(1)

Revised version of:

Multi-# unknotting operations: a new family of local moves on a knot diagram and related invariants of knots

(2)

A note on the #-unknotting operation

N. A. Askitas

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D-53225 Bonn

Germany

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MULTI-# UNKNOTTING OPERATIONS: A NEW FAMILY OF LOCAL MOVES ON A KNOT DIAGRAM AND RELATED INVARIANTS OF KNOTS

N. A. ASKITAS

ABSTRACT. We define new families of (so called multi-#) local moves on knot projections (which contain the #-local move and the ordinary crossing change) and study some of their properties together with related knot invariants. We show that they define unknotting operations and hence resulting unknotting numbers. We use 4-manifold theory as a tool.

1. Introduction

In [M] H. Murakami defines a #-local move on knot diagrams and shows it to be an unknotting operation. The proof that it is indeed an unknotting operation is based on knot projections related to non-orientable spanning surfaces for the knot. Consequently he defines a metric $d_{\#}$ on the space $\mathcal{K}$ of all knots: There are finite length sequences of operations from any knot K_1 to any knot K_2. Take the minimum such length over all knot projections and sequences to be the distance between the K_i's. An #-unknotting number of knots is then defined as a special case (unknotting number of K = the distance to the unknot). Murakami studies some of the properties of these numbers. In [A] I recovered slightly stronger versions of his estimates on these invariants by using input from four-manifold theory (P. Gilmer's thesis [G] is taylored ideally for this and it is the main estimating tool here). There are, besides the ordinary crossing change, a number of unknotting operations, defined and studied by various authors, all of which result in producing metrics on $\mathcal{K}$ and associated knot invariants: the unknotting numbers. In this paper we want to generalize the #-unknotting operation by producing a family of multi-# unknotting operations[1]. These are nothing but local moves on a knot diagram, a finite sequence of which exists, for any given knot which will unknot it. In the next section we define the family of moves study some of their immediate properties and then proceed to establish that they are unknotting operations. The proofs in this section are entirely combinatorial and in fact pictorial. In the next section we introduce some more unknotting operations and in the last section we define the relevant resulting invariants and study some of their properties. It is in

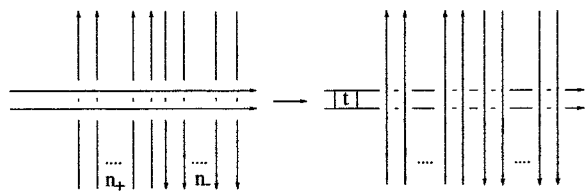
Key words and phrases. 4-manifolds, multi-# local moves, knots, unknotting numbers of knots, singularities.

[1]This is a revised version of my MPI-preprint 97-53. Part 2 remains the same. In Part (1): The main Theorem 2.1 is improved to Theorem 2.1 here. The ideas of the proof of the present stronger theorem were, essentially, contained in our previous proof. Some bugs were eliminated from the proof as well. Finally some more new local moves are introduced clocely related to the ones we had already defined and shown to be unknotting operations as well.

this section that one can see that the definition of these operations is taylored in order to involve four-manifold theory in their study.

2. The Multi-# local move

Define the multi-# local move of type (t, s, d) on a knot projection as in the figure below:



A #-local move of type (t,s,d):
s=n+ + n- and d = n+ - n-
n+ ≥ n-

FIGURE 1

The direction shown in the figure will be called a + to − multi-# crossing change of type (t, s, d) and its inverse a − to + one. The boxed t in the figure is always even and indicates $|t|$ (sign(t))-half twists (we will follow this convention every time we abbreviate a number of twists by boxing them). We will show these local moves to be unknotting operations. Consequently related metrics on the space of knots will be defined, as well as related unknotting numbers.

We begin with some preliminary lemmas which will be used later in the proofs. We will work with non-orientable spanning surfaces of knots as in [M]. It is immediate that a multi-# local move of type $(0, 1, 1)$ is nothing but Murakami's #-local move and that one of type $(2, 0, 0)$ is nothing but the ordinary unknotting operation. The moves of type $(0, s, 1)$ and $(2, s, 0)$ are interesting restricted generalizations of them respectively.

Lemma 2.1. *The effect of a ± to ∓, multi-# move of type (t, s, d) on two like-oriented parallel strings is $\pm(t - 4d)$ half twists.*

Proof : Notice the equality in the figure below.

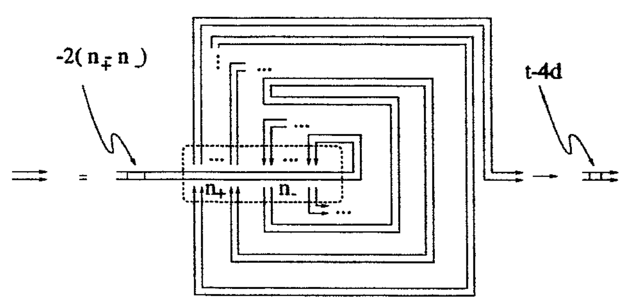


FIGURE 2

Applying the move inside the dotted box finishes the proof in the case of a + to − move. The inverse is entirely similar.

Lemma 2.2. *A + to − multi-# local move of type (t,s,d) can be locally expressed as one + to − multi-# local move of type $(t\pm 4, s+1, d\mp 1)$.*

Proof : Appropriately apply lemma 2.1 above.

