a serious of sets (without 1!) as the set of essential values of the spectral multiplicity function on L_2^0, we need to calculate $D_G(n)$. This construction can be considered as a source of mixing transformations having new (highly nonhomogeneous) spectral multiplicity functions.

2 Elementary properties of $D_G(n)$

Proposition 1. *Let G be a subgroup of $\mathfrak{S}_n$. Then $D_G(n)$ has the following properties:*

(1) $D_G(n) = (1, \dots, 1)$ if $G = \mathfrak{S}_n$.
(2) $D_G(n) = (n, n(n-1), \dots, n!, n!)$ if $G = \{e\}$.
(3) $D_G(n) = (2, 3, \dots, n, n)$ if $G = \{g \in \mathfrak{S}_n : g(n) = n\}$.
(4) $D_k \leq D_m$ if $k < m$.
(5) $D_{n-1} = D_n$.
(6) $[\forall j \; D_j = D_1] \Leftrightarrow [n = 2$ or $G = \mathfrak{S}_n]$.
(7) $D_n = n!/\sharp G$.
(8) $D_G(n) \leq D_{G_1}(n)$ if G_1 is a subgroup of G.

We leave the proof of these simple statements to the reader. Let us only mention that part 6 follows from the obvious claim $D_2 \geq D_1(D_1 - 1)$.

Remark 4. $D_G(n)$ is closely related to the average numbers of fixed points. Indeed, due to the Cauchy-Frobenius lemma (see [11]) if G acts by permutations on a finite set X, and D is the number of its orbits, then

$$D = \frac{1}{\sharp G} \sum_{g \in G} \sharp F(g), \tag{1}$$

where $F(g)$ denotes the set of all fixed points of g. Associate with G the polynomial $P(z) = \sum b_i z^i$, where $b_i = \sharp\{g \in G : \sharp F(g) = i\}$. Obviously, $P(1) = \sharp G$. Using (1), it can easily be checked that

$$D_G(n) = (P'(1), \dots, P^{(n)}(1))/P(1).$$

This expression of $D_G(n)$ does not give extra information concerning possible values of $D_G(n)$, but is more useful for calculations. For example, for a unique nontrivial normal subgroup of $\mathfrak{S}_n$, i.e. it is the case of $n = 4$ and we consider the set of all permutations with two cycles of 2 length equipped by e, we have $P(z) = 3 + z^4$, and $D_G(n) = (1, 3, 6, 6)$.

Remark 5. $D_G(n)$ can be also expressed in probability terms. Namely,

$$D_G(n) = (E(\xi), E(\xi(\xi - 1)), \dots, E(\xi(\xi - 1) \cdots (\xi - n + 1))),$$

where $E(\eta)$ is the expectation of a random variable η, and $\xi(g)$ is the number of all fixed points of g.

3 Multiplicity functions of factors of tensor products

Given a subgroup G of $\mathfrak{S}_n$, and $(D_1, \dots, D_n) = D_G(n)$, let

$$M_G(n) = \{D_1, \dots, D_n\}.$$

Every element g of G naturally defines a permutation of the coordinates viewed as a transformation, say g, of (X^n, μ^n), i.e. $g(x_1, \dots, x_n) = (x_{g(1)}, \dots, x_{g(n)})$, and a unitary operator on $L_2(X^n, \mu^n)$. So we have a certain unitary representation, say V, of G by operators on $L_2(X^n, \mu^n)$. Denote

$$H_{\text{inv}}(G) = \{f \in L_2 : \forall g \in G \ [V_g f = f]\}.$$

Then the subspace of all square integrable functions which are invariant with respect to any permutations of the coordinates, named $H_{\text{sym}}(n)$, is just $H_{\text{inv}}(\mathfrak{S}_n)$. Obviously, for any unitary operator U acting on $L_2(X, \mu)$, $g \in \mathfrak{S}_n$, $U^{(n)} V_g = V_g U^{(n)}$ and $U^{(n)} H_{\text{inv}}(G) = H_{\text{inv}}(G)$, where $U^{(n)} = U \otimes \cdots \otimes U$ (n times) is a tensor product of U.

Definition. We say that a unitary operator $U_G^{(n)}$ is a *factor* of $U^{(n)}$ associated with G, if it is just a restriction of $U^{(n)}$ to $H_{\text{inv}}(G)$.

Note that the choice of this name was done, because this operator looks as a natural generalization of operators adjoint to (measure-theoretical) factors of dynamical systems.

For any Hilbert subspace, say H, of $L_2(Y,\nu)$ we will write H^0 instead of $\{f \in H : \int f d\nu = 0\}$. Obviously, if $\mathbf{1} \in H$, then $H = H^0 \bigoplus \{c\mathbf{1} : c \in \mathbf{C}\}$.

Theorem 2. *Suppose U is a unitary operator preserving $\mathbf{1}$, i.e. $U(\mathbf{1}) = \mathbf{1}$, and $U^{(n)}$ has a simple continuous spectrum on $H^0_{sym}(n)$; then*

$$M(U_G^{(n)}|_{L_2^0}) = M_G(n). \tag{2}$$

Remark 6. It is not difficult to construct a unitary operator U ($U(\mathbf{1}) = \mathbf{1}$) such that $U^{(n)}$ has a simple spectrum on $H_{\text{sym}}(n)$ (and $H^0_{\text{sym}}(n)$), and (2) is not valid for some G. However, in the case of a unitary operator U coming from a dynamical system, the simplicity of the spectrum of $U^{(n)}$ on $H_{\text{sym}}(n)$ implies that U on L_2^0, consequently, $U^{(n)}$ on $H^0_{\text{sym}}(n)$ has a continuous spectrum.

