Max-Planck-Institut für Mathematik
Bonn

Integers without large prime factors: From Ramanujan to de Bruijn

by

Pieter Moree

Max-Planck-Institut für Mathematik
Preprint Series 2012 (70)

Integers without large prime factors: From Ramanujan to de Bruijn

Pieter Moree

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

MPIM 12-70

INTEGERS WITHOUT LARGE PRIME FACTORS: FROM RAMANUJAN TO DE BRUIJN

PIETER MOREE

In memoriam: Nicolaas Govert ('Dick') de Bruijn (1918-2012)

Abstract. A small survey of work done on estimating the number of integers without large prime factors up to around the year 1950 is provided. Around that time N.G. de Bruijn published results that dramatically advanced the subject and started a new era in this topic.

1. Introduction

Let $P(n)$ denote the largest prime divisor of n. We set $P(1) = 1$. A number n is said to be y-*friable*[1] if $P(n) \leq y$. We let $S(x, y)$ denote the set of integers $1 \leq n \leq x$ such that $P(n) \leq y$. The cardinality of $S(x, y)$ is denoted by $\Psi(x, y)$. We write $y = x^{1/u}$, that is $u = \log x/ \log y$.

Fix $u > 0$. In 1930, Dickman [13] proved that

$$\lim_{x\to\infty} \frac{\Psi(x, x^{1/u})}{x} = \rho(u), \tag{1}$$

with

$$\rho(u) = \rho(N) - \int_N^u \frac{\rho(v-1)}{v} dv, \ \ (N < u \leq N+1, \ \ N = 1, 2, 3, \ldots),$$

and $\rho(u) = 1$ for $0 < u \leq 1$ (see Figure 1). It is left to the reader to show that we have

$$\rho(u) = \begin{cases} 1 & \text{for } 0 \leq u \leq 1; \\ \frac{1}{u}\int_0^1 \rho(u-t)dt & \text{for } u > 1. \end{cases} \tag{2}$$

The function $\rho(u)$ in the literature is either called the *Dickman function* or the *Dickman-de Bruijn function*.

In this note I will briefly discuss the work done on friable integers up to the papers of de Bruijn [7, 8] that appeared around 1950 and

2000 *Mathematics Subject Classification.* Primary 11N25, Secondary 34K25.
Key words and phrases. smooth integers, Dickman-de Bruijn function.

[1]In the older literature one usually finds y-smooth. Friable is an adjective meaning easily crumbled or broken.

dramatically advanced the subject. A lot of the early work was carried out by number theorists from India.

De Bruijn [7] improved on (1) by establishing a result that together with the best currently known estimate for the prime counting function (due to I.M. Vinogradov and Korobov in 1958) yields the following result.

Theorem 1. *The estimate*

$$\Psi(x,y) = x\rho(u)\Big\{1 + O^2\Big(\frac{\log(u+1)}{\log y}\Big)\Big\}, \tag{3}$$

holds for $1 \le u \le \log^{3/5-\epsilon} y$, *that is*, $y > \exp(\log^{5/8+\epsilon} x)$.

De Bruijn's most important tool in his proof of this result is the *Buchstab equation* [9],

$$\Psi(x,y) = \Psi(x,z) - \sum_{y<p\le z} \Psi(\frac{x}{p}, p), \tag{4}$$

where $1 \le y < z \le x$. The Buchstab equation is easily proved on noting that the number of integers $n \le x$ with $P(n) = p$ equals $\Psi(x/p, p)$. Given a good estimate for $\Psi(x,y)$ for $u \le h$, it allows one to obtain a good estimate for $u \le h+1$.

