Every holomorphic symplectic manifold admits a Kähler metric

Andrey N. Todorov

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Straße 26
D–5300 Bonn 3

Federal Republic of Germany

MPI/90-106

An explicit method of constructing pluriharmonic maps from compact complex manifold into complex Grassmann manifold

by

Seiichi UDAGAWA

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Department of Mathematics
School of Medicine
Nihon University
Itabashi-Ku, Tokyo 173
JAPAN

MPI/90-96

EVERY HOLOMORPHIC SYMPLECTIC MANIFOLD ADMITS A KÄHLER METRIC

ANDREY N. TODOROV

#0. INTRODUCTION.

It is well known that the class of Kähler manifolds form a very important class of complex manifolds. In dimension two combining the results of Myaoka, Harvey and Lawson, [15] and [16] one can conclude that every two dimensional complex manifold with even first Betti number is a Kähler surface. KODAIRA conjectured this. In higher dimension it is no longer true that if a manifold has an even first BETTI number then the manifold has a Kähler metric. Hence it is important to give some simple conditions in higher dimensions that will imply the Kählerian property of the given manifold. In this paper we give such a condition.

It is well known that every K3 surface admits a Kähler metric (See [15] and [16]). A natural generalization in higher dimension of K3 surfaces are the so called Hyper-Kählerian manifolds. The first examples of compact Hyper-Kählerian manifolds were constructed by Fujiki and later his construction was generalized by Beauville. (See [02]). In [02] Beauville gave second construction of Hyper-Kählerian manifolds, different from the first which generalizes Fujiki's example.

The aim of this article is to generalize the statement that every K3 surface is Kähler one. More precisely the following theorem is proven:

THEOREM. Every holomorphic symplectic manifold admits a Kähler metric.

The definition of a holomorphic symplectic manifold is the following one:

DEFINITION.

Suppose that X is a compact complex manifold such that:

1) There exists a closed holomorphic two form $\omega_X(2,0)$ such that at each point $x \in X$, $\omega_X(2,0)$ is a non-degenerate skew symmetric matrix, i.e. everywhere $\omega_X(2,0)$ has a maximal rank equal to $2n=\dim_{\mathbf{C}} X$.

2) $\dim_{\mathbf{C}} H^2(X,\mathcal{O}_X)=1$

3) $\dim_{\mathbf{C}} X \geq 4$

then X will be called a holomorphic symplectic manifold. If X has a KÄHLER METRIC we will called it HYPER-KÄHLERIAN.

Remark. From Condition 2 it follows that up to a constant we have a unique close holomorphic two form on X.

The proof of the Theorem follows the lines [15] and [16]. The main points of the proof are:

a) On holomorphic symplectic manifold there exists a real closed two form

$$\omega=\omega^{2,0}+\omega^{1,1}+\overline{\omega^{2,0}}$$

where $\omega^{2,0}=\partial\alpha^{1,0}$ and $\omega^{1,1}$ is a positive definite Hermitian form at each point. The construction of ω is done by checking the conditions of Theorem 38 in the beautifull paper by R. Harvey and B. Lawson. (See [10]).

b) Modification of arguments of Bogomolov proves that there exits a non-singular Kuranishi family $\mathfrak{X}\to U$ of symplectoc manifolds such that $\dim_{\mathbb{C}}U=\mathrm{dom}_{\mathbb{C}}\ H^2(X,\mathbb{C})-2$.

c) We prove that "small" deformations of Hyper-Kählerian manifold X are also Hyper-Kählerian manifolds.

d) Next we show an anologue to criterium of Moishezon-Nakai which establishes which Hyper-Kählerian manifold X to be an algebraic one. From this result and the local Torelli Theorem we deduce that in U there exists an open and everywhere dense subset W such that each point $\tau\in U$ corresponds to an algebraic Hyper-Kähler manifold.

e) Using Yau's solution of Calabi conjecture the so called isometric deformations are constructed. (See [16].) The Harvey-Lawson metric ω defines a disk D in U. So there is a family $\mathfrak{X}\to D$ containing X. Moreover it may be supposed that in D there is an everywhere dense subset $W\bigcap D$ corresponding to the algebraic Hyper-Kählerian manifolds. Using the isometric deformations a new family $\mathfrak{X}'\to D$ is constructed such that all its fibres are Hyper-Kähler manifolds. Moreover those two families are isomorphic over an open and everywhere dense subset in D. From Bishop's criterium we conclude that the two families are isomorphic.

Acknowledgements.

This articles was done during my visit to Max-Plank Institüte für Mathematik in Bonn and finished in Rice University. The author would like to thank both institutions for excellent working conditions.

I want to thanks prof. M. Kreck and prof. A. Assadi for their usefull comments conserning references and the proof of the remark on p.15. I should also thanks the referee for the usefull comments.

#1. CONSTRUCTION OF A HARVEY-LAWSON METRIC.

THEOREM.

Let X be a holomorphic symplectic manifold, then X admits a real closed two form

$$\omega=\omega^{2,0}+\omega^{1,1}+\omega^{0,2}$$

such that

a) $\omega^{2,0}=\partial\alpha^{1,0}$, $\omega^{0,2}=\overline{\partial\alpha^{1,0}}$.

b) $\omega^{1,1}$ is positive definite at each point $x\in X$.

PROOF:

The proof is based on the following Theorem of Harvey and Lawson:

THEOREM. (See [10].) Suppose that X is a compact complex manifold, then X admits a real closed two form

$$\omega=\omega^{2,0}+\omega^{1,1}+\omega^{0,2}$$

such that

a) $\omega^{2,0}=\partial\alpha^{1,0}$, $\omega^{0,2}=\overline{\partial\alpha^{1,0}}$

b) $\omega^{1,1}$ is positive definite at each point $x\in X$ if and only if X does not support a non-trivial, d-closed positive current which is the bidimension (1,1) component of a boundary.

We need to check that if X is a holomorphic symplectic manifold then X satisfies the conditions of the Theorem of R. Harvey and B. Lawson Jr.

Let

$$\mu=\sqrt{-1}\sum\mu^{i\bar{j}}\frac{\partial}{\partial z^{i}}\wedge\overline{\frac{\partial}{\partial z^{j}}}$$

be an exact real (1,1) positive current on X. Since on X we have a closed holomorphic form $\omega_X(2,0)$ which is non-degenerate at each point $x\in X$ we get immediately from μ an exact (2n-1,2n-1) current η in the following way:

$$\eta=\mu\wedge((\wedge^{n-1}\omega^*_X(2,0))\wedge((\wedge^{n-1}\omega^*_X(2,0))$$

REMARK. From now on $\perp$ will denote contruction of tensors, i.e. $\frac{\partial}{\partial z^j}\perp dz^i=\delta_{ij}$

Definition of $\omega^*_X(2,0)$.

Since $\omega^*_X(2,0)$ is a non-degenerate closed holomorphic form, the arguments of Darboux Lemma can be repeated(See [01].) to get a local coordinate system $(z^1,...,z^n,...,z^{2n})$ such that locally

$$\omega_X(2,0)=\sum_{i=1}^{n} dz^i\wedge dz^{i+n}$$

then

$$\omega^*_X(2,0):=\sum_{i=1}^{n}\frac{\partial}{\partial z^i}\wedge\frac{\partial}{\partial z^{i+n}}$$

Let $\eta=dj^*$, then clearly

$$\alpha=\eta\perp(\wedge^n(\omega_X(2,0))\wedge(\wedge^n(\omega_X(0,2))=$$
$$d(j^*\perp(\wedge^n(\omega_X(2,0))\wedge(\wedge^n(\omega_X(0,2))))=dj$$

where α is a real two form of type (1,1) with distribution coefficients and j is also a real one form. We can write $j=\beta+\overline{\beta}$ where β is a (1,0)-form on X. Since α is of type (1,1) it follows that

$$\alpha=\overline{\partial}\beta+\partial\overline{\beta} \text{ and } \overline{\partial}\,\overline{\beta}=0$$

So from $\overline{\partial}\,\overline{\beta}=0$ it follows that

$$\overline{\beta}\in H^1(X,\mathcal{O}_X)$$

Proposition 1. If X is a holomorphic symplectic manifold, then

$$H^1(X,\mathcal{O}_X)=0$$

if $\dim_C X\geq 4$.

Proof: Case 1.

Suppose that $\dim_C H^1(X,\mathcal{O}_X)=1$.

Let $H^1(X,\mathcal{O}_X)=C\alpha$, where α is a form of type (0,1) and $\overline{\partial}\alpha=0$. Consider the map:

$$\psi:\ \alpha\rightarrow\alpha\wedge(\wedge^{n-1}\overline{\omega_X(2,0)}\wedge(\wedge^n\omega_X(2,0))$$

Since $\omega_X(2,0)$ is a non degenerate holomorphic two form it follows that ψ gives an isomorphism between

$$H^1(X,\mathcal{O}_X) \text{ and } H^{2n-1}(X,\Omega^{2n})$$

From Serre's duality we know that the pairing $H^1(X,\mathcal{O}_X) \times H^{2n-1}(X,\Omega^{2n}) \to \mathbb{C}$, given by

$$(\alpha,\beta) \to \int_X \alpha\wedge\beta$$

is non-degenerate. On the other hand α generates $H^1(X,\mathcal{O}_X)$ and

$$H^{2n-1}(X,\Omega^{2n}) \doteq \mathbb{C}\alpha\wedge(\wedge^{n-1}\overline{\omega_X(2,0)}\wedge(\wedge^n\omega_X(2,0)))$$

Since

$$\alpha\wedge\alpha\wedge(\wedge^{n-1}\overline{\omega_X(2,0)}\wedge(\wedge^n\omega_X(2,0)))=0$$

we get a contradiction with Serre's duality. Hence if X is a symplectic holomorphic manifold we have two possibilities in case of $\dim_{\mathbb{C}} X \geq 4$; either $H^1(X,\mathcal{O}_X)=0$ or $\dim_{\mathbb{C}} H^1(X,\mathcal{O}_X) \geq 2$.

Case 2.

$$\dim_{\mathbb{C}} H^1(X,\mathcal{O}_X) \geq 2.$$

Sublemma. Suppose X is a holomorphic symplectic manifold and $\dim_{\mathbb{C}} X \geq 4$. Let

$$\alpha,\ \beta \in H^1(X,\mathcal{O}_X)$$

then

$$\alpha\wedge\beta=\overline{\partial}\mu$$

where μ is a (0,1) form.

Proof: Clearly

$$\alpha\wedge\beta \in H^2(X,\mathcal{O}_X)$$

Since

$$\dim_{\mathbb{C}} H^2(X,\mathcal{O}_X)=1 \text{ and } H^2(X,\mathcal{O}_X)=\mathbb{C}\overline{\omega_X(2,0)}$$

it follows that

$$\alpha\wedge\beta=c\overline{\omega_X(2,0)}+\overline{\partial}\mu$$

It is necessary to prove that c=0. Clearly we have

$$\alpha\wedge\beta\wedge\alpha\wedge\beta=0=c^2\wedge^2\overline{\omega_X(2,0)}+c\overline{\partial}\mu\wedge\overline{\omega_X(2,0)}+\overline{\partial}\mu\wedge\overline{\partial}\mu$$

From $\dim_{\mathbb{C}} X \geq 4$ it follows that

$$\int_X \alpha\wedge\beta\wedge\alpha\wedge\beta\wedge(\wedge^{n-2}\overline{\omega_X(2,0)}\wedge(\wedge^n\omega_X(2,0))=$$

$$c^2\int_X (\wedge^n\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))+c\int_X \overline{\partial}\mu\wedge(\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))+$$

$$\int_X \overline{\partial}\mu\wedge\overline{\partial}\mu\wedge(\wedge^{n-2}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))=0.$$

From

$$\overline{\partial}\mu\wedge(\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0)=d(\mu\wedge(\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))$$

$$\overline{\partial}(\mu\wedge\overline{\partial}\mu\wedge(\wedge^{n-2}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))=$$

$$d(\mu\wedge\overline{\partial}\mu\wedge(\wedge^{n-2}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))$$

and Stokes' Theorem we get that

$$\int_X \overline{\partial}\mu\wedge(\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))=0$$

$$\int_X \overline{\partial}\mu\wedge\overline{\partial}\mu\wedge(\wedge^{n-2}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))=0$$

Hence $$c^2\int_X (\wedge^n\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))=0$$

Since $\omega_X(2,0)$ is a non-degenerate form it follows that

$$\int_X (\wedge^n\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))>0$$

Thus

$$c^2\int_X (\wedge^n\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))=0\Rightarrow c=0$$

Q.E.D.

Recall that every element of $H^{2n-1}(X,\Omega^{2n})$ can be expressed as

$$\beta\wedge((\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))$$

where $\beta\in H^1(X,\mathcal{O}_X)$.

By Serre's duality the pairing

$$(\alpha,\beta\wedge((\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))=\int_X \alpha\wedge\beta\wedge((\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))$$

is a non-degenerate bilinear map. Since $\alpha\wedge\beta=\overline{\partial}\mu$ it follows that

$$\alpha\wedge\beta\wedge((\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))=$$
$$d(\mu\wedge((\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))$$

Stokes' Theorem yields

$$\int_X \alpha\wedge\beta\wedge((\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))=$$
$$\int_X d(\mu\wedge((\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))=O$$

which contradicts Serre's duality. This proves that $H^1(X,\mathcal{O}_X)=0$.

Q.E.D.

Proposition 2. Suppose that η is a positive (1,1) current and $\eta=dj^*$, then $\eta=0$.

Proof:

Let $\eta=dj^*$, then we have

$$\alpha=\eta\perp(\wedge^n(\omega_X(2,0))\wedge(\wedge^n(\omega_X(0,2))=$$
$$d(j^*\perp(\wedge^n(\omega_X(2,0))\wedge(\wedge^n(\omega_X(0,2))))=dj$$

where α is a real two form of type (1,1) with distribution coefficients and j is also a real one form. We can write $j=\beta+\overline{\beta}$ where β is a (1,0)-form on X. Since α is of type (1,1) it follows that

$$\alpha=\overline{\partial}\beta+\partial\overline{\beta} \text{ and } \overline{\partial}\overline{\beta}=0$$

Hence $\overline{\partial}\overline{\beta}=0$ yields

$$\overline{\beta}\in H^1(X,\mathcal{O}_X)=0\Rightarrow\overline{\beta}=\overline{\partial}\sigma$$

where σ is a (0,0) current on X. Hence

$$\alpha=\sqrt{-1}\partial\overline{\partial}\tau,\text{ where } \tau=\sqrt{-1}(\overline{\sigma}-\sigma)$$

The positivity of the (1,1) current on X implies that τ is a plurisubharmonic Hence $\alpha=\partial\overline{\partial}\text{const}=0$ and $\eta=0$.

Q.E.D.

Proposition 3.

Let η be a positive closed (1,1) current and $\eta=(d\alpha)(1,1)$(i.e. η is a (1,1) component of a boundary), then $\eta\equiv0$.

Proof: The existence of the closed holomorphic two form $\omega_X(2,0)$ which is a non-degenerate form on X shows η can be considered as a form of type (1,1) on X. Since $d\eta=0$ and $\eta=\overline{\partial}\alpha^{1,0}+\partial\alpha^{0,1}\Rightarrow\partial\overline{\partial}\alpha^{1,0}=-\overline{\partial}\partial\alpha^{1,0}=0$ and the regularity of the $\overline{\partial}$ operator implies that $\partial\alpha^{1,0}$ is a holomorphic form on X. It is easy to see that $\partial\alpha^{1,0}=0$, indeed suppose that

$\partial\alpha^{1,0}\neq 0$ *then*

$$\int_X \partial\alpha^{1,0}\wedge\overline{\partial\alpha^{1,0}}\wedge(\wedge^{n-1}(\omega_X(2,0))\wedge(\wedge^{n-1}(\omega_X(0,2))>0$$

On the other hand we have

$$d(\alpha^{1,0}\wedge\overline{\partial\alpha^{1,0}}\wedge(\wedge^{n-1}(\omega_X(2,0))\wedge(\wedge^{n-1}(\omega_X(0,2))=$$

$$\partial\alpha^{1,0}\wedge\overline{\partial\alpha^{1,0}}\wedge(\wedge^{n-1}(\omega_X(2,0))\wedge(\wedge^{n-1}(\omega_X(0,2))$$

From Stokes' Theorem it follows that

$$0<\int_X \partial\alpha^{1,0}\wedge\overline{\partial\alpha^{1,0}}\wedge(\wedge^{n-1}(\omega_X(2,0))\wedge(\wedge^{n-1}(\omega_X(0,2))=$$

$$\int_X d(\alpha^{1,0}\wedge\overline{\partial\alpha^{1,0}}\wedge(\wedge^{n-1}(\omega_X(2,0))\wedge(\wedge^{n-1}(\omega_X(0,2))=0$$

This contradicts $\partial\alpha\neq 0$. Therefore $\partial\alpha=0$.

Q.E.D.

Since $\partial\alpha^{1,0}=0$, hence $\eta=d\alpha$. From Proposition 3 it follows that $\eta\equiv 0$. So the conditions of The THEOREM of Harvey and Lawson are fulfilled for holomorphic symplectic manifolds.

Q.E.D.

#2. HODGE THEORY OF WEGHT TWO.

LEMMA 2.1.

Let $[\phi]$ be a non-zero element of $H^2(X,\mathbf{R})$, then $[\phi]=c\omega_X(2,0)+\phi^{1,1}+\overline{c\omega_X(2,0)}$, where ϕ is a real closed form of type (1,1).

PROOF: From de Rham's Theorem it follwos that $[\phi]$ can be realized as a real closed two form ϕ. (See [09].) Let

(2.1.1.) $$\phi=\phi^{2,0}+\phi^{1,1}+\overline{\phi^{2,0}}$$

From $d\phi=0$ we get:

(2.1.2.) $$\partial\phi^{2,0}=\overline{\partial\phi^{2,0}}=0\ \overline{\partial}\,\phi^{2,0}+\partial\phi^{1,1}=\partial\phi^{0,2}+\overline{\partial}\,\phi^{1,1}=0$$

From (2.1.2) it follows that $\overline{\phi^{2,0}}=\phi^{0,2}\in H^2(X,\mathcal{O}_X)$. From the condition $H^2(X,\mathcal{O}_X)\approx\mathbf{C}\omega_X(2,0)$ yields

(2.1.3.) $$\phi^{2,0}=c\omega_X(2,0)+\partial\alpha^{1,0}$$

Hence $\overline{\partial}\,\phi^{2,0}+\partial\phi^{1,1}=0$ gives

(2.1.4.) $$\overline{\partial}\,\partial\alpha^{1,0}+\partial\phi^{1,1}=0\ \&\ \partial\overline{\partial}\,\alpha^{1,0}=\partial\phi^{1,1}$$

Combining (2.1.1.) and (2.1.3.) yields

(2.1.5.) $$\phi=c\omega_X(2,0)+\partial\alpha^{1,0}+\omega^{1,1}+\overline{\partial}\ \overline{\alpha^{1,0}}+\overline{c\omega_X(2,0)}$$

Now (2.1.4.) and (2.1.5.) imply

(2.1.6.) $$\phi-d(\alpha^{1,0}+\alpha^{0,1})=c\omega_X(2,0)-\overline{\partial}\,\alpha^{1,0}-\partial\overline{\alpha^{1,0}}+\phi^{1,1}+\overline{c\omega_X(2,0)}=$$

$$c\omega_X(2,0)+\omega^{1,1}+\overline{c\omega_X(2,0)}$$

where

$$\omega^{1,1}=\phi^{1,1}-\overline{\partial}\alpha^{1,0}-\partial\overline{\alpha^{1,0}}$$

and hence $d(\omega^{1,1})=0$. See 2.1.4. Lemma 2.1.1. follows immediately from 2.1.6..

