Certain examples of birationally rigid varieties with a pencil of double quadrics

A.V.Pukhlikov

Institute for Systems Analysis
Prospekt 60-letya Oktyabrya 9
117312 Moscow
RUSSIA
and
Max-Planck-Insitut für Mathematik
Gottfried-Claren-Strasse 26
53225 Bonn
GERMANY

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Abstract

We prove that double covers of $\mathbf{P}^1 \times Q$, where $Q \subset \mathbf{P}^{k+1}$ is a non-degenerate quadric and the branch divisor is cut out by a hypersurface of the type $(2\lambda, 2k-2)$, are birationally superrigid. In particular, they have only one structure of a Fano fibration. Therefore they are non-rational.

Introduction

The aim of the present paper is to prove that certain higher-dimensional varieties with a pencil of Fano double quadrics are birationally superrigid. The double quadrics themselves were studied in [P1]. Informally speaking, we prove a very particular case of the following "general principle":

> if a Fano fibration is "sufficiently twisted" over the base, then "birational geometry of the entire variety reduces to the birational geometry of the generic fiber", that is, a Fano variety over a non-closed field.

All the known examples of birationally rigid Fano fibrations [P2,S] can be considered as realizations of this principle. Here we give one more example.

The paper consists of three sections. In the first section the necessary definitions are collected. We also prove there some elementary general facts. In Section 2 we give an explicit construction of a certain family of algebraic varieties. After that, we formulate the principal result — that is, birational superrigidity of any variety from this family, and start to prove it. Section 3 contains the key part of the proof — that is, exclusion of infinitely near maximal singularities. We do it by means of the technique, which is a higher-dimensional generalization [P2] of the method of V.A.Iskovskikh and Yu.I.Manin [IM].

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1 Fano fibrations and their birational correspondences

Here we introduce a large class of higher-dimensional algebraic varieties which are (at least, in principle) to be studied by means of the method of maximal singularities.

Definition 1. A projective algebraic variety V, equipped with a morphism

$$\pi: V \to S,$$

is said to be a *Fano fibration*, if the following conditions are satisfied:

(i) V is non-singular in codimension 1:

$$\operatorname{codim} \operatorname{Sing} V \geq 2,$$

(ii) the dimension of the base S is strictly smaller than $\dim V$,

(iii) the fiber F_η over the generic (non-closed) point of the base S is irreducible and *canonical adjunction* terminates on S, i.e. for any Weil divisor D on F_η there is $\alpha \in \mathbf{R}_+$ such that the linear system

$$|aD + bK_{F_\eta}|$$

is empty for any nonnegative integers $a, b \in \mathbf{Z}_+$ such that $b > \alpha a$.

Obviously, the (principal) condition (iii) is equivalent to the following condition: for a general point $t \in S$ canonical adjunction terminates on the corresponding fiber $F_t = \pi^{-1}(t)$.

Definition 2. Let $D \subset V$ be a Weil divisor on the Fano fibration (V, π). The least $\alpha \in \mathbf{R}_+$ for which

$$|aD + bK_V| = \emptyset$$

for any $a, b \in \mathbf{Z}_+$ such that $b > \alpha a$, is said to be the *threshold of canonical adjunction* (or, briefly, just threshold) and denoted by $c(D) = c(V, D)$.

Examples of Fano fibration. (i) Let $S = *$ be a point, V be a smooth (or with mild singularities) Fano variety, i.e. the anticanonical class $(-K_V)$ is ample. For instance, $V_M \subset \mathbf{P}^M$ is a smooth hypersurface of degree M. Here $\operatorname{Pic} V = \mathbf{Z}K_V$.