De Bruijn [8] complemented Theorem 1 by an asymptotic estimate for $\rho(u)$. That result has as a corollary that, for $u \ge 3$,

$$\rho(u) = \exp\Big\{-u\Big\{\log u + \log_2 u - 1 + \frac{\log_2 u - 1}{\log u} + O\Big(\big(\frac{\log_2 u}{\log u}\big)^2\Big)\Big\}\Big\}, \tag{5}$$

which will suffice for our purposes. Note that (5) implies that, as $u \to \infty$,

$$\rho(u) = \frac{1}{u^{u+o(u)}},\ \ \rho(u) = \Big(\frac{e+o(1)}{u\log u}\Big)^u,$$

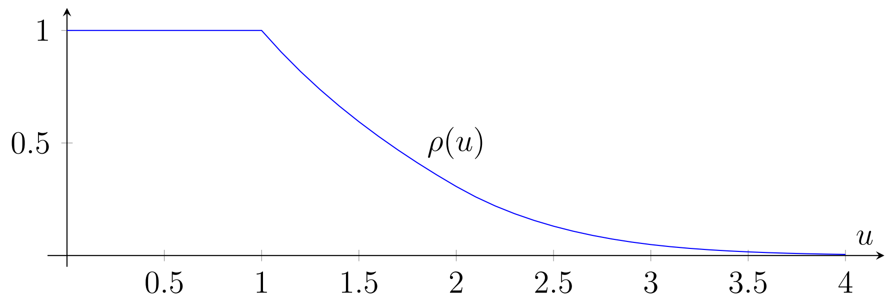
formulas that suffice for most purposes and are easier to remember. For a more detailed description of this and other work of de Bruijn in analytic number theory, we refer to Moree [20].

2. Results on $\rho(u)$

Note that $\rho(u) > 0$, for if not, then because of the continuity of $\rho(u)$ there is a smallest zero $u_0 > 1$ and then substituting u_0 in (2) we easily

[2] The reader not familiar with the Landau-Bachmann O-notation we refer to wikipedia or any introductory text on analytic number theory, e.g., Tenenbaum [35]. Instead of $\log\log x$ we sometimes write $\log_2 x$, instead of $(\log x)^A$, $\log^A x$.

Figure 1. The Dickman-de Bruijn function $\rho(u)$



arrive at a contradiction. Note that for $u > 1$ we have

$$\rho'(u) = -\frac{\rho(u-1)}{u} \tag{6}$$

It follows that $\rho(u) = 1 - \log u$ for $1 \le u \le 2$. For $2 \le u \le 3$, $\rho(u)$ can be expressed in terms of the dilogarithm. However, with increasing u one has to resort to estimating $\rho(u)$ or finding a numerical approximation.

Since $\rho(u) > 0$ we see from (6) that $\rho(u)$ is strictly decreasing for $u > 1$. From this and (2) we then find that $u\rho(u) \le \rho(u-1)$, which on using induction leads to $\rho(u) \le 1/[u]!$ for $u \ge 0$. It follows that $\rho(u)$ quickly tends to zero as u tends to infinity.

Ramaswami [29] proved that

$$\rho(u) > \frac{C}{u4^u\Gamma(u)^2},\ u \ge 1,$$

for a suitable constant C, with Γ the Gamma function. By Stirling's formula we have $\log \Gamma(u) \sim u \log u$ and hence the latter inequality is for u large enough improved on by the following inequality due to Buchstab [9]:

$$\rho(u) > \exp\Big\{ -u\Big\{ \log u + \log_2 u + 6\frac{\log_2 u}{\log u}\Big\}\Big\},\ (u \ge 6). \tag{7}$$

Note that on its turn de Bruijn's result (5) considerably improves on the latter inequality.

3. S. Ramanujan (1887-1920) and the friables

In his first letter (January 16th, 1913) to Hardy (see, e.g. [3]), one of the most famous letters in all of mathematics, Ramanujan claims that

$$\Psi(n,3) = \frac{1}{2}\frac{\log(2n)\log(3n)}{\log 2 \log 3}. \tag{8}$$

The formula is of course intended as an approximation, and there is no evidence to show how accurate Ramanujan supposed it to be. Hardy [17, pp. 69-81] in his lectures on Ramanujan's work gave an account of an interesting analysis that can be made to hang upon the above assertion. I return to this result in the section on the $\Psi(x,y)$ work of Pillai.