Denote by σ_U the measure of the maximal spectral type for U on an invariant subspace L_2^0, and by σ^n the measure $\sigma_U \times \cdots \times \sigma_U$ (n times) on $\mathbf{T^n}$. The image, say $\sigma^{(n)}$, of the measure σ^n under the map

$$\pi : (\lambda_1, \ldots, \lambda_n) \to \lambda_1 \cdots \lambda_n,$$

, i.e. $\sigma^{(n)} = \pi(\sigma^n)$, is named a *convolution* of measures σ_U.

Let us first prove the following lemma.

Lemma 1. *Assume U satisfy the conditions in Theorem 2; then for any $k \le n$ and $i_k \in I'_k$, there exists a (nonzero) Hilbert subspace H_{i_k} in $L_2^0(X^n,\mu^n)$ such that the following properties are hold:*

(1) $L_2^0(X^n,\mu^n) = \bigoplus_{k\le n} \bigoplus_{i_k \in I'_k} H_{i_k}$.
(2) H_{i_k} is orthogonal to H_{j_m} iff $i_k \ne j_m$.
(3) For any $i_k \in I'_k$, $U^{(n)} H_{i_k} = H_{i_k}$, moreover, $U^{(n)}|_{H_{i_k}}$ has a simple spectrum, and $\sigma_{U^{(n)}|_{H_{i_k}}} \sim \sigma^{(k)}$.
(4) For any $g \in \mathfrak{S}_n$, $i_k \in I'_k$, $V_g H_{i_k} = H_{g(i_k)}$, and if $g(i_k) = i_k$, then $V_g|_{H_{i_k}} = E$.

Proof. Consider a map $s : L_2^0(X,\mu) \to H^0_{\text{sym}}(n)$ defined by

$$s(\phi) = \frac{1}{\sqrt{n}} \sum_i \phi(x_i).$$

The map s is a unitary isomorphism between Hilbert spaces $L_2^0(X,\mu)$ and $s(L_2^0(X,\mu))$, and one has $sU = U^{(n)}s$. Therefore a unitary operator U has a simple continuous spectrum on $L_2^0(X,\mu)$. This implies that U has a simple spectrum on $L_2(X,\mu)$, and σ_U is a continuous measure.

By the spectral theorem, there exists a unitary isomorphism between Hilbert spaces $L_2(X,\mu)$ and $L_2(\mathbf{T},\sigma_1)$, say Φ, such that the image of U is just a multiplication by λ, where $\sigma_1 = (\sigma_U + \delta)/2$, and δ is a one point measure concentrated at 1, i.e. $\delta(\{1\}) = 1$. Denote by φ the primage of $\mathbf{1}$ via Φ. Since $\sigma_U(\{1\}) = 0$, $\Phi(L_2^0(X,\mu)) = \mathbf{1}_{\mathbf{T}\setminus\{1\}}L_2(\mathbf{T},\sigma_1)$, and $\Phi(\int \varphi d\mu) = \mathbf{1}_{\{1\}}$, where $\mathbf{1}_B$ is the indicator of B.

We will next repeatedly use a certain unitary isomorphism between Hilbert spaces $L_2(X^n,\mu^n)$ and $L_2(\mathbf{T}^n,\sigma_1^n)$, say Φ_n, defined by Φ. Obviously, $\Phi_n\varphi_n = \mathbf{1}$ $(= \mathbf{1}_{\mathbf{T}^n})$, where $\varphi_n(x_1,\ldots,x_n) = \varphi(x_1)\cdots\varphi(x_n)$.

For any $i_k \in I'_k$ $(1 \le k \le n)$, let

$$B^{i_k} = \{(\lambda_1,\ldots,\lambda_n) \in \mathbf{T}^n : \forall j[\lambda_j \ne 1 \text{ iff } \exists p[j = i_k(p)]]\},$$

where $i_k = (i_k(1),\ldots,i_k(k))$. We will use the order on $\mathbf{T}\setminus\{1\}$ coming from a certain correspondence between $\mathbf{T}\setminus\{1\}$ and $(0,1)$. Put

$$B_{i_k} = \{(\lambda_1,\ldots,\lambda_n) \in B^{i_k} : \lambda_{i_k(1)} < \lambda_{i_k(2)} < \cdots < \lambda_{i_k(k)}\}.$$

It is easy to see that sets B_{i_k} are mutually disjoint ($i_k \in I'_k$, $1 \le k \le n$), and a remaining set to $\mathbf{T}^n\setminus\{(1,\ldots,1)\}$ is a subset of a union of the following sets:

$$R_{j_1,j_2} = \{(\lambda_1,\ldots,\lambda_n) \in \mathbf{T}^n : \lambda_{j_1} = \lambda_{j_2} \ne 1\},\ (j_1 < j_2 \le n).$$

Since σ_U is the continuous measure, for any $j_1 < j_2$ we have

$$\sigma_1^n(R_{j_1,j_2}) = \sigma^2\{(\lambda_1,\lambda_2) \in \mathbf{T}^2 : \lambda_1 = \lambda_2\} = 0.$$

Collecting all above remarks, we conclude that $L_2(\mathbf{T}^n,\sigma_1^n)$ can be represented as

$$L_2(\mathbf{T}^n,\sigma_1^n) = \mathbf{1}_{\{(1,\ldots,1)\}}L_2(\mathbf{T}^n,\sigma_1^n) \bigoplus \bigoplus_{k\le n} \bigoplus_{i_k\in I'_k} \mathbf{1}_{B_{i_k}}L_2(\mathbf{T}^n,\sigma_1^n).$$

Let us check that for any $i_k \in I'_k$ Hilbert spaces $\Phi_n^{-1}(\mathbf{1}_{B_{i_k}}L_2(\mathbf{T}^n,\sigma_1^n))$ can be chosen as H_{i_k}. Indeed, part 1 of Lemma 1 follows from $\Phi_n(c\mathbf{1}) = \mathbf{1}_{\{(1,\ldots,1)\}}$, where $c = (\int \varphi d\mu)^n$. Parts 2 and 4 are obvious, because for any $g \in \mathfrak{S}_n$ an image of V_g via Φ_n is just a unitary operator defined by the permutation g of coordinates on $\mathbf{T}^n$.