(ii) *Standard* conic bundles

$$\pi: V \to S,$$

V, S are smooth, $\mathcal{E} = \pi_* \mathcal{O}_V(-K_V)$ is a locally free sheaf of the rank 3 on S, $V \hookrightarrow \mathbf{P}(\mathcal{E})$, and each fiber $\pi^{-1}(t)$ is a conic in $\mathbf{P}^2 = \mathbf{P}(\mathcal{E}_t^*)$, and the following condition is satisfied:

$$\operatorname{Pic} V = \mathbf{Z} K_V \oplus \pi^* \operatorname{Pic} S.$$

(iii) Del Pezzo fibrations

$$\pi: V \to \mathbf{P}^1,$$

where V is smooth, $\operatorname{Pic} V = \mathbf{Z} K_V \oplus \pi^* \operatorname{Pic} \mathbf{P}^1$ and the generic fiber F_η is a Del Pezzo surface of degree d over the non-closed functional field $\mathbf{C}(\mathbf{P}^1)$.

One more example of a Fano fibration will be given in Section 2 below. It makes the subject of the present paper.

The general task of the our theory is to describe birational correspondences between Fano fibration.

Definition 3. A pair (V', H'), where V' is a Fano fibration, $\pi': V' \to S'$, and H' is a Weil divisor on V', is said to be a *test pair*, if the linear system $|H'|$ is free in codimension 1. The threshold of the divisor H' is said to be the threshold of the pair (V', H'), and denoted by $c(V', H')$.

Examples of test pairs. (i) Let V' be a Fano variety, $|H'| \subset |-rK_V|$ be a linear system with no fixed components, $r \in \mathbf{Q}_+^*$. Obviously, $c(V', H') = r$.

(ii) Let $\pi': V' \to S'$ be a Fano fibration over a positive-dimensional base, $|H'|$ be the inverse image of a linear system on the base, which is free in codimension 1. Obviously, $c(V', H') = 0$.

Assume that there is a birational map

$$\chi: V - - \to V',$$

where V is a Fano fibration and (V', H') is a test pair. The dimensions $\dim V$ and $\dim V'$ are, of course, assumed to be equal. Denote by $|\chi|$ the proper inverse image of the linear system $|H'|$ on V with respect to χ. It is a linear system of divisors on V with no fixed components. Let $D \in |\chi|$ be a general member, so that $|\chi| \subset |D|$. The threshold $n(\chi) = c(V, D)$ is an important characteristic of the birational map χ.

The group of birational automorphisms of the variety V is denoted by $\operatorname{Bir} V$, the group of biregular automorphisms is denoted by $\operatorname{Aut} V$.

Definition 4. For a Fano fibration $\pi: V \to S$ the group of birational automorphisms of the generic fiber

$$\operatorname{Bir} F_\eta \subset \operatorname{Bir} V$$

is said to be the group of *proper* birational automorphisms of the Fano fibration.

The thresholds of canonical adjunction accumulate the most essential, from the viewpoint of birational geometry, information about linear systems. Respectively, if we know how the thresholds change when we apply a birational map, then we know about this map practically everything. Now we introduce the key concept of the theory.

Definition 5. A Fano fibration V is said to be *birationally rigid*, if for any test pair (V', H') and any birational map

$$\chi: V - - \to V',$$

there is a *proper* birational automorphism

$$\chi^* \in \operatorname{Bir} F_\eta \subset \operatorname{Bir} V$$

such that the "twisted" birational map

$$\chi \circ \chi^*\colon V - - \to V'$$

satisfies the following monotonicity condition:

$$n(\chi \circ \chi^*) \leq c(V', H').$$

Provided this condition is always satisfied:

$$n(\chi) \leq c(V', H'),$$

i.e. we can take $\chi^* = \mathrm{id}_V$, the Fano fibration V is said to be birationally *superrigid.*

Proposition 1.1 *Assume that the Fano fibration V satisfies the condition* $\operatorname{Pic} F_\eta \cong \mathbf{Z}$*, where the generic fiber F_η is a smooth algebraic variety over the field $\mathbf{C}(S)$ (which is non-closed if $\dim S \geq 1$), and that for any linear system $|\chi| \subset |D|$, free from fixed components, the following equality is true:*

$$c(V, D) = c(F_\eta, D|_{F_\eta})$$

(obviously, in any case $c(V, D) \leq c(F_\eta, D|_{F_\eta})\big)$. Assume that V is birationally rigid. Then V can not be birationally equivalent to any Fano fibration $\pi'\colon V' \to S'$ over a base of a higher dimension: $\dim S' > \dim S$. For any birational equivalence