In the so-called Lost Notebook [27] we find at the bottom half of page 337:

$\phi(x)$ is the no. of nos of the form

$$2^{a_2}\cdot 3^{a_3}\cdot 5^{a_5}\cdots p^{a_p} \qquad p\le x^{\epsilon}$$

not exceeding x.

$$\tfrac{1}{2}\le\epsilon\le 1,\qquad \phi(x)\sim x\Big\{1-\int_{\epsilon}^{1}\frac{d\lambda_0}{\lambda_0}\Big\}$$

$$\tfrac{1}{3}\le\epsilon\le\tfrac{1}{2},\qquad \phi(x)\sim x\Big\{1-\int_{\epsilon}^{1}\frac{d\lambda_0}{\lambda_0}+\int_{\epsilon}^{\frac{1}{2}}\frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1}\frac{d\lambda_0}{\lambda_0}\Big\}$$

$$\begin{aligned}\tfrac{1}{4}\le\epsilon\le\tfrac{1}{3},\qquad \phi(x)\sim x\Big\{1\quad &-\int_{\epsilon}^{1}\frac{d\lambda_0}{\lambda_0}+\int_{\epsilon}^{\frac{1}{2}}\frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1}\frac{d\lambda_0}{\lambda_0}\\ &-\int_{\epsilon}^{\frac{1}{3}}\frac{d\lambda_2}{\lambda_2}\int_{\lambda_2}^{\frac{1-\lambda_2}{2}}\frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1}\frac{d\lambda_0}{\lambda_0}\Big\}\end{aligned}$$

$$\begin{aligned}\tfrac{1}{5}\le\epsilon\le\tfrac{1}{4},\qquad \phi(x)\sim x\qquad &\Big\{1-\int_{\epsilon}^{1}\frac{d\lambda_0}{\lambda_0}+\int_{\epsilon}^{\frac{1}{2}}\frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1}\frac{d\lambda_0}{\lambda_0}\\ &-\int_{\epsilon}^{\frac{1}{3}}\frac{d\lambda_2}{\lambda_2}\int_{\lambda_2}^{\frac{1-\lambda_2}{2}}\frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1}\frac{d\lambda_0}{\lambda_0}\\ &+\int_{\epsilon}^{\frac{1}{4}}\frac{d\lambda_3}{\lambda_3}\int_{\lambda_3}^{\frac{1-\lambda_3}{3}}\frac{d\lambda_2}{\lambda_2}\int_{\lambda_2}^{\frac{1-\lambda_2}{2}}\frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1}\frac{d\lambda_0}{\lambda_0}\Big\}\end{aligned}$$

and so on.

In the book by Andrews and Berndt [1, §8.2] it is shown that Ramanujan's assertion is equivalent with (1) with

$$\rho(u)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}I_k(u),$$

where

$$I_k(u) = \int_{\substack{t_1,\ldots t_k \geq 1 \\ t_1+\ldots+t_k \leq u}} \frac{dt_1}{t_1} \cdots \frac{dt_k}{t_k}.$$

This is one of many examples where Ramanujan reached with his hand from his grave to snatch a theorem, in this case from Dickman who was at least 10 years later than Ramanujan, cf. Berndt [2]. Chowla and Vijayaragahavan [12] seemed to have been the first to rigorously prove (1) with $\rho(u)$ expressed as a sum of iterated integrals (cf. the section on Buchstab). The asymptotic behaviour of the integrals $I_k(u)$ has been studied by Soundararajan [32].

Ramanujan's claim reminds me of the following result of Chamayou [10]: If $x_1, x_2, x_3, \cdots$ are independent random variables uniformly distributed in $(0, 1)$, and $u_n = x_1 + x_1x_2 + \ldots + x_1x_2\cdots x_n$, then u_n converges in probability to a limit u_∞ and u_∞ has a probability distribution with density function $\rho(t)e^{-\gamma}$, where γ denotes Euler's constant.