In order to prove part 3 we consider a spectral representation of the operator $U^{(n)}$. The operator $U^{j_1} \otimes U^{j_2} \otimes \cdots \otimes U^{j_n}$ on $L_2(X^n, \mu^n)$ is unitarily isomorphic to the operator $\lambda_1^{j_1} \cdots \lambda_n^{j_n} \cdot \widehat{E}$ on $L_2(\mathbf{T}^n, \sigma_1^n)$. Therefore, it is clear that for any $i_k \in I_k'$, $U^{(n)} H_{i_k} = H_{i_k}$. Fix k. By part 4, spectral invariants of $U^{(n)}$ on H_{i_k} do not depend on our choice of H_{i_k}. Let $i_k = (1, \ldots, k)$. Obviously, the Hilbert space $\mathbf{1}_{B_{i_k}} L_2(\mathbf{T}^n, \sigma_1^n)$ is naturally isomorphic to $\mathbf{1}_C L_2(\mathbf{T}^k, \sigma^k)$, where

$$C = \{(\lambda_1, \ldots, \lambda_k) \in \mathbf{T}^k : \forall j[\lambda_j \neq 1] \& \lambda_1 < \lambda_2 < \cdots < \lambda_k\},$$

and the image of $\tilde{U}^{(n)} = \lambda_1 \cdots \lambda_n \cdot \widehat{E}$ via such an isomorphism is $\tilde{U}^{(k)}$. Since σ_U is a continuous measure, we have

$$\mathbf{1}_{\mathbf{T}^k} = \sum_{g \in \mathfrak{S}_k} \mathbf{1}_{gC} \quad \sigma^k \text{ a.e.}$$

Since σ^k is a symmetric measure, we have

$$\forall p \ \langle \tilde{U}^{(k)p} \mathbf{1}_C, \mathbf{1}_C \rangle = \frac{1}{\sharp \mathfrak{S}_k} \langle \tilde{U}^{(k)p} \mathbf{1}, \mathbf{1} \rangle = \frac{1}{k!} \Big(\int \lambda^p d\sigma_U \Big)^k.$$

Using the fact that the Fourier transform of a multiplication is a convolution, we see that the spectral measure of $\mathbf{1}_C \in L_2(\mathbf{T}^k, \sigma^k)$ is $\sigma^{(k)}/k!$. Obviously, $\sigma^{(k)}$ is the measure of the maximal spectral type of $\tilde{U}^{(k)}$ on $L_2(\mathbf{T}^k, \sigma^k)$. Therefore $\sigma^{(k)}$ is the measure of the maximal spectral type of $U^{(n)}$ on H_{i_k}.

To show the simplicity of the spectrum of $U^{(n)}$ on H_{i_k}, consider a symmetrization map $\psi : H_{i_k} \to H^0_{\text{sym}}(n)$ defined by

$$\psi(\phi) = \sum_{i_k' \in I_k'} \phi_{i_k'},$$

where $\phi_{i_k'} \in H_{i_k'}$, and $\phi_{i_k'} = V_g \phi$ for some $g \in \mathfrak{S}_n$. Since $\psi U^{(n)} = U^{(n)} \psi$, $U^{(n)}$ has a simple spectrum on H_{i_k}. Lemma 1 is proved. $\square$

Proof of Theorem 2. Given an $i_k \in I_k'$ $(k \leq n)$, $f \in H_{i_k}$, let $O_{i_k}(G) = \{g(i_k) : g \in G\}$. For any $g \in G$, denote by $f_{i_k'}$ the function $V_g f$, where $i_k' = g(i_k)$. Therefore, by Lemma 1, $f_{i_k'} \in H_{i_k'}$, and for any $i_k' \in O_{i_k}(G)$, $f_{i_k'}$ does not depend on our choice of g. Consider a symmetrization map SG_{i_k} :

$H_{i_k} \to \bigoplus_{i'_k \in O_{i_k}(G)} H_{i'_k}$, defined by

$$SG_{i_k}(f) = \frac{1}{\sqrt{\sharp O_{i_k}(G)}} \sum_{i'_k \in O_{i_k}(G)} f_{i'_k}.$$

Put

$$H_{\text{sym}}(i_k, G) = H_{\text{inv}}(G) \cap \bigoplus_{i'_k \in O_{i_k}(G)} H_{i'_k}.$$

It is easy to see that $SG_{i_k}(H_{i_k})$ is just $H_{\text{sym}}(i_k, G)$. Clearly, SG_{i_k} is a unitary isomorphism. Moreover, SG_{i_k} intertwines $U^{(n)}$ and $U_G^{(n)}$ ($= U^{(n)}$ on $H_{\text{inv}}(G)$), because, for any g, $V_g U^{(n)} = U^{(n)} V_g$. Thus Lemma 1 part 3 implies that $U_G^{(n)}|_{H_{\text{sym}}(i_k,G)}$ has a simple spectrum of the maximal spectral type $\sigma^{(k)}$. Consequently, $U_G^{(n)}|_{\bigoplus_{i_k \in I'_k} H_{i_k}}$ has D_k simple (mutually orthogonal) components of the maximal spectral type $\sigma^{(k)}$, because $H^0_{\text{inv}}(G)$ is a sum of $U_G^{(n)}$ invariant mutually orthogonal subspaces $H_{\text{sym}}(i_k, G)$, $i_k \in I'_k$, $k = 1, \ldots, n$.

