ON LINEAR FORMS WITH COEFFICIENTS IN $\mathbb{N}\,\zeta(1+\mathbb{N})$

The detailed version, **Part 6**

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In memory of Professor Alexander Buchstab

Content

§0. Foreword

As it have been already remarked in the foreword and in the section 3 of the paper [40], the situation in [22] is much simpler than in [34]-[37],[39],[43]. The point $z=-1$ is not a singular point of the differential equation, considered in [22], but the point $z=1$ is a singular point of the differential equation, considered in [34]-[37],[39],[43], and the matrices $B_i(z,\nu)$ in (2.3.5) of the paper [35] are not defined at the point $z=1$ Therefore we must use the passage to the limit $z\to 1$. Many unpleasant things may occure by this passage to the limit. As it have been proved in [37], the six polynomials (3.2.10) are solutions of the difference equation (3.1.106) in the section 1 of [37]. But one of these six polynomials vanishes at the point $z=1$ (precisely $\gamma^*(1;\nu)=0$) and therefore we have only five linear independent solutions of the equation (3.1.106), which has sixth order. Where is one more solution hidden ? The answer was given in [43]. In [43] we proved that coefficient $q_0(z,\nu^{-1})$ vanishes at the point $z=1$ and therefore the specified hidden solution is a trivial solution

$$x(\nu_0)=1,\, x(\nu)=0,\, \nu\in\nu_0+\mathbb{N},$$

where $\nu_0=[1/\varepsilon_2(F)]+1$, and we can not take it in the account, if we pass to the equation (29) in [43], which has the fifth order.

Let us consider the following simple difference equation

$$y(\nu+1)=\frac{\nu}{(z-1)\nu+1}y(\nu).$$

For each $z\neq 1$ it is a linear difference equation of Poincaré type. But for $z=1$ this eqution is not a linear difference equation of Poincaré type .

In the section 1 I prove a some general Theorem (below Lemma 6.1.1), which give sufficient conditions for lineare difference equation

$$\sum_{k=0}^{n} a_k(\nu^{-1})y(\nu+k)=0, \tag{0.1}$$

where $a_k(\nu^{-1})\in\mathbb{C}(\nu)$, $k=0,\ldots,n-1$, $a_n(\nu^{-1})=1$, $a_0(\nu^{-1})\neq 0$, would be a difference equation of a Poincaré type. These conditions may be formulatad as follows:

the space of solutions of equation (0.1) must have a basis

$$\beta_j(\nu),$$

where $j=1,\ldots,n$, for which there exist $C>0$ and $M>0$ such that

$$|\beta_j(\nu)|\leq CM^{\nu}$$

for all the sufficiently big ν. The necessity of these conditions was formulated by A.Poincaré for the general (i.e. not necessary $a_k(\nu^{-1})\in\mathbb{C}(\nu)$) lineare difference equation of Poincaré type and and proved by E.B.Van Vleck [2]. As the followig simple example shows, the reqirement

$$a_k(\nu^{-1})\in\mathbb{C}(\nu)$$

for $k=0,\ldots,n-1$ is essential for the discussed Theorem:

let $a_0(\nu)=4+(-1)^{\nu}2, a_1(\nu)=4+(-1)^{\nu}$; then the equation

$$y(\nu+2)+a_1(\nu)+a_0(\nu)y(\nu)=0$$

is not difference equation of Poincaré type, but for any its solution $y(\nu)$ there exists a constante $C>0$ such that $|y(\nu)|\leq C3^{\nu}$.

In the end of the section 1 I apply the above general Theorem to the considered in these publications case and show that equation (29) in [43] is a linear difference equation of Poincar'e type.

Let us consider the following linear difference equation

$$y(\nu+1) = \frac{(z-1)\nu+2}{(z-1)\nu+1} y(\nu)$$

For each $z \neq 1$ its characteristical polynomial is equal to $\lambda - 1$, but for $z = 1$ its characteristical polynomial is equal to $\lambda - 2$.

Where is an exit from this devil's morass? Suddenly I remarked a small footpath there and I go on this footpath in the next sections.

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§1. Pizzicato

Lemma 6.1.1. *Let $\varepsilon_0 > 0$ and*

$$a_k(\nu^{-1}) \in \mathbb{C}(\nu),\ k = 0, \dots, n-1 \tag{1.1}$$

from (0.1) are well defined for $\nu \in \mathbb{N} + [1/\varepsilon_0]$, and

$$a_0(\nu^{-1}) \in \mathbb{C}(\nu) \neq 0 \tag{1.2}$$

for $\nu \in \mathbb{N} + [1/\varepsilon_0]$, Let

$$\beta_j(\nu), \tag{1.3}$$

where $j = 1, \dots, n$, is solution of the equation (0.1), and for $\nu \in \mathbb{N} + [1/\varepsilon_0]$ solutions (4) compose a basis of the linear space over $\mathbb{C}$ of all the solutions of the equation (0.1). Let further there exists $C > 0$ and $M > 0$ such that

$$|\beta_j(\nu)| \leq CM^\nu \tag{1.4}$$

for $\nu \in \mathbb{N} + [1/\varepsilon_0]$.

Then the rational functions (1.1) are well defined at the point $\nu^{-1} = 0$ and therefore the equatin (0.1) is a difference equation of a Poincaré type.

Proof. Let the assertion of the Lemma is not true. Then

$$0 \in \mathfrak{J} = \{j \in \{0, \dots, n-1\}\colon a_j(\nu^{-1}) \in \mathbb{C}(\nu^{-1}) \neq 0_{\mathbb{C}(\nu)}\} \neq \emptyset.$$

Let

$$a_j(\nu^{-1}) = b_j(\nu^{-1})\nu^{\lambda_j}$$

where $j \in \mathfrak{J}$, $b_j(\nu^{-1}) \in \mathbb{C}(\nu^{-1})$, these $b_j(\nu^{-1})$ are well defined at the point $\nu^{-1} = 0$ and $b_j(0) \neq 0$. Then the equation (1) may be rewritten in the form

$$y(\nu + n) + \sum_{j\in\mathfrak{J}} b_j(\nu^{-1})\nu^{\lambda_j} y(\nu + j) = 0, \tag{1.5}$$

The equation (6) was studied by N.E.Nörlund [19]. Following to N.E.Nörlund we let $y(\nu) = (\nu!)^\alpha w(\nu)$ with $\nu \in \mathbb{N} + [1/\varepsilon_0]$ and

$$\alpha = \sup\{\lambda_j/(n - j) \colon j \in \mathfrak{J}\}. \tag{1.6}$$

If $\alpha \leq 0$ then assertion of the Lemma, of course, is true. Let

$$\alpha > 0.$$

Then equation (1.5) turns into equation

$$w(\nu + n) + \sum_{j\in\mathfrak{J}} c_j(\nu) w(\nu + j) = 0, \tag{1.7}$$

where $c_j(\nu^{-1}) = b_j(\nu^{-1})\nu^{\lambda_j-(n-j)\alpha}$ for $j \in \mathfrak{J}$ and $\nu \in \mathbb{N} + [1/\varepsilon_0]$. Let

$$\mathfrak{J}_1 = \{j \in \mathfrak{J} \colon \lambda_j/(n - j) = \alpha\}.$$

Then

$$\lim_{\nu\to\infty} (c_j(\nu)) = b_j(0) \neq 0 \tag{1.8}$$

for $j \in \mathfrak{J}_1$ and

$$\lim_{\nu\to\infty} (c_j(\nu)) = 0 \tag{1.9}$$

for $j \in \mathfrak{J}\diagdown\mathfrak{J}_1$. In view of (1.8) and (1.9) the equation (1.7) is an equation of the Poincaré type with characteristic polynomial

$$\eta^n + \sum_{j\in\mathfrak{J}_1} b_j(0)\eta^j. \tag{1.10}$$

According to (1.8), the polynomial (1.10) has at least one different from zero root, let η_0 be one of the different from zero roots of the polynomial (1.10). According to the Perron's theorem, the equation (1.7) has a solution $w_0(\nu)$ with the following property:

for any $\varepsilon \in (0,1)$ (for example for $\varepsilon = 1/2$) the inequality

$$|w_0(\nu)| \geq |\eta_0(1-\varepsilon)|^\nu,$$

holds infinitely many times when $\nu \to \infty$. For the corresponding to $w_0(\nu)$ solution

$$y_0(\nu) = w_0(\nu)(\nu!)^\alpha$$

of the equation (6) the inequality

$$|y_0(\nu)| \geq (\nu!)^\alpha |\eta_0(1-\varepsilon)|^\nu,$$

with $\alpha > 0$ holds infinitely many times when $\nu \to \infty$. On the other hand the solution $y_0(\nu)$ is a $\mathbb{C}$-linear combinaion of the solutions (1.3); therefore there exist a constants $C_0 > 0$ and $M_0 > 0$ such that

$$|y_0(\nu)| \leq C_0(M_0)^\nu$$

for $\nu \in \mathbb{N} + [1/\varepsilon_0]$.■

Let us consider the difference equation (29) in [43] now, i.e. the following difference equation:

$$\sum_{j=0}^{4} q_j^{\wedge *}(\nu^{-1}) y(\nu+j) + y(\nu+5) = 0, \tag{2.1}$$

where $q_j^{\wedge *}(\nu^{-1}) \in \mathbb{Q}(\nu^{-1})$, if $j = 0, \ldots, 4$, are well defined for $\nu \in [1/\varepsilon_5] + \mathbb{N}$ with some $\varepsilon_5 > 0$ and $q_0^{\wedge *}(\nu^{-1}) \neq 0$, when $\nu \in [1/\varepsilon_5] + \mathbb{N}$ As it is shown in [43],the equation (2.1) has a solutions

$$y(\nu) = \alpha_k^*(1;\nu), \tag{2.2}$$

where $k = 0, 1, 3, 4, 5$ and $\nu \in \mathbb{N} + [1/\varepsilon_5]$, and the polynomials $\alpha_k^*(z;\nu)$ are defined in (2.21), (2.22), (2.24), (2.26) and (2.28) of the paper [35]; the Lemma 4.1.3 in the paper [39] shows that solutions (2.2) compose a basis of the space of the solutions of the equation (2.1).

Making use of (2.16) - (2.18), (2.21), (2.22), (2.24), (2.26) and (2.28) of the paper [35] it is easy to check the following inequalities:

$$|\alpha_0^*(1;\nu)| \leq 2^{11\Delta\nu} 2\Delta\nu,$$

$$|\alpha_1^*(1;\nu)| \le 2^{11\Delta\nu}12(\Delta\nu)^2,$$
$$|\alpha_3^*(1;\nu)| \le 2^{11\Delta\nu}100(\Delta\nu)^4,$$
$$|\alpha_4^*(1;\nu)| \le 2^{11\Delta\nu}132(\Delta\nu)^4,$$
$$|\alpha_5^*(1;\nu)| \le 2^{11\Delta\nu}168(\Delta\nu)^4.$$

Therefore according to the Lemma 6.1 equation (2.1) is an equation of Poincaré type. So, the equation (2.1) (the equation (29) in [43]) is an equation of Poincaré type. Our nearest aim is to study the characteristic polynomial of this equation.

§2. Metamorphosis

Let $\mathfrak{k}$ be a field of characteristic zero contained in an integral domain R with quotient field K. Let us consider a non-trivial differentiation u on K. Suppose that u is trivial on a subfield K_0, containing $\mathfrak{k}$, and that the ring R contains an eigenvector z of u, whose eigenvalue λ is not equal to 0 and lies K_0. Clearly, the element z is transcendental over $\mathfrak{k}$.