4. I.M. Vinogradov (1891-1983) and the friables

The first to have an application for $\Psi(x, y)$ estimates seems to have been Ivan Matveyevich Vinogradov [36] in 1927. Let $k \geq 2$ be a prescribed integer and $p \equiv 1(\text{mod } k)$ a prime. The k-th powers in $(\mathbb{Z}/p\mathbb{Z})^*$ form a subgroup of order $(p-1)/k$ and so the existence follows of $g_1(p, k)$, the least k-th power non-residue modulo a prime p. Suppose that $y < g_1(p, k)$, then $S(x, y)$ consists of k-th power residues only. It follows that

$$\Psi(x, y) \leq \#\{n \leq x \ : \ n \equiv a^k (\text{mod } p) \text{ for some } a\}.$$

The idea is now to use good estimates for the quantities on both sides of the inequality sign in order to deduce an upper bound for $g_1(p, k)$.

Vinogradov [36] showed that $\Psi(x, x^{1/u}) \geq \delta(u)x$ for $x \geq 1$, $u > 0$, where $\delta(u)$ depends only on u and is positive. He applied this to show that if $m \geq 8$, $k > m^m$, and $p \equiv 1(\text{mod } k)$ is sufficiently large, then

$$g_1(p, k) < p^{1/m}. \tag{9}$$

See Norton [21] for a historical account of the problem of determining $g_1(p, k)$ and original results.

5. K. Dickman (1861-1947) and the friables

Karl Dickman was active in the Swedish insurance business in the end of the 19th century and the beginning of the 20th century. Probably, he studied mathematics in the 1880's at Stockholm University, where the legendary Mittag-Leffler was professor[3].

As already mentioned Dickman proved (1) and in the same paper[4] gave an heuristic argument to the effect that

$$\lim_{x\to\infty}\frac{1}{x}\sum_{2\le n\le x}\frac{\log P(n)}{\log n}=\int_0^\infty\frac{\rho(u)}{(1+u)^2}du. \tag{10}$$

Denote the integral above by λ. Dickman argued that $\lambda\approx 0.62433$. Mitchell [19] in 1968 computed that $\lambda=0.62432998854\ldots$. The interpretation of Dickman's heuristic is that for an average integer with m digits, its greatest prime factor has about λm digits. The constant λ is now known as the *Golomb-Dickman constant*, as it arose independently in research of Golomb and others involving the largest cycle in a random permutation.

De Bruijn [7] in 1951 was the first to prove (10). He did this using his $\Lambda(x,y)$-function, an approximation of $\Psi(x,y)$, that he introduced in the same paper.

6. S.S. Pillai (1901-1950) and the friables

Subbayya Sivasankaranarayana Pillai (1901-1950) was a number theorist who worked on problems in classical number theory (Diophantine equations, Waring's problem, etc.). Indeed, he clearly was very much inspired by the work of Ramanujan. He tragically died in a plane crash near Cairo while on his way to the International Congress of Mathematicians (ICM) 1950, which was held at Harvard University.

Pillai wrote two manuscripts on friable integers, [23, 24], of which [23] was accepted for publication in the Journal of the London Mathematical Society, but did not appear in print. Also [24] was never published in a journal.

In [23], see also [26, pp. 481-483], Pillai investigates $\Psi(x,y)$ for y fixed. Let $p_1,p_2,\ldots,p_k$ denote all the different primes $\le y$. Notice that $\Psi(x,y)$ equals the cardinality of the set

$$\{(e_1,\ldots,e_k)\in\mathbb{Z}^k : e_i\ge 0,\ \sum_{i=1}^k e_i\log p_i\le x\}.$$

Thus $\Psi(x,y)$ equals the number of lattice points in a k-dimensional tetrahedron with sides of length $\log x/\log 2,\ldots,\log x/\log p_k$. This

[3] I have this information from Lars Holst.