To prove Theorem 2, it is remain to show that $\sigma^{(k_1)} \perp \sigma^{(k_2)}$ if $k_1 < k_2 \leq n$. Let us mention first that $U^{(n)}$ invariant subspaces $H_{\text{sym}}(i_{k_j}, \mathfrak{S}_n)$ ($j = 1, 2$) of $H_{\text{sym}}(n)$ are mutually orthogonal. The operator $U^{(n)}$ has a simple spectrum on $H_{\text{sym}}(n)$. Therefore measures of the maximal spectral type ($\sigma^{(k_j)}$) of $U^{(n)}$ on these subspaces ($j = 1, 2$) are mutually disjoint. Theorem 2 is proved. $\square$

4 Applications to dynamical systems

Every subgroup G of $\mathfrak{S}_n$ acts by permutations of coordinates on X^n, and defines an orbital equivalence relation $\sim$ on X^n. Since for any $g \in \mathfrak{S}_n$ transformations g and $T^{(n)}$ commute, $T^{(n)}$ defines a transformation, say $T_G^{(n)}$, on a non-atomic Lebesgue space $(X^n/\sim, \varphi(\mu^n))$, where φ is a canonical map from X^n to $X^n/\sim$. The dynamical system $T_G^{(n)}$ is named a *factor* of $T^{(n)}$. It is easy to see that Koopman's operator $\widehat{T_G^{(n)}}$ is unitarily isomorphic to a factor $\widehat{T}_G^{(n)}$.

Let us recall that the set of all transformations (automorphisms of the σ- algebra of measurable sets) of (X, μ) is a Polish (complete metrizable separable) topological group, noted $\mathbf{Aut}(\mu)$, with respect to the weak (coarse) topology defined by

$$T_n \to T \Leftrightarrow \mu(T_n^{-1} A \Delta T^{-1} A) \to 0 \text{ for each measurable } A$$

(we identify transformations if they are coincide up to a set of measure zero). We say that a property holds for a *typical* element from a topological space D if the set of elements from D with this property contains a dense G_δ subset of D.

Theorem 3. *For a typical dynamical system T, $T_G^{(n)}$ is weakly mixing and*

$$M(\widehat{T_G^{(n)}}|_{L_2^0}) = M_G(n).$$

Remark 7. In particular, if $\sharp G = 1$, we have

$$M(\widehat{T^{(n)}}|_{L_2^0}) = \{n, n(n-1), \ldots, n!\}$$

for a typical T. It was proved in [2] as an answer to Katok's conjecture.

Proof. It is well known that a typical (generic) transformation has a simple continuous spectrum on L_2^0. Therefore every (nontrivial) factor of $T^{(n)}$ (in particular, $T_G^{(n)}$) is weakly mixing. To apply Theorem 2 it remains to show that for a typical transformation T the operator $\widehat{T}^{(n)}$ has a simple spectrum on $H_{\text{sym}}(n)$. The proof of this statement is appeared (not very explicitly) in [2],[3]. In spite of that fact, for the sake of clarity we next give an adapted proof.

It is well known (see, for example, [4, Corollary 2]) that, for a typical T, and any $\alpha \in (0,1)$, there exists a sequence $m_i(\alpha)$ such that

$$\widehat{T}^{m_i(\alpha)} \to Q_\alpha,$$

where $Q_\alpha = \alpha\widehat{E} + (1-\alpha)\widehat{T}$, and the symbol "$\to$" denotes the weak operator convergence. Therefore

$$\widehat{T}^{(n)m_i(\alpha)} \to Q_\alpha \otimes \cdots \otimes Q_\alpha. \tag{3}$$

We will use the notation described in Sect. 3. (3) then can be rewritten as follows.

$$\tilde{T}^{(n)m_i(\alpha)} \to (\alpha + (1-\alpha)\lambda_1) \cdots (\alpha + (1-\alpha)\lambda_n) \cdot \widehat{E}, \tag{4}$$

where $\tilde{T}^{(n)} = \lambda_1 \cdots \lambda_n \cdot \widehat{E}$ acts on $L_2(\mathbf{T}^n, \sigma_1^n)$, and $\Phi_n \widehat{T}^{(n)} = \tilde{T}^{(n)}\Phi_n$. Put

$$P_n(\lambda_1, \ldots, \lambda_n, \Theta) = (\Theta + \lambda_1) \cdots (\Theta + \lambda_n).$$

It is clear that if $\widehat{T}^{n_i} \to Q$ for some linear operator Q, then, for any $f \in L_2$, $Qf \in \overline{\text{span}}\{\widehat{T}^j f : j \in \mathbf{Z}\}$. Taking into account (4), we have

$$P_n(\cdot, \dots, \cdot, \Theta) \in \overline{\text{span}}\{\tilde{T}^{(n)j}\mathbf{1} : j \in \mathbf{Z}\} \text{ for any } \Theta \in \mathbf{R}_+. \tag{5}$$

Consider a system of canonical conditional measures, say $\nu_n(\cdot|c)$, for σ_1^n naturally defined by a partition of $\mathbf{T}^n$ on *fibers*

$$F(c) = \{(\lambda_1, \dots, \lambda_n) \in \mathbf{T}^n : \lambda_1 \cdots \lambda_n = c\}.$$

It is easy to see that, for any $f \in L_2(\mathbf{T}^n, \sigma_1^n)$, f is an element of $\overline{\text{span}}\{\tilde{T}^{(n)j}\mathbf{1} : j \in \mathbf{Z}\}$ if and only if f is equal (σ_1^n a.e.) to an $\tilde{f}$, where for any c $\tilde{f}|_{F(c)} \equiv \text{const}$. (5) then implies that polynomials $P_n(\lambda_1, \dots, \lambda_n, \Theta)$ are equal to constants on the "*typical*" fiber $F(c)$ for a.e. $(\lambda_1, \dots, \lambda_n)$ with respect to the measure $\nu_n(\cdot|c)$, where "*typical*" means for $\sigma_1^{(n)}$ a.e. c, and $\sigma_1^{(n)} = \pi(\sigma_1^n)$.