Now let D_m be a 2-disk bundle over a 2-sphere of Euler class m and let $2D_m$ be its double (the latter is therefore diffeomorphic to a punctured $S^2\times S^2$ or $CP^2\#\overline{CP}^2$ according as m is even or odd). The handlebody picture of D_m is then given in the figure below where we also introduce a homology basis and intersection number conventions:

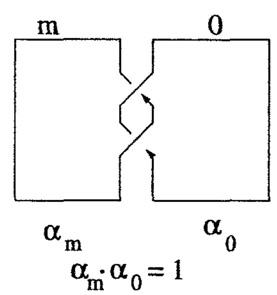


FIGURE 3

Lemma 2.3. *Let K be any knot and suppose that it can be unknotted as a result of applying a ± to ∓ multi-# local move of type (t,s,d) on one of its projections. Then there is a 2-sphere embedded in $2D_{\mp t/2}$ which is smooth everywhere except at one point where it is a cone on K and represents $2(\alpha_{\mp t/2}\pm \frac{d}{2}\alpha_0)$ on homology.*

Proof : The idea is contained in the figure below. Just multiply the strands appropriately and include the t twists.

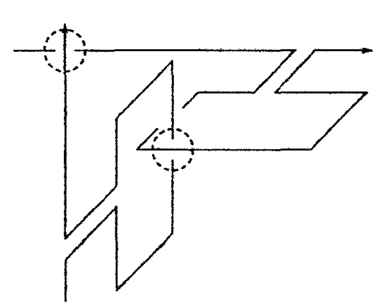


FIGURE 4

Remark 2.1. *It is an easy consequence of lemma 2.3 and theorem 4.2 that if K_1 and K_2 are two knots which differ by an odd (even) number of multi-# moves of type (t,s,d) with $t\equiv 0 \bmod 4$ and $d\equiv 1 \bmod 2$ then they have different (equal) Arf invariants. A move of type $(0,s,d)$ with $d\equiv 0 \bmod 2$ preserves the Arf invariant.*

The following lemma is a kind of inverse to lemma 2.2 above.

Lemma 2.4. *A multi-# local move of type (t, s, d) is locally expressible in terms of 2s pass moves and 4s half twists.*

Proof : The proof is contained in the figure below:

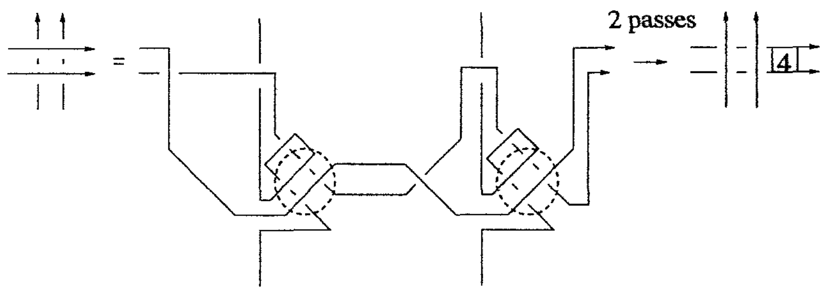


FIGURE 5

We are now ready to state the main theorem of this section:

Theorem 2.1. *Let K be any knot and $t \equiv 0 \bmod 2$, $d \geq 0$ and s any integers. If $t \equiv 2 \bmod 4$ or $t \equiv 0 \bmod 4$ and $d - \frac{t}{4} \equiv 1 \bmod 2$ then a multi-# local move of type (t, s, d) is an unknotting operation. If $t \equiv 0 \bmod 4$ and $d - \frac{t}{4} \equiv 0 \bmod 2$ then a finite sequence of $\pm$ to $\mp$, multi-# local moves of type (t, s, d) leads to the unknot (trefoil) if $Arf(K) = 0$ ($Arf(K) = 1$).*

We will prove this theorem in the remainder of this section by using non-orientable Seifert surfaces for knots and the resulting knot projections as in the example below. We can obviously assume that the feet of each non orientable handle are next to each other and have no other feet in between and that each handle has an odd number of half twists.

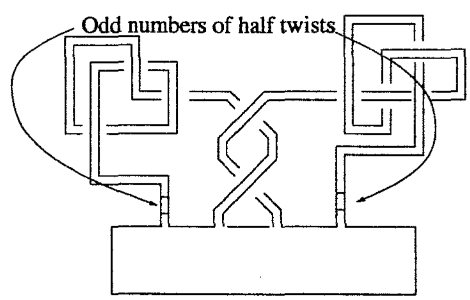


FIGURE 6

Lemma 2.5. *Every knot can be, via a sequence of multi-# local moves of type (t, s, d) for any t, s, d, reduced to a connected sum $\#_{i=1}^{s} K(q_i, 2)$ of torus knots with the q_i's well defined modulo $|t-4d|$ and in case $|t-4d| > 0$ such that $0 \leq q_i \leq |t-4d|$.*

Proof : We will show how to split off a connected summand of the knot which is a torus knot of the desired type and the rest of the knot is again the boundary of a non-orientable surface as before with one less handle. Then we can repeat the proccess as many times as we have handles and we will be done.

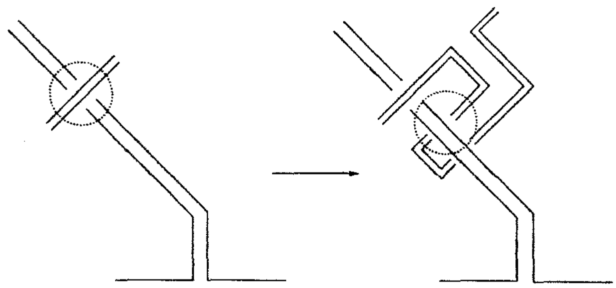


FIGURE 7

The idea is as follows; along each handle there is a number of (over or under) #-crossings which when changed all at once unknot the handle from all other handles as well as from itself so that we do get an unknotted handle with an odd number of half twists. We will show that we can bring all of these together. The first step is to turn all of them to (say) cross our handle from under. The picture above shows that we can do that as the pictures are equal and changing the marked #-crossings in either has the same effect. Now just slide together all these undercrossings by following the handle as in the figure below and you are done. We should only point out that if say we are sliding all the crossings to a given foot of the handle then organize the sliding of the self #-crossings of the handle so that the ones furthest from the foot are slid first.