Q.E.D.

Cor.2.1.1. If $[\omega]\in H^2(X,\mathbf{R})\otimes C$, then $[\omega]=a\omega_X(2,0)+\omega^{1,1}+b\overline{\omega_X(2,0)}$, where $d\omega^{1,1}=0$ and $a\&b\in C$.

Next we show that $H^1_C(\Omega^1)\neq 0$, where

$$H^1_C(\Omega^1):=\left\{[\omega]\in H^2(X,C)\,|\,0\neq[\omega] \text{ contains a form of type } (1,1)\right\}$$

Lemma 2.2.

Let ω be the closed 2-form constructed in THEOREM 1, then

a) ω defines a non-zero class in $H^2(X,\mathbf{R})$.

b) there exists a real closed (1,1) form Θ such that $[\Theta]=[\omega]$ in $H^2(X,\mathbf{R})$.

Proof:

Condition *a)* follows directly from the following proposition and Stokes' Theorem.

Proposition 2.2.1.

$$\int_X \omega^{2n}>0, \text{ where } \omega^{2n}=\wedge^{2n}(\omega) \text{ and } 2n=\dim_C X=2n.$$

Proof:

We need to compute $\omega\wedge\ldots\wedge\omega=(\omega^{2,0}+\omega^{1,1}+\omega^{0,2})\wedge\ldots\wedge(\omega^{2,0}+\omega^{1,1}+\omega^{0,2})=$

$\sum_{k=0}^{2n} c_k(\omega^{2,0})^k\wedge(\omega^{0,2})^k\wedge(\omega^{1,1})^{2n-k}$ and c_k is a positive number.

From the following Lemma:

LEMMA. If η is a primitive form of type (p,q), then

$$*\eta=\frac{(\sqrt{-1}\,)^{p-q}}{(2n-p-q)}(-1)^{\frac{(p+q)(p+q+1)}{2}}L^{2n-p-q}$$

where $*$ is the Hodge star operator and $L=\mathrm{Im}(g_{\alpha,\overline{\beta}})$ and $(g_{\alpha,\overline{\beta}})$ is a Hermitian metric on X.

(For a proof see [05].)

This Lemma yields

$$*(\omega^{2,0})^k=(\omega^{1,1})^{2n-2k}\wedge(\omega^{2,0})^{2k}$$

where $*$ is the **Hodge operator** with respect to the metric defined by $\omega^{1,1}$ on X. Applying the formula above we get:

$$\int_X \omega^{2n}=\sum_{k=1}^{n} c_k\|(\omega^{2,0})^k\|^2+\mathrm{vol}(X),\text{ where } c_k>0$$

where the norm is taken with respect to the Harvey-Lawson metric $\omega^{1,1}$.

Q.E.D.

Proof of a:

Suppose that $\omega=d\eta$, then by Stokes' Theorem gives

$$0=\int_X d(\eta\wedge(\omega)^{2n-1})=\int_X \omega^{2n}>0$$

Therefore we get a contrudiction that proves part *a)* of the Lemma.

Q.E.D.

Proof of b:

Let $\Theta=\omega-\bar{\partial}\alpha^{1,0}-\partial\alpha^{0,1}$. Recall that $\omega=\partial\alpha^{1,0}+\omega^{1,1}+\bar{\partial}\alpha^{0,1}$ and $d\omega=0$. Hence

$$d\Theta=\partial\omega^{1,1}-\partial\bar{\partial}\alpha^{1,0}+\bar{\partial}\omega^{1,1}-\bar{\partial}\partial\alpha^{0,1}=0$$

and

$$\Theta-\omega=\omega^{1,1}-\bar{\partial}\alpha^{1,0}-\partial\alpha^{1,0}-\omega^{1,1}-\bar{\partial}\alpha^{0,1}-\partial\alpha^{0,1}=-d(\alpha^{1,0}+\alpha^{0,1})$$

Q.E.D.

Cor.2.2.2. $\dim_{\mathbf{C}}H^1_{\mathbf{C}}(\Omega^1))>0$, where $H^1_{\mathbf{C}}(\Omega^1)\subset H^2(X,\mathbf{C})$.

Cor.2.2.3. Let ω be the form constructed in THEOREM 1. let Θ be the closed (1,1) form that represents the non-zero class $[\omega]\in H^2(X,\mathbb{R})$ and $C\subset X$ be an irreducible complex subspace in X. Then if $\dim_{\mathbf{C}}C=r$ we have

$$\int_C \Theta^k>0 \text{ and C is a non-zero element in } H_{2k}(X,\mathbf{Z})$$

Proof; From chapter 1 [09] we know that

$$\int_C \Theta^k = \int_{C\text{-sing}C} \Theta^k$$

Hence C can be taken as a non-singular submanifold in X. Repeating the calculations and arguments in (2.2.1.) we get that

$$\omega^k{}_{|C} = (\omega^{2,0})^k{}_{|C} + (\omega^{0,2})^k{}_{|C} + \sum_{i=1}^{k-1} c_i (\omega^{2,0})^i \wedge * \overline{(\omega^{2,0})^k} + \mathrm{vol}(C)$$

Since $(\omega^{2,0})^k{}_{|C}=0$ and $(\omega^{0,2})^k{}_{|C}=0$ we get that

$$\int_C \phi^k = \sum_{i=1}^{k} \|(\omega^{2,0})^i\|^2 + \mathrm{vol}(C) > 0$$

Since $\phi=\Theta+d(\alpha^{1,0}+\alpha^{0,1})$ we get that

$$\int_C \phi^k = \int_C \Theta^k$$

Stokes' theorem yields for $C=\partial B$

$$\int_C \phi^k = \int_B d(\phi^k) = 0$$

Which contradicts $\int_C \phi^k > 0$

Q.E.D.

Remark 2.2.4.

From (2.2.3.) it follows that the cohomology class of $\Theta \in H^{1,1}(X,\mathbf{R})$ behaves like the imaginary part of a Kähler metric.

#3. LOCAL DEFORMATION THEORY OF HOLOMORPHIC SYMPLECTIC MANIFOLDS

First we will make some remarks.

Remark 1. The closed holomorphic non-degenerate two form $\omega_X(2,0)$ induces an isomorphism:

$$i_{\omega_X(2,0)}:\Theta_X \rightarrow \Omega^1$$

where $i_{\omega_X(2,0)}(\alpha)=\alpha \perp \omega_X(2,0)$.

Remark 2.

We know from Kodaira-Spencer-Kuranishi theory that "small" deformations of the complex structure on X are determined by:

$$\phi_t=\sum \phi^i_{\bar{j}}(t)d\bar{z}^j \otimes \frac{\partial}{\partial z^i} \in \Gamma(X,\Theta\otimes\Omega^{0,1}).$$

Using the isomorphism $i_{\omega_X(2,0)}$ we get that:

$$i_{\omega_X(2,0)}\phi_t=\tilde{\phi}_t \in \Gamma(X,\Omega^{1,0}\otimes\Omega^{0,1})$$

Remark 3.

Let $\phi=\sum \phi^i_{\bar{j}}d\bar{z}^j \otimes \frac{\partial}{\partial z^i}$ and $\psi=\sum \psi^i_{\bar{j}}d\bar{z}^j \otimes \frac{\partial}{\partial z^i}$ are elements of $\Gamma(X,\Theta\otimes\Omega^{0,1})$, then we can define $[\phi,\psi] \in \Gamma(X,\Theta\otimes\Omega^{0,2})$, where

$$[\phi,\psi]_{|U}=\sum\Big(\sum_i(\phi^i\partial_i\psi^j-\psi^i\partial_i\phi^j)\Big)\otimes\frac{\partial}{\partial z^j}$$

$$\phi^i=\sum_j \phi^i_{\bar{j}}d\bar{z}^j \text{ and } \psi^i=\sum_j \psi^i_{\bar{j}}d\bar{z}^j.$$

The operator $i_{\omega_X(2,0)}$ transforms the braket operation [,] into a braket operation [,] on $\Gamma(X,\Omega^{1,0}\otimes\Omega^{0,1})$, i.e. we have:

$$[\ ,\ ]: \Gamma(X,\Omega^{1,0}\otimes\Omega^{0,1})\times\Gamma(X,\Omega^{1,0}\otimes\Omega^{0,1})\rightarrow\Gamma(X,\Omega^{1,0}\otimes\Omega^{0,2})$$

Remark 4.

Suppose that ω_1 and $\omega_2 \in \Gamma(X,\Omega^{1,0}\otimes\Omega^{0,1})$ and either $\partial\omega_1=\partial\omega_2=0$ or $d\omega_1=d\omega_2=0$, then it is to easy to see that $\partial[\omega_1,\omega_2]=0$ or $d[\omega_1,\omega_2]=0$. (See [04].)

Remark 5.

We know from Kodaira-Spencer-Kuranishi deformation theory that first order deformations of a complex structure X are contained in $H^1(X,\Theta)$ and if X is a symplectic holomorphic manifold we know that $i_{\omega_X(2,0)}:H^1(X,\Theta_X)\xrightarrow{\sim}H^1(X,\Omega^1)$.

Definition. Let $H^1_d(X,\Omega^1)=\{[\omega]_{Dol}\in H^1(X,\Omega^1)|[\omega]_{Dol}$ contains a closed representative, where $[w]_{Dol}$ denotes the Dolbault class}.

Let $H^1_d(X,\Theta)= i_{\omega_X(2,0)}{}^{-1}(H^1_d(X,\Omega^1))$.

Remark 6.

We should mention that $H^1_d(X,\Omega^1)$ in the case when X is a holomorphic symplectic manifold can be realized as a subspace of $H^2(X,C)$ however more it is not at all dificult to see that we can identify $H^1_d(X,\Omega^1)$ with $H^1_C(\Omega^1)$. From Lemma 2.1. it follows that that $\dim_C H^1_d(X,\Omega^1)=\dim_C H^1_d(X,\Theta)=b_2-2$, where $b_2=\dim_C H^2(X,C)$.

THEOREM 3.1.(Bogomolov)

There are no obstructions for one parameter deformations of complex structures on X that correspond to the elements of $H^1_d(X,\Theta)\doteq H^1_d(X,\Omega^1)$.

Proof of 3.1.:

From Kodaira-Spencer-Kuranishi THEORY it follows that if for each $\phi_1\in H^1_d(X,\Theta)$ we can find a power series:

$$\phi=\phi_1 t+\phi_2 t^2+..+\phi_n t^n+..$$

such that

(3.1.1.) $$\overline{\partial}\phi(t)=\tfrac{1}{2}[\phi(t),\phi(t)]$$

then our LEMMA will be proved. This is so because KURANISHI proved that if (3.1.1.) is fulfilled, then we can find a convergent power series

$$\tilde{\phi}=\tilde{\phi}_1 t+\tilde{\phi}_2 t^2+..+\tilde{\phi}_n t^n+..$$

such that

a) $[\tilde{\phi}_1]=[\phi]\in H^1_d(X,\Theta)$

b) $\tilde{\phi}_1$ is a harmonic representative of the class $[\phi]$ with respect to some Hermitian metric on X.

c) $\overline{\partial}\tilde{\phi}(t)=\tfrac{1}{2}[\tilde{\phi}(t),\tilde{\phi}(t)]$.(See [11].)

PROPOSITION 3.1.2.

If ω_1 and $\omega_2\in H^1_d(X,\Omega^1)$, then $[\omega_1,\omega_2]$ is the zero cohomology class in $H^2_d(X,\Omega^1)$, where $H^2_d(X,\Omega^1)=\{[\omega]\in H^3(X,C)|\ [\omega]\neq 0$ and $[\omega]$ can be represented by de Rham theorem by a form of type (2,1)}

Proof:

Let ω_1 and $\omega_2\in H^1_d(X,\Omega^1)$. From the Definiton of $H^1_d(X,\Omega^1)$ it follows that $d\omega_1=d\omega_2=0$ and remark 4 yields $d[\omega_1,\omega_2]=0$. From here we get that

(3.1.2.1.) $$[\omega_1,\omega_2]\in H^2_d(X,\Omega^1)$$

If we prove that for any three dimensional cycle $\gamma \in H_3(X,\mathbf{Z})$

$$\int_\gamma [\omega_1,\omega_2]=0$$

then we will have

$$[\omega_1,\omega_2]=0 \text{ in } H^2_d(X,\Omega^1)\subset H^3(X,\mathbf{C})$$

<u>REMARK.</u> We will prove that if $[\gamma]\in H^3(X, \mathbf{Z})$, then $[\gamma]$ can be realized as a three dimensional oriented manifold $\gamma\subset X$ such that $H_1(\gamma,\mathbf{R})=H_2(\gamma,\mathbf{R})=H_1(X,\mathbf{R})=0$. (See [04].)

<u>Proof of the remark:</u>

From a THEOREM of R. THOM (See [21] THEOREM II.27) it follows that we can realize each cycle $[\gamma]\in H_3(X,\mathbf{Z})$ by a three dimensional real manifolds $\gamma'\subset X$. This follows from the fact that

$$\dim_{\mathbf{R}}[\gamma]=3<\tfrac{1}{2}\dim_{\mathbf{R}}X\geq 8$$

We will prove that after some surgary we can assume that $H_1(\gamma',\mathbf{R})=0$ since $H_1(X,\mathbf{R})=0$. Indeed let $\beta\in\ker(i_*)$, where

$$0\rightarrow\ker(i_*)\rightarrow\pi_1(\gamma')\rightarrow\pi_1(X).$$

Then β can be realized as S^1xD^2 in γ'. (See Prop. IV.1.4. in the book [22])). Let us do now surgery, i.e. replace S^1xD^2 by D^2xS^1 in γ'. We will obtain a new manifold γ'' imbedded in X. If $0\neq[\beta]\in H_1(\gamma',\mathbf{R})$ then $\dim_{\mathbf{R}}H_1(\gamma'',\mathbf{R})<\dim_{\mathbf{R}}(\gamma',\mathbf{R})$. acording to Proposition IV.2.5. in [22] Now if we continue this process we will get that the three dimensional cycle $[\gamma]$ can be realized as an imbedded three dimensional manifold $\gamma\subset X$ such that $H_1(\gamma,\mathbf{R})=H_1(X,\mathbb{R})=0$ and $H_2(\gamma,\mathbf{R})=0$ by Poincare duality. Q.E.D.

Bogomolov proved the following fact:

<u>LEMMA 3.1.2.2.(See [04].)</u>

For each cycle $[\gamma_i]\in H_3(X,\mathbf{R})$ we can find a nonsingular three dimensional compact manifold γ_i, realizing $[\gamma_i]$ and γ_i fulfills the following conditions:

a) $\gamma_i\cap\gamma_j=\emptyset$ *b)* $H_1(\gamma_i,\mathbf{Z})=H_2(\gamma_i,\mathbf{Z})$ and *c)* For each γ_i there exists a small neighborhood $U(\gamma_i)$, where $U(\gamma_i)$ is a STEIN manifold and $H^2(U(\gamma_i),\mathbf{R})=0$.

Let $U(\gamma_i)$ be a small Stein manifold of γ_i constructed by (3.1.2.2.). Let ω_1 and $\omega_2\in H^1_d(X,\Omega^1)$, then since $d\omega_1=d\omega_2=0$ it follows that $\omega_1|_{U(\gamma_i)}$ and $\omega_2|_{U(\gamma_i)}$ are zero elements in $H^2(U(\gamma_i),\mathbf{R})$, since $H^2(U(\gamma_i),\mathbf{R})=0$. We need the following sublemma, which is proved in [04]:

Sublemma 3.1.2.3.

Let U be a STEIN manifold. Let ω be a (p,q) form (p,q>1) such that $d\omega=0$ and $[\omega]=0$ in $H^{p+q}(U)$. *Then* $\omega=\partial\overline{\partial}\phi$ for some form ϕ.

From (3.1.2.3.) it follows that $\omega_1|_{U(\gamma_i)}=\partial\overline{\partial}\phi_i^1$ and $\omega_2|_{U(\gamma_i)}=\partial\overline{\partial}\phi_i^2$. We can continue ϕ_i^2 and ϕ_i^1 as C^∞ forms to $\tilde{\phi}_i^2$ and $\tilde{\phi}_i^1$ on X. Let

$$\omega_1^1=\omega_1-\partial\overline{\partial}\tilde{\phi}_i^1 \text{ and } \omega_1^2=\omega_1-\partial\overline{\partial}\tilde{\phi}_i^2$$

then clearly we have

$$\textit{a)}\ \omega_1^1|_{U(\gamma_i)}=\omega_2^1|_{U(\gamma_i)}\equiv 0\ \textit{b)}\ [\omega_1^1]=[\omega_1] \text{ and } [\omega_2^1]=[\omega_1] \text{ and } \textit{c)}\ [\omega_1^1,\omega_2^1]\equiv 0 \text{ on } U(\gamma_i)$$

From *c)* we get that $\int_\gamma [\omega_1^1,\omega_2^1]=0$ and this proves 3.2.1..

Q.E.D.

Proposition 3.1.1.

Let X be a symplectic holomorphic manifold. Let U be a STEIN submanifold in X and let ω be a d-closed from of type (1,2), $[\omega]=0$ in $H^3(X,C)$ with $\omega|_U\equiv 0$. *Then* there exists a form ϕ such that ***a)*** $\partial\phi=0$ ***b)*** $\overline{\partial}\phi=\omega$ ***c)*** $\phi|_U\equiv 0$

Proof:

Since ω is such that $[\omega]\equiv 0$ in $H^3(X,C)$ and ω is of type (1,2) we get that

$$\omega=d\alpha^{0,2}+d\beta^{1,1}, \text{ where } \overline{\partial}\alpha^{0,2}=\partial\beta^{1,1}=0$$

So we have $\alpha^{0,2}\in H^2(X,\mathcal{O}_X)\doteq \mathbf{C}\omega_X(0,2)$. If $\alpha^{0,2}\neq 0$ in $H^2(X,\mathcal{O}_X)$, then $\omega_X(0,2)=\alpha^{0,2}+\overline{\partial}\mu^{0,1}$

Since $d\omega_X(0,2)=0$ we get

$$\partial\alpha^{0,2}=\overline{\partial}\partial\mu^{0,1}$$

and therefore

$$\omega=d\alpha^{0,2}+d\beta^{1,1}=\overline{\partial}\partial\mu^{0,1}-\overline{\partial}\beta^{1,1}=\overline{\partial}(\partial\mu^{0,1}-\beta^{1,1})$$

Let $\phi=\partial\mu^{0,1}-\beta^{1,1}$. Then $\partial\phi=\partial\partial\mu^{0,1}-\partial\beta^{1,1}=0$ and hence $\overline{\partial}\phi=\omega$. We have proved *a)* and *b)*. Condition *c)* follows immediately from the fact that $\omega|_U\equiv 0$, therefore $\phi|_U\equiv 0$.