$$\begin{array}{cccc} \chi\colon & V & - - \to & V' \\ & \downarrow & & \downarrow \\ & S & & S' \end{array}$$

where $\dim S' \leq \dim S$, there is a rational map of the base

$$\alpha\colon S - - \to S',$$

which makes a commutative diagram:

$$\pi' \circ \chi = \alpha \circ \pi.$$

Proof. Let

$$\begin{array}{cccc} \chi\colon & V & - - \to & V' \\ & \downarrow & & \downarrow \\ & S & & S' \end{array}$$

be a birational equivalence of Fano fibrations. Take a linear system $|H'|$ on V', which is the inverse image with respect to π' of a very ample linear system on the base S'. Obviously, $c(V', H') = 0$, so that, twisting by a proper birational automorphism $\chi^* \in \operatorname{Bir} F_\eta$, we may assume that $n(\chi) = 0$. From the condition $\operatorname{Pic} F_\eta \cong \mathbf{Z}$ we derive that the divisors in the linear system $|\chi|$ are inverse images of divisors on the base with respect to π. Therefore, χ is a fiber-wise map, with respect to the given structures of Fano fibrations. Q.E.D. for Proposition 1.1.

Corollary 1.1 *Birationally rigid Fano fibrations, satisfying the hypothesis of Proposition 1.1, are non-rational.*

Corollary 1.2 *Birational automorphisms of a rigid Fano fibration are fiber-wise.*

Now we can give a precise formulation of what we intend to do in this paper: we are going to prove that certain families of Fano fibrations (over a one-dimensional base) are birationally superrigid.

2 Varieties with a pencil of double quadrics

2.1 Construction of the family $\mathcal{V}_k(\lambda)$

We study a certain family of smooth projective $(k+1)$-dimensional varieties $V \in \mathcal{V}_k(\lambda)$, fibered into double quadrics of index 1. Let $Q \subset \mathbf{P}^{k+1}$ be a non-singular quadric,

$$X = \mathbf{P}^1 \times Q \subset \mathbf{P}^1 \times \mathbf{P}^{k+1},$$

$$\mathcal{W}(\lambda) \ni W \subset \mathbf{P}^1 \times \mathbf{P}^{k+1}$$

a divisor of the type $(2\lambda, 2k-2)$, $\lambda \in \mathbf{Z}_+$, where the scheme-theoretic intersection $X \cap W$ is a smooth divisor on X. This data determines a variety V, which is a smooth double cover

$$\sigma: V \to X,$$

branched over $X \cap W$. Denote the family just constructed by the symbol $\mathcal{V}(\lambda)$. Projections of V and X onto $\mathbf{P}^1$ we denote by the same symbol π.

Let $L = X \cap \mathbf{P}^1 \times \{\text{hyperplane} \subset \mathbf{P}^{k+1}\}$, $G_t = \pi^{-1}(t) \subset X$. Obviously, $\operatorname{Pic} X = \mathbf{Z}L \oplus \mathbf{Z}G$ and $K_X = -kL - 2G$. Furthermore, $W \in |2(k-1)L + 2\lambda G|$, so that

$$K_V = \sigma^*(-L + (\lambda - 2)G).$$

Obviously, $\operatorname{Pic} V = \mathbf{Z}K_V \oplus \mathbf{Z}F$, where F is the class of a fiber $\pi^{-1}(t)$, $t \in \mathbf{P}^1$.

We assume that the following condition of general position is satisfied:

(G) if $x \in W \cap G_t$ is a singular point, then it is of multiplicity exactly two, the tangent cone $T_x(W \cap G_t)$ is reduced at the generic point, and the intersection of projectivized tangent cones $T_x(T_xQ \cap Q)$ and $T_x(W \cap G_t)$ in $\mathbf{P}(T_xQ) \cong \mathbf{P}^{k-1}$ is a reduced cycle of dimension $(k-3)$ and degree 4.