Consider $P_n(\lambda_1, \dots, \lambda_n, \Theta)$ as a polynomial of degree n in the variable Θ on the fiber $F(c)$. Then both the first and the last coefficients are constant. Since the number of various Θ is at least $n-1$, this polynomial is uniquely defined, i.e. its remaining coefficients are equal to constants for a.e. $(\lambda_1, \dots, \lambda_n)$ with respect to the measure $\nu_n(\cdot|c)$ on the fiber $F(c)$. Thus it has only one collection of zeros for $\nu_n(\cdot|c)$ a.e. $(\lambda_1, \dots, \lambda_n)$, where $\lambda_1 \cdots \lambda_n = c$. However zeros of $P_n(\lambda_1, \dots, \lambda_n, \Theta)$ are $(-\lambda_1, \dots, -\lambda_n)$. It implies that supp $\nu_n(\cdot|c)$ is a single point modulo permutations of coordinates. Besides since σ_1^n is a symmetric measure, for any $g \in \mathfrak{S}_n$, and $\sigma_1^{(n)}$ a.e. c, we have $g(\nu_n(\cdot|c)) = \nu_n(\cdot|c)$, equivalently, $\nu_n(\cdot|c)$ is a symmetric measure.

Collecting all above remarks, we conclude that every function in $\Phi_n H_{\text{sym}}(n)$, say f, is equal (σ_1^n a.e.) to an $\tilde{f}$, where for any c $\tilde{f}|_{F(c)} \equiv \text{const}$, and, consequently,

$$\Phi_n H_{\text{sym}}(n) = \overline{\text{span}}\{\tilde{T}^{(n)j}\mathbf{1} : j \in \mathbf{Z}\}.$$

Therefore $\widehat{T}^{(n)}$ has a simple spectrum on $H_{\text{sym}}(n)$. Theorem 3 is proved. $\square$

5 Applications to mixing transformations

Theorem 1 is a natural corollary of Proposition 1 and the following theorem:

Theorem 4. *For any subgroup G of $\mathfrak{S}_n$, there exists a mixing (of all orders) transformation T satisfying*

$$M(\widehat{T}|_{L_2^0}) = M_G(n).$$

Proof. It is well known that for any $k > 1$ both Cartesian products and factors of k-fold mixing transformations are also k-fold mixing. Therefore, by Theorem 2, to prove Theorem 4 we only need to find a mixing (of all orders) transformation T satisfying the simplicity of the spectrum of $\widehat{T}^{(n)}|_{H_{\mathrm{sym}}(n)}$ condition. However such transformations exist even in the class of staircase constructions of rank one transformations. It was announced by Ryzhikov (see [14, Theorem 5.2]), and proved for $n = 2$. For full completeness, we give another couple of such examples (see Subsect 5.3) using more spectral arguments for proofs. Let us also recall that if a rank 1 transformation is mixing then it is mixing of all orders. $\square$

5.1 Staircase constructions

A transformation T has *rank* 1 if for any m, there exist an integer h_m and a tower (column)

$$A_m, TA_m, \ldots, T^{h_m-1}A_m,$$

such that all levels T^jA_m, $0 \le j < h_m$, and the remaining set form a measurable partition of X, say ξ_m, and $\xi_m \to \varepsilon$, i.e. for any measurable set B there are ξ_m-measurable sets B_m such that $\mu(B\Delta B_m) \to 0$ as $m \to \infty$. It is well known that every rank one transformation can be defined inductively using the procedure *cutting* and *stacking* described below for staircase transformations. Next we will consider realizations of rank one transformations by piece wise shifts on disjoint unions of subintervals in $\mathbf{R}$.

A rank-one transformation T is called a **staircase construction** if there exists a sequence $(r_m)_{m=1}^{\infty}$ of natural numbers $r_m > 1$ such that each column C_{m+1} is obtained by cutting C_m into r_m subcolumns, say $C_m(i)$, of equal width, placing $i-1$ spacers only on the subcolumn $C_m(i)$ for $1 \le i < r_m$, and then stacking the subcolumn $C_m(i+1)$ on top of $C_m(i)$ for $1 \le i < r_m$. For the column C_m, let h_m be its height and let d_m be a measure of its one level, i.e. its width, where $m \ge 1$. Note that is a modification of staircase construction (see [1]) because we do not add spacers on the top of the subcolumn $C_m(r_m)$. The pair $(h_1, (r_m)_{m=1}^{\infty})$ defines a unique staircase transformation up to a metrical isomorphism. Since our next considerations are independent of the initial choice of h_1, we will denote by $T = T_{(r_m)}$ such a transformation.

Let $O_m(0) = C_m \backslash C_m(1)$. Define $O_m(j) = C_m(1) \cap \cdots \cap C_{m+j-1}(1) \backslash C_{m+j}(1)$ $(j = 1, 2, \ldots)$. From the geometry of the construction we immediately get that

(1) Each C_m consists of pairwise disjoint subcolumns $O_m(j)$ $(j = 0, 1, 2, \ldots)$ of height h_m and width $(r_{m+j} - 1)d_{m+j+1}$.
(2) Each $O_m(j)$ consists of pairwise disjoint subcolumns $O_m(j) \cap C_{m+j}(i)$ $(i = 2, \ldots, r_{m+j})$ of height h_m and equal width.
(3) $T^{-h_m}\Delta_k(i) = T^{i-2}\Delta_k(i-1)$, $i = 2, \ldots, r_{m+j}$, where $\Delta_k(i)$ is k^{th} level of $O_m(j) \cap C_{m+j}(i)$ if $i = 2, \ldots, r_{m+j}$, and $\Delta_k(1)$ is k^{th} level of $C_{m+j}(1)$.