If $\alpha^{0,2}$ is zero in $H^2(X,\mathcal{O}_X)$, then $\alpha^{0,2}=\overline{\partial}\mu^{0,1}$. Hence we get that

$$\omega=\partial\alpha^{0,2}+\bar\partial\beta^{1,1}=\partial\bar\partial\mu^{0,1}+\bar\partial\beta^{1,1}=\bar\partial(\beta^{1,1}-\partial\beta^{0,1})$$

Let $\phi=\beta^{1,1}-\partial\beta^{0,1}$. Clearly $\partial\phi=\partial\beta^{1,1}-\partial\partial\mu^{0,1}=0$ and $\bar\partial\phi=\omega$. Therefore condition *a),b)* and *c)* are fulfilled.

Q.E.D.

The end of the proof of THEOREM 3.1.

Suppose that $\omega_1\in H^1_d(X,\Omega^1)$ and $\omega_1|_{U(\gamma_i)}=0$. We have proved that $[\omega_1,\omega_1]=0$ in $H^2_d(X,\Omega^1)\subset H^3(X,\mathbf{C})$, i.e.

(3.1.4.) $\quad [\omega_1,\omega_1]=\bar\partial\omega_2$, where $\partial\omega_2=0$ and $\omega_2|_{U(\gamma_i)}=0$

This follows directly from (3.1.2.) and (3.1.3.). From (3.1.4.), (3.1.2.) and (3.1.3.) it follows that $[\omega_1,\omega_2]=\bar\partial\omega_3$, where $\partial\omega_3=0$ and $\omega_3|_{U(\gamma_i)}\equiv 0$. Since

$$\partial\omega_1=\partial\omega_2=0\Rightarrow\partial[\omega_1,\omega_2]=0.$$

On the other hand we have automatically that $\bar\partial[\omega_1,\omega_2]=0$. This is the Jacobi identity.

By induction we can form the power series $\omega(t)=\omega_1 t+\omega_2 t^2+\ldots+\omega_n t^n+..$ such that 1) $\partial\omega(t)=0$, 2) $\bar\partial\omega(t)=\frac{1}{2}[\omega(t),\omega(t)]$ and 3) $\omega_i|_{U(\gamma_i)}\equiv 0$. Notice that condition 2) is equivelent to

(*) $$\bar\partial\omega_n=\frac{1}{2}\sum_{i=1}^{n-1}[\omega_i,\omega_{n-i}] \text{ and } \partial\omega_n=0$$

Using (3.1.2.), (3.1.3.),

$$\bar\partial\Big(\frac{1}{2}\sum_{i=1}^{n-1}[\omega_i,\omega_{n-i}]\Big)=0 \textit{(Jacobi identity)}$$

and the induction hypothesis (*) can be solved step by step. Hence all obstructions vanish. THEOREM 3.1. is proved. Q.E.D.

Cor. 3.1.A.

From KURANISHI existence THEOREM we can conclude that there exixts a semi-universal family of complex analytic manifolds $\pi:\mathfrak{X}\to U$, where

1) U is a non-singular manifold with $\dim_{\mathbf{C}} U=\dim_{\mathbf{C}} H^1_d(X,\Omega^1)$.

2) The tangent space $T_{o,U}=H^1_d(X,\Omega^1)$.

3) $\begin{array}{ccc} X & \to & \mathfrak{X} \\ \downarrow & & \downarrow \\ o & \in & U \end{array}$

REMARK 1.

We will denote the KURANISHI family of hyper-Kählerian manifolds by $\pi:\mathfrak{X}\to U$.

REMARK 2.

We may suppose that U is a Stein manifold. For each $p\in\mathbf{Z}$ and each coherent sheaf $\mathcal{F}$ on $\mathfrak{X}$ Grauert's "direct image theorem" yields $H^p(\mathfrak{X},\mathcal{F}) \approx H^o(U,R^p\pi_*\mathcal{F})$. Hence $H^p(\mathfrak{X},\mathcal{F})$ is a finitely generated $\Gamma(U,\mathcal{O}_U)$ module. See [08].

THEOREM 3.2.

Every fibre $X_t=\pi^{-1}(t)$ is a holomorphic symplectic manifold in the KURANISHI family $\pi:\mathfrak{X}\to U$ defined in Cor. 3.1.A..

Proof:

Let $D=\{t\in\mathbf{C}|\ |t|<1\}$ be any disk containing $o\in U$. *Theorem 3.2.* will follow if it can be proved for the restriction of $\pi:\mathfrak{X}\to U$ to the family $\pi:\mathfrak{X}_D\to D$

Denote by $\pi:\mathfrak{X}_D\to D$ by $\pi:\mathfrak{X}\to D$. From now on we will consider the family $\pi:\mathfrak{X}\to D$, where $\pi^{-1}(o)=X_o$ is a holomorphic symplectic manifold

The following notation will be used:

DEFINITION 3.2.1. Denote by $\Omega^1_{\mathfrak{X}/D}:=\Omega^1_{\mathfrak{X}}/\pi^*\Omega^1_D$, then by definition $\wedge^k\Omega^1_{\mathfrak{X}/D}:=\Omega^k_{\mathfrak{X}/D}$

DEFINITION 3.2.2.

Let $\omega^k\in\Gamma(\mathfrak{X},\Omega^k_{\mathfrak{X}/D})$. Define $d_{/D}\omega^k\in\Gamma(\mathfrak{X},\Omega^{k+1}_{\mathfrak{X}/D})$ in the following way:

Let $\{\mathcal{U}_i\}$ be a covering of $\mathfrak{X}$, where $\mathcal{U}=U\text{x}D$ and U be an open subset in X_o. Let $(z^1,\dots,z^{2n},t)$ be local coordinates in $\mathcal{U}$, then

$$\omega^k{}_{|\mathcal{U}}=\sum_{i_1<i_2<\dots<i_k}\omega_{i_1,i_2,\dots,i_k}dz^{i_1}\wedge dz^{i_2}\wedge\dots\wedge dz^{i_k}$$

where $\omega_{i_1,\dots,i_k}$ is a complex analytic function of $(z^1,\dots,z^{2n},t)$, then define:

$$d_{/D}\omega^k{}_{|\mathcal{U}}:=\sum_{m=1}^{2n}\frac{d\omega_{i_1,i_2,\dots,i_k}}{dz^m}\,dz^m\wedge dz^{i_1}\wedge\dots\wedge dz^{i_k}$$

For the proof of THEOREM 3.2. we need to prove and recall some auxilaury results:

LEMMA 3.2.3. $\Gamma(D,\mathcal{O}_D)$ is a ring of principal ideals.

PROOF: $\Gamma(D,\mathcal{O}_D)$ is a subring of $\mathbf{C}[[t]]$. It is a well known fact that $\mathbf{C}[[t]]$ is a ring of principal ideals.(See [23].) This implies the lemma.

Q.E.D.

REMARK. We will use later the following THEOREM (See [23].): Let F be a finitely genetated module over $\Gamma(D,\mathcal{O}_D)$, then F is isomorphic to a direct sum of a free module plus a torsion module, i.e. module isomorphic to $\bigoplus_i \mathbb{C}[\![t]\!]/(t^{n_i})$

LEMMA 3.2.4.

$H^p(\mathfrak{X},\pi^*(\mathcal{O}_D))$ is a torsion free finetely generated $\Gamma(D,\mathcal{O}_D)$ module for p=2 and 3.

PROOF: The standart Leray spectral seqence yields

(3.2.4.1.) $$H^p(\mathfrak{X}, \pi^*(\mathcal{O}_D))\approx H^o(\mathfrak{X},R^p\pi_*\pi^*(\mathcal{O}_D))$$

Proving $R^p\pi_*\pi^*(\mathcal{O}_D)$ is a locally free sheaf together with (3.2.4.1.) will give

(3.2.4.2.) $H^p(\mathfrak{X},\pi^*(\mathcal{O}_D))\approx H^o(\mathfrak{X},R^p\pi_*\pi^*(\mathcal{O}_D))$ is a free $\Gamma(D,\mathcal{O}_D)$ module.

In order to prove (3.2.4.2.) we need to prove that $R^p\pi_*\pi^*(\mathcal{O}_D)$ is a free $\Gamma(D,\mathcal{O}_D)$ module. From the Cor. 2. p.50-51 proved in [14] it will be enouph to show that the $\dim_{\mathbb{C}} H^p(X_t,\pi^*(\mathcal{O}_D)|_{X_t})$ does not depends on t, i.e. it is constant. This is so since $\pi^*(\mathcal{O}_D)|_{X_t}$ is just the constant sheaf $\mathbb{C}$ on X_t and hence

(3.2.4.3.) $$H^p(X_t,\pi^*(\mathcal{O}_D)|_{X_t})\approx H^o(X_t,\mathbb{C})$$

since $\mathfrak{X}\approx X_o\mathrm{x}D$. Hence $H^p(\mathfrak{X},\pi^*(\mathcal{O}_D))\approx H^o(\mathfrak{X},R^p\pi_*\pi^*(\mathcal{O}_D))$ is a free $\Gamma(D,\mathcal{O}_D)$ module.

Q.E.D.

LEMMA 3.2.5.

a) $H^i(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$ is a torsion free finetely generated $\Gamma(D,\mathcal{O}_D)$ module, for i=2 and 3.

b) $H^2(\mathfrak{X},d_{/D}\mathcal{O}_{\mathfrak{X}})$ is a free finetely generated $\Gamma(D,\mathcal{O}_D)$ module.

Proof of a:

We need to use the following exact seqeunces in order to prove 3.2.5.a.

$$0\rightarrow\mathcal{O}_{\mathfrak{X}/D}\overset{\otimes t}{\rightarrow}\mathcal{O}_{\mathfrak{X}/D}\rightarrow\mathcal{O}_{X_o}\rightarrow 0$$

(*) $$0\rightarrow H^o(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})\overset{\otimes t}{\rightarrow}H^o(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})\rightarrow H^o(X_o,\mathcal{O}_{X_o})\rightarrow 0$$

$$0\rightarrow H^1(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})\overset{r_1}{\rightarrow}H^1(X_o,\mathcal{O}_{X_o})\rightarrow H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})\overset{\otimes t}{\rightarrow}H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D}))\overset{r_2}{\rightarrow}H^2(X_o,\mathcal{O}_{X_o})\rightarrow$$

$$\rightarrow H^3(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})\overset{\otimes t}{\rightarrow}H^3(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})\rightarrow\ldots$$

If we prove that r_1 and r_2 are maps onto and because $H^i(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})$ i=2&3 are finitely generated modules over $\Gamma(D,\mathcal{O}_D)$ and the multiplication by t is injective it implies that $H^i(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})$ i=2&3 are zero or finetely generated free modules over $\Gamma(D,\mathcal{O}_D)$. (This is proved in Proposition 3 on p.22.) We already proved that $H^1(X_o,\mathcal{O}_{X_o})=0$ hence we can conclude that $H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})$ is either zero or a finitely generated free $\Gamma(D,\mathcal{O}_D)$ module. If we prove that the rank of $H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})$ over $\Gamma(D,\mathcal{O}_D)$ is ≥ 1 and since we assumed that $\dim_{\mathbb{C}} H^2(X_o,\mathcal{O}_{X_o})=1$ we get automatically that r_2 is a map onto. Hence $H^i(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})$ i=2&3 are finitely generated free modules over $\Gamma(D,\mathcal{O}_D)$. In order to finish the proof of LEMMA 3.2.5. we need to prove that the rank of $H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})\geq 1$.

Proposition 1. The rank of $H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})\geq 1$.

PROOF OF Proposition 1.:

It is enouph to prove that for each $t\in D$ $\dim_{\mathbb{C}} H^2(X_t,\mathcal{O}_t)\geq 1$. (See Cor. 2 p.50-51 in [14].) We will prove this fact using Dalbault cohomology, i.e. that each class in $H^2(X_t,\mathcal{O}_t)$ can be represented by $\bar{\partial}$ closed (0,2) form on X_t.

Choose a $\mathbb{C}^\infty$ trivialization of $\pi:\mathfrak{X}\to D$, i.e. $\mathfrak{X}\approx X_o x D$ as $\mathbb{C}^\infty$ manifolds. Since $\omega_{X_o}(2,O)$ is a non-zero class of cohomology in $H^2(X_o,\mathbb{C})$ we will get that for each $t\in D$ $\omega_{X_o}(2,O)|_{X_t}=\omega_t=\omega_t(2,O)+\gamma_t(1,1)+\eta_t(O,2)$. Since ω_t is a $d_{/D}$ closed form it follows that

$$\partial_t\omega_t(2,O)=O \text{ on } X_t, \text{ where } d=\partial_t+\bar{\partial}_t \text{ on } X_t.$$

Hence $\bar{\partial}_t\omega_t(O,2)=O$, where $\omega_t(O,2)$ is the complex conjugate of $\omega_t(2,O)$, i.e. $\omega_t(O,2)=\overline{\omega_t(2,O)}$. Q.E.D.

Next we will prove the following statement:

Proposition 2. For each $t\in D$, $\omega_t(O,2)$ is a non-zero class in $H^2(X_t,\mathcal{O}_{X_t})$.

Proof:

Suppose that $\omega_t(O,2)$ is the zero class in $H^2(X_t,\mathcal{O}_{X_t})$, i.e. $\omega_t(O,2)=\bar{\partial}_t\phi$. Then we must get a contradiction. The contradiction will be obtained in the following way; Consider $\wedge^{n}\omega_t(O,2)$. Then we will prove that $\wedge^{n}\omega_t(O,2)$ is a non-zero Dalbault class in $H^{2n}(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})$. We get a contradiction. Hence we need to prove that:

<u>Step 1.</u> $\wedge^n\omega_t(0,2)$ is a non-zero element of $H^{2n}(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})$.

<u>Remark.</u> $\omega_t(0,2)$ is defined on page 21.

<u>Proof:</u> Note that $\wedge^n\omega_t(0,2)\in\Gamma(\mathfrak{X},\Omega^{2n}_{\mathfrak{X}/D})$. Recall that it was shown that $\overline{\partial}_{/D}(\wedge^n\omega_t(0,2))=0$. Since $\wedge^n\omega_t(0,2)$ for t=o is an antiholomorphic 2n form $\omega_{X_o}(0,2n)$ on X_o which has no zeroes, i.e. we get that $\wedge^n\omega_o(0,2)\neq 0$ in $H^{2n}(X_o,\mathcal{O}_{X_o})$. From

$$\wedge^n\omega_t(0,2)|_{X_o}=\wedge^n\omega_o(0,2)\neq 0 \text{ in } H^{2n}(X_o,\mathcal{O}_{X_o})$$

and the exact sequence (*) it follows that $\wedge^n\omega_t(0,2)$ is a non-zero section of $H^{2n}(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})$. Q.E.D.

<u>Step 2.</u> $H^{2n}(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})$ is a free $\Gamma(D,\mathcal{O}_D)$ module of rank 1.

<u>Proof:</u>

Since $H^1(X_o,\mathcal{O}_{X_o})=0$ Serre's duality implies that $H^{2n-1}(X_o,\mathcal{O}_{X_o})=0$. From the exact sequences:

$$0\rightarrow\mathcal{O}_{\mathfrak{X}/D}\xrightarrow{\otimes t}\mathcal{O}_{\mathfrak{X}/D}\rightarrow\mathcal{O}_{X_o}\rightarrow 0$$

(*) $$0=H^{2n-1}(X_o,\mathcal{O}_{X_o})\rightarrow H^{2n}(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})\xrightarrow{\otimes t}H^{2n}(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})\xrightarrow{r}H^{2n}(X_o,\mathcal{O}_{X_o})\rightarrow 0$$

we get that $H^{2n}(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})$ is a free $\Gamma(D,\mathcal{O}_D)$ module since by Serre's duality and the fact that the canonical bundle of X_o is trivial implies that

$$a)\ H^{2n}(X_o,\mathcal{O}_{X_o})\doteq H^o(X_o,\Omega^{2n}_{X_o})\quad b)\ \Omega^{2n}_{X_o}\doteq\mathcal{O}_{X_o}$$

and therefore

(**) $$\dim_{\mathbb{C}}H^{2n}(X_o,\mathcal{O}_{X_o})=\dim_{\mathbb{C}}H^o(X_o,\mathcal{O}_{X_o})=1$$

(**) and (*) implies that $H^{2n}(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})$ is a free $\Gamma(D,\mathcal{O}_D)$ module, since $\Gamma(D,\mathcal{O}_D)$ is a ring of principle ideals. Q.E.D.

End of the proof of Proposition 2.

Suppose that $\omega_t(0,2)=\overline{\partial}_t\mu_t$ for each $t\in D\setminus\{o\}$ then it implies that $\wedge^n\omega_t(0,2)=\overline{\partial}_t(\mu_t\wedge(\wedge^{n-1}\omega_t(0,2))$. So $\wedge^n\omega_t(0,2)=0$ in $H^{2n}(\mathfrak{X},\mathcal{O}_{\mathfrak{X}/D})$. On the other hand we have that $r(\wedge^n\omega_t(0,2))=\omega_o(0,2n)\neq 0$ in $H^{2n}(X_o,\mathcal{O}_{X_o})$. This contradicts Step 2. Q.E.D.

Proposition 2. yields that for each $t\in D$, $H^2(X_t,\mathcal{O}_t)\neq O$, hence $H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$ as finetely generated module over principal ideal ring $\Gamma(D,\mathcal{O}_D)\subset\mathbb{C}[[t]]$ is a direct sum of a free module of rank 1 and a torsion part. This follows from the structure theorem of finetely generated modules over principal ideal rings and the fact that $\dim_{\mathbb{C}} H^2(X_o,\mathcal{O}_o)=1$. (See Lang "Algebra"[23].)

Proposition 3. The torsion part of the finitely generated $\Gamma(D,\mathcal{O}_D)$ module $H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$ is zero.

Proof:

The following exact sequences will be used:

(3.2.5.1.) $$O\to\mathcal{O}_{\mathfrak{X}}\xrightarrow{t}\mathcal{O}_{\mathfrak{X}}\to\mathcal{O}_{X_o}\to O$$

(3.2.5.2.) $$..\to H^1(X_o,\mathcal{O}_{X_o})\to H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})\xrightarrow{t}H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})\to H^2(X_o,\mathcal{O}_o)\to H^3(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})\xrightarrow{t}H^3(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$$

Since $H^1(X_o,\mathcal{O}_{X_o})=O$ we get that the map

$$j: H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})\xrightarrow{t}H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$$

in (3.2.5.1.) is an injection and moreover

$$j(\omega)=t\omega.$$

Hence multiplication by t is monomorphism. Since

$$\mathrm{Tor}\, H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})\approx\bigoplus_i\mathbb{C}[[t]]/(t^{n_i})$$

and the multiplication by t is a monomorphism j

$$j: \mathrm{Tor}\, H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})\xrightarrow{t}\mathrm{Tor}\, H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$$

it follows that $\mathrm{Tor}\, H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})=O$. Q.E.D.

End of the proof of LEMMA 3.2.5.a:

Hence we have proved that $H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$ is a free $\Gamma(D, \mathcal{O}_D)$ module of rank one.

Claim: Either $H^3(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$ is a free $\Gamma(D, \mathcal{O}_D)$ module or it is the zero module.

Proof of the claim: From the exact sequence (3.2.5.2.). , the fact that the map in it

$$r_2:H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})\rightarrow H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$$

is surjective implies that the map

$$H^3(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})\xrightarrow{t} H^3(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$$

which is a multiplication by t is injective. From the arguments of Proposition 3 the claim follows. LEMMA 3.2.5.a. is proved. Q.E.D.