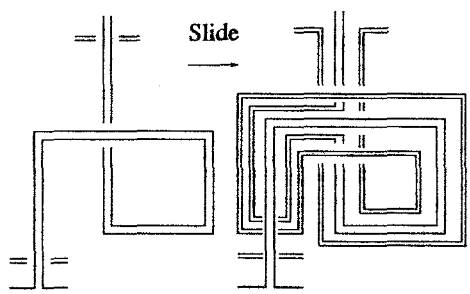


FIGURE 8

It is now clear that some (t, s, d) move will (**at once!**) unknot the handle from all the other handles and will also untie it. We can achieve this by any desired move (t, s, d) as long as we are willing to apply it many times and add twists to the handle which is to be split. In the figure below we see that by adding curls we can change d and s at will. The effect is that of adding $\pm 4x$-half twists to the handle.

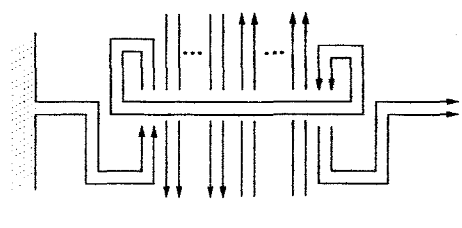


FIGURE 9

This establishes the reduction to a connected sum of torus knots of type $K(-,2)$. That the first parameter can be changed modulo $t-4d$ is a consequence of lemma 2.1. This finishes the proof of this lemma.

Now we have to deal with the case of a connected sum of $K(-,2)$'s. For that we need some preparatory lemmas which we will use as pieces of a puzzle in our proof. The lemma below is without a doubt, as the careful reader can quickly realize, the central observation which makes the rest of the proof possible. It and its many offspring lemmas are the ones which will allow us to desolve connected sums.

Lemma 2.6. *We can pass from (a) to any of (b) or (c) in the figure below by applying two multi-# moves of type* (t,s,d) *for any* t,s,d.

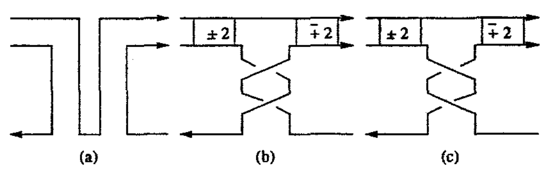


FIGURE 10

Proof : We do the case $s=0$. The case of an arbitrary s will then be clear from the proof. The latter is based on observing the following equalities:

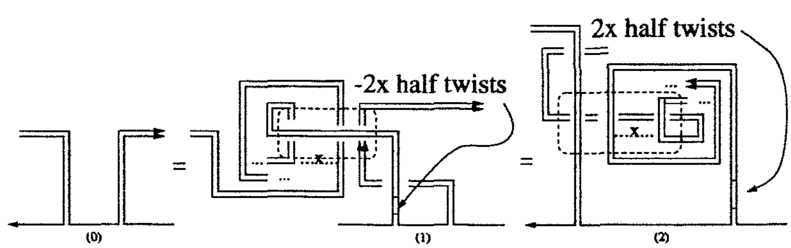


FIGURE 11

Applying a + to − multi-# move of type $(t,0,x+1)$ in the figure above inside the dotted box of (1) and a − to + multi-# move of type $(t,0,x+1)$ in (2), we get respectively the figure below. Applying our lemma 2.1 to each of these inside the dotted boxes for $d=x+1$ finishes the proof.

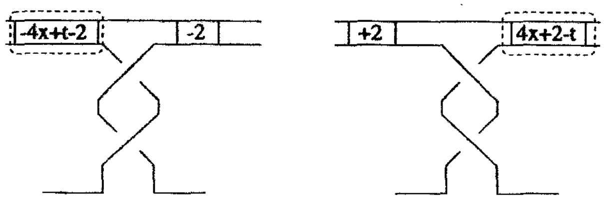


FIGURE 12

Lemma 2.7. *We can pass from $K(y,2)\#K(3,2)$ to $K(y+4,2)$ for any y, via one multi-# move of type (t,s,d) for any t,s,d.*

Proof : The proof is given in the figure below where the arrow means application of lemma 2.6.

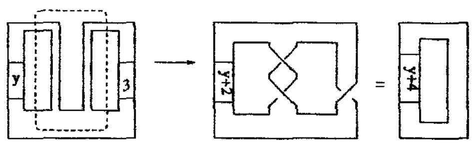


FIGURE 13

Lemma 2.8. *We can pass from $K(3,2)$ to $K(-3,2)$ via two (t,s,d) moves any t,s,d.*

Proof : In the figure below apply lemma 2.6 as follows. When $(x,y)=(3,-1)$ go from (a) to (b) with the lower signs and when $(x,y)=(-3,1)$ go from (a) to (c) with the upper signs.

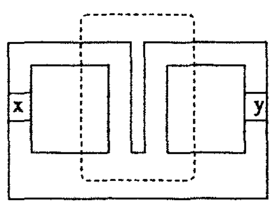


FIGURE 14

The results are mirror images of each other but in either case it is nothing but the (amphiceiral) figure eight knot! Now apply one of these moves followed by the inverse of the other and you can connect $K(3,2)$ to $K(-3,2)$.

An immediate consequence of lemma 2.7 is:

Lemma 2.9. *We can pass from $K(4x\pm 1,2)$ to $\#_{i=1}^{|x|}K(\frac{x}{|x|}3,2)$ via a sequence of (t,s,d) moves any t,s,d. We can pass from $K(-3,2)\#K(3,2)$ to the unknot.*

Now we are finally ready to prove our punch-line lemma:

Lemma 2.10. *Every connected sum* $\#_{i=1}{}^{n}K(q_i,2)$ *of torus knots can be unknotted via any multi-# move of type* (t,s,d) *if* $t\equiv 2 \bmod 4$ *or if* $t\equiv 0 \bmod 4$ *and* $\frac{t}{4}-d\equiv 1 \bmod 2$ *and can be reduced to either the unknot or the trefoil if* $t\equiv 0 \bmod 4$ *and* $\frac{t}{4}-d\equiv 0 \bmod 2$.