Let $y \in F = F_t$ be a singular point of the fiber. In other words, $x = \sigma(y) \in G_t$ is a singularity of the branch divisor $W \cap G_t$ on the fiber. Let $\varphi: \widetilde{F} \to F$ be the blowing up of the point y, $\widetilde{E} \subset \widetilde{F}$ be the exceptional divisor. Set $D = T_xQ \cap Q$. It is a quadratic cone with the vertex at x.

Proposition 2.1 *$\sigma^{-1}(D)$ is an irreducible Weil divisor. The multiplicity of its proper inverse image $\widetilde{D} \subset \widetilde{F}$ at any point in $\widetilde{D} \cap E$ is at most 2.*

Proof. Let $\psi: \widetilde{Q} \to Q$ be the blowing up of the point x, $E = \psi^{-1}(x) \subset \widetilde{Q}$ be the exceptional divisor. Note that there is the following commutative diagram of morphisms:

$$\begin{array}{ccccc} \widetilde{E} & \subset & \widetilde{F} & \stackrel{\varphi}{\longrightarrow} & F \\ \sigma\downarrow & & \downarrow\sigma & & \downarrow\sigma \\ E & \subset & \widetilde{Q} & \stackrel{\psi}{\longrightarrow} & Q, \end{array}$$

Here the vertical arrows are double covers. It is necessary to prove that the diagram does exist — that is, that the original double cover σ extends to $\widetilde{F}$. But this can be done easily by local coordinate computations, taking into consideration the condition (G).

Consider the projectivized tangent cone to the divisor $T_x Q \cap Q$. It is a non-singular quadric in E. Moreover, its image on $\widetilde{E}$ is irreducible (since its intersection with the ramification divisor is reduced). Consequently, $\sigma^{-1}(D)$ is also irreducible.

Finally, $\widetilde{D} \cap E$ is realized as a double cover of the smooth quadric. Therefore multiplicity of any singular point of this irreducible variety cannot be higher than two. Q.E.D. for the Proposition.

As in [P2], we call an effective cycle *horizontal*, if each of its components covers the base, and *vertical*, if each of its components lies in a fiber of π.

2.2 K^2-condition. Formulation of the principal result

Assume that the variety $V \in \mathcal{V}(\lambda)$ satisfies the following K^2-*condition* (see [P2]):

(K^2) the numerical class of codimension 2 cycles

$$MK_V^2 + (K_V \cdot F)$$

is not effective for any $M \in \mathbf{Z}_+$.

Obviously, the K^2-condition is certainly satisfied, if the following inequality

$$\left(K_V^2 \cdot \sigma^* L^{k-1}\right) \leq 0,$$

is true, because the system $|L|$ is free. It is easy to see that the left-hand side is equal to $4(2-\lambda)$. Thus, it is sufficient to assume that $\lambda \geq 2$.

Theorem 1 *If the conditions (G) and (K^2) are satisfied, then the Fano fibration (V, π) is birationally superrigid.*

Remark. If $\lambda = 0$, then any variety $V \in \mathcal{V}(\lambda)$ is a direct product: $V = V^* \times \mathbf{P}^1$, where V^* is a smooth double quadric of index 1. Thus V has a large group of birational automorphisms $\operatorname{Bir} V$.

2.3 Start of the proof. Maximal singularities

Fix a test pair (V', H') and a birational map

$$\chi: V - - \to V'$$

(provided there are any). Then there are a nonnegative integer $n \in \mathbf{Z}_+$ and an integer $l \in \mathbf{Z}$ such that

$$|\chi| \subset |-n(\chi)K_V + lF|.$$

Lemma 2.1 $l \geq 0$

Proof. Indeed, let $D_1, D_2 \in |\chi|$ be generic divisors, then $Z = (D_1 \bullet D_2)$ is an effective cycle of codimension 2 on V. Its numerical class

$$z = n^2 K_V^2 + 2nl(-K_V \cdot F)$$

is effective. Now the K^2-condition implies that $l \in \mathbf{Z}_+$.

Corollary 2.1 *The integer $n = n(\chi)$ is equal to the threshold of canonical adjunction $c(D, V)$ of the system $|\chi|$, $D \in |\chi|$.*

Now let us assume that the monotonicity condition is not satisfied:

$$n > c(V', H').$$

Our aim is to show that this inequality implies a contradiction, proving the theorem.