Apriori staircase transformations constructed above act on the collection of mutually disjoint intervals (levels) of $\mathbf{R}$. Note that all properties we are interesting are independent of the mutual location of these intervals on $\mathbf{R}$. We will consider only *finite* staircase transformations, i.e. staircase transformations acting on the space X of finite measure. Choosing the length of the first column level (d_1) appropriately , we can assume that all finite staircase transformations are defined on the unit interval in $\mathbf{R}$ under standard Lebesgue measure. Obviously, *bounded* staircase transformations, i.e. transformations $T = T_{(r_m)}$ such that $\limsup_{m\to\infty} r_m < \infty$, form a natural subclass of finite staircase transformations.

5.2 Subsets of limit polynomials

Next we will use the compactness of the set $\{U \in \mathcal{L}(L_2) : \|U\| \leq 1\}$ with respect to the weak operator topology, where $\mathcal{L}(L_2)$ is the space of bounded operators on L_2.

Denote

$$P_k(z) = \frac{1}{k-1}\sum_{j=0}^{k-2} z^j \ (k > 1),$$

and

$$B_{p,\alpha,\beta}(z) = \frac{1}{1+\alpha+\beta}(P_p(z) + \alpha P_{p+1}(z) + \beta P_{p+2}(z)) \ (\alpha, \beta \in \mathbf{R}).$$

Proposition 2. *For any $p > 1$ there is a sequence of positive integers r_m such that $|r_m - p - 1| \leq 1$ and for any pair $(\alpha, \beta) \in I_1 \times I_2$*

$$\widehat{T}^{-h_{m_k}} \to B_{p,\alpha,\beta}(\widehat{T}) \text{ as } k \to +\infty, \tag{6}$$

where $T = T_{(r_m)}$, $m_k = m_k(\alpha, \beta) \to +\infty$, *and* I_i $(i = 1, 2)$ *is a countable subset of* $\mathbf{R}$.

Proof. It is enough to show (6) on pairs of functions running independently over some dense set in $L_2(X, \mu)$. Therefore we can assume that f and g are constant, say $f_m(j)$ and $g_m(j)$, on each j^{th} level of C_m for every sufficiently large m.

Obviously, if $\mu(X_i) \to 0$, then $\mathbf{1}_{X_i}\widehat{T}^{k_i} \to 0$ for any sequence k_i . Thus

$$\widehat{T}^{-h_m} - \sum_j \mathbf{1}_{O_m(j)}\widehat{T}^{-h_m} \to 0 \textit{ as } m \to +\infty.$$

It is easy to see that on every set $O_m(j) \cap C_{m+j}(i)$ the function $\widehat{T}^{-h_m}f$ is "almost" $\widehat{T}^{i-2}f$, and then $\langle \mathbf{1}_{O_m(j)}\widehat{T}^{-h_m}f, g\rangle$ is "almost" $\mu(O_m(j))\langle P_{r_{m+j}}(\widehat{T})f, g\rangle$. Moreover

$$\sum_j \langle \mathbf{1}_{O_m(j)}\widehat{T}^{-h_m}f, g\rangle - \sum_j \mu(O_m(j))\langle P_{r_{m+j}}(\widehat{T})f, g\rangle \to 0 \textit{ as } m \to +\infty.$$

Therefore,

$$\widehat{T}^{-h_m} - \sum_j \mu(O_m(j))P_{r_{m+j}}(\widehat{T}) \to 0 \textit{ as } m \to +\infty.$$

Note that for any j

$$\mu(O_m(j)) = \frac{r_{m+j} - 1}{r_m \cdots r_{m+j}}\mu(C_m),$$

and $\mu(C_m) \to 1$ as $m \to +\infty$.

Collecting all above remarks we conclude that a sequence r_m we need is, for example, any *weakly normal* sequence of symbols $\{p, p+1, p+2\}$, i.e. a sequence satisfying the following property: for any finite couple of elements of $\{p, p+1, p+2\}$, say $(\alpha_0, \dots, \alpha_k)$, there exists a countable set of positive integers i_j such that, for any j, $r_{i_j+q} = \alpha_q$, $q = 0, \dots, k$. Proposition 2 is proved. □

Every rank 1 transformation has a simple spectrum (see [5]). Therefore, under the notation described in Sect. 3, for any finite staircase transformation, Φ_n is an isomorphism between Hilbert spaces $L_2(X^n, \mu^n)$ and $L_2(\mathbf{T}^n, \sigma_1^n)$.

Proposition 3. *Let* $n \in \mathbf{N}$, $p > 1$, *and* $T = T_{(r_m)}$ *for any sequence* r_m *coming* *from Proposition 2; then*

$$\prod_{i=1}^{n} B_{p,\alpha,\beta}(\lambda_i) \in \overline{span}\{\tilde{T}^{(n)j}\mathbf{1} : j \in \mathbf{Z}\} \tag{7}$$

for all pairs (α, β) *of real numbers.*

Proof. We will follow the proof of Theorem 3. (6) then implies that (7) is valid for any pair $(\alpha, \beta) \in I_1 \times I_2$, and, consequently, $\prod_{i=1}^{n} B_{p,\alpha,\beta}(\lambda_i)$ is equal to the const ($\nu_n(\cdot|c)$ a.e.) on a "typical" fiber $F(c)$ for any pair $(\alpha, \beta) \in I_1 \times I_2$.