Proof : By repeated applications of lemma 2.7 (in reverse) and its mirror image we can reduce any $\#_{i=1}nK(q_i,2)$ to a connected sum of trefoil knots which by lemma 2.8 we can assume to be all right-handed: $\#_{i=1}^{m}K(3,2)$. We can change m by any even number as follows: Introduce a summand of type $K(3,2)\#K(-3,2)$ via lemma 2.9. Turn this to a summand of the form $K(3,2)\#K(3,2)$ by lemma 2.8. So now we are left with dealing with $K(3,2)$. We separate cases. Suppose that $t\equiv 0 \bmod 4$ and let $t=4T$. Then we can pass from $K(3,2)$ to $K(4d-4T+3,2)=K(4(d-T+1)-1,2)$ by lemma 2.1, and from the latter to $\#_{i=1}^{|d-T+1|}K(\frac{d-T+1}{|d-T+1|}3,2)$ by lemma 2.9. We can then pass from the latter to the unknot in case $d-T$ is odd. Now suppose that $t=4T+2$. Since we can always pass from $K(3,2)$ to $K(-3,2)$ we can pass from $K(3,2)$ to any of $K(4d-4T-2\pm 3,2)$. From the latter ones we can, by applying 2.9, pass to $\#_{i=1}^{|d-T|}K(\frac{d-T}{|d-T|}3,2)$ or to $\#_{i=1}^{|d-T-1|}K(\frac{d-T-1}{|d-T-1|}3,2)$ i.e. we can pass to a connected sum of an even number ($|d-T|$ or $|d-T-1|$) of trefoils and hence we are finished. This finishes the proof of this lemma as well as that of Theorem 2.1.

3. Some more local moves

We would like in this section to introduce some more unknotting operations. We need a definition first which we make as general as possible for the purpose of future study as well. The moves in this section are really an outgrowth of the ideas involved in defining and proving the multi-# unknotting operations.

Definition 3.1. *A grid local move of type* $(d_1,r_1)\times(d_2,r_2)$ *will be a local move as in the picture below.*

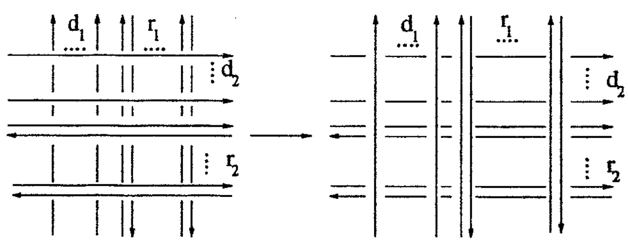


FIGURE 15

Observe that in this language a multi-# local move of type $(t,0,d)$ is locally expressible as a grid move of type $(2,0)\times(2d,2(s-d))$. We then want to observe:

Theorem 3.1. *A grid move of type* $(2,0)\times(2d+1,r)$ *is an unknotting operation.*

Proof : The proof is a consequence of the fact (see picture below where the left hand side is nothings but the left hand side of fingure 1 projected in a slightly different way)

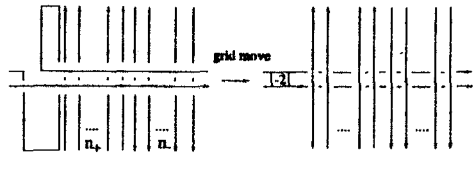


FIGURE 16

that a multi-# local move of type $(-2, s, d)$ is locally expressed as a grid move of type $(2, 0) \times (2d + 1, 2(s - d))$.

We write down one last unknotting operation:

Theorem 3.2. *A ±-full twist on four parallel like-oriented strings is an unknotting operation.*

Proof: It is clear that we can unknot the handles from each other by applying the desired moves so that the result is the connected sum of torus knots of type $K(-, 2)$. Since the moves we are discussing can obviously induce ± 8 half twists on two parallel like-oriented strings (just double the band by curling) we can assume that we have a connected sum of $K(3, 2)$'s and $K(-3, 2)$'s. Now unknot these latter ones as in the figure below where what we see pictured is nothing but a projection of the right-handed trefoil. Applying the moves indicated in the picture the reader can easily see that results in the unknot. The unknotting of a $K(5, 2)$ is the same (by our discussion above) as that of a $K(-3, 2)$ which is achieved by taking the mirror image of our figure 2.

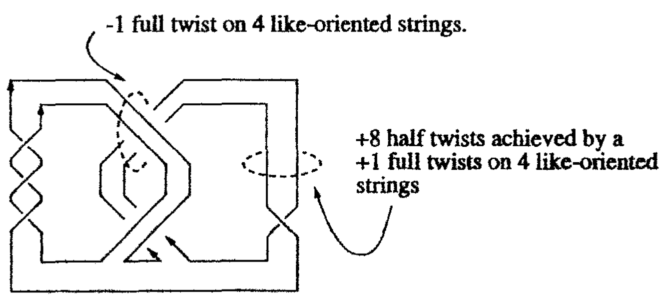


FIGURE 17

This finishes the proof of our theorem 3.2.

Note 3.1. *Y. Ohyama's result in* [O] *(Corollary 2.14 to his Theorem 2.13)* **does not** *prove our theorem 3.2 (see his Remark 2.5)!*

4. Multi-# Knot Invariants, 4-manifolds and Estimates

We can now define knot invariants resulting from the multi-#, grid and twist unknotting operations we defined in the previous sections. For efficiency of the write up we will write $v = (t, s, d)$. Similarly w will stand for the parameters of a grid move. As in [M] we can define for every move of type $v = (t, s, d)$ a metric $d^{\#}_v$ on the space $\mathcal{K}$ of all knots, which we will some times abbreviate as d_v. For $i = 0, 1$, $\mathcal{K}_i$ will denote the subsets of $\mathcal{K}$ of Arf invariant equal to i.