We use the language of discrete valuations in the form of [P2]. For a quasi-projective variety X the symbol $\mathcal{N}(X)$ denotes the oriented graph of geometric discrete valuations ν, which have centres $Z(X, \nu) \not\subset \operatorname{Sing} X$ of codimension ≥ 2. Recall that a discrete valuation $\nu \in \mathcal{N}(V)$ is said to be a *maximal singularity* of the linear system $|\chi|$, if the *Nöther-Fano inequality* is true:

$$\nu(|\chi|) > a(V, \nu)n(\chi).$$

For a valuation $\nu \in \mathcal{N}(V)$ set

$$e(\nu) = \nu(|\chi|) - n\delta(\nu),$$

where $\delta(\nu) = a(V, \nu)$ is the discrepancy of ν. Thus, ν is a maximal singularity of the system $|\chi|$ if and only if $e(\nu) > 0$.

Abusing the notations, we sometimes write T instead of ν, if $\nu = \nu_T$, where T is a prime Weyl divisor on a model $\widetilde{V}$ of the field $\mathbf{C}(V)$, $T \not\subset \operatorname{Sing} \widetilde{V}$, realizing the discrete valuation ν.

Proposition 2.2 *A maximal singularity does exist. Moreover, one of the following two cases takes place: (i) there is a maximal singularity $\nu \in \mathcal{N}(V)$ such that its centre on V covers the base: $\pi(Z(V, \nu)) = \mathbf{P}^1$;*

(ii) there is a finite set of maximal singularities $\mathcal{M}\subset\mathcal{N}(V)$, whose centres on V lie in the fibers of π,

$$\pi(Z(V,\nu))=t(\nu)\in\mathbf{P}^1,$$

and the following inequality is satisfied:

$$\sum_{t\in\mathbf{P}^1}\left(\max_{\{\nu\in\mathcal{M}|t(\nu)=t\}}\frac{e(\nu)}{\nu(F_t)}\right)>l.$$

Proof follows the regular scheme [P2] almost word for word. There exist a smooth quasi-projective variety W and a birational morphism $\varphi: W\to V$ such that for the proper inverse image $\widetilde{D}\subset W$ of a general divisor $D\in|\chi|$ we have

$$|a\widetilde{D}+bK_W|=\emptyset,$$

if $b>c(V',H')a$. Writing down

$$\widetilde{D}=\varphi^*D-\sum_{T\in\mathcal{L}}\nu_T(D)T,$$

$$K_W=\varphi^*K_V+\sum_{T\in\mathcal{L}}\delta(T)T,$$

where $\mathcal{L}$ is the set of all those Weil divisors on W, which lose dimension when they are put down onto V, and taking into account that $n>c(V',H')$ by hypothesis, we get that the linear system

$$\left|l\varphi^*F-\sum_{T\in\mathcal{L}}e(T)T\right|$$

is empty. Setting

$$\mathcal{M}=\{T\in\mathcal{L}|e(T)>0\},$$

we see that $\mathcal{M}\neq\emptyset$. It was shown in [P2], that for any point $t\in\mathbf{P}^1$ the Weil divisor

$$\varphi^*F_t-\sum_{\{T\in\mathcal{M}|t(T)=t\}}\nu_T(F_t)T$$

is effective, so that the $\mathbf{Q}$-divisor

$$\max_t\left(\frac{e(T)}{\nu_T(F_t)}\right)\varphi^*F_t-\sum\nolimits_t e(T)T$$

is effective, too (where the lower index t means that the operation max or $\sum$ is taken over those valuations ν_T, whose centres lie in the fiber F_t). Now, if the statement (i) is not true, then obviously (ii) is true. Q.E.D. for Proposition.

Proposition 2.3 *The case (i) of Proposition 2.2 does not realize.*

Proof. Assume the converse: the centre $Z(V,\nu)$ of a maximal singularity $\nu \in \mathcal{N}(V)$ covers the base of our Fano fibration. It is easy to see that on a general fiber F_t the valuation ν induces a maximal singularity $\nu_t \in \mathcal{N}(F_t)$ of the linear system $|\chi|_t = |\chi||_{F_t}$. However, in [P1] it was proved that this is impossible. Q.E.D.