PROOF OF LEMMA 3.2.5.b.:

Recall the following exact sequences:

$$0\rightarrow \pi^*(\mathcal{O}_D)\rightarrow \mathcal{O}_{\mathfrak{X}}\rightarrow d_{/D}\mathcal{O}_D\rightarrow 0$$

(3.2.5.b.1.)

$$\ldots\rightarrow H^2(\mathfrak{X},\pi^*(\mathcal{O}_D))\xrightarrow{\mu} H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})\rightarrow H^2(\mathfrak{X},d_{/D}\mathcal{O}_D)\rightarrow H^3(\mathfrak{X},\pi^*(\mathcal{O}_D))\rightarrow\ldots$$

Claim. The map μ in (3.2.5.b.1.) is a surjective map.

Proof of the claim:

From the standart resolutions of the sheaves $\pi^*(\mathcal{O}_D)$ and $\mathcal{O}_{\mathfrak{X}}$ we get that:

(3.2.5.b.2.)

$$H^2(\mathfrak{X},\pi^*(\mathcal{O}_D)):=\left\{\omega\in\Gamma(\mathfrak{X},\Omega^2_{\mathfrak{X}})|d_{/D}\omega=0\right\}/d_{/D}\left(\Gamma(\mathfrak{X},\Omega^{0,1}_{\mathfrak{X}})\right)$$

$$H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}):=\left\{\omega(0,2)|\bar{\partial}\,\omega(0,2)=0\right\}/\bar{\partial}\left(\Gamma(\mathfrak{X},\Omega^{0,1}_{\mathfrak{X}})\right)$$

and the map μ is given by the following formula:

(3.2.5.b.3.) $\mu(\omega=\omega(2,0)+\omega(1,1)+\omega(0,2))=\omega(0,2)$

Since $H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$ is a free finetely generated module of rank 1 and the generator is defined as follows:

Choose a $\mathcal{C}^{\infty}$ trivialization of $\pi:\mathfrak{X}\rightarrow D$, i.e. $\mathfrak{X}\approx X_o x D$ as $\mathcal{C}^{\infty}$ manifolds. Since

$$\omega_{X_o}(0,2):=\overline{\omega_{X_o}(2,O)}$$

is a non-zero class of cohomology in $H^2(X_o,C)$ we will get that for each $t\in D$

$$\omega_{X_o}(0,2)|_{X_t}=\omega_t=\eta_t(2,O)+\gamma_t(1,1)+\omega_t(O,2).$$

Since $\omega_{X_o}(0,2)$ is a d closed form on $\mathfrak{X}\doteq X_o x D$ it follows that $\omega_t(0,2)$ is a $\overline{\partial}$ closed form on $\mathfrak{X}$. We already proved that $\omega_t(0,2)$ generates the free module $H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$. This implies that μ is a surjective map. The claim is proved. Q.E.D.

<u>End of the proof of 3.2.5.b.:</u>

The surjectivity of the map μ and the exact sequence (3.2.5.b.2.) imply that $H^2(\mathfrak{X},d_{/D}\mathcal{O}_{\mathfrak{X}})$ is a submodule of a the free $\Gamma(D,\mathcal{O}_D)$ module $H^3(\mathfrak{X},\pi^*(\mathcal{O}_D))$ (See Lemma 3.2.4.) These fact yields that $H^2(\mathfrak{X},d_{/D}\mathcal{O}_{\mathfrak{X}})$ is a free $\Gamma(D,\mathcal{O}_D)$ module since $\Gamma(D,\mathcal{O}_D)$ is a ring of principal ideals.(See [23].) So (3.2.5.b.) is proved. Q.E.D.

<u>LEMMA 3.2.6.</u> The following equality holds for symplectic holomorphic manifolds:

(3.2.6.1.) $$\dim_{\mathbb{C}}H^o(X_o,d\Omega^1_{X_o})=\dim_{\mathbb{C}}H^o(X_o,\Omega^2_{X_o})=1$$

<u>PROOF OF LEMMA 3.2.6.:</u>

It is sufficient to show that a symplectic holomorphic manifold does not admit a non-closed holomorphic two form. Suppose that $\kappa_o(2,0)$ is a holomorphic two form on X_o such that $d(\kappa_o(2,0))=w_o(3,0)\neq 0$. Then :

$$\int_{X_o} w_o(3,0)\wedge\overline{w_o(3,0)}\wedge\omega^{2n-3}=\int_{X_o} d(\kappa_o(2,0))\wedge\overline{w_o(3,0)}\wedge(\wedge^{2n-3}\omega^{1,1})>0$$

where ω is the form constructed in THEOREM 1, $\omega=\omega^{2,0}+\omega^{1,1}+\omega^{0,2}$ and $\omega^{1,1}$ is positive definite at each point of X_o. Since Stokes' Theorem implies

$$\int_{X_o} d(\kappa_o(2,0)\wedge\overline{w_o(3,0)}\wedge(\wedge^{2n-3}w^{1,1})=0$$

A contradiction is reached and equality (3.2.[illegible]1.) is proved.

Q.E.D.

<u>LEMMA 3.2.7.</u> $H^1(\mathfrak{X},\Omega^1_{\mathfrak{X}/D})$ is a torsion free $\Gamma(D,\mathcal{O}_D)$ module.

<u>PROOF OF 3.2.7.:</u> We have the following exact sequence

(A1) $$O \to \Omega^1_{\mathfrak{X}/D} \xrightarrow{t} \Omega^1_{\mathfrak{X}/D} \to \Omega^1_{X_o} \to O$$

(A2) $$O \to H^o(\mathfrak{X},\Omega^1_{\mathfrak{X}/D}) \xrightarrow{t} H^o(\mathfrak{X},\Omega^1_{\mathfrak{X}/D}) \to H^o(X_o,\Omega^1_{X_o}) \to H^1(\mathfrak{X},\Omega^1_{\mathfrak{X}/D}) \xrightarrow{t} H^1(\mathfrak{X},\Omega^1_{\mathfrak{X}/D}) \to ..$$

<u>Proposition 1.</u> $H^o(X_o,\Omega^1_{X_o})=O$.

<u>Proof:</u>

We need to prove that there are no holomorphic one forms on X_o. Since $H^1(X_o,\mathcal{O}_{X_o})=O$ there are no one holomorphic forms that are closed.

Suppose that α is a holomorphic one form on X_o such that $d\alpha\neq O$. Since $\omega_o(2,O)$ is a nondegenerate form on X_o we get that:

$$\int_{X_o} \partial\alpha\wedge\bar{\partial}\bar{\alpha}\wedge(\omega_o(2,O))^n\wedge(\omega_o(O,2))^n>O,$$

$$\partial\alpha\wedge\bar{\partial}\bar{\alpha}\wedge(\omega_o(2,O))^n\wedge(\omega_o(O,2))^n=d(\alpha\wedge\bar{\partial}\bar{\alpha}\wedge(\omega_o(2,O))^n\wedge(\omega_o(O,2))^n)=d\psi,$$

(*) $$O<\int_{X_o} \partial\alpha\wedge\bar{\partial}\bar{\alpha}\wedge(\omega_o(2,O))^n\wedge(\omega_o(O,2))^n=\int_{X_o} d(\alpha\wedge\bar{\partial}\bar{\alpha}\wedge(\omega_o(2,O))^n\wedge(\omega_o(O,2))^n)=O$$

Hence (*) implies that $d\alpha=O$ and thus $\alpha=O$. Q.E.D.

<u>End of the proof of LEMMA 3.2.7.:</u>

The map $H^1(\mathfrak{X},\Omega^1_{\mathfrak{X}/D}) \xrightarrow{t} H^1(\mathfrak{X},\Omega^1_{\mathfrak{X}/D})$ in (A2) which is a multiplication by t has zero kernal. From here it follows that $H^1(\mathfrak{X},\Omega^1_{\mathfrak{X}/D})$ is a torsion free finitely generated $\Gamma(D,\mathcal{O}_D)$ module. *Thus Lemma 3.2.7.* is proved. Q.E.D.

LEMMA 3.2.8.

$H^1(\mathfrak{X},\Omega^1_{\mathfrak{X}/D})\big/j\big(H^1(\mathfrak{X},d_{/D}\mathcal{O}_D)$ is a torsion free $\Gamma(D,\mathcal{O}_D)$ sub-module in $H^1(\mathfrak{X}, d_{/D}\Omega^1_{\mathfrak{X}/D})$, where j is defined from the following exact sequences:

$$(**)\qquad 0\to d_{/D}\mathcal{O}_D\to\Omega^1_{\mathfrak{X}/D}\xrightarrow{d_{/D}} d_{/D}\Omega^1_{\mathfrak{X}/D}\to 0$$

$$..\to H^1(\mathfrak{X}, d_{/D}\mathcal{O}_{\mathfrak{X}/D})\xrightarrow{j} H^1(\mathfrak{X},\Omega^1_{\mathfrak{X}/D})\xrightarrow{\rho} H^1(\mathfrak{X}, d_{/D}\Omega^1_{\mathfrak{X}/D})\to H^2(\mathfrak{X}, d_{/D}\mathcal{O}_{\mathfrak{X}/D})\to ..$$

REMARK.

$H^1(\mathfrak{X},\Omega^1_{\mathfrak{X}/D}):=\big\{\ \omega_t(1,1)\in\Gamma(\mathfrak{X},\Omega^{1,1}_{\mathfrak{X}/D})\ |\bar\partial_{/D}\ \omega_t(1,1)=O\ \big\}\big/\bar\partial_{/D}(\Gamma(\mathfrak{X},\Omega^{O,1}_{\mathfrak{X}/D}))$

We have proved that $H^1(\mathfrak{X},\Omega^1_{\mathfrak{X}/D})$ is a finetely generated free $\Gamma(D,\mathcal{O}_D)$ module. (This is LEMMA 3.2.7.)

Proof:

Lemma 3.2.8. follows directly from the following two propositions:

PROPOSITION 1.

Let $\omega_t(1,1)\in H^1(\mathfrak{X},\Omega^1_{\mathfrak{X}/D})$ be a fixed nonzero class and suppose that there does not exists two form $\omega_t(2,0)$ such that $d_{/D}\ \omega_t(2,0)=d_{/D}\omega_t(1,1)$, then for each $n\in\mathbf{Z}$ and $n>0$, there does not exists a two form $\psi^n_t(2,O)$ such that

$$d_{/D}\ t^n\ \omega_t(1,1) = d_{/D}\ \psi^n_t(2,O)$$

Proof of Proposition 1.:

Suppose that for some $n>O$ there exists $\psi^n_t(2,O)$ such that

$$d_{/D}(t^n\ \omega_t(1,1)) = d_{/D}(\psi^n_t(2,O))$$

and $\omega_t(1,1)$ satisfies the conditions in the Proposition 1.

From Taylor expensions of $\omega_t(1,1)$

$$\text{(A)}\qquad \omega_t(1,1) = \omega_O(1,1)+t\omega_1(1,1)+..+t^n\omega_n(1,1)+..$$

and of $\psi^n_t(2,O)$

(B) $\psi_t^n(2,O) = \psi_o + t\psi_1 + .. + t^n\psi_n +$

and since $d_{/D}$ and t^n comute the following formulas are obtained::

(C) $d_{/D}(\psi_o + t\psi_1 + .. + t^{n-1}\psi_{n-1}) = O$

(D) $t^n d_{/D}\omega_t(1,1) = d_{/D}(t^n\psi_n + t^{n+1}\psi_{n+1} + ..) = t^n d_{/D}(\psi_n + t\psi_{n+1} + .. + t^k\psi_{n+k} + ..) =$
$= t^n d_{/D}\omega_t(2,O)$

From (D) we get that

(E) $d_{/D}\,\omega_t(1,1) = d_{/D}\omega_t(2,O)$

But (E) contradicts the assumtion of $\omega_t(1,1)$. Thus the PROPOSITION 1. is proved. Q.E.D.

Recall that:

PROPOSITION 2.

a) $H^1(\mathfrak{X}, d_{/D}\mathcal{O}_{\mathfrak{X}/D}) := \left\{\omega_t(2,0)+\omega_t(1,1) | d_{/D}\left(\omega_t(2,0)+\omega_t(1,1)\right)=0\right\} / d_{/D}\left(\Gamma(\mathfrak{X},\Omega^1_{\mathfrak{X}/D})\right)$, where $\Omega^1_{\mathfrak{X}/D}$ is the sheaf of C^∞ relative one-forms.

b) $H^1(\mathfrak{X}, d_{/D}\,\Omega^1_{\mathfrak{X}/D}) := \left\{\omega_t \in \Gamma(\mathfrak{X},\Omega^{3,O}_{\mathfrak{X}/D} \oplus \Omega^{2,1}_{\mathfrak{X}/D}) | \, d_{/D}\,\omega_t = O\right\} \Big/ \, d_{/D}(\Gamma(\mathfrak{X},\Omega^{2,O}_{\mathfrak{X}/D}))$

c) the map ρ: $H^1(\mathfrak{X},\Omega^1_{\mathfrak{X}/D}) \rightarrow H^1(\mathfrak{X}, d_{/D}\Omega^1_{\mathfrak{X}/D})$ in the exact sequence (**) is just the map $d_{/D}$, i.e. $\rho(\omega_t) = d_{/D}(\omega_t)$.

PROOF OF PROPOSITION 2.a:

PROPOSITION 2.a. follows directly from the following resolution of the holomorphic sheaf $d_{/D}\mathcal{O}_{\mathfrak{X}/D}$:

$$0 \rightarrow d_{/D}\mathcal{O}_{\mathfrak{X}/D} \rightarrow \Omega^{1,0}_{\mathfrak{X}/D} \xrightarrow{d/D} \Omega^{2.0}_{\mathfrak{X}/D} \dot{+} \Omega^{1,1}_{\mathfrak{X}/D} \rightarrow .....$$ Q.E.D.

PROOF OF PROPOSITION 2.b. & 2.c:

It is verry easy to prove Proposition b. and Proposition c. Just use the standart resolutions of the sheaves $d_{/D}\mathcal{O}_{\mathfrak{X}}$, $\Omega^1_{\mathfrak{X}/D}$ and $d_{/D}\,\Omega^1_{\mathfrak{X}/D}$. Q.E.D.

COR.a. The map j in the formulation of LEMMA 3.8. is defined as follows: $j(\omega_t(2,0)+\omega_t(1,1)) = \omega_t(1,1)$.

End of the proof of LEMMA 3.2.8.:

Lemma 3.2.8. follows directly from PROPOSITION 1, PROPOSITION 2.c. and the definition of a torsion element of the submodule $\rho(H^1(\mathfrak{X},\Omega^1_{\mathfrak{X}/D})/j(H^1(\mathfrak{X},d_{/D}\mathcal{O}_{\mathfrak{X}})$ in $H^1(\mathfrak{X},d_{/D}\Omega^1_{\mathfrak{X}/D})$

Q.E.D.

Theorem 3.2. is a consequence of the following lemma:

MAIN LEMMA 3.2.9. $H^1(\mathfrak{X},d_{/D}\ \Omega^1_{\mathfrak{X}/D})$ is a torsion free $\Gamma(D,\mathcal{O}_D)$ module.

PROOF OF LEMMA 3.2.9.:

The following exact seqences will imply 3.2.3.:

$$(*)\qquad O \to d_{/D}\mathcal{O}_{\mathfrak{X}} \to \Omega^1_{\mathfrak{X}/D} \to d_{/D}\Omega^1_{\mathfrak{X}/D} \to O$$

$$(**)\quad \ldots\to H^1(\mathfrak{X},d_{/D}\mathcal{O}_{\mathfrak{X}}).\xrightarrow{j} H^1(\mathfrak{X},\Omega^1_{\mathfrak{X}/D}) \to H^1(\mathfrak{X},d_{/D}\ \Omega^1_{\mathfrak{X}/D}) \to H^2(\mathfrak{X},d_{/D}\mathcal{O}_{\mathfrak{X}}) \to ..$$

The map j: $H^1(\mathfrak{X},d_{/D}\mathcal{O}_{\mathfrak{X}}) \to H^1(\mathfrak{X},\Omega^1_{\mathfrak{X}/D})$ in (**) is given by the formula:

$$(***)\qquad j(\omega_t(2,O)+\omega_t(1,1))=\omega_t(1,1)\text{ (See Cor. a. on the same page.)}$$

From (**) we get :

$$(3.2.9.1)\quad O\to H^1(\mathfrak{X},\Omega^1_{\mathfrak{X}/D})\Big/j\Big(H^1(\mathfrak{X},d_{/D}\mathcal{O}_{\mathfrak{X}})\Big) \to H^1(\mathfrak{X},d_{/D}\Omega^1_{\mathfrak{X}/D}) \to H^2(\mathfrak{X},\pi^*(\mathcal{O}_D)) \to ..$$

It was proved that $H^1(\mathfrak{X},\Omega^1_{\mathfrak{X}/D})\Big/j\Big(H^1(\mathfrak{X},d_{/D}\mathcal{O}_D)$ and $H^2(\mathfrak{X},\pi^*(\mathcal{O}_D))$ are free $\Gamma(D,\mathcal{O}_D)$ modules, then (3.2.9.1.) implies lemma (3.2.9.). Q.E.D.

The end of the proof of THEOREM 3.2.:

COR. 3.2.9.a. From Lemma 3.2.9. follows *theorem 3.2.*

PROOF OF 3.2.9.a.:

We have the following exact sequences :

$$(3.2.9.1.)\qquad O\to d_{/D}\Omega^1_{\mathfrak{X}/D}\xrightarrow{t} d_{/D}\Omega^1_{\mathfrak{X}/D}\to d_{/D}\ \Omega^1_{\mathfrak{X}/D}|X_o \to O$$

$$O\to H^o(\mathfrak{X},d_{/D}\Omega^1_{\mathfrak{X}/D})\xrightarrow{t}H^o(\mathfrak{X},d_{/D}\Omega^1_{\mathfrak{X}/D})\to H^o(X_o,d\Omega^1_{X_o})\to H^1(\mathfrak{X},d_{/D}\Omega^1_{\mathfrak{X}/D})\xrightarrow{t}H^1(\mathfrak{X},d_{/D}\Omega^1_{\mathfrak{X}/D}$$

In the long exact sequence (3.2.3.1.1.) the map

$$i\colon H^1(\mathfrak{X},d_{/D}\Omega^1_{\mathfrak{X}/D})\xrightarrow{t}H^1(\mathfrak{X},d_{/D}\Omega^1_{\mathfrak{X}/D})$$

is given by multiplication by t. Recall that $H^1(\mathfrak{X}, d_{/D}\Omega^1_{\mathfrak{X}/D})$ is a torion free finitely generated $\Gamma(D,\mathcal{O}_D)$ module and $\Gamma(D,\mathcal{O}_D)\subset\mathbf{C}[\![t]\!]$. Hence the map

$$H^1(\mathfrak{X}, d_{/D}\Omega^1_{\mathfrak{X}/D}) \xrightarrow{t} H^1(\mathfrak{X}, d_{/D}\Omega^1_{\mathfrak{X}/D}$$

is injective. The exact sequence

$$O \to H^o(\mathfrak{X}, d_{/D}\Omega^1_{\mathfrak{X}/D}) \xrightarrow{t} H^o(\mathfrak{X}, d_{/D}\Omega^1_{\mathfrak{X}/D}) \to H^o(X_o, d\Omega^1_{X_o}) \to 0,$$

the fact that $\dim_{\mathbf{C}} H^o(X_o, d\Omega^1_{X_o})=1$ and the structure Theorem of finetely generated modules and LEMMA 3.2.6. implies that the $H^o(\mathfrak{X}, d_{/D}\Omega^1_{\mathfrak{X}/D})$ is a free $\Gamma(D,\mathcal{O}_D)$ module of rank one. Let $\omega_{\mathfrak{X}/D}(2,O)$ be the generator of $H^o(\mathfrak{X}, d_{/D}\Omega^1_{\mathfrak{X}/D})$ such that $\omega_{\mathfrak{X}/D}(2,O)|_{X_o}$ is a non-zero holomorphic form on X_o. The restriction of

$$\omega_{\mathfrak{X}/D}(2,O)|_{X_t} = \omega_t(2,O) \quad \text{where} \quad X_t = \pi^{-1}(t)$$

will be a non-zero closed holomorphic two-form. This follows from the definition of the sheaf $d_{/D}\Omega^1_{\mathfrak{X}/D}$. Hence each X_t will be a holomorphic symplectic manifold, after possibly shrinking the disk D. Continuity arguments yield that the form $\omega_t(2,O) := \omega_{\mathfrak{X}/D}(2,O)|_{X_t}$ is a non-degenerate one on X_t. Hence 3.2.3.a. is proved and *Theorem 3.2.* is proved. Q.E.D.