Definition 4.1. *For any knots K_1, K_2 define their multi-# distance* $\mathrm{d}_v^{\#}(K_1, K_2)$ *to be the minimum, over all projections and multi-# sequences, number of multi-# moves necessary to go from K_1 to K_2. In case $t \equiv 0 \bmod 4$ and $d \equiv 0 \bmod 2$ these metrics are defined only on $\mathcal{K}_i$. The multi-# unknotting number $u_v^{\#}$ of a knot is nothing but its distance from the unknot.*

We can also define the various unknotting sets:

Definition 4.2. *By $\mathcal{U}_v^{\#}(p,n)$ we will denote the set of all knots which can be unknotted via p + to - and n - to + multi-# moves of type v. In other words these are nothing but the circles of radius $p+n$ in $(\mathcal{K}, \mathrm{d}_v^{\#})$.*

We make some further definitions which we find interesting:

Definition 4.3. *A local move will be called a* **slicing operation** *if every knot leads to a slice knot after a finite number of moves. We will say that the operation slices the knot. Every unknotting operation is a slicing operation.*

We find the concept an interesting one because of the following definition which can be given for every unknotting operation:

Definition 4.4. *By $\mathcal{S}_v^{\#}(p,n)$ we will denote the set of all knots which can be sliced via p + to - and n - to + multi-# moves of type v. In other words these are nothing but the subset of $(\mathcal{K}, \mathrm{d}_v^{\#})$ whose distance from the subset of slice knots is $p+n$. We can similarly to unknotting numbers define slicing numbers $s_v^{\#}$.*

Notice the inclusion $\mathcal{U}_v^{\#}(p,n) \subset \mathcal{S}_v^{\#}(p,n)$. We can also define such invariants as multi-# distances of knots from their mirror images, ribbon sets and "ribboning numbers" etc. Similar entities can be defined for the grid unknotting operations: $d_w^{\natural}$, $u_w^{\natural}$ $\mathcal{U}_w^{\natural}(p,n)$ etc and the twist on fours like-oriented strings: d^4, u^4 $\mathcal{U}^4(p,n)$ etc.

Question 4.1. *Do there exist local moves on knot diagrams which are slicing operations but are not unknotting operations?*

What is interesting here is first that slicing numbers seem to, in principle, be smaller than unknotting numbers and second the difference seems undetectable by the various methods available. The reader can see that definitions of slicing sets and numbers of knots are motivated by proposition 4.1 and the question that follows it.

We now quote the various 4-manifold theoretic results which will be used in giving estimates regarding the various unknotting numbers. The following theorem is proven in [G] by P. Gilmer (from where we copy it here) and as mentioned there a special case of it (in the case $d = 2$ and one singular point) is also due to O. Ya. Viro. The version we quote here is only a special case of Gilmer's result (Theorem 4.1, Corollary 4.2).

Theorem 4.1. *Suppose $A \subset M^4$ is an embedded sphere in a simply connected 4-manifold which is smooth everywhere except at q points where it is the cone on knots K_i, $i = 1, \ldots, q$ and is such that on homology $[A] = \alpha = d\beta \in H_2(M^4)$, with d a power of a prime p. Then for every $0 < j < d$, $j \not\equiv 0 \bmod p$:*

$$b_2(M^4) + q - 1 \geq |\sigma(M^4) - \frac{2}{d^2} j(d-j)\alpha^2 - \sum_i \sigma_{j/d}(K_i)|$$

For the following theorem see p. 66 of [K] but beware because it is written down incorrectly there! See also theorem 1.2 on p. 31 of [K]:

Theorem 4.2. *Suppose M^4 is closed, smooth, simply connected and $\omega \in H_2(M^4)$ is characteristic. If ω is represented by an embedded sphere which is smooth everywhere except at q singular points where it is a cone on knots K_i, $i = 1, \dots, q$ then it satisfies*

$$\frac{\sigma(M^4) - \omega \cdot \omega}{8} \equiv \sum_i Arf(K_i) \bmod 2$$

Such an ω is always representable by an embedded sphere which is smooth everywhere except at one singular point where it is a cone on a knot K.

I quote the following theorem from [K] where it is attributed to D. Ruberman.

Theorem 4.3. *The minimal genus of a surface in $S^2 \times S^2$ representing the class (a, b) with respect to basis $S^2 \times q$, $p \times S^2$ with $ab \neq 0$ is $(|a|-1)(|b|-1)$. The classes $(a, 0), (0, b)$ are represented by embedded spheres. The minimal genus of a surface in $CP^2 \# \overline{CP^2}$ which represents (a, b) in the obvious basis is*

$$\binom{|a|-1}{2} - \binom{|b|}{2}$$

if $|a| > |b|$. If $|a| < |b|$, the roles of a and b are reversed in the formula. If $|a| = |b|$ the class is represented by an embedded sphere.

These are the tools I know of that can be used to estimate the multi-# unknotting numbers defined above. Lemma 2.3 and its analogues for the other unknotting operations are the connecting tools. We have the following basic estimate:

Theorem 4.4. *If $K \in \mathcal{U}_v^{\#}(p, n)$ then:*

$$2(n-p)(2d - \frac{t}{2} + 1) - 4n \leq \sigma(k) \leq 2(n-p)(2d - \frac{t}{2} + 1) + 4p$$

Proof : By a repeated application of lemma 2.3 if $K \in \mathcal{U}^{\#}_{(t,d)}(p, n)$ then there is a 2-sphere embedded in $\#_p(2D_{\frac{-t}{2}})\#_n(2D_{\frac{t}{2}})$ smooth everywhere except at one point where it is the cone on K representing $\oplus_p(\alpha_{\frac{-t}{2}} + d\alpha_0) \oplus_n (\alpha_{\frac{t}{2}} - d\alpha_0)$ on homology. Now apply Theorem 4.1.