Given an $\alpha \in I_1$, and a "typical" fiber $F(c)$, consider $\prod_{i=1}^{n} B_{p,\alpha,\beta}(\lambda_i)$ as a polynomial of degree $\leq n$ in the variable β on the fiber $F(c)$. Since the number of various $\beta \in I_2$ is countable, this polynomial is uniquely defined, i.e. all its coefficients are equal to constants for a.e. $(\lambda_1, \ldots, \lambda_n)$ with respect to the measure $\nu_n(\cdot|c)$ on the fiber $F(c)$. Therefore, for any $\beta \in \mathbf{R}$, it is equal to the const for $\nu_n(\cdot|c)$ a.e. $(\lambda_1, \ldots, \lambda_n)$, where $\lambda_1 \cdots \lambda_n = c$. This implies that for any $\beta \in \mathbf{R}$ there exists an $\tilde{f} \in L_2(\mathbf{T}^n, \sigma_1^n)$ such that $\prod_{i=1}^{n} B_{p,\alpha,\beta}(\lambda_i) = \tilde{f}$ (σ_1^n a.e.) and, for any c, $\tilde{f}|_{F(c)} \equiv$ const. Therefore (7) is valid for any real β.

Given an arbitrary real β, by repeating the proof, we have that (7) is valid for any real α. Proposition 3 is proved. $\square$

Proposition 4. *For any sequence* r_m *coming from Proposition 2, for any* n, $T^{(n)}_{(r_m)}$ *has a simple spectrum on* $H_{sym}(n)$*. Moreover, one has*

$$\forall \varphi \in L_2 \ [\overline{span}\{\widehat{T}^{j}_{(r_m)}\varphi : j \in \mathbf{Z}\} = L_2(X, \mu) \Rightarrow$$

$$\forall n \ \overline{span}\{\widehat{T}^{(n)j}_{(r_m)}\varphi_n : j \in \mathbf{Z}\} = H_{sym}(n)],$$

where $\varphi_n(x_1, \ldots, x_n) = \varphi(x_1) \cdots \varphi(x_n)$*.*

Proof. Given a $\beta \in \mathbf{R}$ and $p > 1$, choose an α satisfying

$$\frac{\alpha}{p} + \frac{\beta}{p+1} = -\frac{1}{p-1}.$$

Then we have

$$B_{p,\alpha,\beta}(z) = (\frac{1}{1-p} + \frac{1}{1+p}\beta z) z^{p-1}.$$

Using Proposition 3, we then conclude that for any positive number n and

$\beta \in \mathbf{R}$,

$$\prod_{i=1}^{n} \lambda_i^{p-1}\left(\frac{1+p}{1-p} + \beta\lambda_i\right) \in \overline{\text{span}}\{\tilde{T}^{(n)j}\mathbf{1} : j \in \mathbf{Z}\},$$

or, equivalently,

$$\prod_{i=1}^{n}(\Theta + \lambda_i) \in \overline{\text{span}}\{\tilde{T}^{(n)j}\mathbf{1} : j \in \mathbf{Z}\} \ for\ any\ \Theta \in \mathbf{R}.$$

It is remain to repeat exactly all arguments in the proof of Theorem 3 starting with (5). Proposition 4 is proved. □

5.3 Approximations by finite staircase transformations

Note that staircase transformations introduced in Subsect 5.1 can be non-weakly mixing. However, using, for example, Remark 6 and Proposition 4, we have that staircase transformations constructed in Subsect. 5.2 are weakly mixing. It is easy to see that they are not mixing. Moreover, it is well known that if a rank 1 transformation is mixing, then a sequence of cuts $\{r_m\}$ in their definition (by cutting and stacking) tends to infinity.

By repeating with obvious changes of Adams' proof (done for standard generalizations of classical staircase transformations (see [1])) , it is not difficult to conclude that if r_m tends to $+\infty$ (r_m is "divergent" in the sense of [1]) and the growth is not to fast (i.e. $\lim_{n\to\infty} r_m^2/h_m = 0$)), then $T_{(r_m)}$ is also mixing.

Let $p_k \to +\infty$. For any $p = p_k$ choose a sequence, say $r_m(k)$, as in Proposition 2. We wish to consider limits of finite staircase $T_{(r_m(k))}$. It is convenient to construct every $T_{(r_m(k+1))}$ starting with an appropriate column of $T_{(r_m(k))}$, where without loss of generality we shall assume that the first column of $T_{(r_m(1))}$ has height 1 and weight 1. Therefore $h_1(k+1) = h_{m(k)}(k)$, where a sequence $m(k)$ will be chosen later. Let $C_{m,k}$ be the m^{th} column of $T_{(r_m(k))}$. We will assume next that $T_{(r_m(k+1))}$ is equal to $T_{(r_m(k))}$ on all levels of the column $C_{m(k),k}$, but the top one. It is convenient to consider a union of levels of the column $C_{m,k}$ as an interval, say $[0, a(m,k))$ of $\mathbf{R}$. Note that, in general, $T_{(r_m(k))}$ act on different spaces (X_k, μ), where $X_k = [0, a_k)$, $a_k = \lim_{m\to\infty} a(m,k)$. However if $m(k)$ tends to $+\infty$ sufficiently fast, then the difference is small, and $\mu(X_k) \to c$ for some $0 < c < \infty$.

Given a sequence m_k of positive integers, let us finally define a staircase transformation $T_{(r_m)}$ so that for any k $T_{(r_m)}$ is equal to $T_{(r_m(k))}$ on all but top levels of the column $C_{m(k),k}$. It is easy to see that $T_{(r_m)}$ is well defined, acts on $X=[0,a)$, $a=\lim_{k\to\infty} a(m(k),k)$, $h_1=h_1(1)$, and

$$(r_1,r_2,\dots,r_m,\dots)=(r_1(1),\dots,r_{m(1)-1}(1),r_1(2),\dots,r_{m(2)-1}(2),\dots). \quad (8)$$

Proposition 5. *If the sequence $m(k)$ tends to $+\infty$ sufficiently fast, then for any n the (mixing) transformation $T^{(n)}_{(r_m)}$ has a simple spectrum on $H_{sym}(n)$, where $T_{(r_m)}$ is the mixing staircase transformation constructed above.*

Proof. Obviously, $r_m\to+\infty$, and for a sequence $m(k)$ having the sufficiently fast growth with respect to p_k, $\mu(X)=a<\infty$, $r_m^2/h_m\to\infty$. Therefore $T_{(r_m)}$ is mixing, and $T^{(n)}_{(r_m)}$ is mixing on every invariant subspace of $L_2(X^n,\mu^n)$.