#4. THE PERIOD MAP FOR HOLOMORPHIC SYMPLECTIC MANIFOLDS.

Let $\pi:\mathfrak{X}\to U$ be the family of symplectic manifolds constructed in #3. Remember that $\dim_{\mathbf{C}} U=b_2-2=\dim_{\mathbf{C}} H^1_d(X,\Omega^1)$ and U is a non-singular complex manifold. Since $\pi:\mathfrak{X}\to U$ in the catagory of C^∞ manifolds is diffeomorphic to the trivial familyπ: $U\mathrm{x}X\to U$, hence if we fix a basis $(\delta_1,\ldots,\delta_{b_2})$ of $H_2(X,\mathbf{Z})$, then $(\delta_1,\ldots,\delta_{b_2})$ will be a basis of $H_2(X_t,\mathbf{Z})$ for all t $\in U$. From now on let us fix the basis $(\delta_1,\ldots,\delta_{b_2})$ of $H_2(X,\mathbf{Z})$.

Definition 4.1. The period map $\rho:U\to\mathbf{P}(H_2(X,\mathbf{Z})\otimes\mathbf{C})$ is defined as follows:

$$\rho(t):=\left(\ldots\ldots,\int_{\delta_i}\omega_t(2,0),\ldots\right)$$

where $\omega_t(2,0)$ is the only holomorphic two form defined up to a constant on $X_t=\pi^{-1}(t)$ and $d(\omega_t(2,0))=0$.

Let $\dim_{\mathbf{C}} X=2n$, $\int_{X_t}(\omega_t(2,0)^n\wedge\omega_t(0,2)^n)=1$ and $\omega_X(2,0)=\omega_o(2,0)$.

Definition 4.2.

For every $\alpha\in H^2(X,\mathbf{C})$ define

$$q(\alpha):=\frac{n}{2}\int_X(\omega_o(2,0)\wedge\omega_o(0,2))^{n-1}\wedge\alpha^2+$$

$$+(1-n)\left(\int_X(\omega_o(2,0)^{n-1}\wedge\omega_o(0,2))^n\wedge\alpha\right)\left(\int_X(\omega_o(2,0)^n\wedge\omega_o(0,2))^{n-1}\wedge\alpha\right)$$

Proposition 4.3.

A) The quadratic form $q(\alpha)$ is a non-degenerate one and is defined over $\mathbf{Z}$ taking into account that $H^2(X,\mathbf{C})=H^2(X,\mathbf{Z})\otimes\mathbf{C}$

B) Let Ω be a subvariety in $\mathbb{P}(H^2(X,\mathbf{Z})\otimes\mathbf{C})$ defined by $q(\alpha)=0$ and $q(\alpha+\bar{\alpha})>0$ then $\rho:U\to\Omega$ is an isomorphism on its image and $\rho(U)\subset\Omega$.

REMARK. 4.3.*B*. is the so called local TORELLI THEOREM for holomorphic symplectic manifolds, which says that the differential of the period map at each point of U has a maximal rank equal to $\dim_{\mathbf{C}} U$.

Proof of *A and B*: For the proof of *a)* and *b)* see [02].

Q.E.D.

Lemma 4.3.1.

The classes of cohomologies $[\omega]$ of the forms that are constructed in THEOREM 1. form an open and convex cone in $H^{1,1}(X,\mathbf{R}) \subset H^2(X,\mathbf{R})$, where $H^{1,1}(X,\mathbf{R}) := \Big\{$all $[w] \in H^2(X,\mathbf{R}) \mid [w]$ contains a closed form of type $(1,1)\Big\}$.

Proof: From 2.1. follows that $\dim_{\mathbf{R}} H^{1,1}(X,\mathbf{R}) = b_2 - 2$. Let $\delta_1, \ldots, \delta_{b_2-2}$ be a basis of $H^{1,1}(X,\mathbf{R})$, then if N is a positive sufficiently large real number and $\epsilon_1, \ldots, \epsilon_{b_2-2}$ are sufficiently small rael positive numbers, then the compactness of X implies that

$$N\omega + \sum \epsilon_i \delta_i \in H^{1,1}(X,\mathbf{R})$$

and the form

$$N\omega + \sum \epsilon_i \delta_i$$

will fulfill the properties stated in THEOREM 1. Thus 4.3.1. follows.

Q.E.D.

From 4.3.1. it follows that we can choose a basis of $H^2(X,\mathbf{R})$ in the following way: {Re $\omega_0(2,0)$, Im $\omega_0(2,0)$, $\delta_1, \ldots, \delta_{b_2-2}$} where δ_i for all i are in the convex cone defined in 4.3.1.. Clearly we have $q(\mathrm{Re}\ \omega_0(2,0)) > 0$ $q(\mathrm{Im}\ \omega_0(2,0)) > 0$. From the way we defined δ_i, it follows that δ_i can be realized by a form ω that fulfills the conditions of THEOREM 1., then

$$q(\omega) = \frac{n}{2} \int_X (\omega_0(2,0) \wedge \omega_0(0,2))^{n-1} \wedge \omega^2 =$$

$$\frac{n}{2} \int_X (\omega_0(2,0) \wedge \omega_0(0,2))^{n-1} \wedge \partial \alpha^{1,0} \wedge \bar{\partial} \alpha^{0,1} +$$

$$\frac{n}{2}\int_X (\omega_o(2,0)\wedge\omega_o(0,2))^{n-1}\wedge\omega^{1,1}\wedge\omega^{1,1}$$

Clearly $q(\omega)>0$, where $q(\delta_i)=q(\omega)$. This proves that q is a non-degenerate form. In [02] it was proved that q is defined over $\mathbf{Z}$ up to a constant.

Q.E.D.

Definition 4.4. $K(X):=\left\{\omega\in H^{1,1}(X,\mathbf{R}) \mid \int_{C_k}\omega^k>0\text{, where } C_k \text{ is any k-dimensional complex analytic subspace in } X\right\}$. We will call K(X) the Kähler cone of X.

Remark. Note that from 4.3.1. it follows that if X is a symplectic holomorphic manifold, then K(X) is an open convex cone in $H^{1,1}(X,\mathbf{R})$.

Proposition 4.5.

Let $\pi:\mathfrak{X}\to U$ be the family constructed in 3.1.A., where $\pi^{-1}(0)=X$, then in U we can find an everywhere dense subset $U^1\subset U$ such that *a)* U^1 is an open subset in U. *b)* for each $\tau\in U^1$ $\pi^{-1}(\tau)=X_\tau$ is a Kähler manifold.

Proof: From local Torelli Theorem it follows that we can suppose that $U\subset\Omega\subset\mathbf{P}(H^2(X,\mathbf{Z})\otimes\mathbf{C})$, where $\dim_{\mathbf{C}}\Omega=b_2-2$ and $\dim_{\mathbf{C}}U=b_2-2$ and thus U is an open subset in Ω.

DEFINITION.

i) Let $\mathfrak{Q}\subset\mathbf{P}(H^2(X,\mathbf{R})$ be the open set defined by $q(u)>0$ for $u\in\mathfrak{Q}$ and $u\neq0$..

ii) Let $\mathfrak{L}\subset H^2(X,\mathbf{R})$ be the set defined as $\{$the union of all lines in $H^2(X,\mathbf{R})$ that corresponds to points in$\mathfrak{Q}\}$.

iii) Let $W:\stackrel{\text{def}}{=}$ be the union of all $K(X_\tau)$ for $\tau\in U$.

LEMMA 4.5.1. W is an open subset in $H^2(X,\mathbf{R})$.

PROOF OF 4.5.1.:

Note that $\mathfrak{L}$ is an open subset in $H^2(X,\mathbf{R})$. Let $\tau\in U$, $X_\tau=\pi^{-1}(\tau)$ be a holomorphic symplectic manifold in the KURANISHI family $\pi:\mathfrak{X}\to U$. Let $\omega_\tau\in H^{1,1}(X_\tau,\mathbf{R})$, where

$$H^{1,1}(X_\tau,\mathbf{R}):=\left\{[w]\in H^2(X_\tau,\mathbf{R})|[w] \text{ contains a close form of type } (1,1)\right\}$$

and ω_τ be cohomological to Harvey-Lawson form on X_τ, i.e ω_τ is cohomological to $w_\tau=\partial\alpha^{1,0}(\tau)+w^{1,1}(\tau)+\overline{\partial\alpha^{1,0}}(\tau)$ where $w^{1,1}(\tau)$ is positive definite at each point of X_τ. From LEMMA 2.2. we know that w_τ exists for each $\tau\in U$, represents a non-zero class in $H^{1,1}(X_\tau,\mathbf{R})$ and belongs to $K(X_\tau)$, hence $q([\omega_\tau],[\omega_\tau])>0$. This implies that $\omega_\tau\in\mathfrak{L}$. Let $[w_\nu]$ be in the open subset $\mathfrak{L}\subset H^2(X,\mathbf{R})$ and sufficiently close to $[w_\tau]$. The local TORELLI THEOREM implies that $U\subset\mathbf{P}(H^2(X,\mathbf{C}))$. Let $U_{[w_\nu]}:=\{t\in U \mid q(t,[w_\nu])=0$. From the local TORELLI THEOREM it follows that that for each $t\in U_{[w_\nu]}$ $[w_\nu]\in H^{1,1}(X_t,\mathbf{R})$. (For more details see [02].) Continuity argument yields that if $t\in U_{[w_\nu]}$ is suficiently close to τ then $[w_\nu]\in K(X_t)$. Since $\mathfrak{L}$ is an open subset in $H^2(X,\mathbf{R})$ and the above arguments imply that W is an open subset in $H^2(X,\mathbf{R})$.

Q.E.D.

Let $W(\mathbf{Q}) \stackrel{\text{def}}{=} W\cap H^2(X,\mathbf{Q})$. $H^2(X,\mathbf{Q})$ is an everywhere dense subset in $H^2(X,\mathbf{R})$. It implies $W(\mathbf{Q})$ is an everywhere dense subset in W.

From the definition of $W(\mathbf{Q})$ it follows that if $l\in W(\mathbf{Q})$, hence there exists $\tau\in U$ such that $l\in K(X_\tau)$. Thus X_τ is a symplectic holomorphic manifold such that $l\in H^{1,1}(X,\mathbf{R})\cap H^2(X,\mathbf{Z})$ and for every complex analytic subspace $C_k\subset X_\tau$ we have

$$\int_{C_k} l^k>0$$

Hence by a THEOREM 5.1. from the next section X_τ is an algebraic manifold. , The points $\tau\in U$ for which $W(\mathbf{Q})\cap K(X_\tau)\neq\emptyset$ is an everywhere dense subset in U since $W(\mathbf{Q})$ is an everywhere dense subset in W. Let us denote this subset by U". Thus every point τ of U" corresponds to a projective holomorphic symplectic manifold. Proposition 4.5. follows from a theorem of Kodaira [11a], which says that the Kāhlerian property is an open property.

Q.E.D.

#5. NAKAI-MOISHEZON CRITERIUM.

THEOREM 5.1.

Let X_o be a holomorphic symplectic manifold. If L be a line bundle such that for any complex analytic subspace $C_k \subset X_o$ of dimension k $\int_{C_k} (c_1(L))^k > 0$, then X_o is a projective algebraic manifold.

PROOF: The proof is based on the following LEMMA :

LEMMA 5.1.1.

Let X_o be a holomorphic symplectic manifold and let L be a line bundle on X_o such that $c_1(L) \neq 0$, then there exists a divisor D such that $L \cong \mathcal{O}_{X_o}(D)$.

PROOF: Let $\mathcal{M}_{X_o}$ be the sheaf of meromorphic functions on X_o. We have the following exact sequences:

(*) $$0 \to \mathcal{O}^*_{X_o} \to \mathcal{M}_{X_o} \to \mathcal{D}_{X_o} \to 0$$

(**) $$0 \to H^o(\mathcal{O}^*_{X_o}) \to H^o(\mathcal{M}_{X_o}) \to H^o(\mathcal{D}_{X_o}) \xrightarrow{\delta} H^1(\mathcal{O}^*_{X_o}) \to H^1(\mathcal{M}_{X_o}) \xrightarrow{i} H^1(\mathcal{D}_{X_o}) \to ..$$

where $\mathcal{D}_{X_o}$ is the sheaf of divisors on X_o and hence $H^o(\mathcal{D}_{X_o})$ are all divisors on X_o, i.e $\delta(D) = \mathcal{O}_{X_o}(D) \in H^1(\mathcal{O}^*_{X_o})$.

Proposition 1. The map $i: H^1(\mathcal{M}_{X_o}) \to H^1(\mathcal{D}_{X_o})$ is an inclusion.

PROOF:

Step 1.

There exists a family of holomorphic symplectic manifolds $\pi: \mathfrak{X}_D \to D$ such that $H^1(\mathcal{O}^*_{\mathfrak{X}_D}) = 0$.

PROOF of Step 1:

Recall that the base of the Kuranishi family $\{\pi: \mathfrak{X} \to U, \pi^{-1}(o) = X_o\}$ constructed in 3.1.A. can be viewed as a submanifold $U \subset \mathbf{P}(H^2(X,\mathbf{Z}) \otimes \mathbf{C})$ and U is contained in an open set Ω of a quadric in $\mathbf{P}(H^2(X,\mathbf{Z}) \otimes \mathbf{C})$ defined over $\mathbf{Z}$. We suppose that $o \in \Omega$.

A plane $\mathbf{P}^2 \subset \mathbf{P}(H^2(X,\mathbf{Z}) \otimes \mathbf{C})$ can be chosen such that $\mathbf{P}^2$ intresects Ω transversally at o and

$\mathbf{P}^2$ is not contained in any hyperplane H_L, for any $L \in H^2(X,\mathbf{Q})$, where

$$(*) \qquad H_L=\left\{u\in\mathbf{P}(H^2(X,\mathbf{Z})\otimes C)\mid <u,L>=o,\ L\in H^2(X,\mathbf{Q})\right\}$$

Let D be a disk, such that $D\subset\mathbf{P}^2\cap\Omega$ and $0\in D$. From the definition of the quadratic form q, that defines Ω (See #4.) it follows

Condition $H^2(X_\tau,\mathbf{Z})\cap H^{1,1}(X_\tau,\mathbf{R})\neq\emptyset$ for $\tau\in D$ and $\pi^{-1}(\tau)=X_\tau$ is equivalent to $\tau\in H_L$, where $L\in H^2(X_\tau,\mathbf{Z})\cap H^{1,1}(X_\tau,\mathbf{R})$. The proof is straightforward. See [02].

From the definition of $\mathbf{P}^2$ and that of D we get that the set of points $\tau\in D$ such that

$$H^2(X_\tau,\mathbf{Z})\cap H^{1,1}(X_\tau,\mathbf{R})=\emptyset$$

is a non-empty set which is everywhere dense set in D. Let us denote this set bu $\mathfrak{D}$.

Suppose that $H^1(\mathfrak{X}_D,\mathcal{O}^*_{\mathfrak{X}_D})\neq 0$, where $\mathfrak{X}_D\rightarrow D$ is the restriction of $\mathfrak{X}\rightarrow U$ over D. Let

$$\alpha(L)\neq 0 \text{ and } \alpha(L)\in H^1(\mathfrak{X}_D,\mathcal{O}^*_{\mathfrak{X}_D})$$

We know that $\alpha(L)$ corresponds to a line bundle $\mathcal{L}$ on $\mathfrak{X}_D$. We will prove that $c_1(\mathcal{L})\neq 0$ in $H^2(\mathfrak{X}_D,\mathbf{Z})$, where $c_1(\mathcal{L})$ is the first Chern class of $\mathcal{L}$. We have the following exact sequences:

$$0\rightarrow\mathbf{Z}\rightarrow\mathcal{O}_{\mathfrak{X}_D}\xrightarrow{\exp}\mathcal{O}^*_{\mathfrak{X}_D}\rightarrow 1$$

$$\ldots\rightarrow H^1(\mathcal{O}_{\mathfrak{X}_D})\rightarrow H^1(\mathcal{O}^*_{\mathfrak{X}_D})\xrightarrow{\delta}H^2(\mathfrak{X}_D,\mathbf{Z})\rightarrow\ldots$$

We have proved that $H^1(\mathcal{O}_D)=0$. (See #3.) Since $\delta(\alpha(L))=c_1(\mathcal{L})$(See [05].) Thus we get that $c_1(\mathcal{L})\neq 0$ since a) δ is an inclusion, b) $\alpha(L)\neq 0$ $\alpha(L)\in H^1(\mathfrak{X}_D,\mathcal{O}^*_{\mathfrak{X}_D})$ and c) $c_1(\mathcal{L})=\delta(\alpha(L))\neq 0$.

Since $\mathfrak{X}_D$ is a strong retract of X_τ it follows that for each $\tau\in D$ $H^2(\mathfrak{X}_D,\mathbf{Z})=H^2(X_\tau,\mathbf{Z})$. From this fact and since $\mathcal{L}$ is a non-trivial line bundle on $\mathfrak{X}_D$ we get that the Chern class of $\mathcal{L}_\tau$ on X_τ is $\neq 0$. Hence on each X_τ for $\tau\in D$ $\mathcal{L}_\tau=\mathcal{L}_{|X_\tau}$ is a non-trivial line bundle. Hence we get a non-zero element $c_1(\mathcal{L}_\tau)\in H^2(X_\tau,\mathbf{Z})\cap H^{1,1}(X_\tau,\mathbf{R})$ for each $\tau\in D$. On the other hand we know that on a dense subset $\mathfrak{D}\subset D$ we have

$$H^2(X_\tau,\mathbf{Z})\cap H^{1,1}(X_\tau,\mathbf{R})=\emptyset$$

for $\tau\in\mathfrak{D}$. We get a contradiction. Hence $H^1(\mathcal{O}^*_{\mathfrak{X}_D})=0$.

Q.E.D.

<u>Step 2.</u>

The map $i:H^1(X_o,\mathcal{O}^*_{X_o})\to H^1(X_o,\mathcal{M}_{X_o})$ induced from the exact sequence

$$(*)\qquad 0\to\mathcal{O}^*_{X_o}\to\mathcal{M}_{X_o}\to\mathfrak{D}_{X_o}\to 0$$

is such that $i(\phi)=1$ for every $\phi\in H^1(X_o,\mathcal{O}^*_{X_o})$.