Remark 4.1. *It is clear that we need only assume that $K \in \mathcal{S}_v^{\#}(p, n)$ in the theorem above and we would get the same estimates. This method produces estimates on slicing numbers rather than unknotting ones.*

Noting the equalities $u_v^{\#}(K) = min\{p+n \colon K \in \mathcal{U}_v^{\#}(p, n)\}$ and $s_v^{\#}(K) = min\{p+n \colon K \in \mathcal{S}_v^{\#}(p, n)\}$ we can now get:

Corollary 4.1. $|\sigma(K)| \leq 2u_v^{\#}(K)(|2d - \frac{t}{2}| + 1)$. *Same for $s_v^{\#}(K)$.*

We can also, by applying Theorem 4.2 get:

Theorem 4.5. *If $t \equiv 0 \bmod 4$ and $K \in \mathcal{U}_v^{\#}(p, n)$ then*

$$u_v^{\#}(K)(\frac{t}{4} - d) \equiv Arf(K) \bmod 2.$$

Applying Theorem 4.3 in the obvious way we obtain:

Theorem 4.6. *If $K \in \mathcal{U}_v^{\#}(1, 0)$ or $K \in \mathcal{U}^{\#}_{(t,d)}(0, 1)$ imply*

$$g_{B^4}(K) \geq |2d - \frac{t}{2}| - 1.$$

Since any unknotting operation produces an immersed disk in B^4 we can get obstructions for a knot to have multi-# unknotting number equal to one in terms of unknotting numbers of other operations as in the following:

Corollary 4.2. *Suppose* $K \in \mathcal{U}_v^{\#}(p,n)$ *with* $p+n=1$. *Then* $K \in \mathcal{U}(k,l)$ *implies that* $\max\{k,l\} \geq |2d - \frac{t}{2}| - 1$. *Similarly if for example* $K \in \mathcal{U}_{(0,1)}^{\#}(k,l)$ *then* $4\max\{k,l\} \geq |2d - \frac{t}{2}| - 1$.

The proof of the following proposition can easily be extracted from the proof of lemma 2.5:

Proposition 4.1. *Any two-strand cable knot (or equivalently any knot which is the boundary of non-orientable Seifert surface with rank of first homology equal to one) has multi-# unknotting number equal to one for infinitely many unknotting operations of type* (t,s,d).

Any knot which is the boundary of non-orientable Seifert surface with rank of first homology equal to one has multi-# slicing number equal to one for infinitely many unknotting operations of type (t,s,d).

There are of course knots whose slicing (and hence also unknotting) numbers cannot be one. $K(7,4)$ is one such. Its signature is $4 \mod 8$ and it cannot be sliced or unknotted at once by any multi-# unknotting operation!

We finish our discussion of multi-# moves with some questions:

Question 4.2. *1. Can knots with multi-# unknotting number equal to one be characterized in some nice way? Corollary 4.2 provides characterizations of some kind. What if we ask to characterize "knots with multi-# unknotting number of type* (t,s,d) *equal to one for some* t,s,d*"? Characterization, up to concordance, of knots with given slicing numbers are possible in terms of "canonical projections".*

2. How can knots with slicing number smaller than unknotting number be detected?

3. For multi-# unknotting operations of type (t,s,d) *with* $4d-t=$*const. the estimates of their unknotting numbers coincide. Is there some way to tell them apart for some knots or are they the same?*

Similar estimates can easily be given for the 4-twist and grid unknotting operations. We write down only the ones for the 4-twist one.

Theorem 4.7. *If* $K \in \mathcal{U}^4(p,n)$ *then:*

$$-8p+6n \leq \sigma(K) \leq 8n-6p, \quad -6p+4n \leq \sigma_{\frac{1}{4}}(K) \leq 6n-4p, \quad \text{and}$$

$$\max\{\frac{1}{8}|\sigma(K)|, \frac{1}{6}|\sigma_{\frac{1}{4}}(K)|\} \leq u^4(K)$$

A basic estimate for the grid move with $d_1 = 2$ amd $d_2 = 2d+1$, involving $\sigma_{\frac{1}{3}}$ can easily be worked out using P. Gilmer's Theorem and the fact that such a move is locally expressed in terms of one 3-twist and Murakami's # move and hence then in terms of $\pm$3-twists (cf. [O]). In the same manner estimates for all multi-# moves involving $\sigma_{\frac{1}{3}}$ could be worked out in more than one ways.

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A NOTE ON THE #-UNKNOTTING OPERATION

N. A. ASKITAS

ABSTRACT. In this note we use a connection between local moves on knots and the singularity type of certain PL-spheres in 4-manifolds and use it to recover (stronger in principle versions of) known results as well as obtain new ones regarding certain #-unknotting numbers of knots.

1. Introduction

In [M] H. Murakami defines the #-local move below on a projection of a knot (notice how it differs from a *pass move* only by orientation):

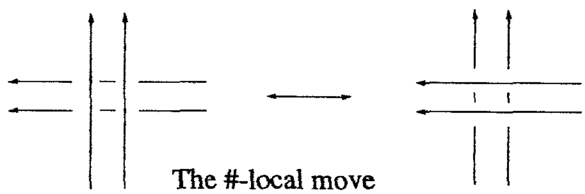

FIGURE 1

The move shown in the figure below will be refered to as a $\pm - \Xi$ move (plus for right to left in the figure, minus for reverse):

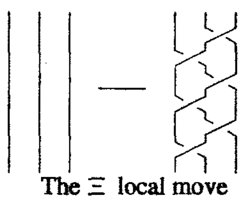

FIGURE 2

Y. Ohyama shows in [O] (Proposition 2.11) that a $\pm-\Xi$ move is locally equivalent to a #-local move. He shows infact that a $(+ : -)$, #-local move can be locally expressed in terms of three successive Ξ-operations (first two $+\Xi$ and then a $-\Xi$ operation) and that a $+\Xi$-local move is given locally by a single $(+ : -)$ #-local move.