Given a positive integer n and $L_2([0,1)^n,\mu^n)$, fix a countable dense subset $\{f_{n,j}\}$ ($j\in\mathbf{N}$) of $H_{\mathrm{sym}}(n)\subseteq L_2([0,1)^n,\mu^n)$. Let $b>0$. The map $\psi_{1,b}:[0,1)\to[0,b)$, $\psi_{1,b}(x)=bx$ defines a certain (non-unitary) isomorphism, say $\psi_{n,b}$, between Hilbert spaces $L_2([0,1)^n,\mu^n)$ and $L_2([0,b)^n,\mu^n)$, and corresponding symmetric spaces. Let $g_{n,j,b}=\psi_{n,b}f_{n,j}$ ($n,j\in\mathbf{N}$). It is convenient to consider every $g_{n,j,b}$ as an element of $L_2(\mathbf{R}^n,\mu^n)$, where supp $g_{n,j,b}\subseteq[0,b)^n$.

Fix a sequence $0<\varepsilon_k\to0$ as $k\to+\infty$. For any $T_{(r_{m(k)})}$ as above ($k\in\mathbf{N}$), define $0<\delta_k<\delta_{k-1}/2$ ($\delta_0=1$) satisfying

$$|b-a_k|<\delta_k\Rightarrow\forall n,j\le k\ \| g_{n,j,b}-g_{n,j,a_k}\|_{L_2}<\varepsilon_k. \quad (9)$$

Let us use the following well-known technical proposition:

Proposition 6. *Let T be a weakly mixing rank 1 transformation of the Lebesgue space (X,μ), and ξ_m be the corresponding sequence of partitions (see the definition in Subsect. 5.1). If ξ_m is monotonic, then*

$$\overline{span}\{\widehat{T}^j\varphi: j\in\mathbf{Z}\}=L_2(X,\mu),$$

where φ is the indicator of every level of every column.

Suppose χ is the indicator of the first level of the column $C_{1,1}$, $\chi_n(x_1,\dots,x_n)=\chi(x_1)\cdots\chi(x_n)$. Using Proposition 4, for any $T_{(r_{m(k)})}$ we choose numbers $c_q(n,j,k)$ so that

$$\forall n,j\le k\ \| g_{n,j,a_k}-\sum_q c_q(n,j,k)\widehat{T}^{(n)q}_{(r_m(k))}\chi_n\|_{L_2}<\varepsilon_k. \quad (10)$$

Given k, $m(k-1)$, $T_{(r_{m(s)})}$, a_s, δ_s ($s\le k$), it is easy to see that we can choose

$m(k)$ so that

$$0 < a_k - a(m(k), k) < \frac{\delta_k}{6}, \tag{11}$$

$$|a_{k+1} - a_k| < \frac{\delta_k}{6}, \tag{12}$$

$$\forall n, j \le k \parallel \sum_q c_q(n, j, k)\widehat{T}^{(n)q}_{(r_m(k))}\chi_n - \sum_q c_q(n, j, k)\widehat{T}^{(n)q}_{(r_m)}\chi_n \parallel_{L_2} < \varepsilon_k \tag{13}$$

uniformly over all $m(s)$ $(s > k)$ (, and over transformations $T_{(r_m(s))}$ $(s > k)$, $T_{(r_m)}$ defined by these $m(s)$).

Let us consider a sequence $m(k)$ (and $T_{(r_m)}$) constructed above. Note that $|a - a_k| < \delta_k$. Indeed, by (11) and (12) we have

$$|a - a_k| = |a(m(k), k) - a_k + \sum_{i \ge k}(a(m(i+1), i+1) - a(m(i), i))| <$$

$$\frac{\delta_k}{6} + \sum_{i \ge k} |a_{i+1} - a_i| + \frac{\delta_i}{6} + \frac{\delta_{i+1}}{6} < \delta_k.$$

Therefore, using (9),(10), and (13), we conclude

$$\forall k \forall n, j \le k \parallel g_{n,j,a} - \sum_q c_q(n, j, k)\widehat{T}^{(n)q}_{(r_m)}\chi_n \parallel_{L_2} < 3\varepsilon_k.$$

This implies that

$$\forall n, j \ g_{n,j,a} \in \overline{\text{span}}\{\widehat{T}^{(n)q}_{(r_m)}\chi_n : q \in \mathbf{Z}\},$$

or, equivalently,

$$\forall n \quad H_{\text{sym}}(n) = \overline{\text{span}}\{\widehat{T}^{(n)q}_{(r_m)}\chi_n : q \in \mathbf{Z}\}.$$

Proposition 5 is proved. □

Remark 8. Since every staircase transformation is defined by the pair $(h_1, \{r_m\})$ uniquely up to a metrical isomorphism, Proposition 5 is valid for any staircase $T_{(r_m)}$ having the same sequence $\{r_m\}$ as above. Moreover, it is easy to check that Proposition 5 is also valid for slightly more large subclass of staircase $T_{(r_m)}$, where r_m is constructed in the same manner as in (8) but we admit to place between blocks $(r_1(k), \dots, r_{m(k)-1}(k))$ finite blocks B_k under certain conditions on B_k.

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