For $f \in R$, let $\mathfrak{m}_f$ denote the operator of multiplication by f in R, so that

$$\mathfrak{m}_f\colon a \mapsto fa$$

for $a \in R$, and let

$$\mathfrak{M}_R = \{\mathfrak{m}_f\colon f \in R\}.$$

Since R is an integral domain, the ring $\mathfrak{M}_R$ is isomorphic to R. Let $\mathfrak{M}^\vee$ denotes the ring of all the $\mathfrak{k}-$endomorphisms of the additive group of the $\mathfrak{k}-$algebra R. For any $\phi \in \mathfrak{M}^\vee$ and $\psi \in \mathfrak{M}^\vee$, let us denote by $\phi\circ\psi$ the composition of the operators ϕ and ψ, so that $(\phi\circ\psi)f = \phi(\psi f)$ for $f \in R$. The above differentiation u is a $\mathfrak{k}-$linear operator on R. Let $R < u >$ stand for the subring of the ring $\mathfrak{M}^\vee$ generated by u and $\mathfrak{M}_R$; an element of the ring $R < u >$ will be called a differential polynomial. Since

$$u \circ \mathfrak{m}_f = \mathfrak{m}_{uf} + \mathfrak{m}_f \circ u, \tag{2.1}$$

it follows that every element D of $R < u >$ may be represented in the form

$$D = \sum_{k=0}^{n(D)} \mathfrak{m}_{\mathfrak{c}_k(D)} \circ u^k \tag{2.2}$$

with $n(D) \in \mathbb{N}-1, u^0 = \mathfrak{m}_{1_R}$ and $\mathfrak{c}_k(D) \in R$ for $k = 0, \ldots, n(D)$. We may rewrite the equality (2.2) in the form

$$D = \sum_{k=0}^{\infty} \mathfrak{m}_{\mathfrak{c}_k(D)} \circ u^k, \tag{2.3}$$

where $\mathfrak{c}_k(D)=0$ for $k\in n(D)+\mathbb{N}$. For $n\in\mathbb{N}-1$, let $R|_n<u>$ be the set of those elements D in $R<u>$ with $\mathfrak{c}_k(D)=0$ for $k>n$ in (2.3). Clearly,

$$R<u>=\bigcup_{n=0}^{\infty}R|_n<u>\,.$$

Applying an operator D in $R|_n<u>$ to $1,z\,,\dots,z^n$, one can see that, for $k\in\mathbb{N}-1$, the element $\mathfrak{c}_k(D)$ in (2.3) is uniquely defined and $\mathfrak{c}_k$ is a $\mathfrak{k}-$linear map from the ring $R<u>$ to R. For $D\neq 0$, let

$$\deg(D):=\sup\{k\in\mathbb{N}-1\colon \mathfrak{c}_k(D)\neq 0\}$$

denote the degree of the differential polynomial D. Let

$$uD=\sum_{k=0}^{\infty}\mathfrak{m}_{u\mathfrak{c}_k(D)}\circ v^k, \tag{2.4}$$

then

$$u\circ D=uD+D\circ u=uD+\sum_{k=0}^{\infty}\mathfrak{m}_{\mathfrak{c}_k(D)}\circ u^{k+1} \tag{2.5}$$

for $D\in R<u>$.

For $\mu\in\mathbb{Q}$ let $\mathfrak{O}^{(\mu)}$ denote the ring those elements in $\mathbb{Q}(z,\nu^{-1})$, which are well-defined in the two points $(1,\mu)$ and $(1,0)$; the ring $\mathfrak{O}^{(0)}$ will be simply denoted by $\mathfrak{O}$. Let $\mathfrak{B}$ be the ring of those elements in $\mathbb{Q}(z)$, which are well-defined at the point $z=1$. The field $\mathbb{Q}(z)$ can be regarded as a subfield of the field $\mathbb{Q}(z,\nu^{-1})$, so that $\mathfrak{B}\subset\mathfrak{O}^{(\mu)}$ for $\mu\in\mathbb{Q}$. Clearly, the ring $\mathfrak{O}^{(\mu)}$ is an integral domain. First let $R=\mathfrak{O}^{(\mu)}$, then $K=\mathbb{Q}(z;\nu^{-1})$; let $K_0=\mathbb{Q}(\nu^{-1})$. For $\delta=z\frac{\partial}{\partial z}$, let $v=\nu^{-1}\delta$, and we take $u=v$. Then the independent variable z will play the role of the eigenvector z, the corresponding eigenvalue λ being equal to ν^{-1}, and $\mathfrak{k}=\mathbb{Q}$; by definition, $\nu^{-1}\in K_0$. Thereby we have constructed the rings $\mathfrak{O}^{(\mu)}<v>$ and, in particular, the ring $\mathfrak{O}<v>$. We shall consider also the ring $\mathfrak{B}<\mu\delta>$, which we obtain, on letting $R=\mathfrak{B},\ K=\mathbb{Q}(z),\ K_0=\mathbb{Q}$ and taking $u=\mu\delta$. In this case, the independent variable z will play the role of the eigenvector z, the corresponding eigenvalue λ being equal to μ, and $\mathfrak{k}=\mathbb{Q}$. Thus $K_0=\mathfrak{k}=\mathbb{Q}$ in this case.

Let $\mathfrak{O}^*<v>$ stand for the set of those differential polynomials D in $\mathfrak{O}<v>$, for which $\mathfrak{c}_0(D)(1,0)\neq 0$, and let, for $n\in\mathbb{N}-1$,

$$\mathfrak{O}|_n^*<v>=\mathfrak{O}^*<v>\ \cap\mathfrak{O}|_n<v>\,.$$

For $k \in \mathbb{N} - 1$ and $D \in \mathfrak{O}^* < v >$, let

$$\mathfrak{b}_k(D) = \mathfrak{c}_k(D) + v\mathfrak{c}_{k+1}(D) - (v\mathfrak{c}_0(D))\mathfrak{c}_{k+1}(D)/\mathfrak{c}_0(D).$$

Then $\mathfrak{b}_k(D) \in \mathfrak{O}$ for $k \in \mathbb{N} - 1$ and $D \in \mathfrak{O}^* < v >$,

$$\mathfrak{b}_k(D) = \mathfrak{c}_k(D) \tag{2.6}$$

for $k \in \mathbb{N} - 1 + n$ and $D \in \mathfrak{O}|_n^* < v >$,

$$\mathfrak{b}_k(D)(z, 0) = \mathfrak{c}_k(D)(z, 0) \tag{2.7}$$

for $k \in \mathbb{N} - 1$ and $D \in \mathfrak{O}^* < v >$.

Let

$$\mathfrak{d} \colon \mathfrak{O}^* < v > \to \mathfrak{O}^* < v >$$

denote the map defined by the following equality

$$\mathfrak{c}_k(\mathfrak{d}D) = \mathfrak{b}_k(D)$$

for all $D \in \mathfrak{O}^* < v >$ and $k \in \mathbb{N} - 1$. It follows from (2.6) and (2.7) that, for $n \in \mathbb{N} - 1$,

$$\mathfrak{d}(\mathfrak{O}|_n^* < v >) \subset \mathfrak{O}|_n^* < v > . \tag{2.8}$$

Lemma 6.2.1. *If $D \in \mathfrak{O}^* < v >$, then*

$$(v - \mathfrak{m}_{v\mathfrak{c}_0(D)/\mathfrak{c}_0(D)}) \circ D = (\mathfrak{d}D) \circ v. \tag{2.9}$$

Proof. It follows from (2.5) that

$$(v - \mathfrak{m}_{v\mathfrak{c}_0(D)/\mathfrak{c}_0(D)}) \circ D = v \circ D = D \circ v + vD - \mathfrak{m}_{v\mathfrak{c}_0(D)/\mathfrak{c}_0(D)}) \circ D =$$

$$\sum_{k=0}^{\infty} \mathfrak{m}_{\mathfrak{c}_k(D)} \circ v^{k+1}) + \mathfrak{m}_{v\mathfrak{c}_0(D)} + \sum_{k=0}^{\infty} \mathfrak{m}_{v\mathfrak{c}_{k+1}(D)} \circ v^{k+1} - \sum_{k=0}^{\infty} \mathfrak{m}_{v\mathfrak{c}_0(D)\mathfrak{c}_k(D)/\mathfrak{c}_0(D))} \circ v^k =$$

$$\sum_{k=0}^{\infty} (\mathfrak{m}_{\mathfrak{c}_k(D)} + \mathfrak{m}_{v\mathfrak{c}_{k+1}(D)} - \mathfrak{m}_{v\mathfrak{c}_0(D)\mathfrak{c}_{k+1}(D)/\mathfrak{c}_0(D))}) \circ v^{k+1}.$$

■

Since $v\mathfrak{O} \subset \nu^{-1}\mathfrak{O}$, it follows from (2.5) that

$$D \circ v^n - v^n \circ D \in \nu^{-1}\mathfrak{O} < v >$$

for $D \in \mathfrak{O} < v >, n \in \mathbb{N}-1$ and that

$$D_1 \circ D_2 - D_2 \circ D_1 \in \nu^{-1}\mathfrak{O} < v >$$

for $D_1 \in \mathfrak{O} < v >$, $D_2 \in \mathfrak{O} < v >$. Therefore, for each $D \in \mathfrak{O} < v >$, the right ideal $D \circ \mathfrak{O} < v > +\nu^{-1}\mathfrak{O} < v >$ is a two-sided ideal; here

$$D \circ \mathfrak{O} < v > := \{D \circ D_1 \colon D_1 \in \mathfrak{O} < v >\}.$$

In particular, it follows that the right ideal

$$\mathfrak{m}_{(z-1)}\mathfrak{O} < v > +\nu^{-1}\mathfrak{O} < v >$$

is a two-sided ideal.

Lemma 6.2.2. *If* $D \in \mathfrak{O}^* < v >$, $k \in \mathbb{N}-1$, *then*

$$v^k Ker(D) \subset \mathrm{Ker}\,(\mathfrak{d}^k D),\ \mathfrak{d}^k D - D \in \nu^{-1}\mathfrak{O} < v > \tag{2.10}$$

Proof. We apply induction on $k+1$. For $k=0$ relations (2.10) are obvious. Let relations (2.10) are true for some $k \in \mathbb{N}-1$ and let $D_1 = \mathfrak{d}^k D$. In view of (2.8) and (2.9),

$$(v - \mathfrak{m}_{v\mathfrak{c}_0(D_1)/\mathfrak{c}_0(D_1)}) \circ D_1 = (\mathfrak{d}D_1) \circ v$$

and therefore

$$vKer(D_1) \subset \mathrm{Ker}\,(\mathfrak{d}D_1) = \mathrm{Ker}\,(\mathfrak{d}^{k+1}D).$$

Accoding to assumpion of induction, $v^k\,\mathrm{Ker}\,(D) \subset \mathrm{Ker}\,(D_1)$. Therefore

$$v^{k+1}\,\mathrm{Ker}\,(D) \subset v\,\mathrm{Ker}\,(D_1) \subset \mathrm{Ker}\,(\mathfrak{d}^{k+1}D).$$

In wiew of (2.7), $\mathfrak{d}D_1 - D_1 \in \nu^{-1}\mathfrak{O} < v >$. Accoding to assumpion of induction,

$$D_1 - D \in \nu^{-1}\mathfrak{O} < v > .$$

Therefore $\mathfrak{d}^{k+1}D - D = \mathfrak{d}D_1 - D \in \nu^{-1}\mathfrak{O} < v >$. So,the relaitions (2.10) are true also for $k+1$. ■

Let us consider the projection $\mathfrak{r}_n$ defined, for every $n \in \mathbb{N}$ and $D \in \mathfrak{O} < v >$, by the equality

$$\mathfrak{r}_n(D) = D - \mathfrak{m}_{\mathfrak{c}_n}(D) \circ v^n.$$

It follows that

$$\mathfrak{r}_n\mathfrak{O}^* < v > \subset \mathfrak{O}^* < v > .$$

Let $\mathfrak{c}_n(D) \in (z-1)\mathfrak{O}$ for some n in $\mathbb{N}$ and some D in $\mathfrak{O}|_n^* < v >$, and let $k \in \mathbb{N}-1$. Then it follows from (2.6) that

$$\mathfrak{d}^k D - \mathfrak{r}_n(\mathfrak{d}^k D) = \mathfrak{m}_{c_n(\mathfrak{d}^k D)} \circ v^n = \mathfrak{m}\mathfrak{c}_n(D) \circ v^n \in (z-1)\mathfrak{O} < v >, \tag{2.11}$$

$$\mathfrak{r}_n(\mathfrak{d}^k D) Ker(\mathfrak{d}^k D) \subset (z-1)\mathfrak{O}, \tag{2.12}$$

and it follows from (2.10) that

$$(\mathfrak{r}_n(\mathfrak{d}^k D) \circ v^k) Ker(D) \subset \mathfrak{r}_n(\mathfrak{d}^k D) Ker(\mathfrak{d}^k D) \subset (z-1)\mathfrak{O}. \tag{2.13}$$