H. Murakami proves in [M], by making use of non-orientable Seifert surfaces, that the #-local move is an unknotting operation (i.e. every knot has a projection with a finite number of local # crossings so that, when changed, a projection of the unknot is obtained). This is easily seen to define a metric $d_G^{\#}$ on the space of all knots, simply by defining $d_G^{\#}(K_1, K_2)$ to be the minimum number (over all projections and sequences of moves) of $\#$-crossing changes necessary to pass from a projection of K_1 to one of K_2. The $\#$-unknotting number $u^{\#}$ of a knot K is then nothing but $d_G^{\#}(K, O)$, where O is the unknot. As a consequence of Ohyama's aforementioned theorem the Ξ local move is also an unknotting operation and we can define similarly d_G^{Ξ} and u^{Ξ}. We wish to use a connection between local moves on knot diagrams and certain PL-embedded spheres in 4-manifolds and use it to recover (stronger in principle versions of) Murakami's Theorems 3.2, 3.5 and Corollary 3.3 as an application. This observation together with Ohyama's theorem allows us to get some new estimates on the $\#$-unknotting number which involve $\sigma_{\frac{1}{3}}$. The aforementioned connection is inspired by a simple geometric idea of S. Suzuki in [S]. The results will be obtained by invoking results of Viro and Gilmer and Kervaire-Milnor, Freedman-Kirby.

We now introduce some language which will improve the write-up of the rest of this note. By $\mathcal{U}_K^{\#}(p, n)$ we will denote (in analogy with [CL] where $\mathcal{U}_K(p, n)$ are defined for the ordinary crossing change) the set of all knots which have a projection which can be turned to one for K by p $(+\colon -)$ (left to right in Figure 1) and n $(-\colon +)$ (reverse direction of figure 1) moves. To say then that the $\#$-local move is an unknotting operation can be translated into saying that every knot K belongs to $\mathcal{U}_O^{\#}(p, n)$ for some p, n non-negative integers. The same discussion holds for the Ξ-local move and we can define $\mathcal{U}_K^{\Xi}(p, n)$ etc. Ohyama's Proposition 2.11 of [O] can then be stated as saying that $\mathcal{U}_O^{\#}(p, n) \subset \mathcal{U}_O^{\Xi}(2p + n, p + 2n)$ and $\mathcal{U}_O^{\Xi}(p, n) \subset \mathcal{U}_O^{\#}(p, n)$. Notice that these inclusions imply $u^{\#} \leq u^{\Xi} \leq 3u^{\#}$. It should also be pointed out that:

$$u^{\#}(K) = \min\{p + n\colon K \in \mathcal{U}_O^{\#}(p, n)\},$$
$$u^{\Xi}(K) = \min\{p + n\colon K \in \mathcal{U}_O^{\Xi}(p, n)\}.$$

2. From local moves to 4-manifolds

The main lemma which establishes the claimed connection is essentially nothing but what is called *fusion* and *fision* (the two directions). We were motivated by Suzuki's idea in [S].

Lemma 2.1. *If $K \in \mathcal{U}_O^{\#}(p, n)$ then there is an embedded 2-sphere $A \subset M_{p,n} = \#_{p+n} S^2 \times S^2$, which is smooth everywhere except at one point where it is the cone on K and is such that on homology $[A] = 2(1, p - n) \oplus 0 \in H_2(M_{p,n}) = H_2(S^2 \times S^2 \#_{p+n-1} S^2 \times S^2) = H_2(S^2 \times S^2) \oplus_{p+n-1} H_2(S^2 \times S^2)$.*

Sketch of Proof Take the projection which becomes one of the unknot after applying the p $(+\colon -)$ and n $(-\colon +)$, $\#$ local moves on it. Figure 3 below contains the idea of the proof. In it one can see how a $-$ to $+$ move results in expressing the resulting knot as the original one band-connect-summed (four bands) with the double of a Hopf link with negative linking number. The reverse direction is completely analogous.

We can then see that this leads to a "canonical" projection of the knot just as in [S] (the canonical form says essentially that every knot has a projection which is obtained from one of the unknot by "fusions" and "fisions" with the four component link obtained from the hopf link by doubling each of its two components). On the other hand the Kirby calculus picture of a punctured connected sum of k copies

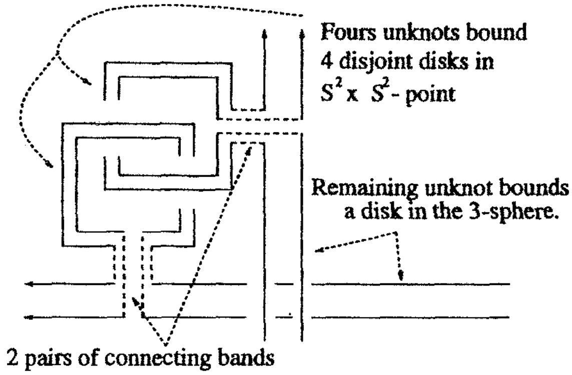

FIGURE 3

of $S^2 \times S^2$ is k copies of the Hopf link all of whose components are zero framed. This means that in a punctured $\#_{p+n} S^2 \times S^2$ our knot bounds a disk which on homology represents $\oplus_p (2,2) \oplus_n (2,-2)$. The disk consists of two copies of each of the core disks of the $2(p+n)$ 2-handles union with the bands (in the figure the bands are between the dotted lines) union with a disk that the remaining unknot bounds in the boundary 3-sphere. C. T. C. Wall's Theorems of [W1] AND [W2] on the transitivity of the automorphism group of the intersection form of such a manifold and its realizability by self-diffeomorphisms lead us to the desired class. Now close the manifold by adding a 4-ball and cone off the knot to get the singular sphere.