Remark. We have actually proved the following fact. If (V,π) is an arbitrary Fano fibration, whose generic fiber F_η (or a general fiber F_s, where $s \in S$ is a general point) is a birationally superrigid variety (that is, linear systems cannot have maximal singularities), then linear systems $|\chi|$ on V do not permit maximal singularities $\nu \in \mathcal{N}(V)$, whose centre $Z(V,\nu)$ covers the base S of the Fano fibration. Similarly, in the general context all the maximal singularities of this type come from maximal singularities on the general fiber.

Proposition 2.4 *The centre of any maximal singularity is of codimension ≥ 2 with respect to the fiber, in which it lies.*

Proof. Let $B = Z(V,\nu) \subset F$ be a Weil divisor. Then for a general divisor $D \in |\chi|$ we get $\sigma_* D_F \in |2nH|$, where H is a hyperplane section. $Q = G_t\sigma(F)$, and moreover $\sigma_* B$ comes into $\sigma_* D_F$ with multiplicity $> n$. But Q is a factorial variety with $\mathrm{Pic}\, Q = \mathbf{Z}H$, so that $\sigma_* B$ is a hyperplane section.

Lemma 2.2 *Let $Y = R_1 \cap \ldots \cap R_k \subset \mathbf{P}^N$ be a complete intersection, $H \subset \mathbf{P}^N$ be a hyperplane. Then*

$$\dim \mathrm{Sing}(Y \cap H) \leq \dim \mathrm{Sing}\, Y + 1.$$

Proof. See [IP], also [P1]. This fact seems to be very old and classical (as Professor F.L.Zak told the author), but I have no idea where and when it first appeared.

Corollary 2.2 $B \in |-K_F|$

Indeed, the intersection of $\sigma_*(B)$ with W cannot be non-reduced everywhere. But $|\chi|_F \subset |-nK_F|$, so that B can come into this linear system with multiplicity at most n. A contradiction. Q.E.D.

To complete the proof, we have to exclude the *infinitely near case*, when the centres of all the maximal singularities are of codimension ≥ 2 in the fibers.

This will be done in the next section.

3 The infinitely near case

3.1 Supermaximal singularity

For a pair of general divisors $D_i \in |\chi|$, $i = 1,2$, consider the effective codimension 2 cycle $Z = (D_1 \bullet D_2)$ of their scheme-theoretic intersection. We have the decomposition

$$Z = Z^v + Z^h = Z^h + \sum_{t \in \mathbf{P}^1} Z^v_t,$$

where Z^h is a horizontal cycle, Z^v is a vertical cycle, and the support of the cycle Z^v_t is contained in the fiber F_t. For a vertical cycle $Y \subset F_t$ of codimension k (with respect to V) set

$$\deg Y = \left(Y \cdot L_V^k\right) \in \mathbf{Z}.$$

Now we get the following fact.

Lemma 3.1

$$\sum_{t\in\mathbf{P}^1} \deg Z^v_t \leq 8nl.$$

Proof: just repeat word for word the argument of Section 4, Lemma 4.1 in [P2].

Corollary 3.1 *There exists a maximal singularity $T \in \mathcal{M}$, $t(T) = t \in \mathbf{P}^1$, such that*

$$8ne(T) > \nu_T(F_t) \deg Z^v_t.$$

This singularity is said to be *supermaximal.* Fix it. We write F instead of F_t, Z^v instead of Z^v_t (now this will cause no misunderstanding), e instead of $e(T)$, ν instead of ν_T, δ instead of $\delta(T)$.