<u>Proof of Step 2:</u>

<u>DEFINITION.</u>

Let $\mathfrak{U}$ be an open subset in $\mathfrak{X}_D$, define $\mathcal{M}^1_{\mathfrak{X}_D}(\mathfrak{U}):=\{$all meromorphic functions $f\neq 0$ on $\mathfrak{U}|$ $(f)_o\neq\mathfrak{U}\cap X_o$ and $(f)_\infty\neq\mathfrak{U}\cap X_o\}$. We denote the zero set of f by $(f)_o$ and $(f)_\infty=(\frac{1}{f})_o$. Let $\mathcal{M}^1_{\mathfrak{X}_D}$ be the sheaf obtain from the presheaf $\mathcal{M}^1_{\mathfrak{X}_D}(\mathfrak{U})$.

<u>REMARK.</u> Let $\mathfrak{U}$ be an open subset in $\mathfrak{X}_D$ and let $(t,\zeta^1,\ldots,\zeta^{2n})$ be local coordinates in $\mathfrak{U}$, then if $f_{\mathfrak{U}}(t,\zeta^1,\ldots,\zeta^{2n})\in\mathcal{M}^1_{\mathfrak{X}_D}(\mathfrak{U})$ it ca be expressed as:

$$f_{\mathfrak{U}}(t,\zeta^1,\ldots,\zeta^{2n})=\phi^o_{\mathfrak{U}}(\zeta^1,\ldots,\zeta^{2n})+\sum_{i=1}^{\infty}t^i\phi^i_{\mathfrak{U}}(\zeta^1,\ldots,\zeta^{2n})$$

where $\phi^i_{\mathfrak{U}}(\zeta^1,\ldots,\zeta^{2n})$ are meromorphic functions.

Hence the sheaf $\mathcal{M}^1_{\mathfrak{X}_D}$ corresponding to the presheaf $\mathcal{M}^1_{\mathfrak{X}_D}(\mathfrak{U})$ is a subsheaf of the sheaf of the meromorphic functions on $\mathfrak{X}_D$. We have the following exact sequence:

$$0\to\mathcal{O}^*_{\mathfrak{X}_D}\to\mathcal{M}^1_{\mathfrak{X}_D}\to\mathfrak{D}^1_{\mathfrak{X}_D}\to 0$$

Since $H^1(\mathcal{O}^*_{\mathfrak{X}_D})=0$ the following inclusion is obtained:

$$(5.2.1.)\qquad 0\to H^1(\mathcal{M}^1_{\mathfrak{X}_D})\to H^1(\mathfrak{D}^1_{\mathfrak{X}_D})\to$$

<u>Proposition 2.</u>

The restriction map $r:H^1(\mathfrak{X}_D,\mathcal{M}^1_{\mathfrak{X}_D})\to H^1(X_o,\mathcal{M}_{X_o})$, induced from the restriction of the sheaf $\mathcal{M}^1_{\mathfrak{X}_D}$ on X_o, i.e.

$$r:\mathcal{M}^1_{\mathfrak{X}_D}\to\mathcal{M}_{X_o}\to 1$$

where $r(\phi^o_{\mathcal{U}}(\zeta^1,\ldots,\zeta^{2n})+\sum_{i=1}^{\infty}t^i\phi^i_{\mathcal{U}}(\zeta^1,\ldots,\zeta^{2n}))=\phi^o_{\mathcal{U}}(\zeta^1,\ldots,\zeta^{2n})|_{\mathcal{U}\cap X_o}$ is a map onto.

<u>Proof:</u> We have the following exact sequence of sheaves:

$$1\to\mathcal{M}^1_{\mathfrak{X}_D}(1)\to\mathcal{M}^1_{\mathfrak{X}_D}\to\mathcal{M}_{X_o}\to 1$$

where $\mathcal{M}^1_{\mathfrak{X}_D}(1)(\mathcal{U}):=\{f_{\mathcal{U}}(\zeta^1,\ldots,\zeta^{2n},t)\in\mathcal{M}^1_{\mathfrak{X}_D}\mid f_{\mathcal{U}}|_{\mathcal{U}\cap X_o}=1\}$.

<u>REMARK.</u>

It is easy to prove that if $f_{\mathcal{U}}\in\mathcal{M}^1_{\mathfrak{X}_D}(1)(\mathcal{U})$, then

$$f_{\mathcal{U}}=1+\sum_{i=1}^{\infty}t^i f^i_{\mathcal{U}}(\zeta^1,\ldots,\zeta^{2n}))$$

where $f^i_{\mathcal{U}}(\zeta^1,\ldots,\zeta^{2n})$ are meromorphic functions depending only on $\{\zeta^1,\ldots,\zeta^{2n}\}$, i.e. for each i $f^i_{\mathcal{U}}$ is such that $(f^i_{\mathcal{U}})_o\neq\mathcal{U}\cap X_o$ and $(f^i_{\mathcal{U}})_\infty\neq\mathcal{U}\cap X_o$.

We have the long exact sequence:

(5.2.2.1.) $$\to H^1(\mathcal{M}^1_{\mathfrak{X}_D})\to H^1(\mathcal{M}_{X_o})\to H^2(\mathcal{M}^1_{\mathfrak{X}_D}(1))\xrightarrow{i^*}H^2(\mathcal{M}^1_{\mathfrak{X}_D})\to..$$

If we prove that the map

$$H^2(\mathcal{M}^1_{\mathfrak{X}_D}(1))\xrightarrow{i^*}H^2(\mathcal{M}^1_{\mathfrak{X}_D})$$

has kernal equal to 1, then

$$H^1(\mathcal{M}^1_{\mathfrak{X}_D})\to H^1(\mathcal{M}_{X_o})$$

is surjective. Thus we need to prove the following sublemma:

<u>Sublemma 2.1.</u> The map

$$i^*:H^2(\mathcal{M}^1_{\mathfrak{X}_D}(1))\to H^2(\mathcal{M}^1_{\mathfrak{X}_D})$$

has no kernal, i.e. ker $i^*=1$.

<u>Proof:</u> Suppose that $\phi\in H^2(\mathcal{M}^1_{\mathfrak{X}_D}(1))$, then from the definition of Cheh's cohomology, it follows that

$$\phi=\{\phi_{ijk}\}\in\prod_{i<j<k}\Gamma(U_i\cap U_j\cap U_k,\mathcal{M}^1_{\mathfrak{X}_D}(1))$$

and

$$\{(\delta\phi)_{ijkl}\}=\{\phi_{ijk}\phi_{jkl}\phi_{kli}\phi_{lij}=1\},\text{ where}$$
$$\phi_{ijk}=1+\sum_{i=1}^{\infty}t^i\phi^i_{ijk}$$

Suppose that $i^*\phi=1$, hence

(5.2.2.2.1) $$\{\phi_{ijk}\}=\{\phi_{ij}\phi_{jk}\phi_{ki}\},\text{ where}$$

$$\phi_{ij}=\phi^o_{ij}+\sum_{\mu=1}^{\infty}t^{\mu}\phi^{\mu}_{ij}\in\Gamma(U_i\cap U_j,\mathcal{M}^1_{\mathfrak{X}_D})\text{ and}$$

$$\phi_{ijk}=1+\sum_{i=1}^{\infty}t^i\phi^i_{ijk}$$

From (5.2.2.2.1) we get that

(5.2.2.2.2) $$\phi^o_{ij}\phi^o_{jk}\phi^o_{ki}=1$$

Let

$$\{\phi^1_{ij}\}=\{\phi_{ij}(\phi^o_{ij})^{-1}\}=\{1+\sum_{\mu=1}^{\infty}t^{\mu}(\phi^{\mu}_{ij}(\phi^o_{ij})^{-1})\}$$

Clearly $\{\phi^1_{ij}\}\in\prod_{i<j}\Gamma(U_i\cap U_j,\mathcal{M}^1_{\mathfrak{X}_D}(1))$. Since

$$\{\phi^1_{ij}\phi^1_{jk}\phi^1_{ki}\}=\{\phi_{ij}(\phi^o_{ij})^{-1}\phi_{jk}(\phi^o_{jk})^{-1}\phi_{ki}(\phi^o_{ki})^{-1}\}=\{(\phi_{ij}\phi_{jk}\phi_{ki})(\phi^o_{ij}\phi^o_{jk}\phi^o_{ki}=1)^{-1}\}=\{\phi_{ijk}\}$$

Hence $\{\phi_{ijk}\}\in\prod_{i<j}\Gamma(U_i\cap U_j,\mathcal{M}^1_{\mathfrak{X}_D}(1))$ is a coboundary. From here we get that $\ker i^*=1$. Q.E.D.

Proposition 2. follows from (5.2.2.1.). Q.E.D.

<u>End of the proof of Step 2.</u>

Let $\{\mathfrak{U}^o_i\}$ be a covering of X_o. Since $\mathfrak{X}_D$ as C^{∞} manifold is diffeomorphic to $X_o xD$, then $\{\mathfrak{U}_i=\mathfrak{U}^o_i xD\}$ is a covering of $\mathfrak{X}_D$. We may suppose that $\mathfrak{U}_i$ are polycilinders. If $\mathfrak{U}_i$ is a Stein manifold then $H^k(\mathcal{U}_i,\mathcal{M}_{\mathcal{U}_i})=1$ for $k>0$. Since Cheh's cohomologies with respect to the usual topology are isomorphic to the Cheh's cohomologies with respect to the Zariski topology (See [07]) hence

Sublemma 2.2. $H^k(\mathcal{U}_i,\mathcal{M}^1_{\mathfrak{S}_D}|\mathcal{U}_i)=1$ for $k>0$.

Proof:

If $\{\mathcal{W}_j\}$ is a covering of $\mathcal{U}_i$ by Zariski open sets, then if $\{\phi_{i_o i_1,i_2,..,i_k}\}$ is k-cocyle, i.e.

$$\{\phi_{i_o,i_1,i_2,..,i_k}\}\in\prod_{i_o<..<i_k}\Gamma(\mathcal{W}_{i_o}\cap..\cap\mathcal{W}_{i_k},\mathcal{M}^1_{\mathfrak{S}_D}|\mathcal{U}_i)$$

From the definition of a ***Zariski open set*** it follows that we can consider $\phi_{i_o,i_1,i_2,..,i_k}$ as a section of $\Gamma(\mathcal{W}_{i_o}\cap..\cap\hat{\mathcal{W}}_{i_j}.\cap\mathcal{W}_{i_k},\mathcal{M}^1_{\mathfrak{S}_D}|\mathcal{U}_i)$, where intersection with $\mathcal{W}_{i_j}$ is missing. This holds since $\phi_{i_o,i_1,i_2,..,i_k}$ is a meromorphic function. We can define from the k-cocycle $\{\phi_{i_o,i_1,i_2,..,i_k}\}$, $k-1$-cocycle in the following way: Let $I_k=(i_o,...,i_k)$ be a $(k+1)$ multiindex and let I_{k-1} be a k-1 multiindex. If $I_k=(\gamma,I_{k-1})$, then

$$\{\sigma_{I_{k-1}}\}=\{\prod_\gamma\phi_{(\gamma,I_{k-1})}\}$$

Then from $\delta\{\phi_{i_o,i_1,i_2,..,i_k}\}=1$ it follows that $\delta\{\sigma_{I_{k-1}}\}=\{\phi_{i_o,i_1,i_2,..,i_k}\}$. (See [07].). Q.E.D.

Remark.

From Sublemma 2.2. and a Theorem of Leray, it follows that $H^k(\mathfrak{S}_D,\mathcal{M}^1_{\mathfrak{S}_D})$ are defined from a Cheh's complex, arizing from a finite covering of $\mathfrak{S}_D$ by $\{\mathcal{U}_i\}$. The last statemnt is true since X_o is compact.

Now we are ready to prove that the map $i:H^1(X_o,\mathcal{O}^*_{X_o})\rightarrow H^1(X_o,\mathcal{M}_{X_o})$ induced from the exact sequence

$$(*)\qquad 0\rightarrow\mathcal{O}^*_{X_o}\rightarrow\mathcal{M}_{X_o}\rightarrow\mathcal{D}_{X_o}\rightarrow 0$$

is such that $i(\phi)=1$ for every $\phi\in H^1(X_o,\mathcal{O}^*_{X_o})$.

Let $\{\phi^o_{ij}\}\in H^1(X_o,\mathcal{O}^*_{X_o})$, where

$$\{\phi^{o}_{ij}\}\in\prod_{i<j}\Gamma(\mathfrak{U}_i\cap\mathfrak{U}_j,\mathcal{O}^{*}_{X_o})\subset\prod_{i<j}\Gamma(\mathfrak{U}_i\cap\mathfrak{U}_j,\mathfrak{D}_{X_o})$$

i.e. we may consider $\{\phi^{o}_{ij}\}\in H^1(X_o,\mathfrak{D}_{X_o})$. Since the map

$$H^1(\mathfrak{X}_D,\mathfrak{D}^1_{\mathfrak{X}_D})\to H^1(X_o,\mathfrak{D}_{X_o})\to 0$$

is surjective an element $\{\phi_{ij}\}\in H^1(\mathfrak{X}_D,\mathfrak{D}^1_{\mathfrak{X}_D})$ can be found such that

$$\{\phi_{ij}\}\in\prod_{i<j}(\mathfrak{U}_i\cap\mathfrak{U}_j,\mathcal{M}^1_{\mathfrak{X}_D})\ \delta(\{\phi_{ij}\}=1$$

and

$$\{\phi_{ij}\}|_{X_o}=\{\phi^{o}_{ij}\}$$

Here we can think of $\{\phi_{ij}\}$ as an element of $\prod_{i<j}(U_i\cap U_j,\mathcal{M}_{X_o})$ since

$$0\to\prod_{i<j}(\mathfrak{U}_i\cap\mathfrak{U}_j,\mathcal{O}^{*}_{X_o})\to\prod_{i<j}(\mathfrak{U}_i\cap\mathfrak{U}_j,\mathcal{M}_{X_o})\to$$

From the defintion of $\mathfrak{D}^1_{\mathfrak{X}_D}$ it follows that for each (i,j,)

$$\phi_{ij}=\tilde{\phi}^{o}_{ij}+\sum_{\mu=1}^{\infty}t^{\mu}\phi^{\mu}_{ij},\text{ where}$$

$\tilde{\phi}^{o}_{ij}=\phi^{o}_{ij}(\xi^1,....,\xi^{2n})$, $\phi^{\mu}_{ij}=\phi^{\mu}_{ij}(\xi^1,....,\xi^{2n})$. Since $\phi_{ij}|_{X_o}=\tilde{\phi}^{o}_{ij}|_{X_o}$ we may suppose that

$$\phi_{ij}|_{X_o}=\tilde{\phi}^{o}_{ij}$$

Notice that $\tilde{\phi}^{o}_{ij}\tilde{\phi}^{o}_{jk}\tilde{\phi}^{o}_{ki}=1$ and hence $\{\tilde{\phi}^{o}_{ij}\}$ is a cocycle. Since $\{\tilde{\phi}^{o}_{ij}\}$ are only finite numbers,

$$\tilde{\phi}^{o}_{ij}|_{X_o}=\phi^{o}_{ij}$$

and all ϕ^{o}_{ij} has no zeroes and no poles in $\mathfrak{U}^{o}_i\cap\mathfrak{U}^{o}_j$ we may suppose that $\tilde{\phi}^{o}_{ij}$ has no zeroes and no poles in $(\mathfrak{U}^{o}_i x D)\cap(\mathfrak{U}^{o}_j x D)=\mathfrak{U}_i\cap\mathfrak{U}_j$. The inclusion $H^1(\mathcal{M}^1_{\mathfrak{X}_D})\subset H^1(\mathfrak{D}^1_{\mathfrak{X}_D})$ and the definition of $\mathfrak{D}^1_{\mathfrak{X}_D}:=\mathcal{M}^1_{\mathfrak{X}_D}/\mathcal{O}^{*}_{\mathfrak{X}_D}$ imply that

$$\{\tilde{\phi}^{o}_{ij}\}=1\text{ in }H^1(\mathcal{M}^1_{\mathfrak{X}_D}).$$

Thus $\{\tilde{\phi}^{o}_{ij}\}$ is 1 in $H^1(\mathfrak{D}^1_{\mathfrak{X}_D})$ which means that $\{\tilde{\phi}^{o}_{ij}\}|_{X_o}=\{\phi^{o}_{ij}\}$ is 1 in $H^1(\mathcal{M}_{X_o})$. and Step 2. is established

Q.E.D.

From Step 2 it follows that the coboundary map

$$\delta: H^o(X_o, \mathfrak{D}_{X_o}) \to H^1(X_o, \mathcal{O}^*_{X_o})$$

is a map onto. Hence if $\mathcal{L}$ is a line bundle on X_o, then $\mathcal{L} = \mathcal{O}_{X_o}(Y)$, where Y is a divisor on X_o. This is true, since $\mathcal{L} \in H^1(X_o, \mathcal{O}^*_{X_o})$ and hence $\mathcal{L} = \delta(Y)$, $Y \in H^o(X_o, \mathfrak{D}_{X_o})$.

Q.E.D.

<u>End of the proof of the THEOREM 5.1.</u>

The proof is standart and is based on the following induction hypothesis:

Let Y be a compact analytic manifold such that $\dim_{\mathbf{C}} Y < \dim_{\mathbf{C}} X$ *and* $\mathcal{L} \doteq \mathcal{O}_Y(D')$ *where* D' *is a divosor on* Y. *Suppose that for any analytic submanifold* C_k *of* dim=k *we have:*

$$\int_{C_k} c_1(\mathcal{L})^k > 0$$

then Y is a projective manifold and $\mathcal{L}^n$ *is defining a holomorphic birational map* $Y \to \mathbb{P}^n$. Let X_o satisfies the assumptions of the Theorem, i.e. X_o is a holomorphic symplectic manifold and let $\mathcal{L}$ be a holomorphic line bundle on X_o such that for any analytic submanifold $C_k \subset X_o$ of $\dim_{\mathbf{C}} = k$ we have:

$$\int_{C_k} c_1(\mathcal{L})^k > 0$$

From LEMMA 5.1.1. it follows that $\mathcal{L} \doteq \mathcal{O}_{X_o}(D')$, where D' is a divisor on X_o. Let X'_o be obtained from X_o by blowing up one point on X_o. Notice that if we prove our THEOREM for X'_o it will also hold for X_o.