Remark 2.1. *1. Similarly as above we can see that if $K_1 \in \mathcal{U}^{\#}_{K_2}(p,n)$ then there is an embedded sphere in $M_{p,n}$ representing the same homology as in the lemma with two singular points where it is a cone on K_1 and the mirror image of K_2.*

2. Notice that $2u^{\#}(K)$ becomes the minimum second Betti number of $M_{p,n}$ as in the Lemma.

Similar considerations as in lemma 2.1 above imply then the analogous results in the case of a Ξ-local move. Because of this similarity and the geometric nature of the argument we ommit a sketch of proof in this case:

Lemma 2.2. *If $K \in \mathcal{U}^{\Xi}_{O}(p,n)$ then there is an embedded 2-sphere $A \subset N_{p,n} = \#_p CP^2 \#_n \overline{CP}^2$, which is smooth everywhere except at one point where it is the cone on K and is such that on homology $[A] = 3(\oplus_p (1) \oplus_n (1)) \in H_2(N_{p,n}) = \oplus_p H_2(CP^2) \oplus_n H_2(\overline{CP}^2)$.*

3. The estimates

Now we quote the 4-manifold theoretic results we need to achieve the promised estimates. The following theorem is proven in [G] by P. Gilmer (from where we copy it here) and as mentioned there a special case of it ($d = 2$ and one singular point) is also due to O. Ya. Viro. The version we quote here is only a special case of Gilmer's result (Theorem 4.1, Corollary 4.2) but it is more than what we need:

Theorem 3.1. *Suppose $A \subset M^4$ is an embedded sphere in a simply connected 4-manifold which is smooth everywhere except at q points where it is the cone on knots*

K_i, $i = 1, \dots, q$ and is such that on homology $[A] = \alpha = d\beta \in H_2(M^4)$, with d a power of a prime p. Then for every $0 < j < d$, $j \not\equiv 0 \bmod p$:

$$b_2(M^4) + q - 1 \geq |\sigma(M^4) - \frac{2}{d^2} j(d-j)\alpha^2 - \sum_i \sigma_{j/d}(K_i)|$$

For the following theorem see p. 66 of [K] but beware because it is written down incorrectly there! See also theorem 1.2 on p. 31 of [K]:

Theorem 3.2. *Suppose M^4 is closed, smooth,simply connected and $\omega \in H_2(M^4)$ is characteristic. If ω is represented by an embedded sphere which is smooth everywhere except at q singular points where it is a cone on knots K_i, $i = 1, \dots, q$ then it satisfies*

$$\frac{\sigma(M^4) - \omega \cdot \omega}{8} \equiv \sum_i Arf(K_i) \bmod 2$$

Such an ω is always representable by an embedded sphere which is smooth everywhere except at one singular point where it is a cone on a knot K.

Our Lemma 2.1 and Theorem 3.1 above easily imply:

Proposition 3.1. *If $K_1 \in \mathcal{U}^{\#}_{K_2}(p,n)$ then $-6p + 2n - \delta \leq \sigma(K_1) - \sigma(K_2) \leq 6n - 2p + \delta$, where δ is zero or one according as $K_2 = O$ or not.*

This easily implies Theorem 3.2 and Corollary 3.3 of [M] whereas Theorem 3.5 falls out of 3.2. In analogy we have the corresponding statement in the case of the Ξ move.

Proposition 3.2. *If $K \in \mathcal{U}^{\Xi}_{O}(p,n)$ then $-4p + 2n \leq \sigma_{1/3}(K) \leq -2p + 4n$ and $p + n \equiv Arf(K) \bmod 2$; in particular $u^{\Xi}(K) \equiv Arf(K) \bmod 2$.*

Using $\mathcal{U}^{\#}_{O}(p,n) \subset \mathcal{U}^{\Xi}_{O}(2p+n, p+2n)$ and $\mathcal{U}^{\Xi}_{O}(p,n) \subset \mathcal{U}^{\#}_{O}(p,n)$ one can get the following cross information about ($\#$, Ξ)-unknotting numbers which as far as we knot is new as we do not know of any estimates regarding $u_{\#}$ which involve $\sigma_{\frac{1}{3}}$:

Theorem 3.3. *If $K \in \mathcal{U}^{\#}_{O}(p,n)$ then $-6p \leq \sigma_{1/3}(K) \leq 6n$ and hence $|\sigma_{1/3}(K)| \leq 6u^{\#}(K)$. If $K \in \mathcal{U}^{\Xi}_{O}(p,n)$ then $-6p + 2n \leq \sigma(K) \leq 6n - 2p$.*

Remark 3.1. *Note that similar arguments can be applied for the classical unknotting number u. Changing p positive (n negative) crossings corresponds to p negative (n positive) full twists. Hence our knot is then the singularity type of a PL-sphere in $\#_p CP^2 \#_n \overline{CP}^2$ which represents the direct sum of twice the generator of each $\pm CP$. Applying Theorem 3.1 then shows $-2p \leq \sigma(K) \leq 2n$. This in particular is stronger than the well known $|\sigma(K)| \leq 2u(K)$, when $u(K) = p + n$. The general idea can be applied to any local move on knot diagrams. In fact we will do that for a new kind of unknotting operation in subsequent work. In that work we generalize the $\#$-local move and the observation here about its unknotting numbers is logically contained there. However the emphasis in that paper is the new local moves whereas here we want to bring out the possibility of estimating unknotting numbers via 4-manifold theory. The present note contains in addition Proposition 3.2 and Theorem 3.3. It should be noted that input from gauge theory for such unknotting operations are only possible for studying unknotting number one. The only possible input then for higher unknotting numbers is P. Gilmer's thesis. For carefully chosen knots higher unknotting numbers can be studied via branched covers as in [CL].*

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(N. Askitas) MAX-PLANCK-INSTITUT FÜR MATHEMATIK, GOTTFRIED-CLAREN-STR. 26, D-53225, BONN, GERMANY

E-mail address: askitas@mpim-bonn.mpg.de