[4] Several sources falsely claim that Dickman wrote only one mathematical paper. He also wrote [14].

tetrahedron has volume

$$\frac{1}{k!}\prod_{p\le y}\left(\frac{\log x}{\log p}\right).$$

Pillai shows that

$$\Psi(x,y)=\frac{1}{k!}\prod_{p\le y}\left(\frac{\log x}{\log p}\right)\Big(1+(1+o(1))\frac{k\log(p_1p_2\dots p_k)}{2\log x}\Big).$$

If $\rho_1,\dots,\rho_k$ are positive real numbers and ρ_1/ρ_2 is irrational, then the same estimate with $\log p_i$ replaced by ρ_i holds for

$$\{(e_1,\dots,e_k)\in\mathbb{Z}^k : e_i\ge 0,\ \sum_{i=1}^{k}e_i\rho_i\le x\}.$$

This was proved by Specht [33] (after whom the Specht modules are named), see also Beukers [4]. A much sharper result than that of Pillai/Specht was obtained in 1969 by Ennola [15] (see also Norton [21, pp. 24-26]). In this result Bernoulli numbers make their appearance.

Note that Pillai's result implies that

$$\Psi(x,3)=\frac{1}{2}\frac{\log(2x)\log(3x)}{\log 2\log 3}+o(\log x), \tag{11}$$

and that the estimate

$$\Psi(x,3)=\frac{\log^2 x}{2\log 2\log 3}+o(\log x)$$

is false. Thus Ramanujan's estimate (8) is more precise than the trivial estimate $\log^2 x/(2\log 2\log 3)$. Hardy [17, §5.13] showed that the error term $o(\log x)$ in (11) can be replaced by $o(\log x/\log_2 x)$. In the proof of this he uses a result of Pillai [22], see also [25, pp. 53-61], saying that given $0<\delta<1$, one has $|2^x-3^y|>2^{(1-\delta)x}$ for all integers x and y with $x>x_0(\delta)$ sufficiently large.

In [24], see also [26, pp. 515-517], Pillai claims that, for $u\ge 6$, $B/u<\rho(u)<A/u$, with $0<B<A$ constants. He proves this result by induction assuming a certain estimate for $\rho(6)$ holds. However, this estimate for $\rho(6)$ does not hold. Indeed, the claim contradicts (5) and is false.

Since Pillai reported on his work on the friables at conferences in India and stated open problems there, his influence on the early development of the topic was considerable. E.g., one of the questions he raised was whether $\Psi(x,x^{1/u})=O(x^{1/u})$ uniformly for $u\le(\log x)/\log 2$. This question was answered in the affirmative by Ramaswami [29].

7. R.A. Rankin (1915-2001) and the friables

In his work on the size of gaps between consecutive primes Robert Alexander Rankin [31] in 1938 introduced a simple idea to estimate $\Psi(x,y)$ which turns out to be remarkably effective and can be used in similar situations. This idea is now called 'Rankin's method' or 'Rankin's trick'. Starting point is the observation that for any $\sigma > 0$

$$\Psi(x,y) \le \sum_{n\in S(x,y)} (\frac{x}{n})^\sigma \le x^\sigma \sum_{P(n)\le y} \frac{1}{n^\sigma} = x^\sigma \zeta(\sigma,y), \tag{12}$$

where

$$\zeta(s,y) = \prod_{p\le y}(1-p^{-s})^{-1},$$

is the partial Euler product up to y for the Riemann zeta function $\zeta(s)$. Recall that, for $\Re s > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}.$$