If, besides, $(\mathfrak{c}_{n-1}(D))(1;0) \neq 0$, then it follows from (2.7) that

$$\mathfrak{c}_{n-1}(\mathfrak{d}^k D)(1;0) \neq 0 \tag{2.14}$$

for $k \in \mathbb{N}-1$. Moreover, for each $s \in \mathbb{N}-1$ there exists $\varepsilon(D,s) > 0$ such that the coefficients

$$\mathfrak{c}_i(\mathfrak{d}^k D)(z;\nu^{-1}),\ 0 \le k \le s,\ 0 \le i \le n,$$

the coefficient

$$(\mathfrak{c}_0(\mathfrak{d}^k D)(z;\nu^{-1}))^{-1},$$

and the coefficients

$$(\mathfrak{c}_{n-1}(\mathfrak{d}^k D)(z;\nu^{-1}))^{-1},\ 0 \le k \le s,$$

are well-defined on the set

$$\{(z;\nu^{-1}): |z-1| < \varepsilon(D,s),\ |\nu^{-1}| < \varepsilon(D,s)\}.$$

Thus

$$\mathfrak{c}_i(\mathfrak{d}^k D) \in \mathfrak{O}^{(\mu)},\ (\mathfrak{c}_0(\mathfrak{d}^k D))^{-1} \in \mathfrak{O}^{(\mu)},\ (\mathfrak{c}_{n-1}(\mathfrak{d}^k D))^{-1} \in \mathfrak{O}^{(\mu)}$$

for $0 \le k \le s,\ 0 \le i \le n$ and $\mu \in [0, \varepsilon(D,s))$.

Let $\theta_\mu: \mathfrak{O}^{(\mu)} \to \mathfrak{B}$ be the homomorphism defined by the equality

$$(\theta_\mu f)(z) = f(z;\mu)$$

for $f \in \mathfrak{O}^{(\mu)}$. By a slight abuse of notation, we denote by θ_μ both the natural extension of this mapping to the set of the matrices with entries in $\mathfrak{O}^{(\mu)}$ and the homomorphism

$$\theta_\mu: \mathfrak{O}^{(\mu)} < v > \to \mathfrak{B} < \mu\delta >$$

defined by the equality

$$\theta_\mu(D) = \sum_{i=0}^{\infty} \mathfrak{m}_{\theta_\mu(\mathfrak{c}_i(D))}(\mu\delta)^i$$

for $D \in \mathfrak{O}^{(\mu)} < v >$ (it is clear that $\theta_\mu(D) \in \mathfrak{B} < \mu\delta >$ for $D \in \mathfrak{O}^{(\mu)} < v >$). The homomorphism θ_μ defines a structure of an unitary module over $\mathfrak{O}^{(\mu)} < v >$ on the ring $\mathfrak{B}$.

Let $\mathfrak{r}_{\mu,n}$ be the projection defined, for every $\mu \in \mathbb{Q}$, $n \in \mathbb{N}$ and $D \in \mathfrak{B} < \mu\delta >$, by the equality $\mathfrak{r}_{\mu,n}(D) = D - \mathfrak{m}_{\mathfrak{c}_n}(D) \circ (\mu\delta)^n$. Then $\mathfrak{r}_n(\mathfrak{B} < \mu\delta >) \subset \mathfrak{B} < \mu\delta >$ for $\mu \in \mathbb{Q}$, $n \in \mathbb{N}$, and the diagram

$$\begin{array}{ccc} \mathfrak{O}^{(\mu)} < v > & \xrightarrow{r_n} & \mathfrak{O}^{(\mu)} < v > \\ {\scriptstyle \theta^{(\mu)}}\Big\downarrow & & \Big\downarrow{\scriptstyle \theta^{(\mu)}} \\ \mathfrak{B} < \mu\delta > & \xrightarrow[r_{\mu,n}]{} & \mathfrak{B} < \mu\delta > \end{array} \tag{2.15}$$

is commutative.

Let $\mathfrak{B}^* < \mu\delta >$ denote the set of those differential polynomials D in $\mathfrak{B} < \mu\delta >$ for which $\mathfrak{c}_0(D)(1) \neq 0$, and let

$$\mathfrak{B}^*|_n < \mu\delta > := \mathfrak{B}^* < \mu\delta > \cap \mathfrak{B}|_n < \mu\delta >$$

for $n \in \mathbb{N} - 1$. Let $D \in \mathfrak{B}^* < \mu\delta >$; since $c_k(D) \in \mathfrak{B}$, $\mathfrak{B} \subset \mathfrak{O}$, and $c_0(D)(1,0) = c_0(D)(1) \neq 0$, for each $k \in \mathbb{N} - 1$ there is an element

$$\mathfrak{b}_k(D) = \mathfrak{c}_k(D) + v\mathfrak{c}_{k+1}(D) - (v\mathfrak{c}_0(D))\mathfrak{c}_{k+1}(D)/\mathfrak{c}_0(D)$$

in $\mathfrak{O}^\mu$ as above. Let

$$\mathfrak{b}_{\mu,k}(D) = \theta_\mu(\mathfrak{b}_k(D)),$$

then

$$b_{\mu,k}(D) = \mathfrak{c}_k(D) + \mu\delta\mathfrak{c}_{k+1}(D) - (\mu\delta\mathfrak{c}_0(D))\mathfrak{c}_{k+1}(D)/\mathfrak{c}_0(D) \in \mathfrak{B}$$

for $k \in \mathbb{N} - 1$. Let, further,

$$\mathfrak{d}_\mu \colon \mathfrak{B}^* < \mu\delta > \longrightarrow \mathfrak{B} < \mu\delta >$$

be the mapping defined by the equality

$$\mathfrak{c}_k(\mathfrak{d}_\mu D) = \mathfrak{b}_{\mu,k}(D) = \theta_\mu(\mathfrak{b}_k(D)) \tag{2.16}$$

for $D \in \mathfrak{B}^* < \mu\delta >$ and $k \in \mathbb{N} - 1$. Since

$$\mathfrak{B} \subset \bigcap_{\mu \in \mathbb{Q}} \mathfrak{O}^{(\mu)}$$

and θ_μ is identical on $\mathfrak{B}$, it follows that homomorphism θ_μ is surjective. Let

$$\mathfrak{O}^{(\mu*)} < v >= \{D \in \mathfrak{O} < v >: \mathfrak{c}_0(D)|_{z=1,(\nu)^{-1}=mu} \neq 0\}.$$

Clearly

$$\theta_\mu(O^{(\mu*)} < v >) \subset \mathfrak{B}^* < \mu\delta > . \tag{2.17}$$

Lemma 6.2.3 *The diagram*

$$\begin{tikzcd} \mathfrak{O}^{(\mu*)} < v > \arrow[r, "\mathfrak{d}"] \arrow[d, "\theta^{(\mu)}"'] & \mathfrak{O}^{(\mu)} < v > \arrow[d, "\theta^{(\mu)}"] \\ \mathfrak{B} < \mu\delta > \arrow[r, "\mathfrak{d}_\mu"'] & \mathfrak{B} < \mu\delta > \end{tikzcd} \tag{2.18.}$$

is commutative.

Proof. Let

$$D \in \mathfrak{O}^{(\mu*)} < v > .$$

Then $\mathfrak{d}(D) \in \mathfrak{O}^{(\mu)} < v >$ and

$$\theta_\mu(\mathfrak{d}(D)) \in \mathfrak{B} < \mu\delta >$$

It follows from (2.9) that $\mathfrak{d}_\mu(\theta_\mu(D))$ is defined, belongs to $B^* < \mu\delta >$ and

$$\mathfrak{d}_\mu(\theta_\mu(D)) = \theta_\mu(\mathfrak{d}(D)). \tag{2.19}$$

■

Lemma 6.2.4. *Suppose that*

$$D \in \mathfrak{O}^* < v >,\ s \in \mathbb{N} - 1,\ 0 \leq k \leq s,$$

$\mu \in [0, \varepsilon(D, s))$. *Then* $(\mathfrak{d}_\mu)^k(\theta_\mu(D))$ *is defined, belongs to* $B^* < \mu\delta >$ *and*

$$(\mathfrak{d}_\mu)^k(\theta_\mu(D)) = \theta_\mu(\mathfrak{d}^k(D)). \tag{2.20}$$

Proof. We apply induction on $k + 1$. For $k + 1 = 1$ the assertion of the Lemma is obvious. Let $s \in \mathbb{N},\ 0 \leq k < s$ and assertion of the Lemma is true for k Accoding to above definition of $\varepsilon(D, s)$,

$$D_1 = \mathfrak{d}^k(D) \in \mathfrak{O}^{(\mu*)} < v > . \tag{2.21}$$

According to the Lemma 6.2.3, $\mathfrak{d}_\mu(\theta_\mu(D_1))$ is defined, belongs to $B^* < \mu\delta >$ and

$$\mathfrak{d}_\mu(\theta_\mu(D_1)) = \theta_\mu(\mathfrak{d}(D_1)) = \theta_\mu(\mathfrak{d}^{k+1}(D)). \tag{2.22}$$

According to asumption of induction $(\mathfrak{d}_\mu)^k(\theta_\mu(D))$ is defined, belongs to $\mathfrak{B}^* < \mu\delta >$ and

$$(\mathfrak{d}_\mu)^k(\theta_\mu(D)) = \theta_\mu(\mathfrak{d}^k(D)).$$

Therefore $\mathfrak{d}_\mu((\mathfrak{d}_\mu)^k(\theta_\mu(D)))$ is defined, belongs to $\mathfrak{B} < \mu\delta >$ and

$$\mathfrak{d}_\mu((\mathfrak{d}_\mu)^k(\theta_\mu(D))) = \mathfrak{d}_\mu(\theta_\mu(\mathfrak{d}^k(D)) = \mathfrak{d}_\mu(\theta_\mu(D_1)) \tag{2.23}$$

So, $(\mathfrak{d}_\mu)^{k+1}(\theta_\mu(D))$ is defined, belongs to $\mathfrak{B} < \mu\delta >$ and, according (2.22) and (2.23)

$$(\mathfrak{d}_\mu)^{k+1}(\theta_\mu(D)) = \theta_\mu(\mathfrak{d}^{k+1}(D)). \tag{2.24}$$

Therefore

$$\mathfrak{c}_0((\mathfrak{d}_\mu)^{k+1}(\theta_\mu(D))) = \mathfrak{c}_0(\theta_\mu(\mathfrak{d}^{k+1}(D))). \tag{2.25}$$

According to above definition of $\varepsilon(D,s)$, and to inequality $k+1 \le s$, it follows from (2.25) that $\mathfrak{c}_0((\mathfrak{d}_\mu)^{k+1}(\theta_\mu(D)))|_{z=1} \neq 0$ and therefore $(\mathfrak{d}_\mu)^{k+1}(\theta_\mu(D))$ is defined, belongs not only to $\mathfrak{B} < \mu\delta >$ but also to $\mathfrak{B}^* < \mu\delta >$ ■

Lemma 6.2.5. *Let are fulfilled the conditions of the Lemma 6.2.4. Let further*

$$D \in \mathfrak{O}_n^* < v >,\ \mathfrak{c}_{n-1}(D))(1;0) \neq 0,\ \mathfrak{c}_n(D) \in (z-1)\mathfrak{O}.$$