We can consider on the blown up manifold X'_o the exceptional divisor $\mathbf{CP}^{2n-1} = Y$ and the line bundle $\mathcal{L}(D)$ on X'_o, where $D = ND' - Y$, $N \in \mathbf{Z}_+$ and N big enouph.. Clearly $\mathcal{L}$ on X'_o fulfills the hypothesis of the Theorem and Y fulfills the induction huposesis, i.e. for some big N, $\mathcal{L}^N|_Y$ gives a holomorphic map: $Y \to \mathbf{P}^{N^1}$ which is birational. Thus

(***) $H^i(\mathcal{L}^N|_Y)=0$ for $i\geq 1$

The exact sequence:

(****) $$0\to H^o(\mathcal{L}^N(-Y))\to H^o(\mathcal{L}^N)\to H^o(\mathcal{L}^N|_Y)\to H^1(\mathcal{L}^N(-Y))\to H^1(\mathcal{L}^N)\to 0$$

yields that $\dim_{\mathbf{C}} H^1(\mathcal{L}^N(-Y))\geq \dim_{\mathbf{C}} H^1(\mathcal{L}^N)$

The exact sequence:

$$0\to\mathcal{L}^N(-Y)\to\mathcal{L}^N\to\mathcal{L}^N|_Y\to 0$$

Hirzebruch-Riemann-Roch THEOREM and the fact that $\dim_{\mathbf{C}} H^1(\mathcal{L}^N(-Y))\geq \dim_{\mathbf{C}} H^1(\mathcal{L}^N)$ we conclude that:

$$\dim_{\mathbf{C}} H^o(\mathcal{L}^N)-\dim_{\mathbf{C}} H^o(\mathcal{L}^N(-Y))\geq\chi(\mathcal{L}^N)-\chi(\mathcal{L}^N(-Y))=\chi(\mathcal{L}^N|_Y)= aN^{\dim X-1}-\dots$$

where a is positive integer. Thus $\dim_{\mathbf{C}} H^o(\mathcal{L}^N)>0$. Let $\mathcal{L}=\mathcal{L}^N=\mathcal{O}_{X_o}(D')$ and hence D' is effective. From the exact sequences, where M is big enouph positive integer:

$$0\to\mathcal{L}^{M-1}\to\mathcal{L}^M\to\mathcal{L}^M|_{D'}\to 0$$

$$0\to H^o(\mathcal{L}^{M-1})\to H^o(\mathcal{L}^M)\to H^o(\mathcal{L}^M|_{D'})\to H^1(\mathcal{L}^{M-1})\to H^1(\mathcal{L}^M)\to H^1(\mathcal{L}^M|_{D'})\to\dots$$

By the induction hypothesis it follows that $H^i(\mathcal{L}^M|_{D'})=0$ hence

$$\dim_{\mathbf{C}} H^1(\mathcal{L}^M)\leq\dim_{\mathbf{C}} H^1(\mathcal{L}^{M-1})\leq\dots\leq\dim_{\mathbf{C}} H^1(\mathcal{L})$$

and

$$\dim_{\mathbf{C}} H^i(\mathcal{L}^M)=\dim_{\mathbf{C}} H^i(\mathcal{L}^{M+1}) \text{ for } i\geq 2$$

From Hirzebruch-Riemann-Roch THEOREM (See [12]) and the facts that *a)* $\int_X c_1(\mathcal{L})^n>0$ where $2n=\dim_{\mathbf{C}} X$ and *b)* $\dim_{\mathbf{C}} H^i(X,\mathcal{L}^M)\leq b_i$, where $b_i\in\mathbf{Z}_+$ we get that

$$\dim_{\mathbf{C}} H^o(X,\mathcal{L}^M)=aM^{\dim_{\mathbf{C}} X=2n}+\dots$$

where $a\in\mathbf{Z}_+$. This equality shows that X is a Moishezon space. Now our THEOREM follows from Moishezon-Nakai THEOREM. (See [13].). Q.E.D.

REMARK. Combining the results of #5 & #4 we get that in the Kuranishi family $\mathfrak{X} \rightarrow U$, more precisely in U, there is an open and everywhere dense subset U' such that the points of U' corresponds to Kähler holomorphic symplectic manifolds.

#6. REVIEW OF THE ISOMETRIC DEFORMATIONS .

Definition 6.1.

A **Kähler metric** $(g_{\alpha\overline{\beta}})$ on a holomorphic symplectic manifold X will be called a CALABI-YAU metric if

$$\mathrm{Ricci}(g_{\alpha\overline{\beta}})=\partial\overline{\partial}\log(\det(g_{a\overline{\beta}}))=0$$

The existence of a CALABI-YAU metric follows from the deep work of YAU. (See [19].) The CALABI-YAU metric $(g_{a\overline{\beta}})$ induces a covariant differenciation ∇ on $\wedge^2 T^*X\otimes C$. (See [02].)

LEMMA 6.2.

$\mathrm{Re}\,\omega_X(2,0),\mathrm{Im}\,\omega_X(2,0)$ and $\mathrm{Im}(g_{a\overline{\beta}})$ are parallel sections of $\Gamma(X,\wedge^2 T^*X)$ with respect to ∇.

Proof: See [02]. This is the so called BOCHNER principle.

Q.E.D.

Suppose that $*$ is the HODGE star operator with respect to the CALABI-YAU metric and

$$\int_X \mathrm{Re}\,\omega_X(2,0)\wedge*\mathrm{Re}\,\omega_X(2,0)=\int_X \mathrm{Im}\,\omega_X(2,0)\wedge*\mathrm{Im}\,\omega_X(2,0)=\int_X \mathrm{Im}(g_{\alpha\overline{\beta}})\wedge*\mathrm{Im}(g_{\alpha\overline{\beta}})=1$$

$\mathrm{Re}\,\omega_X(2,0),\mathrm{Im}\,\omega_X(2,0)$ and $\mathrm{Im}(g_{a\overline{\beta}})$ define a three dimensional subspace $E_X(L)$ in $\Gamma(X,\wedge^2 T^*X)$ and since $\mathrm{Re}\,\omega_X(2,0),\mathrm{Im}\,\omega_X(2,0)$ and $\mathrm{Im}(g_{a\overline{\beta}})$ are harmonic forms with respect to the CALABI-YAU metric $E_X(L)$ is a three dimensional subspace in $H^2(X,\mathbf{R})$. It is easy to see that $q|_{E_X(L)}$ is positive definite. (See [16].)

Let $\gamma=a\mathrm{Re}\omega_X(2,0)+b\mathrm{Im}\omega_X(2,0)+c\mathrm{Im}(g_{a\overline{\beta}})$, where a,b and $c\in\mathbf{R}$ and $a^2+b^2+c^2=1$. Since $\gamma\in E_X(L)$, then $\nabla\gamma=0$. Locally γ can be written in the following way:

$$\gamma=\sum\gamma_{\mu\sigma}dx^{\mu}\wedge dx^{\sigma}$$

If $\sum g_{\nu\mu}dx^{\nu}\otimes dx^{\mu}$ is the Riemannian Ricci flat metric on X defined by the CALABI-YAU metric $(g_{\alpha\overline{\beta}})$ on X, then

$$J(\gamma)=(J(\gamma)^{\alpha}_{\beta})\overset{\mathrm{def}}{=}(\sum_{v}g^{\alpha v}\gamma_{v\beta})\in\Gamma(X,T^{*}\otimes T)$$

LEMMA 6.3.

a) $J(\gamma)$ defines a new integrable complex structure on X.

b) γ is an imaginary part of a CALABI-YAU metric with respect to the complex structure $J(\gamma)$. The CALABI-YAU metrics defined by γ and $J(\gamma)$ are equivelent to the CALABI-YAU metric $g_{\alpha\overline{\beta}}$, that we started with.

c) Suppose that $(X;\delta_1,...,\delta_{2n})$ is a marked Hyper-Kählerian manifold and suppose that

$$p(X;\delta_1,...,\delta_{2n})=x_0\in\Omega\subset\mathbf{P}(H^2(X,\mathbf{C}))\ (\text{p is the period map})$$

then there is a one to one map via p between the complex structures $J(\gamma)$ on X, where

$$\gamma=a\mathrm{Re}\omega_X(2,0)+b\mathrm{Im}\omega_X(2,0)+c\mathrm{Im}(g_{a\overline{\beta}}),\ a,b \text{ and } c\in\mathbf{R},\ a^2+b^2+c^2=1$$

and the points of the non-singular quadric

$$\mathbf{P}(E_X(L)\otimes\mathbf{C})\cap\Omega=\mathbf{P}^1_X(L)$$

Proof: See [16] or [17].

Q.E.D.

Remark.

Notice that $J(\mathrm{Im}g_{\alpha\overline{\beta}})$ is the original complex structure on X, Lemma 6.3.c. yields that $x_0 \in \mathbf{P}^1_X(L)$.

#7. CONSTRUCTION OF A SPECIAL FAMILY OF KÄHLER MANIFOLDS..

Definition 7.1.

$N \stackrel{def}{=} \cup\{E \subset H^2(X,\mathbf{R}) \mid$ E is spanned by $Re\omega_X(2,0)$, $Im\omega_X(2,0)$ and ϕ, where $\phi \in K(X)\}$. K(X) is defined in 6.4.. N is a subset in $H^2(X,\mathbf{R})$. Suppose that K(X) is spanned by all $\omega \in \Gamma(X,\wedge^2 T^*X)$, where ω were constructed in THEOREM 1..

N as a subset in $H^2(X,\mathbf{R})$ is diffeomorphic to $E_X x K(X)$, where $E_X := \{Re\omega_X(2,0), Im\omega_X(2,0)\} \subset H^2(X,\mathbf{R})$ therefore N is an open subset in $H^2(X,\mathbf{R})$.(See [16].)

Remark 7.2.a.

In #4.2. we introduce a quadratic form q over $\mathbf{Z}$. This quadratic form has a signature $(3,b_2-3)$. This was proved by Beauville. (See [02].) Let $<\ ,\ >$ be the scalar product defined by q on $H^2(X,\mathbf{R})$.

Remark 7.2.b.

From the definition of N it follows that N is the union of three dimensional subspaces $E \subset H^2(X,\mathbf{R})$ which have the following properties:

1) $<\ ,\ >$ on E is positive definite.

2) E contains E_X, where E_X is spanned by $\{Re\omega_X(2,0), Im\omega_X(2,0)\}$

In [16] the following PROPOSITION was proved:

PROPOSITION 7.3.

There is a one to one map between the points of Ω and all oriented two planes in $H^2(X,\mathbf{R})$ on which $<\ ,\ >$ is positive.

PROPOSITION 7.4.

Let $\pi:\mathfrak{X}\to U$ be the Kuranishi family constructed in #3. and let $\pi^{-1}(o)=X$, then there exists a disc $D\subset U$ such that $o\in D$ Le $\mathfrak{X}'_D\to D$ be the family of Kähler holomorphic symplectic manifolds such that over an open and everywhere dense subset $\mathfrak{D}\subset D$ the restriction $\mathfrak{X}_{\mathfrak{D}}\to\mathfrak{D}$ of $\pi:\mathfrak{X}\to U$ over $\mathfrak{D}$ is isomorphic to the restriction of $\mathfrak{X}'_D\to D$ over $\mathfrak{D}$.

Proof: Let $N(\mathbf{Q}):=\cup\{E\subset N|\ E\cap H^2(X,\mathbf{Q})\neq\emptyset\}$. Since N is an open subset in $H^2(X,\mathbf{R})$, it follows that $N(\mathbf{Q})$ is an everywhere dense subset in N. By the continuity argument we can choose $L\in N(\mathbf{Q})$ such that

a) $L=a\mathrm{Re}\omega_X(2,0)+b\mathrm{Im}\omega_X(2,0)+c\omega$ with $L^{1,1}$ is positive definite. Recall that ω is a form such that $\omega^{1,1}$ is positive definite.

b) If E_t is the orthogonal complement to L in the subspace E spanned by L, $\mathrm{Re}\omega_X(2,0)$ and $\mathrm{Im}\omega_X(2,0)$ in $H^2(X,\mathbf{R})$, then via the period map E_t corresponds to a point $t\in U$, where $\pi:\mathfrak{X}\to U$ is the Kuranishi space constructed in #3., i.e. E_t is spanned by $\mathrm{Re}\omega_t(2,0)$ and $\mathrm{Im}\omega_t(2,0)$, where $\omega_t(2,0)$ is the holomorphic closed two form on $X_t=\pi^{-1}(t)$.

From 2.2., 2.2.3. and Moishezon-Nakai criterium it follows that X_t is an algebraic manifold, hence $t\in U'\subset U$ defined in 4.5.. Let $(g_{\alpha\overline{\beta}}(t))$ be the CALABI-YAU metric on X_t which corresponds to L. Now we can define the isometric deformation $\mathfrak{X}(L)\to S^2$ of X_t with respect to $g_{\alpha\overline{\beta}}(t)$, therefore this family is mapped by the period map p onto $\mathbf{P}(E\otimes\mathbf{C})\cap\Omega$, according to #6.3.. Notice that $\mathbf{P}(E\otimes\mathbf{C})\cap\Omega$ is a projective non-singular plane curve of degree two, contained in Ω. (See [16].) On the other hand from the definition of E, i.e. $E\subset E(\mathbf{Q})$ and # 7.2. it follows that

$$U\cap(\mathbf{P}(E\otimes\mathbf{C})\cap\Omega)=D$$

is an open disk. Since $E_X:=\{\mathrm{Re}\omega_X(2,0),\ \mathrm{Im}\omega_X(2,0)\}\subset E(L)$ it follows that

$$p(X,\delta_1,\dots,\delta_{b_2})=o\in D=U\cap(\mathbf{P}(E\otimes\mathbf{C})\cap\Omega)$$

In [02] it was proved that for Kähler holomorphic symplectic manifolds there is an everywhere dense subset in Ω such that each point of this everywhere dense subset corresponds to algebraic holomorphic symplectic manifolds This subset is of the form, some union $H_L \cap \Omega$, where $H_L := \{u \in \Omega \mid \langle u, L \rangle = 0\}$ and L are vectors in $H^2(X,\mathbf{Q})$. Since $H_L \cap \mathbf{P}(E \otimes \mathbf{C}) \cap \Omega \neq \emptyset$ we get in D an everywhere dense subset of algebraic holomorphic symplectic manifolds, hence from here we get that $\mathfrak{D} = U' \cap D$ (U' is defined in # 6 and every point of U' corresponds to a HYPER-KÄHLERIAN MANIFOLD) is an open and everywhere dense subset in D.

Over $D = U \cap (\mathbf{P}(E \otimes \mathbf{C}) \cap \Omega)$ there are two families. The first one is the restriction of $\pi : \mathfrak{X} \to U$ and the second family $\mathfrak{X}'_D \to D$ is obtained by the isometric deformations. From local TORELLI THEOREM it follows that both these families are isomorphic over $\mathfrak{D}$.

Q.E.D.

<u>Cor. 7.4.1.</u>

There exists a biholomorphic mapping

$$\begin{array}{ccc} \mathfrak{X}'_{\mathfrak{D}} & \to & \mathfrak{X}_{\mathfrak{D}} \\ f: \downarrow & & \downarrow \\ \mathfrak{D} & \to & \mathfrak{D} \end{array}$$

such that f induces the identity on $H^2(X,\mathbf{Z})$.

<u>Proof:</u> The existence of f was established in Proposition 7.4. since f is an isomorphism of marked Holomorphic symplectic manifodls it follows that f induces identity on $H^2(X,\mathbf{Z})$.

Q.E.D.

Next we will prove, using a THEOREM of BISHOP and the existence of KÄHLER-EINSTEIN-CALABI-YAU metric that f can be extended to an isomorphism over D. This will imply that X is a Kählerian manifold. The idea of using BISHOP'S THEOREM in extending isomorphisms belongs to DELIGNE as it is pointed out in the paper of D. BURNS and M. RAPOPORT. See [20].

#8 APPLICATION OF BISHOP'S CRITERIUM.

LEMMA 8.1. (Burns and Rapoport, Siu)

Let $\pi:\mathfrak{X}\to U$ and $\pi':\mathfrak{X}'\to U$ be two holomorphic families of symplectic manifolds with a complex manifold as a parameter space so that both are diffeomorphically identified with a trivial family $X\times U\to U$. Let $X_s=\pi^{-1}(s)$ and $X'_s=\pi'^{-1}(s)$ for $s\in U$. Let s_o be a point of U and let A be a subset of U such that s_o is an accumulation point of A. Assume the following two conditions. *i)* X_{s_o} is Kähler. *ii)* For $s\in A$ the two symplectic holomorphic manifolds X_s and X'_s are biholomorphic under a map f_s which induces $\tau=id$ on $H^2(X,C)$.

THEN X_s and X'_s are ***biholomophic.*** (See [15].)

Proof: First we will prove that X_s and X'_s bimeromorphic.

From THEOREM 1. we know that there exists a real d close 2-form ω on the underlying differentiable structure X such that (1,1)-component $\omega^{1,1}$ of ω with respect to the complex structure of X'_{s_o} is positive definite at every point of X'_{s_o}. By continuity arguments there exists an open neighborhood W of s_o in U such that for $s\in W$ the (1,1) component $\omega^{1,1}(s)$ of $\omega(s)$ with respect to the complex structure of X'_s is positive definite at every point of X'_s.

Since X_{s_o} is assumed Kähler, (after shrinking W if necessary) we have for every $s\in W$ a Kähler form $\theta(s)$ which depends smoothly on s. Let η be a positive definite (1,1)-form on W. The collection of (1,1) forms $\omega^{1,1}(s)$ on X'_s, $\theta(s)$ and η, define a Hermitian metric H on $\mathfrak{X}\times_W\mathfrak{X}'$. Then the pullback of H to the submanifold $X_s\times X'_s$ of $\mathfrak{X}\times_W\mathfrak{X}'$ is equal to

$$\theta(s)+\omega^{1,1}(s)$$

where for notational complicity we use $\theta(s)+\omega^{1,1}(s)$ to denote their pullbacks under the projections from $X_s\times X'_s$ to X_s and X'_s respectively.

For $s\in W\cap A$ let $\Gamma_s\subset X_s\times X'_s$ be the graph of the holomorphic map $f_s:X_s\to X'_s$. We want to compute the volume of Γ_s with respect to H on $\mathfrak{X}\times_W\mathfrak{X}'$ and show that it is bounded as s approaches s_o hence that we can apply BISHOP'S THEOREM to conclude the convergence of the subvariety Γ_s in $\mathfrak{X}\times_W\mathfrak{X}'$ as s approaches s_o.

PROPOSITION 8.1.1. $vol(\Gamma_s)<C$ for every $s\in A$.

Proof: It is easy to see that:

$$vol(\Gamma_s)=\int (f_s^*\omega^{1,1}(s)+\theta(s))^{2n}$$

Let $\phi(s)=\int_{X_s} (f_s^*\omega(s)+\theta(s))^{2n}$. Recall that $f_s^*\omega(s)+\theta(s)$ is a class of cohomology and τ=id on $H^2(X,C)$. We will prove that the following inequalities hold:

$$\text{vol}(\Gamma_s)<\phi(s)=\int_{X_s}(f_s^*\omega^{1,1}(s)+\theta(s))^{2n}<C$$

First we will show that $\phi(s)<C$. Indeed

$$\phi(s)=\int_{X_s}(\tau[\omega(s)]+[\theta(s)])^{2n}$$

From here it follows that

$$\phi(s)<C$$

Hence we need to prove that

$$(*)\qquad \text{vol}(\Gamma_s)<\phi(s)$$

Proof of (*): Let $f_s^*\omega(s)=\omega^{2,0}(s)+\omega^{1,1}(s)+\omega^{0,2}(s)$, then

$$(8.1.1.1.)\qquad (\omega^{2,0}(s)+\omega^{1,1}(s)+\omega^{0,2}(s)+\theta(s))^{2n}=$$

$$(\omega^{2,0}(s))^{2n}\wedge(\omega^{0,2}(s))^{2n}+\sum c_k(\omega^{2,0}(s))^k\wedge(\omega^{0,2}(s))^k\wedge(\omega^{1,1}(s)+\theta(s))^{2n-2k}+$$

$(\omega^{1,1}(s)+\theta(s))^{2n}$, where $c_k\in\mathbf{Z}_+$, i.e. $c_k>0$

Notice that $*(\omega^{2,0}(s))^k=((\omega^{2,0}(s))^k\wedge(\omega^{1,1}(s)+\theta(s))^{2n-k}$, where $*$ is Hodge star operator with respect to the Hermitian metric H, where $\text{Im}H=\omega^{1,1}(s)+\theta(s)$ on $X_s x X_s'$. By integrating (8.1.1.1.) we get

$$(8.1.1.1.)\qquad \phi(s)=\|(\omega^{2,0}(s))^n\|^2+\sum c_k\|(\omega^{2,0}(s))^k\|^2+\text{vol}(\Gamma_s)$$

Therfore from (8.1.1.1.) we get that $\text{vol}(\Gamma_s)\leq\phi(s)<C$

Q.E.D.