By making an appropriate choice for σ and estimating $\zeta(\sigma,y)$ using analytic prime number theory, a good upper bound for $\Psi(x,y)$ can be found. E.g., the choice $\sigma = 1-1/(2\log y)$ leads to

$$\zeta(\sigma,y) \ll \exp\Big\{\sum_{p\le y}\frac{1}{p^\sigma}\Big\} \le \exp\Big\{\sum_{p\le y}\frac{1}{p} + O\Big((1-\sigma)\sum_{p\le y}\frac{\log p}{p}\Big)\Big\} \ll \log y,$$

which gives rise to

$$\Psi(x,y) \ll x\mathrm{e}^{-u/2}\log y. \tag{13}$$

8. A.A. Bukhshtab (1905-1990) and the friables

Aleksandr Adol'fovich Bukhshtab[5]'s most important contribution is the equation (4) now named after him. A generalization of it plays an important role in sieve theory. Buchstab [9] in 1949 proved (1) and gave both Dickman's differential-difference equation as well as the result (14)

$$\rho(u) = 1 + \sum_{n=1}^{N}(-1)^n \int_n^u \int_{n-1}^{t_1-1}\int_{n-2}^{t_2-1}\cdots\int_1^{t_{n-1}-1} \frac{dt_n dt_{n-1}\cdots dt_1}{t_1 t_2\cdots t_n},$$

for $N \le u \le N+1$ and $N \ge 1$ an integer, simplifying Chowla and Vijayaragahavan's expression (they erroneously omitted the term $n = N$). Further, Buchstab established inequality (7) and applied his results

[5]Buchstab in the German spelling.

to show that the exponent in Vinogradov's result (9) can be roughly divided by two.

9. V. Ramaswami and the friables

V. Ramaswami[6] [28] showed that

$$\Psi(x, x^{1/u}) = \rho(u)x + O_U(\frac{x}{\log x}),$$

for $x > 1$, $1 < u \le U$, and remarked that the error term is best possible. He sharpened this result in [29] and showed there that, for $u > 2$,

$$\Psi(x, x^{1/u}) = \rho(u)x + \sigma(u)\frac{x}{\log x} + O(\frac{x}{\log^{3/2} x}), \tag{15}$$

with $\sigma(u)$ defined similarly to $\rho(u)$. Indeed, it turns out that

$$\sigma(u) = (1-\gamma)\rho(u-1),$$

but this was not noticed by Ramaswami. In [30] Ramaswami generalized his results to $B_l(m, x, y)$ which counts the number of integers $n \le x$ with $P(n) \le y$ and $n \equiv l(\text{mod } m)$ [7]. Norton [21, pp. 12-13] points out some deficits of this paper and gives a reproof [21, §4] of Ramaswami's result on $B_l(m, x, x^{1/u})$ generalizing (15).

From de Bruijn's paper [7, Eqs. (5.3), (4.6)] one easily derives the following generalization of Ramaswami's results[8]:

Theorem 2. *Let $m \ge 0$, $x > 1$, and suppose $m + 1 < u < \sqrt{\log x}$. Then*

$$\Psi(x, y) = x\sum_{r=0}^{m} a_r \frac{\rho^{(r)}(u)}{\log^r y} + O_m(\frac{x}{\log^{m+1} y}),$$

with $\rho^{(r)}(u)$ the r-th derivative of $\rho(u)$ and $a_0, a_1, \ldots$ are the coefficients in the power series expansion

$$\frac{z}{1+z}\zeta(1+z) = a_0 + a_1 z + a_2 z^2 + \ldots,$$

with $|z| < 1$.

It is well-known (see, e.g., Briggs and Chowla [5]) that around $s = 1$ the Riemann zeta function has the Laurent series expansion

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\gamma_k (s-1)^k,$$

---

[6] He worked at Andhra University until his death in 1961. I will be grateful for further bibliographical information.

[7] Buchstab [9] was the first to investigate $B_l(m, x, y)$.