Let further $s \in \mathbb{N}-1,\ 0 \le k \le s\ \mu \in [0, \varepsilon(D,s)),$ *and* $0 \le i \le n.$ *Then*

$$\mathfrak{c}_i(\mathfrak{d}^k D) \in \mathfrak{O}^{(\mu)}, (\mathfrak{c}_0(\mathfrak{d}^k D))^{-1} \in \mathfrak{O}^{(\mu)} \tag{2.26}$$

$$(\mathfrak{c}_{n-1}(\mathfrak{d}^k D))^{-1} \in \mathfrak{O}^{(\mu)}, \tag{2.27}$$

$$(\theta_\mu(v) - \mathfrak{m}_{\theta_\mu(v\mathfrak{c}_0(D)/\mathfrak{c}_0(D))}) \circ \theta_\mu(D) = (\mathfrak{d}_\mu\theta_\mu(D)) \circ \theta_\mu(v), \tag{2.28}$$

$$(\theta_\mu(v))^k Ker(\theta_\mu(D)) \subset \mathrm{Ker}\,((\mathfrak{d}_\mu)^k\theta_\mu(D)), \tag{2.29}$$

$$(\mathfrak{d}_\mu)^k\theta_\mu(D) - \mathfrak{r}_{n,\mu}((\mathfrak{d}_\mu)^k\theta_\mu(D)) \in (z-1)\mathfrak{O}^{(\mu)}, \tag{2.30}$$

$$\mathfrak{r}_{n,\mu}((\mathfrak{d}_\mu)^k\theta_\mu(D))Ker((\mathfrak{d}_\mu)^k\theta_\mu(D)) \subset (z-1)\mathfrak{B}, \tag{2.31}$$

$$(\mathfrak{r}_{n,\mu}((\mathfrak{d}_\mu)^k\theta_\mu(D)) \circ \theta_\mu(v^k))\,\mathrm{Ker}\,(\theta_\mu(D)) \subset \tag{2.32}$$

$$\mathfrak{r}_{n,\mu}((\mathfrak{d}_\mu)^k\theta_\mu(D))\,\mathrm{Ker}\,((\mathfrak{d}_\mu)^k\theta_\mu(D)) \subset (z-1)\mathfrak{B}$$

for $0 \le k \le s$ *and* $\mu \in [0, \varepsilon(D, s))$.

Proof. The includings (2.26) directly follow from the Lemma 6.2.3. The including (2.27) directly follows from the above definition of $\varepsilon(D, s)$. It follows from (2.9) that

$$(\theta_\mu(v) - \mathfrak{m}_{\theta_\mu(v\mathfrak{c}_0(D)/\mathfrak{c}_0(D))})\theta_\mu(\circ D) = \theta_\mu((\mathfrak{d}D)) \circ \theta_\mu(v). \tag{2.33}$$

¿From (2.33) and (2.20) follows (2.28). To prove (2.29) we use induction on k. for For $k = 0$ the including (2.29) is obvious. Let $k < s$ and including (2.25) is true for k. Let $D_1 = \mathfrak{d}^k D$. In view of (2.8), (2.21) and (2.28),

$$(\theta_\mu(v) - \mathfrak{m}_{\theta_\mu(v\mathfrak{c}_0(D_1)/\mathfrak{c}_0(D))}) \circ \theta_\mu(D_1) = (\mathfrak{d}_\mu\theta_\mu(D_1)) \circ \theta_\mu(v), \tag{2.34}$$

It follows from (2.34) and (2.20) that

$$\theta_\mu(v)Ker(\theta_\mu(D_1)) \subset \mathrm{Ker}\,(\mathfrak{d}_\mu\theta_\mu(D_1)) \tag{2.35}$$

In view of (2.35) and (2.20)

$$\theta_\mu(v)Ker(\mathfrak{d}_\mu^k\theta_\mu(D)) \subset \mathrm{Ker}\,(\mathfrak{d}_\mu^{k+1}\theta_\mu(D)) \tag{2.36}$$

According to induction assumption,

$$(\theta_\mu(v))^k Ker(\theta_\mu(D)) \subset \mathrm{Ker}\,((\mathfrak{d}_\mu)^k\theta_\mu(D)), \tag{2.37}$$

and therefore

$$(\theta_\mu(v))^{k+1} Ker(\theta_\mu(D)) \subset \theta_\mu(v)\,\mathrm{Ker}\,((\mathfrak{d}_\mu)^k\theta_\mu(D)), \tag{2.38}$$

It follows from (2.37) and (2.38) that (2.29) is true for $k+1$. So, (2.29) is proved. Clearly, $\theta_\mu((z-1)\mathfrak{O}^{(\mu)} < v >)) = (z-1)\mathfrak{B}(\mu\delta)$. In view of (2.11)

$$\theta_\mu(\mathfrak{d}^k D) - \theta_\mu(\mathfrak{r}_n(\mathfrak{d}^k D)) = \theta_\mu(\mathfrak{m}_{c_n(\mathfrak{d}^k D)} \circ v^n) = \tag{2.39}$$

$$\theta_\mu(\mathfrak{m}_{\mathfrak{c}_n(D)} \circ v^n) \in (z-1)\mathfrak{B} < \mu\delta >$$

Applying to (2.39) the equality (2.22) and making use of the commuativity of the diagram (2.15), we obtain (2.30). So, (2.30) is proved. The including (2.31) directly follows from (2.30). Finally, (2.32) directly follows from (2.29) and (2.31). ■

Let are fulfilled the conditions of the Lemma 6.2.5. Then, in view of (2.32),

$$(\theta_\mu(\mathfrak{r}_{n,\mu}(\mathfrak{d}^k D)))((\theta_\mu(v^k))f) \in (z-1)\mathfrak{B}$$

and

$$0 = (\theta_\mu((\mathfrak{c}_0(\mathfrak{d}^k D))(1;\mu))(\theta_\mu(v^{k+0})f)(1) + \ldots +$$

$$(\theta_\mu((\mathfrak{c}_{n-1}(\mathfrak{d}^k D))(1;\mu))(\theta_\mu(v^{k+n-1})f)(1) \tag{2.40}$$

for $f(z) \in \mathbb{Q}[z] \cap \mathrm{Ker}\,(\theta_\mu(D))$. Let $n \geq 2$. In view of (2.27), let $B_k^\vee(D;z;\nu^{-1})$ denote the $(n-1)\times(n-1)-$ matrix, which has

$$-(\mathfrak{c}_{j-1}(\mathfrak{d}^k D))(z;\nu^{-1})/(\mathfrak{c}_{n-1}(\mathfrak{d}^k D))(z;\nu^{-1})$$

on the intersection of the last row with the j-th column for $1 \leq j \leq n-1$, has, in the case $n > 2$, the unit matrix on the intersection of the first $n-2$ rows with the the last $n-2$ columns and, if $n > 2$, has 0 in first $n-2$ rows of the first column. For $k \in \mathbb{N}-1$ let

$$C_k^\vee(D;z;\nu^{-1}) = \prod_{i=1}^{k} B_{k-i}^\vee(D;z;\nu^{-1}).$$

It follows from (2.7) that, for $k \in \mathbb{N}-1$,

$$B_k^\vee(D;z;0) = B_0^\vee(D;z;0) \text{ and } (B_0^\vee(D;z;0))^k = C_k^\vee(D;z;0); \tag{2.41}$$

therefore, for $s \in \mathbb{N},\ 0 \leq k \leq s$, and $\nu^{-1} \in [0,\varepsilon(D,s))$, if $\mu = \nu^{-1}$, then

$$\{B_k^\vee(D;z;\nu^{-1}), C_k^\vee(D;z;\nu^{-1})\} \subset \mathrm{Mat}_{\,n-1}(\mathfrak{O}^{(\mu)}).$$

For $T = T(z) \in \mathbb{C}[z])$ and $m \in \mathbb{N}$ we denote by means $X_{m,T} = X_{m,T}(z;\nu^{-1})$ the column in $(\mathbb{C}[z;\nu^{-1}])^m$, whose i-th element, where $i \in [1,m]\cap\mathbb{N}$, is equal to $v^{i-1}T$. Given $H = \mathfrak{c}_0(H)v^0 + \ldots + \mathfrak{c}_s(H)v^s$ in $\mathfrak{O} < v >$, let

$$F(H;D;z;\nu^{-1}) = \mathfrak{c}_0(H)C_0^\vee(D;z;\nu^{-1}) + \ldots + \mathfrak{c}_s(H)C_s^\vee(D;z;\nu^{-1});$$

since $C_k^\vee(D;z;0) = (B_0^\vee(D;1;0))^k$ for $k \in \mathbb{N}-1$, it follows that

$$F(H;D;1;0) = \mathfrak{c}_0(H)(1;0)(B_0^\vee(D;1;0))^0 + \ldots + \mathfrak{c}_s(H)(1;0)(B_0^\vee(D;1;0))^s. \tag{2.42}$$

If $H \in \mathfrak{O}^{(\mu)} < v >$ and $f(z) \in \mathbb{Q}[z] \cap Ker(\theta_\mu(D))$, then, in view of (2.40)

$$\theta_\mu(B_k^\vee(D; z; \nu^{-1})v^k X_{n-1,f})(1) = \theta_\mu(v^{k+1} X_{n-1,f})(1),$$

$$\theta_\mu(C_k^\vee(D; z; \nu^{-1}) X_{n-1,f})(1) = \theta_\mu(v^k X_{n-1,f})(1),$$
$$(\theta_\mu(H) X_{n-1,f})(1) = (F(H; D; 1; \mu) X_{n-1,f})(1)$$

for $0 \le k \le s$.

Suppose that $i \in \{0, 1\}$. Let us define the differential polynomials

$$D_i = b_{i,0}^\vee(z, \nu^{-1})v^0 + \ldots + b_{i,5}^\vee(z, \nu^{-1})v^5 + (z-1)v^6$$

and

$$P_i = p_{i,0}(\nu^{-1})v^0 + \ldots + p_{i,3\Delta+3i}(\nu^{-1})v^{3\Delta+3i},$$

corresponding to the polynomials $D_i^*(z, \nu^{-1}, w)$ and $P_i^*(\nu^{-1}, w)$ in (2.3.3)-(2.3.4) of the paper [36]. We have

$$\mathfrak{c}_6(D_i) \in (z-1)\mathfrak{O},\ (\mathfrak{c}_5(D_i))(1; 0) \neq 0,\ D_i \in \mathfrak{O}_6^* < v >,$$

and

$$P_i \in \mathfrak{O}_{3\Delta+3} < v > \cap \mathfrak{O}^{(\mu)} < v >$$

for $\mu \in \mathbb{Q}$. Let us take, in the formulae of this section,

$$D = D_i,\ H = P_i,\ n = 6,\ s = 3\Delta + 3i,\ f = f_1^*(z; \nu + i),$$

and let

$$\varepsilon_{13} = \min(\varepsilon(D_0, 3\Delta), \varepsilon(D_1, 3\Delta + 3)).$$

Suppose, moreover, that $\nu \in \mathbb{N}$ and $\nu^{-1} < \varepsilon_{13}$. Then

$$(\theta_{\nu^{-1}}(P_i) X_{5,f_1^*(z;\nu+i)})(1) = (F(P_i; D_i; 1; \nu^{-1}) X_{5,f_1^*(z;\nu+i)})(1).$$

Since the ring $Q[\nu^{-1}] < v >$ is commutative and $P_i \in (Q[\nu^{-1}] < v >$, it follows that

$$P_i X_{5,f_1^*(z;\nu+i)} = X_{5,P_i f_1^*(z;\nu+i)}$$

and, in of (2.3.6) of the paper [36],

$$(F(P_0; D_0; 1; \nu^{-1}) X_{5,f_1^*(z;\nu)})(1) = (F(P_1; D_1; 1; \nu^{-1}) X_{5,f_1^*(z;\nu+1)})(1).$$

§3. The repeating of the main theme

We can apply Lemmata 2.4.5 and 2.4.4. of the paper [36] now. If η runs through the set of all the different from 1 roots of the polynomial $D^{\wedge}(1,\eta)$ defined in the equality (2.4.56) of the paper [36], then the function $w=(\eta d_2+d_1)/(\eta-1)$ ranges over all the roots of the polynomial $D^{\sim}(1,w)$ defined in (2.4.3) of the paper [36]. Therefore there exist $\varepsilon_{14}>0$ and $q_j^{\wedge}(1,\nu^{-1})\in\mathfrak{H}(\{1\},\varepsilon_{14})$ with $j\in[0,4]\cap\mathbb{N}$ such that $y(\nu)=f_1^*(1;\nu)$ satisfies for $\nu\in\mathbb{N}+[1/\varepsilon_{14}]$, the equation of the Poincaré type

$$y(\nu+5)+q_4^{\wedge}(1,\nu^{-1})y(\nu+4)+\ \dots\ +q_0^{\wedge}(1,\nu^{-1})y(\nu+0)=0, \tag{3.1}$$

for $\nu\in\mathbb{N}+[1/\varepsilon_14]$. The polynomial

$$F_1^{\vee}(\eta)=(\eta-h^{\sim}(\gamma_1))\prod_{k=0}^{1}\prod_{l=0}^{1}(\eta-h^{\sim}(\eta_k^{\wedge}(1,(2l-1)\pi/3,\delta_0))) \tag{3.2}$$

is the characteristic polynomial of equation (3.1).