3.2 Resolution of the supermaximal singularity.

Let

$$\begin{array}{rccc} \varphi_{i,i-1}: & V_i & \to & V_{i-1} \\ & \cup & & \cup \\ & E_i & \to & B_{i-1} \end{array}$$

be the resolution of ν: $\varphi_{i,i-1}$ blows up the irreducible cycle B_{i-1}, which is the centre $Z(V_{i-1}, \nu)$ of the valuation ν on V_{i-1}, $E_i = \varphi^{-1}_{i,i-1}(B_{i-1})$ is the exceptional divisor, $1 \leq i \leq K$ and the triplet $(V_K, \varphi_{K,0}, E_K)$ is a realization of ν, i.e. $\nu_{E_K} = \nu$. Recall that for $i > j$ the symbol $\varphi_{i,j}$ denotes the composition

$$\varphi_{j+1,j} \circ \ldots \circ \varphi_{i,i-1} : V_i \to V_j,$$

$\varphi_{i,i} = \mathrm{id}_{V_i}$ and $V_0 = V$. On the set of indices $\{1, \ldots, K\}$ we introduce a natural oriented graph structure, joining the vertices i and j by an oriented edge (an arrow), if $i > j$ and

$$B_{i-1} \subset E^{i-1}_j$$

(the upper index means, as usually, the operation of taking the proper inverse image on V_{i-1}). Notation: $i \to j$. We set

$$p(i, j) = (\text{ number of paths from } i \text{ to } j\ ),$$

if $i \neq j$, and $p(i, i) = 1$, otherwise. Finally, instead of $p(K, i)$ we write p_i.

Since $\varphi_{i,i-1}(B_i) = B_{i-1}$, the dimensions $\dim B_i$ do not decrease. Following [P2], we decompose the succession of blowing ups into the *lower part*, when $\operatorname{codim} B_{i-1} \geq$

3, corresponding to the indices $1 \leq i \leq L$, $L \leq K$, and the *upper part*, when $\operatorname{codim} B_{i-1} = 2$, $i \geq L+1$, if $L \neq K$. We have got the following presentations:

$$\delta \geq 2\sum_{i=1}^{L} p_i + \sum_{i=L+1}^{K} p_i,$$

$$\nu(|\chi|) = \sum_{i=1}^{K} p_i \operatorname{mult}_{B_{i-1}} |\chi|^{i-1}.$$

Set

$$N = \max\{i | 1 \leq i \leq L, B_{i-1} \subset F^{i-1}\},$$

$$\mu_i = \operatorname{mult}_{B_{i-1}} F^{i-1}$$

for $i = 1, \ldots, N$. In terms of these multiplicities we get the following inequality:

$$\nu(F) \geq \sum_{i=1}^{N} \mu_i p_i.$$

To make our formulae more compact, set

$$\Sigma_l = \sum_{i=1}^{L} p_i, \quad \Sigma_u = \sum_{i=L+1}^{K} p_i.$$

Finally, for $i = 1, \ldots, L$ set

$$m_i^h = \operatorname{mult}_{B_{i-1}} (Z^h)^{i-1},$$

$$m_i^v = \operatorname{mult}_{B_{i-1}} (Z^v)^{i-1},$$

$$m_i = \operatorname{mult}_{B_{i-1}} Z^{i-1} = m_i^h + m_i^v.$$

Note at once that since the support of the 1-cycle Z^v is contained in F, all the integers m_i^v vanish for $L \geq i \geq N+1$.

3.3 Counting multiplicities.

In accordance with the standard techniques of counting multiplicities [P2], which goes back to the *test class* techniques of V.A.Iskovskikh and Yu.I.Manin [IM], we get the inequality

$$\sum_{i=1}^{L} p_i m_i \geq \frac{(2\Sigma_l n + \Sigma_u n + e)^2}{\Sigma_l + \Sigma_u}.$$

Rewrite this inequality in the expanded form:

$$(\Sigma_l + \Sigma_u)\left(\sum_{i=1}^{L} p_i m_i^h + \sum_{i=1}^{N} p_i m_i^v\right) \geq$$

$$\geq (2\Sigma_l + \Sigma_u)^2 n^2 + 2(2\Sigma_l + \Sigma_u) ne + e^2.$$

Note, first of all, that

$$m_1^h \leq \deg(Z^h \cap F) = 4n^2.$$

Since the multiplicities $m_i^{h,v}$ do not increase as functions of the index i, our inequality will be true when we replace m_i^h by m_1^h. In this way we get the following inequality (*):

$$(\Sigma_l + \Sigma_u) \sum_{i=1}^{N} p_i m_i^v \geq \Sigma_u^2 n^2 + 2(2\Sigma_l + \Sigma_u) ne + e^2.$$

To get a contradiction and complete the proof, we need to study two cases. Set $B = B_0 = Z(V, \nu) \subset F$.