[8] The notation O_m indicates that the implied constant might depend on m.

with γ_k the k-th Stieltjes constant and with $\gamma_0 = \gamma$ Euler's constant. Using this we find that $a_0 = 1$ and $a_1 = \gamma - 1$. Thus Theorem 2 yields (15) with $\sigma(u) = (1-\gamma)\rho(u-1)$ for the range $2 < u < \sqrt{\log x}$. For $u > \sqrt{\log x}$ the estimate (15) in view of (5) reduces to

$$\Psi(x, x^{1/u}) \ll x \log^{-3/2} x,$$

which easily follows from (13).

10. S. Chowla (1907-1995) and the friables

The two most prominent number theorists in the period following Ramanujan were S.S. Pillai and Sarvadaman Chowla. They kept in contact through an intense correspondence [34]. Chowla in his long career published hunderds of reseach papers.

Chowla and Vijayaragahavan [12] expressed $\rho(u)$ as an iterated integral and gave a formula akin to (14). De Bruijn [6] established some results implying that $\Psi(x, \log^h x) = O(x^{1-1/h+\epsilon})$ for $h > 2$. An easier reproof of the latter result was given by Chowla and Briggs [11].

11. Summary

It seems that the first person to look at friable integers was Ramanujan, starting with his first letter to Hardy (1913), also Ramanujan seems to have been the first person to arrive at the Dickman-de Bruijn function $\rho(u)$. Pillai generalized some of Ramanujan's work and spoke about it on conferences in India, which likely induced a small group of Indian number theorists to work on friable integers. Elsewhere in the same period (1930-1950) only incidental work was done on the topic. Around 1950 N.G. de Bruijn published his ground-breaking papers [7, 8]. Soon afterward the Indian number theorists stopped publishing on friable integers.

It should also be said that the work on friable integers up to 1950 seems to contain more mistakes than more recent work. Norton [21] points out and corrects many of these mistakes.

`Further reading`. As a first introduction to friable numbers I can highly recommend Granville's 2008 survey [16]. It has a special emphasis on friable numbers and their role in algorithms in computational number theory. Mathematically more demanding is the 1993 survey by Hildebrand and Tenenbaum [18]. Chapter III.5 in Tenenbaum's book [35] deals with $\rho(u)$ and approximations to $\Psi(x, y)$ by the saddle point method.

`Acknowledgement`. I thank R. Thangadurai for helpful correspondence on S.S. Pillai and the friables and providing me with a PDF file of Pillai's collected works. B.C. Berndt kindly sent me a copy of [1]. K.K. Norton provided helpful comments on an earlier version. His research monograph [21], which is the most extensive source available on the early history of friable integer counting, was quite helpful to me. In [21], by the way, new results (at the time) on $g_1(p,k)$ and $\Psi_m(x,y)$, the number of y-friable integers $1 \le n \le x$ coprime with m are established. Figure 1 was kindly created for me by Jon Sorenson and Alex Weisse (head of the MPIM computer group).

References

[1] G.E. Andrews and B.C. Berndt, Ramanujan's Lost Notebook, Part IV, Springer-Verlag, preprint, to appear.

[2] B.C. Berndt, Ramanujan reaches his hand from his grave to snatch your theorems from you, *Asia Pac. Math. Newsl.* **1** (2011), no. 2, 8–13.

[3] B.C. Berndt and R.A. Rankin, *Ramanujan. Letters and commentary*, AMS, Providence, RI; London Mathematical Society, London, 1995.

[4] F. Beukers, The lattice-points of n-dimensional tetrahedra, *Indag. Math.* **37** (1975), 365–372.

[5] W.E. Briggs and S. Chowla, The power series coefficients of $\zeta(s)$, *Amer. Math. Monthly* **62** (1955), 323–325.

[6] N.G. de Bruijn, On the number of uncancelled elements in the sieve of Eratosthenes, *Indag. Math.* **12** (1950), 247-256.

[7] N.G. de Bruijn, On the number of positive integers $\le x$ and free of prime factors $> y$, *Indag. Math.* **13** (1951), 50–60.