§4. The first culmination.

Let now $R=\mathbb{Q}(\nu^{-1})$, where ν^{-1} is an independent variable. As above, in the section 3, let $\mathfrak{m}_f\colon a\mapsto fa,\ a\in R$, denotes the operator of multiplication by the element $f\in R$. Since R is a commutative ring R with a unit element 1_R, it follows that the ring $\mathfrak{M}_R=\{\mathfrak{m}_f\colon f\in R\}$ of all the $R-$ endomorphisms of R is isomorphic to R. Let $\mathfrak{M}^{\vee}$ denotes the ring the $\mathbb{Q}-$endomorphisms of the additive group of the $\mathbb{Q}-$algebra R. As above, for $\phi\in\mathfrak{M}^{\vee}$ and $\psi\in\mathfrak{M}^{\vee}$, the element $\phi\circ\psi$ in $\mathfrak{M}^{\vee}$ is defined by the formula $(\phi\circ\psi)f=\phi(\psi f)$ for $f\in R$. The action of $\mathbb{Q}-$linear operator $\bigtriangledown\colon R\to R$ on $R=\mathbb{Q}(\nu^{-1})$ is given by the formula

$$\bigtriangledown\colon f(\nu^{-1})\mapsto f\left(\frac{\nu^{-1}}{\nu^{-1}+1}\right),(\bigtriangledown)^{-1}\colon f(\nu^{-1})\mapsto f\left(\frac{\nu^{-1}}{-\nu^{-1}+1}\right) \tag{4.1}$$

for $f(\nu^{-1})\in R$; clearly, the operator $\bigtriangledown$ turns ν into $\nu+1$ and the operator $(\bigtriangledown)^{-1}$ turns ν into $\nu-1$. Let $R<\bigtriangledown>$ stand for the subring of the ring $\mathfrak{M}^{\vee}$, generated by $\bigtriangledown$ and $\mathfrak{M}_R$. Since

$$\bigtriangledown\circ\mathfrak{m}_f=\mathfrak{m}_{\bigtriangledown f}\circ\bigtriangledown, \tag{4.2}$$

an element D of the ring $R<\bigtriangledown>$ can be represented in the form

$$D=\sum_{k=0}^{n(D)}\mathfrak{m}_{\mathfrak{c}_k(D)}\circ\bigtriangledown^k, \tag{4.3}$$

with $n(D) \in \mathbb{N}-1, \bigtriangledown^0 = \mathfrak{m}_{1_R}$ and $\mathfrak{c}_k(D) \in R$ for all the numbers $k \in [0, n(D)] \cap \mathbb{N}$; let $\mathfrak{c}_k(D) = 0$ for $k \in n(D) + \mathbb{N}$, then

$$D = \sum_{k=0}^{\infty} \mathfrak{m}_{\mathfrak{c}_k(D)} \circ \bigtriangledown^k. \tag{4.4}$$

For $n \in \mathbb{N} - 1$, let $R|_n < \bigtriangledown >$ denotes the set of those elements $D \in R < \bigtriangledown >$, which can represented in the form (4.4) with $\mathfrak{c}_k(D) = 0$ if $k > n$ or, equivalently, in the form

$$D = \sum_{k=0}^{n} \mathfrak{m}_{\mathfrak{c}_k(D)} \circ \bigtriangledown^k. \tag{4.5}$$

Clearly, $R < \bigtriangledown >$ may be regarded as the subring of the ring $\mathfrak{M}^{\vee}$ generated by the ring $\mathfrak{M}_R$ and the operator $\bigtriangleup := \bigtriangledown - \mathfrak{m}_{1_R}$. An element D of the set $R|_n < \bigtriangledown >$ can be represented in the form

$$D = \sum_{k=0}^{n} \mathfrak{m}_{\mathfrak{a}_k(D)} \circ \bigtriangleup^k, \tag{4.6}$$

where $\bigtriangleup^0 = \mathfrak{m}_{1_R}$ and $\mathfrak{a}_k(D) \in R$ for $0 \leq k \leq n$. On applying the operator D to the elements $1, \nu, \dots, \nu^n$, one concludes that both the coefficients $\mathfrak{a}_k(D),\ 0 \leq k \leq n$, in the equality (4.6) and the coefficient $\mathfrak{c}_k(D) = \sum\limits_{\kappa=k}^{n} \binom{\kappa}{k} \mathfrak{a}_\kappa(D),\ 0 \leq k \leq n$, in (4.4) define $\mathbb{Q}-$linear mappings

$$\mathfrak{a}_k : R|_n < \bigtriangledown > \to R,\ \mathfrak{c}_k : R|_n < \bigtriangledown > \to R$$

respectively, and, clearly, those mappings are uniquely defined. An element of the ring $R < \bigtriangledown >$ will be called a difference polynomial. For $D \neq 0_{R<\bigtriangledown>}$, let

$$\deg(D) := \sup\{k \in \mathbb{N} - 1 \colon \mathfrak{c}_k(D) \neq 0\}.$$

For any $D \in R < \bigtriangledown >$, let us define a difference polynomial $\bigtriangledown D$ by means of the equality

$$\bigtriangledown D = \sum_{k=0}^{\infty} \mathfrak{m}_{\bigtriangledown \mathfrak{c}_k(D)} \circ \bigtriangledown^k.$$

It follows that

$$\bigtriangledown \circ D = \bigtriangledown D \circ \bigtriangledown = \sum_{k=0}^{\infty} \mathfrak{m}_{\bigtriangledown c_k(D)} \circ \bigtriangledown^{k+1}$$

for any $D \in R < u >$. Let $\mathfrak{O}$ denotes the ring consisting of all the elements of the field $R = \mathbb{Q}(\nu^{-1})$, which are well-defined in the point $\nu^{-1} = 0$. Let $\mathfrak{O} < \bigtriangledown >$ be the ring of difference polynomials $P \in R < \bigtriangledown >$ with $\mathfrak{c}_k(P) \in \mathfrak{O}$ for $k \in \mathbb{N} - 1$. For $n \in \mathbb{N} - 1$, let

$$\mathfrak{O}|_n < \bigtriangledown > := \mathfrak{O} < \bigtriangledown > \cap R_n < \bigtriangledown > .$$

It is clear that the mappings $f \mapsto \bigtriangledown f$ and $f \mapsto (\bigtriangledown)^{-1} f$ are automorphisms of the ring $\mathfrak{O}$. Therefore $\mathfrak{O} < \bigtriangledown >$ is a subring of the ring $R < \bigtriangledown >$.

Lemma 6.4.1. *Suppose that*

$$A \in R < \bigtriangledown >,\ B \in R < \bigtriangledown >, \deg(A) = m \geq \deg(B) = n.$$

Then there exist the unique $D_1 \in R < \bigtriangledown >$ *and* $A_1 \in R < \bigtriangledown >$ *such that*

$$\deg(D_1) = m - n,\ D_1 - \mathfrak{c}_{m-n}(D_1^{\vee}) \circ (\bigtriangledown)^{m-n} = 0_{R<\bigtriangledown>}, \tag{4.7}$$

and or $A_1^{\vee} = A - D_1 \circ B$ *is equl to* $0_{R<\bigtriangledown>}$, *or* $\deg(A_1) < \deg(A)$. *If* $c_m(B) = 1_{\mathbb{Q}(\nu)}$, *then* $\mathfrak{c}_{m-n}(D_1) = \mathfrak{c}_m(A)$. *Moreover,*

$$c_m(B) = 1_{\mathbb{Q}(\nu)},\ A \in \mathfrak{O} < \bigtriangledown >, B \in \mathfrak{O} < \bigtriangledown >$$

then $A_1 \in \mathfrak{O} < \bigtriangledown >$ *and* $D_1 \in \mathfrak{O} < \bigtriangledown >$.

Proof. Clearly,

$$D_1 = \mathfrak{m}_{\mathfrak{c}_m(A)((\bigtriangledown)^{m-n} c_m(B))^{-1}} \circ (\bigtriangledown)^{m-n}.$$

■

Lemma 6.4.2. *Suppose that*

$$A \in R < \bigtriangledown >,\ B \in R < \bigtriangledown >, \deg(A) = m \geq \deg(B) = n.$$

Then there exist the unique $D_0 \in R < \bigtriangledown >$ *and* $A_0 \in R < \bigtriangledown >$ *such that*

$$\deg(D_0) = m - n$$

and or $A_0 = A - D_0 B$ *is equl to* $0_{R<\bigtriangledown>}$, *or* $\deg(A_0) < \deg(B)$. *If* $c_m(B) = 1_{\mathbb{Q}(\nu)}$, *then* $\mathfrak{c}_{m-n}(D_0) = \mathfrak{c}_m(A)$. *Moreover, if*

$$c_m(B) = 1_{\mathbb{Q}(\nu)},\ A \in \mathfrak{O} < \bigtriangledown >, B \in \mathfrak{O} < \bigtriangledown >$$

then $A_0 \in \mathfrak{O} < \bigtriangledown >$ *and* $D_0 \in \mathfrak{O} < \bigtriangledown >$.