For a subvariety Z of pure codimension in a complex manifold X, we denote by [Z] the current X defined by Z. Now we invoke BISHOP'S THEOREM (Sec [03]) to conclude that for some subsequence $\{s_\nu\}\subset A$ converging to s_0 the current $[\Gamma_{s_\nu}]$ over $\mathfrak{X}x_W\mathfrak{X}'$ converges weakly to a current on $\mathfrak{X}x_W\mathfrak{X}'$ of the form

$$\sum_{i=1}^{k} m_i[\Gamma^i]$$

where m_i is positive integer and Γ^i is an irreducible subvariety of complex dimension 2n on $X_{s_O} x X'_{s_O}$.

For any closed 4n-current θ on $X_{s_O} x X'_{s_O}$, define a linear map:

$$\theta_*: H^*(X_s, \mathbf{C}) \rightarrow H^*(X'_s, \mathbf{C})$$

of cohomology rings as follows: a cohomology class defined by a closed p-form α on X_s is mapped by θ_* to the cohomology class defined by the p-current

$$(pr_2)_*(\theta \wedge (pr_1)^* \alpha) \text{ on } X'_s$$

where pr_1 and pr_2 are respectively the projections of $X_{s_O} x X'_{s_O}$ onto the first and the second factors, and $pr_1{}^*$ and pr_{2*} are the corresponding pushforward and pullback maps. By reversing the rules of X_{s_O} and X'_{s_O}, we define analogously a linear map

$$\theta^*: H^*(X'_s, \mathbf{C}) \rightarrow H^*(X_s, \mathbf{C})$$

The map $[\Gamma_s]_*$ defined by the 4n-current $[\Gamma_s]$ in $X_{s_O} x X'_{s_O}$ clearly agrees with the map defined by f_* from $H^*(X_s, \mathbf{C})$ to $H^*(X'_s, \mathbf{C})$. Since f_s defines an isomorphism of $H^2(X, \mathbf{C})$ equal to id, by passing to the limit along the subsequence $\{s_n\}$ we conclude that

$$(\sum_{i=1}^{k} m_i[\Gamma^i])_*$$

is just the identity on $\wedge H^2(X, \mathbf{C})$.

Let

$$\omega_O(2n,0) := \wedge^{2n}(\omega_O(2,0))$$

be the non-zero holomorphic 2n form, which has no zeroes on X_{s_O}. Since

$$(\sum_{i=1}^{k} m_i[\Gamma^i])_* \text{is an isomorphism of } \wedge H^2(X, \mathbf{C})$$

it follows that 2n-current

$$(pr_2)_*(\sum_{i=1}^{k} m_i[\Gamma^i] \wedge (pr_1)^* \omega_O(2n,0))$$

on X'_{s_O} (which is automatically a holomorphic 2n-form on X'_{s_O}) can not be zero. Hence there must be some Γ^j which is projected both onto X_{s_O} and X'_{s_O}. There can be only one such Γ^j

and moreover $m_j=1$ and both projection maps are onto X_{s_0} and X'_{s_0} and are of degree one, because both

$$(\sum_{i=1}^{k} m_i[\Gamma^i])_* \text{ and } (\sum_{i=1}^{k} m_i[\Gamma^i])^*$$

must leave fixed the class in $H^0(X,C)$ which is defined by the function on X with constant values. From here we deduce that X_{s_0} and X'_{s_0} are bimeromorphically equivelent. To finish the proof of LEMMA 8.1. we need the following PROPOSITION:

PROPOSITION 8.1.2.

Let X_{s_0} be a holomorphic symplectic structure and X'_{s_0} be a HYPER-KÄHLERIAN structure on a C^∞ manifold X. Let $\Gamma \subset X_{s_0} x X'_{s_0}$ be a complex analytic subspace such that *a)* $\dim_C \Gamma = \dim_C X_{s_0}$, *b)* The projection maps $pr_1:\Gamma \rightarrow X_{s_0}$ and $pr_2:\Gamma \rightarrow X'_{s_0}$ are holomorphic maps of degree 1 and *c)* $(\Gamma)_*$ and $(\Gamma)^*$ induce the identity map on $\wedge H^2(X,C)$ *then* Γ induces a biholomorphic map between X_{s_0} and X'_{s_0}.

Proof of PROPOSITION 8.1.2.: The proof is based on the following Proposition:

PROPOSITION 8.1.2.1.

$f^*(Im(g_{\alpha\overline{\beta}}))$ is a globally defined C^∞-form of type (1,1) on X_{s_0} and it is a nondegenerate at each point $x \in X_{s_0}$.

Proof:

Rememmber that f was a bimeromorphic map therefore we can find $U=X_{s_0}\backslash\mathcal{A}$ and $U'=X'_{s_0}\backslash\mathcal{A}'$ such that $f:U \rightarrow U'$ is an isomorphism. $\mathcal{A}$ and $\mathcal{A}'$ are complex analytic subspaces in X_{s_0} and X'_{s_0} of complex codimension ≥ 2. Hence $f^*(\omega'(1,1))$ is well defined form on U, where $\omega'(1,1)$ is the imaginary part of the CALABI-YAU metric with respect to which we are making ***the isometric deformations.*** From the definition ***of isometric deformations*** it follows that we can find $A \in SO(3)$ such that A will define a new complex structure on X'_{s_0}, which we will denote by $X'^{A}_{s_0}$, with the following properties:

a) $Re\omega_{X'_{s_0}}(2,0)$ is a form of type (1,1) on $X'^{A}_{s_0}$

b) $\omega'^{A}(2,0)=\mathrm{Im}\,\omega_{X'_{s_o}}(2,0)+i\mathrm{Im}(g_{\alpha\overline{\beta}})$ is a holomorphic two form on $X'^{A}_{s_o}$.

Clearly the pullback of the integrable complex structure I^A that defines $X'^{A}_{s_o}$ via f defines an integrable complex structure on $U=X_{s_o}\backslash\mathcal{A}$, where $\mathcal{A}$ is a complex analytic subspace of $\mathrm{codim}_{\mathbf{C}}\mathcal{A}\geq 2$.

Sublemma I.

The complex structure I^A can be prolonged to an integrable complex structure on X_{s_o}.

Proof:

Let $(z^1,...,z^{2n})$ be a complex analytic system of local coordinates around a point $z_o\in\mathcal{A}'\subset X'^{A}_{s_o}$ and defined in a policylinder $\mathcal{W}$. Let $\zeta^i=f^*(z^i)$ for $1\leq i\leq 2n$. It is easy to show that ζ^i can be prolonged through $\mathcal{A}$. (See [09]). Indeed let $\{z_n\}$ be a sequense of points such that:

a) $\lim_{n\to\infty} z_n=z_o\in\mathcal{A}'$

b) $z_n\in\mathcal{W}\backslash(\mathcal{W}\backslash\mathcal{A}')$ for each $n\in\mathbf{Z}_+$

Let $\zeta_n:=f^{-1}(z_n)$. Clearly $\zeta^n\in U=X_{s_o}\backslash\mathcal{A}$. We may suppose that:

$$\lim_{n\to\infty}\zeta_n=\zeta_o\in\mathcal{A}$$

Since

$$\zeta^i(\zeta_n)=z^i(z_n)$$

(this follows from the defionitions of ζ^i and ζ_n). Therfore we can define:

$$\zeta^i(\zeta_o)=z^i(z_o)$$

Let $\{y^i\}$ be local complex-analytic coordinates in a "small" plicylinder $\mathcal{P}$ with respect to the complex structure X_{s_o} such that $\zeta_o\in\mathcal{P}$. Then clearly ζ^i as functions of the coordinates $\{y^i\}$ are continuous functions.

Miniproposition.

ζ^i as functions of $\{y^i\}$ are real analytic functions of $(y^1,..,y^{2n})$.

Proof of the miniproposition:

The *PROOF OF THE MINIPROPOSITION CONSISTS OF TWO STEPS:*

STEP 1. Let $(g_{\alpha\overline{\beta}})$ be a CALABI-YAU METRIC on X, *then* $g_{\alpha\overline{\beta}}$ is a real analytic function with respect to

$$\{\mathrm{Re}z^1,\mathrm{Im}z^2,...,\mathrm{Re}z^{2n},\mathrm{Im}z^{2n}\}$$

where $\{z^1,...z^{2n}\}$ is any local holomorphic coordinate system at any point $x_o\in X$.

Proof of Srep 1:

J. Kajdan and D. Deturk have proved in [06] that if (g_{ij}) is an Einstein metric, *then* with respect to the harmonic coordinates it is real analytic, i.e. for each i and j the function g_{ij} is a real analytic with respect to the harmonic coordinates. Recall that $(x^1,....,x^{4n})$ are called ***harmonic coordinates*** if with respect to these coordinates $\Gamma^i_{kl}=0$ at x_o for all i,j and k. Let me remind You that Γ^i_{kl} are the *Cristoffel symbols* for the Levi-Chevita connection of g_{ij}.

Let $(g_{\alpha\overline{\beta}})$ be a CALABI-YAU metric on X. Since $(g_{\alpha\overline{\beta}})$ is a Ricci flat Kähler metric it follows from one of the definitions of Kähler metric that we can find local holomorphic coordinates $(z^1,...,z^{2n})$ such that $\Gamma^i_{j\overline{k}}=0$ at z_o, i.e.

$$g_{\alpha\overline{\beta}}(z,\overline{z})=\delta_{\alpha\overline{\beta}}+O(2)$$

Hence *for any holomorphic local coordinates* $(z^1,...,z^{2n})$ $g_{\alpha\overline{\beta}}$ *are real analytic functions of* $(\mathrm{Re}z^1,\mathrm{Im}z^1,...,\mathrm{Re}z^{2n},\mathrm{Im}z^{2n})$.

Q.E.D.

Step 2.

Let $\{\xi^i\}$ and $\{\tau^i\}$ for $1\leq i\leq 2n$ be complex analytic coordinates for two different isometric complex structures with respect to a CALABI-YAU metric at point $x_o\in X$, where X is a HYPER-KÄHLERIAN manifold. Then ξ^i for each i is a real analytic function if $\mathrm{Re}\tau^i$ and $\mathrm{Im}\tau^i$ for i=1,....,2n.

Proof of Step 2:

From the KADAIRA-SPENCER-KURANISHI DEFORMATION THEORY and the Definition of *ISOMETRIC DEFORMATIONS* it follows that the KADAIRA-SPENCER class that defines the new complex structure X^A, where $A\in SO(3)$ is just

(*) $$\phi(z,\overline{z})=(a\mathrm{Re}\omega_X(2,0)+b\mathrm{Im}\omega_X(2,0)+c\mathrm{Im}(g_{\alpha\overline{\beta}}))\perp\omega^*_X(2,0)$$

$$a^2+b^2+c^2=1$$

$\omega^*_X(2,0)\in\Gamma(X,\wedge^2\Theta_X)$ (Θ_X is the holomorphic tangent bundle) and

$$<\omega^*_X(2,0),\omega_X(2,0)>=1$$

where < , > is induced by the natural pairing

$$\wedge^2\Theta_X x\Omega^2\rightarrow\mathcal{O}_X.\ (\wedge^2\Theta_X \text{ and } \Omega^2 \text{ are dual sheaves}).$$

From STEP 1 and $(*)$ it follows that the coefficients of $\phi(z,\overline{z})$ are real analytic functions with respect

$$\{\mathrm{Re}z^1,\mathrm{Im}z^2,\ldots,\mathrm{Re}z^{2n},\mathrm{Im}z^{2n}\}$$

where $\{z^1,\ldots z^{2n}\}$ is any local holomorphic coordinate system at any point $x_o\in X$.

In [18] it is proved that

a) $[\phi,\phi]\equiv 0$ and

b) $\phi\in\mathbf{H}^1(X,\Theta_X)$, i.e. ϕ is a harmonic Dalbault class with respect to the CALABI-YAU METRIC.

c) $\overline{\partial}\phi=\frac{1}{2}[\phi,\phi]$, i.e. ϕ defines the complex structure X^A.

Let $(z^1,\ldots,z^{2n})$ be any local holomorphic coordinates in some open subset $U\subset X$ and let

$$\phi|_U=\sum\phi^i_{\overline{j}}\overline{dz^j}\otimes\frac{\partial}{\partial z^i}$$

then the solutions of the following equations:

$$(**)\qquad \frac{\overline{\partial}\zeta^i}{\partial\overline{z}^j}=\sum_{\alpha}\phi^{\alpha}_{\overline{j}}\frac{\partial\zeta^i}{\partial z^{\alpha}}$$

are local coordinates in the KURANISHI family.

From $(**)$ and the famous NEWLANDER-NIRENBERG THEOREM it follows that $(**)$ has real analytic solutions with respect

$$\{\mathrm{Re}z^1,\mathrm{Im}z^2,\ldots,\mathrm{Re}z^{2n},\mathrm{Im}z^{2n}\}$$

where $\{z^1,\ldots z^{2n}\}$ is any local holomorphic coordinate system at any point $x_o\in X$.

Q.E.D.

The Miniproposition is proved.

Q.E.D.

<u>End of the proof of Sublemma I:</u>

ζ^i are bounded in $\mathcal{P}\backslash\mathcal{A}$ and since $\mathcal{A}$ is a complex analytic space of complex codim≥ 2 it follows ζ^i are real analytic functions of $(y^1,\ldots,y^{2n})$. This shows that on X_{s_o} we can define a new complex structure $X^A_{s_o}$.

Q.E.D.

Since $f^*(\omega'^A(2,0))=\mathrm{Im}\,\omega_{X'_{s_o}}(2,0)+i\mathrm{Im}(g_{\alpha\overline{\beta}}))$ is a holomorphic form on $U=X_{s_o}\setminus\mathcal{A}$, where the complex codimension of $\mathcal{A}\geq 2$. Standart technique implies $f^*(\omega'^A(2,0))$ can be prolonged to a global holomorphic form on $X^A_{s_o}$.

From the DEFINITION of $f^*(\omega'^A(2,0))$, i.e. from the fact that

$$\omega'^A(2,0)=\mathrm{Im}\,\omega_{X'_{s_o}}(2,0)+i\mathrm{Im}(g_{\alpha\overline{\beta}})$$

it follows that $f^*(\mathrm{Im}(g_{\alpha\overline{\beta}}))$ is a well defined C^∞ form on X_{s_o} of type (1,1). Since

$$\wedge^{2n}(\mathrm{Im}(g_{\alpha\overline{\beta}}))=\omega_{X_{s_o}}(2n,0)\wedge\omega_{X_{s_o}}(0,2n)$$

it follows that

$$f^*(\wedge^{2n}(\mathrm{Im}(g_{\alpha\overline{\beta}})))=f^*(\omega_{X_{s_o}}(2n,0)\wedge\omega_{X_{s_o}}(0,2n))=\omega_{X'_{s_o}}(2n,0)\wedge\omega_{X'_{s_o}}(0,2n)$$

And hence $f^*(\mathrm{Im}(g_{\alpha\overline{\beta}}))$ is a non-degenerate form of type (1,1) on X'_{s_o}.

Q.E.D.

PROPOSITION 8.1.2.1. implies that $f^*(\mathrm{Im}(g_{\alpha\overline{\beta}}))$ defines a Ricci-flat KÄHLER metric on X_{s_o}. This followss since $(g_{\alpha\overline{\beta}})$ is a KÄHLER metric on $X_{s_o}\setminus\mathcal{A}$ and $f^*(\mathrm{Im}(g_{\alpha\overline{\beta}}))$ is a non-degenerate form of type (1,1) on X'_{s_o}. Now we know that f is a bimormorphic map which map one KÄHLER metric to another. Then it is a standart fact that f will be a biholomorphic map. See [15].

Q.E.D.

REFERENCES.

[01]V. I. ARNOLD *"Mathematical Methods in Classical Mechanics"*, Springer -Verlag, New-York, Heidelberg and Berlin. 1981.

[02]A. Beauville, *"Varietes Kählerian dont la premiere clas de Chern est null"*, J. of Differential Geometry 18(1983), 755-782.

[03]Bishop E., *"Conditions for the analyticity of certain sets"*, Michigan Math. J. 11(1964), 235-274.

[04]Bogomolov F.A. *"Kählerian varieties with trivial canonical class"*, Preprint, I.H.E.S. February (1981)

[05]Chern S.S. "Complex manifolds" Chicago University, 1957.

[06]De Turck D. and Kazdan J. *"Some regularity theorems in Riemannian geometry"* A.E.N.S. 14(1982) 249-260.

[07]Godemant R. *"Algebraic topology and Theory of Sheaves"* Mir, Moscow 1961.

[08]Grauert H. *"Ein Theorem der Analytischen Theorie und die Modulraüme Komplexer Structuren"* Publ. Math. I.H.E.S. vol. 5 , 5-84.

[09]Griffiths Ph. and Harris J. *"Principles of Algebraic Geometry"* John Wiley and Sons, New-York, Toronto, 1978.

[10]Harvey R. and Blain Lawson Jr., *"Intrinsic characterization of Kähler manifolds"*, Inven. Math. 74(1983), 139-150

[11]Kodaira K. and Morrow J. *"Complex Manifolds"* Holt, Rinehart and Winston, Inc., New-York, Chicago (1971)

[12]Hirzebruch F. *"Topological methods in Algebraic Geometry"* Springer-Verlag 1957

[13]Moishezon B.G. *"The algebraic analogue of compact complex spaces with sufficiently large field of meromorphic functions I, II and III"* Izv. Acad Nauk USSR 33 (1969).

[14]Mumford D. *"Abelian varieties"* Oxford University Press(1969)

[15]Siu Y.T. *"Every K3 surface is Kähler"* Inv. Math. 73(1983), 251-265

[16]Todorov A.N. *"Aplications of Kähler-Einstein-Calabi-Yau metric to moduli of K3 surfaces"* Inv. Math. vol. 61(1980) 251-265.

[17]Todorov A.N. *"Moduli of Hyper-Kählerian Manifolds I and II"*, Preprint Max-Plank Institute fur Mathematik, Preprint 1985

[18]Todorov A.N. *"Weil-Petersson geometry of Teichnüller space of SU(n≥3)(CALABI-YAU) manifolds(TORELLI PROBLEM) II"* to appear in Comm. in Math. Physics.

[19]Yau S.T., *"On Ricci curvature of a compact Kähler manifold and complex Monge-Amper equation I,"* Comm. Pure and Applied Math. 31 (1978), 229-441.

[20]D. Burns and M. Rapoport, "On the Torelli Problem for Kahlerian K3 surfaces", Ann. Sci. ENS 12 (1975), 269-274.

[21]R. THOM, *"Quelques proprietes global des varietes differentiables",* Commentarii Mathematici Helvetici (1954) 17-86.

[22]W. BROWDER, *"Surgary on simply connected manifolds"* Springer-Verlag, 1971.

[23]S. Lang, *"Algebra"*, Russian translation, Izd. "Mir", Moscow(1971).

Andrey N. Todorov,
MAX-PLANK INSTITUT FÜR MATHEMATIK
Gottfried Claren str. 26
5300 Bonn 3