We say that our case is *non-singular*, if $B \not\subset \operatorname{Sing} F$, and *singular*, otherwise. In accordance with the condition (G), in the singular case B is a point, $\sigma(B) \in W \cap G_t$ is a singularity of the branch divisor.

3.4 Exclusion of the non-singular case

Let $x \in F \setminus \operatorname{Sing} F$ be a smooth point of the fiber F.

Lemma 3.2 *For any cycle $T \subset F$ of codimension 1 the following inequality holds:*

$$\operatorname{mult}_x T \leq \frac{1}{2} \deg T.$$

Proof. Consider the image $Y = \sigma(T)$ of the Weil divisor T on Q. Obviously,

$$\frac{\operatorname{mult}_z Y}{\deg Y} \geq \frac{\operatorname{mult}_x T}{\deg T}.$$

Let us assume that the assertion of the lemma is not true for T. Consequently,

$$\operatorname{mult}_z Y > \frac{1}{2} \deg Y.$$

This implies that Y lies entirely in the tangent hyperplane $T_z Q$. Since Q is non-singular, $Y = Q \cap T_z Q$. If $z \notin W$, then we come to an obvious contradiction:

$$\operatorname{mult}_x T = 2, \quad \deg T = 4.$$

If $z \in W$, then we take a general 3-plane P in $\mathbf{P}^{k+1} \supset Q$, containing z. Obviously, $P \cap Y$ is a pair of (distinct) lines, meeting W transversally at the point z. Consequently,

$$\operatorname{mult}_x T \leq \operatorname{mult}_x (T \cap \sigma^{-1}(P \cap Q)) = 2,$$

and we get a contradiction again. Q.E.D. for the Lemma.

By the Lemma we get the following estimate:

$$\operatorname{mult}_x Z^v < \frac{4ne(T)}{\nu(F)}.$$

Now we complete the proof in the usual way [P2]:

$$\sum_{i=1}^{N} p_i m_i^v < \frac{4ne}{\nu(F)} \sum_{i=1}^{N} p_i \leq 4ne,$$

so that

$$0 > \Sigma_u^2 n^2 - 2\Sigma_u ne + e^2,$$

which is impossible. The non-singular case has been excluded.

3.5 Exclusion of the singular case

Let $B = x$ be a singular point of the fiber F, $z = \sigma(x)$ be its image on the quadric $G_t = Q$.

Set $T^* = \sigma^{-1}(T_z Q \cap Q)$. We know (Section 2), that $T^* \subset F$ is an irreducible Weil divisor. Set

$$Z^v = aT^* + Z_1^v,$$

where Z_1^v does not come into T^* as a component, $a \in \mathbf{Z}_+$. As above (subsection 4), for any irreducible cycle $T \neq T^*$ of codimension 1 with respect to F we have the estimate

$$\operatorname{mult}_x T \leq \frac{1}{2} \deg T.$$

By Proposition 2.1 we get for T^* the following inequality

$$\operatorname{mult}_y \widetilde{T}^* \leq 2 = \frac{1}{2} \deg T^*,$$

where $y \in E_1$ is an arbitrary infinitely near point over x. Using these two inequalities, applied to the two components, into which the cycle Z^v was decomposed, we get

$$\sum_{i=1}^{N} p_i m_i^v \leq \left(2p_1 + \sum_{i=2}^{N} p_i\right) (\frac{1}{2} \deg Z^v).$$

But since x is a singularity of the fiber, we have the following estimate:

$$\nu(F) \geq 2p_1 + \sum_{i=2}^{N} p_i.$$

Combining these inequalities, we get what we need:

$$\sum_{i=1}^{N} p_i m_i^v < 4ne.$$

Now we complete the proof, repeating the "non-singular" arguments above word for word.

Q.E.D. for the Theorem.

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