Proof. We apply an induction on $k = m - n + 1$. If $k = m - n + 1 = 1$, then the assertion of the Lemma coincides with the asserion of the Lemma 6.4.1. Let further $k = m - n + 1 > 1$ and assertion of the Lemma is true for all $\kappa \in \mathbb{N}$, which are less than k First we shall prove existence. According to the Lemma 6.4.1, there exist the unique $D_1 \in R < \bigtriangledown >$, and $A_1 \in R < \bigtriangledown >$ such that $deg(D_1) = m - n$ and $A_1 = A - D_1 B$ or is equl to $0_{R<\bigtriangledown>}$, or $\deg(A_1) < \deg(A)$. If $c_m(B) = 1_{\mathbb{Q}(\nu)}$, then $\mathfrak{c}_{m-n}(D_1) = \mathfrak{c}_m(A)$. Moreover,

$$c_m(B) = 1_{\mathbb{Q}(\nu)},\ A \in \mathfrak{O} < \bigtriangledown >, B \in \mathfrak{O} < \bigtriangledown >$$

then $A_1 \in \mathfrak{O} < \bigtriangledown >$ and $D_1 \in \mathfrak{O} < \bigtriangledown >$. If $A_1 = 0_{R<\bigtriangledown>}$, or $\deg(A_1) < \deg(B)$, then we can take $D_0 = D_1$ and $A_0 = A_1$. Let $\deg(A_1) \geq \deg(B)$. Then

$$1 \leq \kappa = \deg(A_1) - \deg(B) + 1 < \deg(A) - \deg(B) + 1 = m - n + 1.$$

But the induction assumtion implies the existence of the unique $D_0^\vee \in R < \bigtriangledown >$ and $A_0 \in R < \bigtriangledown >$ such that $deg(D_0^\vee) = \deg(A_1) - \deg(B)$ and or $A_0^\vee = A_1 - D_0^\vee B$ is equl to $0_{R<\bigtriangledown>}$, or $\deg(A_0) < \deg(B)$. Moreover,

$$c_m(B) = 1_{\mathbb{Q}(\nu)},\ A_1 \in \mathfrak{O} < \bigtriangledown >, B \in \mathfrak{O} < \bigtriangledown >$$

then $A_0 \in \mathfrak{O} < \bigtriangledown >$ and $D_0^\vee \in \mathfrak{O} < \bigtriangledown >$. We can take

$$D_0 = D_1 + D_0^\vee \tag{4.8}$$

now. We prove uniqueness now. Let a desired D_0 is represented in the form (4.8), where

$$D_1 = \mathfrak{c}_{m-n}(D) \circ (\bigtriangledown)^{m-n}$$

and $D_0^\vee = D_0 - D_1$ or is equl to $0_{R<\bigtriangledown>}$, or $\deg(D_0^\vee) < \deg(D_0)$. Then D_1 satisfies to the condition (4.7) and $A_1 = A - D_1 \circ B = D_0^\vee \circ B + A_0$. Clearly, or A_1 is equl to $0_{R<\bigtriangledown>}$, or $\deg(A_1) < \deg(A)$. According to the Lemma 6.4.1, D_1 is is defined uniquely. Therefore A_1 is defined uniquely. Consequently, $D_0^\vee$ is defined uniquely. ■

Lemma 6.4.3. *The ring $R < \bigtriangledown >$ is the ring of principal left (and also right) ideals.*

Proof. Let $\mathfrak{A}$ is some left nonzero ideal of the ring $R < \bigtriangledown >$. Then $\mathfrak{A}$ contains a nonzero element B, which has

$$\deg(B) = \inf\{\deg(C) \colon C \in \mathfrak{A},\ C \neq 0_{R<\bigtriangledown>}\} = n. \tag{4.9}$$

Let $A \in \mathfrak{A} \diagdown \{0_{R<\bigtriangledown>}\}, m = \deg(A)$. According to the Lemma 6.4.2, there exist the unique $D_0 \in R < \bigtriangledown >$ and $A_0 \in R < \bigtriangledown >$ such that $\deg(D_0) = m - n$ and or the difference polynomial $A_0 = A - D_0B$ is equl to $0_{R<\triangledown>}$, or $\deg(A_0) < \deg(B)$. Clearly, $A_0 \in \mathfrak{A}$. The inequality $\deg(A_0) < \deg(B)$ contradicts to (4.9). ■

Lemma 6.4.4. *Suppose that*

$$B \in R|_m < \triangledown >, Q \in R|_s < \triangledown >, \mathfrak{c}_m(B) = \mathfrak{c}_s(Q) = 1, \tag{4.10}$$

and

$$Q \circ B \in \mathfrak{O} < \triangledown >, \tag{4.11}$$

then $B \in \mathfrak{O} < \triangledown >$ *and* $Q \in \mathfrak{O} < \triangledown >$.

The first proof. Clearly,

$$\mathrm{Ker}\,(B) \in \mathrm{Ker}\,(Q \circ B). \tag{4.12}$$

According to the E.B.Van Vleck's theorem for any $y \in \mathrm{Ker}\,(Q \circ B)$ there exist the real numbers $C > 0$, $M > 0$ and $\nu_0 \in \mathbb{N}$ such that

$$|y_0(\nu)| \leq C(M)^{\nu}$$

for all the $\nu \in \nu_0 + \mathbb{N}$. In view of (4.10) and according to the Lemma 6.1.1, we have inclusion $B \in \mathfrak{O} < \triangledown >$. According to the Lemma 6.4.2, $Q \in \mathfrak{O} < \triangledown >$. ■

The second (pure algebraically) Proof. This proof is based on the same idea, as well known Gauß's Lemma, which they used for the proof that the ring of the polynomials of several variables over arbirary field is a factorial ring. Clearly, the ring $\mathfrak{O}$ is a local ring with the (single) prime ideal $\mathfrak{p} = (\nu)^{-1}\mathfrak{O}$. Since, in view of (4.1), $\triangledown\mathfrak{p} \subset \mathfrak{p}$, $(\triangledown)^{-1}\mathfrak{p} \subset \mathfrak{p}$, it follows that $\nu^{-1}\mathfrak{O} < \triangledown >$ is a two-sided ideal of the ring $\mathfrak{O} < \triangledown >$ and $\triangledown$ determines an automorphism of the factor-ring $\mathfrak{O}/\mathfrak{p}\mathfrak{O}$; this automorphism we will also denote by $\triangledown$. Let α be a natural epimorphism of the ring $\mathfrak{O} < \triangledown >$ on the factor-ring $\mathfrak{K} = \mathfrak{O} < \triangledown > /\nu^{-1}\mathfrak{O} < \triangledown >$. Clearly, the map $a \mapsto \mathfrak{m}_a$, where $a \in \mathfrak{O}$, defines an isomorphism $\alpha^{\vee}$ of the ring $\mathfrak{O}$ onto the ring

$$\mathfrak{O}^{\vee} = \mathfrak{M}_{\mathfrak{O}} = \{\mathfrak{m}_f \colon f \in \mathfrak{O}\}.$$

Let $\mathfrak{p}^{\vee} = \alpha^{\vee}(\mathfrak{p})$. Then restriction of α on $\mathfrak{O}^{\vee}$ defines a natural epimorphism of the ring $\mathfrak{O}^{\vee}$ on the factor-ring $\mathfrak{K}_0 = \mathfrak{O}^{\vee}/\mathfrak{p}^{\vee}$; we denote this restriction also by α. Clearly, if $D \in \mathfrak{O} < \triangledown >$ is represented in the form (4.4) with $\mathfrak{c}_k(D) \in \mathfrak{O}$, then

$$\alpha(D) = \sum_{k=0}^{\infty} \alpha(\mathfrak{m}_{\mathfrak{c}_k(D)})(\alpha(\triangledown))^k.$$

If $\alpha(D) \neq 0_{\mathfrak{K}}$, then let $\deg(\alpha(D)) = \sup\{k \in \mathbb{N} - 1, \alpha(m_{\mathfrak{c}_k(D)}) \neq 0_{\mathfrak{K}_0}\}$. Cleary, when $D_k \in \mathfrak{O}|_n < \triangledown >$, $\alpha(D_k) \neq 0_{\mathfrak{K}}$, where $k = 1, 2$, then, in view of (4.2),

$$\deg(\alpha(D_1 \circ D_2)) = \deg(\alpha(D_1)) + \deg(\alpha(D_2)) \tag{4.13}$$

and therefore $\alpha(D_1 \circ D_2)) \neq 0_{\mathfrak{K}}$. If $D \in \mathfrak{O} < \bigtriangledown >$, is represented in the form (4.4) with $\mathfrak{c}_k(D) \in \mathfrak{O}$, and $\gamma \in \mathbb{Z}$, then then $D \circ \mathfrak{m}_{\nu^\gamma} = m_{\nu^\gamma} D_1$ where

$$D_1 = \sum_{k=0}^{\infty} \mathfrak{m}_{\mathfrak{c}_k(D)(1+k\nu^{-1})^\gamma} \bigtriangledown^k \in \mathfrak{O} < \bigtriangledown >,$$

$$\alpha(D_1) = \sum_{k=0}^{\infty} \alpha(\mathfrak{m}_{\mathfrak{c}_k(D)(1+k\nu^{-1})^\gamma})(\alpha(\bigtriangledown))^k \in \mathfrak{O} < \bigtriangledown >, \tag{4.14}$$

the equality$\alpha(D) = 0_{\mathfrak{K}}$ holds if and only if $\alpha(D_1) = 0_{\mathfrak{K}}$, moreover, if $\alpha(D) \neq 0_{\mathfrak{K}}$, then

$$\deg(\alpha(D)) = \deg(\alpha(D_1)). \tag{4.15}$$

Let now

$$B = \mathfrak{m}_{\nu^\beta} \circ D,\ Q = \mathfrak{m}_{\nu^\gamma} \circ D_2, \tag{4.16}$$

where $D \in \mathfrak{O}|_n < \bigtriangledown >$, $D_2 \in \mathfrak{O}|_n < \bigtriangledown >$, $\alpha(D) \neq 0_{\mathfrak{K}}$, $\alpha(D_2) \neq 0_{\mathfrak{K}}$. Then in view of (4.11),(4,13) - (4.15), $B \circ Q = \mathfrak{m}_{\nu^{\beta+\gamma}} \circ D_1 \circ D_2 \in \mathfrak{O} < \bigtriangledown >$. In view of (4.10) and (4.16) $\beta \geq 0$, $\gamma \geq 0$. And, in view of (4.11) and (4.15), $\beta + \gamma = 0$. ■

In accordance with (2.1), we consider $D^{\wedge *} \in R|_5 < \bigtriangledown >$, which has $\mathfrak{c}_5(D^{\wedge *}) = 1$ and $\mathfrak{c}_j(D^{\wedge *}) = q_j^{\wedge *}(\nu^{-1})$ for $0 \leq j \leq 4$. Let, in accordance with (3.1), $D^{\wedge}$ denotes the element of $R|_5 < \bigtriangledown >$ with $\mathfrak{c}_5(D^{\wedge *}) = 1$ and $\mathfrak{c}_j(D^{\wedge *}) = q_j^{\wedge}(\nu^{-1})$ for $0 \leq j \leq 4$. Then the equations (2.1) and (3.1), for a sufficiently small $\varepsilon > 0$ and $\nu \in [1/\varepsilon] + \mathbb{N}$, have $f_1^*(1;\nu)$ as their common solution. Let us consider the left ideal

$$R < \bigtriangledown > D^{\wedge *} + R < \bigtriangledown > D^{\wedge}.$$

According to the Lemma 6.4.3. there exist

$$B \in R < \bigtriangledown >, A^{\wedge *} \in R < \bigtriangledown >, A^{\wedge} \in R < \bigtriangledown >, Q^{\wedge *} \in R < \bigtriangledown >, Q^{\wedge} \in R < \bigtriangledown >,$$

such that

$$\mathfrak{c}(B)_{\deg B} = 1_R, \tag{4.17}$$

$$D^{\wedge *} = Q^{\wedge *} \circ B,\ D^{\wedge} = Q^{\wedge} \circ B, \tag{4.18}$$

$$A^{\wedge *} \circ D^{\wedge *} + A^{\wedge *} \circ D^{\wedge *} = B, \tag{4.19}$$

For a sufficiently small $\varepsilon > 0$ the fractions

$$\mathfrak{c}_k(D^{\wedge *}), \mathfrak{c}_k(A^{\wedge *}), \mathfrak{c}_k(Q^{\wedge *}), \mathfrak{c}_k(D^{\wedge}), \mathfrak{c}_k(A^{\wedge}), \mathfrak{c}_k(Q^{\wedge}), \mathfrak{c}_k(B),$$

are well defined for $\nu \in [1/\varepsilon]+\mathbb{N}$ and $k \in \mathbb{N}-1$. Let $n = \deg(B)$ and let us consider the cooresponding to B difference equation

$$\sum_{k=0}^{n} \mathfrak{m}_{\mathfrak{c}_k(B)} y(\nu + k) = 0. \tag{4.20}$$

Since the equations (2.1) and (3.1), for a sufficiently small $\varepsilon > 0$ and $\nu \in [1/\varepsilon]+\mathbb{N}$, have $f_1^*(1;\nu)$ as their common solution it follows from (4.19) that equation (4.20) for a sufficiently small $\varepsilon > 0$ and $\nu \in [1/\varepsilon] + \mathbb{N}$, have $f_1^*(1;\nu)$ as its solution. Therefore

$$n = deg(B) > 0 \tag{4.21}$$

In view of (4.17)-(4-18) and according to the Lemma 6.4.4,

$$B \in \mathfrak{O} < \bigtriangledown >, Q^{\wedge *} \in \mathfrak{O} < \bigtriangledown >, Q^{\wedge} \in \mathfrak{O} < \bigtriangledown >, \tag{4.22}$$

$$\mathfrak{c}(Q^{\wedge *})_{\deg(Q^{\wedge *})} = 1_R, \mathfrak{c}(Q^{\wedge})_{\deg(Q^{\wedge})} = 1_R, \tag{4.23}$$

For each $D \in \mathfrak{O} < \bigtriangledown >$, let $P(D, \eta)$ denote the polynomial

$$D = \sum_{k=0}^{\infty} \mathfrak{c}_k(D)\bigg|_{\nu^{-1}=0} \eta^k. \tag{4.24}$$

In view of (4.2),

$$P(\bigtriangledown \circ D \circ (\bigtriangledown)^{-1}, \eta) = P(D, \eta).$$

Therefore the map

$$D \mapsto P(D, \eta)$$

is a homomomorphism (of course, an epimorphism) of the ring $\mathfrak{O} < \bigtriangledown >$ onto $\mathbb{Q}[\eta]$. Consequently, in view of (4.18) we obtain the equality

$$P(D^{\wedge}, \eta) = P(Q^{\wedge}, \eta)P(B, \eta), \tag{4.25}$$

$$P(D^{\wedge *}, \eta) = P(Q^{\wedge *}, \eta) P(B, \eta). \tag{4.26}$$

In view of (4.17),

$$\deg(P(B, z)) = \deg(B) > 0 \tag{4.27}$$

But the polynomial $P(D^{\wedge}, \eta)$ coincides with the polyomial (3.2). In view of (3.1.28) in [37]

$$h^{\sim}(\eta) = (\eta^2 - 1)^3 (1 - \delta_0)^{-3d_1} 2^{-6} \eta^{3d_1} \tag{4.28}$$

$$|h^{\sim}(\eta_1^{\wedge}(1, \pi, \delta_0)| = |h^{\sim}(\gamma_1)| = (\delta_0)^3 / (1 + (\delta_0))^{3\Delta + 3} < 1 \tag{4.29}$$

But $f_1^*(1; \nu) \in \mathbb{Z}$ for $\nu \in \mathbb{N} - 1$, and in view of the Lemma 4.1.4 in [39],

$$\limsup_{\nu \to \infty} |f_1^*(1; \nu)|^{1/\nu} \geq 1, \tag{4.30}$$

In view of (4.29)-(4.30) and according to the Perron's Theorem,

$$\limsup_{\nu \to \infty} |f_1^*(1; \nu)|^{1/\nu} = |h^{\sim}(\eta)|,$$

where η is one of the different from γ_1 roots of the polynomial (3.2), i.e. one of the roots of the polynomial $F_0(\eta)$ in the Lemma 3.1.6 of the paper [37]. But according to the Lemma 3.1.6 of the paper [37], this $F_0(\eta)$ is irreducible over $\mathbb{Q}$. Therefore, in view of (4.25) and (4.27), the polynomial $F_0(\eta)$ is a divisor of the polynomial $P(B, \eta)$. In view of (4.26), we have proved the following important result:

Lemma 6.4.5. *The polynomial $F_0(\eta)$ in the Lemma 3.1.6 of the paper [37] is a divisor of the characteristical polynomial $P(D^{\wedge *}, \eta)$ of the equation (3.2)*

Therefore

$$P(D^{\wedge *}, \eta) = F_0(\eta)(\eta - \lambda_0), \tag{4.31}$$

where $\lambda_0 \in \mathbb{Q}$. Sice the polynomial $F_0(\eta)$ is irreducible over $\mathbb{Q}$, it follows that the multiplicity of the root $\lambda_0 \in \mathbb{Q}$ of the polynomial $P(D^{\wedge *}, \eta)$ is equal to 1.

§5. The second culmination.

According (3.1.61) in [37], if $r = 1, |\psi| \leq \pi$, then $s = cos(\psi/2)$

Lemma 6.5.1. *Let*

$$\delta_0 > 0, \psi^* = 2 \arccos(\delta_0/4),$$

$$\rho_0(\delta_0) = \left|\frac{\eta_1(1^w edge, \psi^*, \delta_0)}{\eta_1^{\wedge}(1, \pi, \delta_0)}\right|^{1-\delta_0)} \left|\frac{\eta_1^w edge(1, \psi^*, \delta_0) + 1}{\eta_1^{\wedge}(1, \pi, \delta_0) + 1}\right|^{\delta_0} \times$$

$$\left|\frac{\eta_1^w edge(1, \psi^*, \delta_0) - 1}{\eta_1^w edge(1, \pi, \delta_0) - 1}\right|^{\delta_0}.$$

If $\delta_0 \leq 1/7$, *then*

$$\rho_0(\delta_0) < 1. \tag{5.1}$$

Proof. According to (3.1.71) in [37],

$$q_0(r, \psi^*, \delta_0) = \frac{4(\delta_0)^2}{(1+\delta_0)^2}.$$

According to the Lemmata 3.1.2 and 3.1.4 in [37],

$$p_0(1, \psi^*, \delta_0) = 2|\eta_0^{\wedge}(1, \psi^*, \delta_0) - 1|^2 = 2|\eta_1^{\wedge}(1, \psi^*, \delta_0) - 1|^2 = \tag{5.2}$$

$$= 2\sqrt{q_0(1, \psi^*, \delta_0)} = \frac{4\delta_0}{1+\delta_0}.$$

In view of (3.1.41), (3.1.42) and (3.1.70) in [37],

$$p_1(1, \psi^*, \delta_0) = \frac{8}{(1+\delta_0)^2}\left(\frac{(\delta_0)^2}{8} + \frac{1}{4} + |D_0(1, \psi^*, \delta_0)|\right) \tag{5.3}$$

$$p_2(1, \psi^*, \delta_0) = \frac{8}{(1+\delta_0)^2}\left(1 - \delta_0\left(\frac{(\delta_0)^2}{8} - 1\right) + \frac{(\delta_0)^2}{4} + |D_0(1, \psi^*, \delta_0)|\right) \tag{5.4}$$

$$p_0(1, \psi^*, \delta_0) = \frac{8}{(1+\delta_0)^2}\left((2+\delta_0)\frac{(\delta_0)^2}{8} + \frac{(\delta_0)^2}{4} + |D_0(1, \psi^*, \delta_0)|\right) \tag{5.5}$$

Therefore

$$p_1(1, \psi^*, \delta_0) - p_0(1, \psi^*, \delta_0) = \frac{8}{(1+\delta_0)^2}\left(-(1+\delta_0)\frac{(\delta_0)^2}{8} + \frac{1-(\delta_0)^2}{4}\right) = \tag{5.6}$$

$$\frac{2}{1+\delta_0}\left(1 - \delta_0 - \frac{(\delta_0)^2}{2}\right)$$

$$p_2(1,\psi^*,\delta_0)-p_0(1,\psi^*,\delta_0)=\frac{8}{(1+\delta_0)^2}\left(-(1+\delta_0)\frac{(\delta_0)^2}{4}+(1+\delta_0)\right)=\tag{5.7}$$

$$\frac{2}{1+\delta_0}(4-(\delta_0)^2)$$

In view of (3.1.43) in [37], (5.2), (5.6) and (5.7),

$$\frac{p_1(1,\psi^*,\delta_0)}{2}=\frac{1}{1+\delta_0}\left(1+\delta_0-\frac{(\delta_0)^2}{2}\right),$$

$$\frac{p_2(1,\psi^*,\delta_0)}{2}=\frac{1}{1+\delta_0}(4+2\delta_0-(\delta_0)^2)),$$

$$\frac{p_1(1,\psi^*,\delta_0)}{2}-\sqrt{q_1(1,\psi^*,\delta_0)}=\frac{1}{1+\delta_0}\left(2\delta_0-\frac{(\delta_0)^2}{2}\right);$$

$$\frac{p_1(1,\psi^*,\delta_0)}{2}+\sqrt{q_1(1,\psi^*,\delta_0)}=\frac{1}{1+\delta_0}\left(2-\frac{(\delta_0)^2}{2}\right);$$

$$\left(\frac{p_1(1,\psi^*,\delta_0)}{2}\right)^2-q_1(1,\psi^*,\delta_0)=\frac{\delta_0}{(1+\delta_0)^2}\left(2-\frac{(\delta_0)^2}{2}\right)\left(2-\frac{\delta_0}{2}\right)\geq$$

$$\frac{9\delta_0}{4(1+\delta_0)^2},$$

$$|\eta_0(1,\psi^*,\delta_0)|^2\geq\frac{1}{1+\delta_0}\left(1+\frac{\delta_0}{2}+\frac{3\sqrt{\delta_0}}{2}\right)\tag{5.8}$$

$$\frac{p_2(1,\psi^*,\delta_0)}{2}-\sqrt{q_2(1,\psi^*,\delta_0)}=\frac{1}{1+\delta_0}(2\delta_0-(\delta_0)^2);$$

$$\frac{p_2(1,\psi^*,\delta_0)}{2}+\sqrt{q_2(1,\psi^*,\delta_0)}=\frac{1}{1+\delta_0}\left(8+2\delta_0-(\delta_0)^2\right);$$

$$\left(\frac{p_2(1,\psi^*,\delta_0)}{2}\right)^2-q_2(1,\psi^*,\delta_0)=\frac{\delta_0}{(1+\delta_0)^2}(2-\delta_0)(8+2\delta_0-(\delta_0)^2)\geq$$

$$\frac{9\delta_0}{(1+\delta_0)^2},$$

because for $s=1-\delta_0$ we have $0<s<1$ and

$$(2-\delta_0)(8+2\delta_0-(\delta_0)^2)=(1+s)(9-s^2)=9+s(9-s-s^2)>9.$$

Further we have

$$|\eta_0(1,\psi^*,\delta_0)+1|^2 \geq \frac{4}{1+\delta_0}\left(1+\frac{\delta_0}{4}+\frac{3}{2}\sqrt{\delta_0}\right). \tag{5.9}$$

In view of (3.1.56) , (3.1.43) in [37] and (5.2)

$$|\eta_1(1,\pi,\delta_0)|^2 = q_1(r,\psi,\delta_0),\ 4|\eta_1(1,\pi,\delta_0)+1|^2 = q_2(1,\psi,\delta_0), \tag{5.10}$$

$$\frac{|\eta_1(1,\psi^*,\delta_0)-1|^2}{|\eta_1(1,\pi,\delta_0)-1|^2} = \frac{1+\delta_0}{2\delta_0}$$

In view of (5.8) and (5.10),

$$\frac{|\eta_1(1,\psi^*,\delta_0)|^2}{|\eta_1(1,\pi,\delta_0)|^2} = \frac{1}{|\eta_0(1,\psi^*,\delta_0)|^2} \leq \tag{5.11}$$

$$\frac{1+\delta_0}{1+\frac{\delta_0}{2}+\frac{3\sqrt{\delta_0}}{2}}.$$

In view of (5.9) and (5.10),

$$\frac{|\eta_1(1,\psi^*,\delta_0)+1|^2}{|\eta_1(1,\pi,\delta_0)+1|^2} = \frac{4}{|\eta_0(1,\psi^*,\delta_0)+1|^2} \leq \frac{1+\delta_0}{1+\frac{\delta_0}{4}+\frac{3\sqrt{\delta_0}}{2}}. \tag{5.12}$$

Therefore, in view of (5.10) - (5.12)

$$(\rho_0(\delta_0))^2 \leq \left(\frac{1}{2\delta_0}\right)^{\delta_0} \frac{(1+\delta_0)^{1+\delta_0}}{1+\frac{3\sqrt{\delta_0}}{2}},$$

$$2\ln(\rho_0(\delta_0)) \leq -\delta_0\ln(\delta_0) - \delta_0 ln(2) + \delta_0 + (\delta_0)^2 - \frac{3\sqrt{\delta_0}}{2} + \frac{9\delta_0}{8} <$$

$$(\delta_0(\ln(1/\delta_0) - \sqrt{1/\delta_0}) + \delta_0 + (\delta_0)^2 - \sqrt{\delta_0}/2 < 0,$$

because $x + x^3 < 1/2$, if $0 \leq x \leq 2/5$, $\ln(t) - \sqrt(t)$ decreases, if $t > 4$ and $\ln(6,25) - \sqrt{6,25} < 0.$ ■

Corollary 1. *If* $\delta_0 \leq 1/7$ *and* $s = cos(\psi/2) > \delta_0/4$ *then*

$$\left|\frac{\eta_1^\wedge(1,\psi,\delta_0)}{\eta_1^\wedge(1,\pi,\delta_0)}\right|^{1-\delta_0} \left|\frac{\eta_1^\wedge(1,\psi,\delta_0)+1}{\eta_1^\wedge(1,\pi,\delta_0)+1}\right|^{\delta_0} \left|\frac{\eta_1^\wedge(1,\psi,\delta_0)-1}{\eta_1^\wedge(1,\pi,\delta_0)-1}\right|^{\delta_0} < 1. \tag{5.13}$$

Proof. According to the Lemma 4.21 in [42],

$$\left|\frac{\eta_1^\wedge(1,\psi,\delta_0)}{|\eta_1^\wedge(1,\psi^*,\delta_0)}\right|^{1-\delta_0)}\left|\frac{\eta_1^\wedge(1,\psi,\delta_0)+1}{\eta_1^\wedge(1,\psi^*,\delta_0)+1}\right|^{\delta_0}\left|\frac{\eta_1^\wedge(1,\psi,\delta_0)-1}{\eta_1^\wedge(1,\psi^*,\delta_0)-1}\right|^{\delta_0}<1. \tag{5.14}$$

Multyplying (5.1) and (5.14) we obtain (5.13). ■

Corollary 2. *If* $\delta_0 \le 1/7$ *then*

$$\left|\frac{\eta_1^\wedge(1,\pi/3,\delta_0)}{|\eta_1^\wedge(1,\pi,\delta_0)}\right|^{1-\delta_0)}\left|\frac{\eta_1^\wedge(1,\pi/3,\delta_0)+1}{\eta_1^\wedge(1,\pi,\delta_0)+1}\right|^{\delta_0}\left|\frac{\eta_1^\wedge(1,\pi/3,\delta_0)-1}{\eta_1^\wedge(1,\pi,\delta_0)-1}\right|^{\delta_0}<1. \tag{5.15}$$

Proof. Clearly, if $\psi=\pi/3$, then $s=cos(\psi/2)=\frac{\sqrt{3}}{2}>1/28\ge\delta_0/4$. Therefore, according to the Corollary 1, (5.15) holds. ■

Corollary 3. *The following inequality holds:*

$$|h^\sim(\eta_1^\wedge(1,\pi,\delta_0)|>|h^\sim(\eta_1^\wedge(1,\pi/3,\delta_0)|. \tag{5.16}$$

Proof. In view of (4.28),

$$|h^\sim(\eta)|=2^{-6}(1-\delta_0)^{-3\Delta+3}\left(|\eta^2-1|^{\delta_0}|\eta|^{1-\delta_0}\right)^{3/\delta_0} \tag{5.17}$$

Therefore (5.16) follows from the Corrollary 2. ■

Lemma 6.5.2. *Let*

$$\rho_1(\delta_0)=\frac{(2\delta_0)^{\delta_0}\left(1+\frac{3\sqrt{\delta_0}}{2}\right)}{(1+\delta_0)^{1+\delta_0}(1-\delta_0)^{1-\delta_0}} \tag{5.18}$$

If $0<\delta_0\le 1/7$, *then*

$$\rho_1(\delta_0)>1. \tag{5.19}$$

Proof . In view of (5.18),

$$\ln(\rho_1(\delta_0)=\delta_0\ln(\delta_0)+\delta_0\ln(2)+\ln\left(1+\frac{3\sqrt{\delta_0}}{2}\right)-$$

$$(1-\delta_0)\ln(1-(\delta_0)^2)-2\delta_0\ln(1+\delta_0)>$$

$$\delta_0(-\ln(1/\delta_0)+\sqrt{1/\delta_0})+\sqrt{\delta_0}-9\delta_0/8-2\delta_0^2>0,$$

if $0<\delta_0\le 1/7$. ■

Corollary 1. *If* $0 < \delta_0 \leq 1/7$, *then*

$$|h^{\sim}(\eta_0^{\wedge}(1,\psi^*,\delta_0))| > 1. \tag{5.20}$$

Proof. In view of (5.2), (5.8) and (5.9),

$$|\eta_0^{\wedge}(1,\psi^*,\delta_0)|^{2(1-\delta_0)} \times |(\eta_0^{\wedge}(1,\psi^*,\delta_0)-1)(|\eta_0^{\wedge}(1,\psi^*,\delta_0)+1)|^{2\delta_0} \geq \tag{5.21}$$
$$\left(\frac{2\delta_0}{1+\delta_0}\right)^{\delta_0}\left(\frac{1+\frac{\delta_0}{2}+\frac{3\sqrt{\delta_0}}{2}}{1+\delta_0}\right)^{1-\delta_0}\left(\frac{4\left(1+\frac{\delta_0}{4}+\frac{3\sqrt{\delta_0}}{2}\right)}{1+\delta_0}\right)^{\delta_0} >$$
$$4^{\delta_0}(1-\delta_0)^{1-\delta_0}\rho_1(\delta_0).$$

The inequality (5.20) follows from (5.21) and (5.17). ■

Corollary 2. If $0 < \delta_0 \leq 1/7$ and $s = cos(\psi/2) \geq \delta_0/4$, then

$$|h^{\sim}(\eta_0^{\wedge}(1,\psi,\delta_0))| > 1. \tag{5.22}$$

Proof. The inequality (5.22) follows from (5.20), (5.17) and Corollary to the Lemma 4.20 in [42]. ■

Corollary 3. If $0 < \delta_0 \leq 1/7$, then

$$|h^{\sim}(\eta_0^{\wedge}(1,\pi/3,\delta_0))| > 1. \tag{5.23}$$

Proof. Clearly, if $\psi = \pi/3$, then $s = cos(\psi/2) = \frac{\sqrt{3}}{2} > 1/28 \geq \delta_0/4$. Therefore, according to the Corollary 2, (5.23) holds. ■

According to (4.29), (5.16) and (5.23),

$$|h^{\sim}(\eta_0^{\wedge}(1,\pi/3,\delta_0))| = |h^{\sim}(\eta_0^{\wedge}(1,\pi/3,\delta_0))| > 1 > \tag{5.24}$$
$$|h^{\sim}(\eta_1^{\wedge}(1,\pi,\delta_0)| = |h^{\sim}(\gamma_1)| >$$
$$|h^{\sim}(\eta_1^{\wedge}(1,\pi/3,\delta_0))| = |h^{\sim}(\eta_1^{\wedge}(1,\pi/3,\delta_0))|.$$

In view of (4.30), (5.24), and according to the Perron's Theorem,

$$\limsup_{\nu\to\infty} |f_1^*(1;\nu)|^{1/\nu} = |h^{\sim}(\eta_0^{\wedge}(1,\pi/3,\delta_0))|. \tag{5.25}$$

According to the Lemma 4.2.1 in [39],

$$\lim_{\nu\to\infty} |f_2^*(1;\nu)|^{1/\nu} = |h^{\sim}(\eta_0^{\wedge}(1,\pi,\delta_0))|, \tag{5.26}$$

$$\lim_{\nu\to\infty} |f_{2+2k}^*(1;\nu)/f_2^*(1;\nu)| = 0, \tag{5.26}$$

where $k = 1, 2$. Therefore

$$\limsup_{\nu\to\infty} |f_{2+2k}^*(1;\nu)|^{1/\nu} = |h^{\sim}(\eta_1^{\wedge}(1,\pi/3,\delta_0))|, \tag{5.27}$$

where $k = 1, 2$.

§6. Some corrections to the previous parts.

During of the work on this part I found some misprints in the previous parts. On the page 16, line 7 from boden in [35] must stand

$$R_0(t;\nu)^3 = \sum_{k=0}^{\nu\Delta} \alpha^*_{\nu,k}(t+k)^{-3} + \sum_{k=0}^{\nu\Delta} \beta^*_{\nu,k}(t+k)^{-2} + \sum_{k=0}^{\nu\Delta} \gamma^*_{\nu,k}(t+k)^{-1}$$

nstead

$$R_0(t;\nu)^3 = \sum_{k=0}^{\nu\Delta} \alpha^*_{\nu,k}(t+k)^{-3} + \sum_{k=0}^{\nu\Delta} \beta^*_{\nu,k}(t+k)^{-2} + \sum_{k=0}^{\nu\Delta} \gamma^*_{\nu,k}(t+k)^{-3}$$

On the page 18, line 7 from the boden in [35] must stand

$$= 3\alpha^*(z;\nu)L_4(z^{-1}) + 2\beta^*(z;\nu)L_3(z^{-1}) + \gamma^*(z;\nu)L_2(z^{-1}) - \psi^*(z;\nu),$$

instead

$$= 3\alpha^*(z;\nu)L_4(z^{-1}) + 2\beta^*(z;\nu)L_3(z^{-1}) + \gamma^*(z;\nu)L_2(z)^{-1} - \psi^*(z;\nu),$$

On the page 9 line 7-8 in [39] from the top must stand
If $z = 1$, then condition of the Lemma 4.1.3 is fulfilled for $J = \{0, 1, 3, 4, 5\}$.
instead
If $z = 1$, then condition of the Lemma 4.1.3 is fulfilled for $J = \{0, 1, 2, 4, 5\}$.
On the page 16, inequality (120) in [42] must stand

$$|\eta_0^\wedge(r,\psi,\delta_0)|^{1-\delta_0}|(\eta_0^\wedge(r,\psi,\delta_0))^2 - 1|^{\delta_0} > \tag{120}$$

instead

$$|\eta_0^\wedge(r,\psi,\delta_0)|^{1-\delta_0}|(\eta_0^\wedge(r,\psi,\delta_0))^2 - 1|^{\delta_0} \geq \tag{120}$$

On the page 24 in [36], equalities (2.4.7) must stand

$$D_1^\sim(z,w) = (1/(z-1))D^\sim(z,w) = w^6 - \sum_{k=1}^{6} b_(^\sim z)w^{k-1} \tag{2.4.7}$$

insead

$$D_1^\sim(z,w) = (1/(z-1))D^\sim(z,w) = (\nu^{-1}w)^6 - \sum_{k=1}^{6} b_(^\sim z)w^{k-1} \tag{2.4.7}$$

On the page 24 in [36], equalities (2.4.8) must stand

$$\det(B^{\sim}(z) - d_2E) = D_1^{\sim}(z, d_2) = (-1/(z-1))(\Delta(\Delta+1))^3. \tag{2.4.8}$$

insead

$$\det(B^{\sim}(z) - d_2E) = D_1^{\sim}(z, d_2) = (1/(z-1))(\Delta(\Delta+1))^3. \tag{2.4.8}$$

On the page 27 in [36], equality (2.4.11) must stand

$$\det(Q^{\sim}(B^{\sim}(z))) = ((-1/(z-1))(\Delta(\Delta+1))^3)^{3d_2}d_1^{3d_1}, \tag{2.4.14}$$

instead

$$\det(Q^{\sim}(B^{\sim}(z))) = (-1/(z-1))(\Delta(\Delta+1))^3)^{3d_2}d_1^{3d_1}, \tag{2.4.14}$$

On the page 27, line 3 from bottom, equality (3.1.92) in [40] must stand

$$D^{\sim}(z, w) = -(\eta-1)^{-m-n}(d^{\wedge})^{m+n}((\eta+\delta)^{m+n} - (1+\delta)^{m+n}\theta_0(z)\eta^n) \tag{3.1.92}$$

instead

$$D^{\sim}(z, w) = -(\eta-1)^{-m-n}(\delta^{\wedge})m + n((\eta+\delta_1)^{m+n} - (1+\delta)^{m+n}\theta_0(z)z\eta^n \tag{3.1.92}$$

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