[8] N.G. de Bruijn, The asymptotic behaviour of a function occurring in the theory of primes, *J. Indian Math. Soc. (N.S.)* **15** (1951). 25-32.

[9] A.A. Buchstab, On those numbers in arithmetical progression all of whose prime factors are small in order of magnitude (Russian), *Dokl. Akad. Nauk. SSSR* **67** (1949), 5–8.

[10] J.-M.-F. Chamayou, A probabilistic approach to a differential-difference equation arising in analytic number theory, *Math. Comp.* **27** (1973), 197–203.

[11] S.D. Chowla and W.E. Briggs, On the number of positive integers $\le x$ all of whose prime factors are $\le y$, *Proc. Amer. Math. Soc.* **6** (1955), 558–562.

[12] S.D. Chowla and T. Vijayaraghavan, On the largest prime divisors of numbers, *J. Indian Math. Soc. (N.S.)* **11** (1947). 31–37.

[13] K. Dickman, On the frequency of numbers containing prime factors of a certain relative magnitude, *Ark. Mat. Astr. Fys.* **22** (1930), 1–14.

[14] K. Dickman, Om maximiantalet konsekutiva summander till ett helt tal (Swedish), 8. Skand. Mat. Kongr., Stockholm (1934), 385–388.

[15] V. Ennola, On numbers with small prime divisors, *Ann. Acad. Sci. Fenn. Ser.* A I **440** (1969), 16 pp.

[16] A. Granville, Smooth numbers: computational number theory and beyond, *Algorithmic number theory: lattices, number fields, curves and cryptography*,

Math. Sci. Res. Inst. Publ. **44**, Cambridge Univ. Press, Cambridge, (2008), 267-323.

[17] G.H. Hardy, *Ramanujan: twelve lectures on subjects suggested by his life and work*, Chelsea Publishing Company, New York, 1959.

[18] A. Hildebrand and G. Tenenbaum, Integers without large prime factors, *J. Théor. Nombres Bordeaux* **5** (1993), 411–484.

[19] W.C. Mitchell, An evaluation of Golomb's constant, *Math. Comp.* **22** (1968), 411–415.

[20] P. Moree, Nicolaas Govert de Bruijn, the enchanter of friable integers, preprint, arXiv:1212.1579.

[21] K.K. Norton, *Numbers with small prime factors, and the least kth power non-residue*, Memoirs of the American Mathematical Society **106**, American Mathematical Society, Providence, R.I. 1971.

[22] S.S. Pillai, On the inequality $0 < a^x - b^y \le n$, *J. Indian Math. Soc.* **19** (1931), 1–11.

[23] S.S. Pillai, Concerning a statement of Ramanujan, unpublished manuscript, written around 1931.

[24] S.S. Pillai, The number of positive integers $\le x$ and free of prime divisors $> x^{1/r}$, unpublished manuscript, written around 1947.

[25] S.S. Pillai, *Collected works of S. Sivasankaranarayana Pillai* **I**, (Eds.) R. Balasubramanian and R. Thangadurai, Ramanujan Mathematical Society Collected Works Series **1**, Ramanujan Mathematical Society, Mysore, 2010.

[26] S.S. Pillai, *Collected works of S. Sivasankaranarayana Pillai* **II**, Published/unpublished papers and letters, (Eds.) R. Balasubramanian and R. Thangadurai, Ramanujan Mathematical Society Collected Works Series **1**, Ramanujan Mathematical Society, Mysore, 2010.

[27] S. Ramanujan, *The lost notebook and other unpublished papers*, Narosa, New Delhi, 1988.

[28] V. Ramaswami, On the number of positive integers less than x and free of prime divisors greater than x^c, *Bull. Amer. Math. Soc.* **55** (1949), 1122–1127.

[29] V. Ramaswami, The number of positive integers $\le x$ and free of prime divisors $> x^c$, and a problem of S. S. Pillai, *Duke Math. J.* **16** (1949). 99–109.

[30] V. Ramaswami, Number of integers in an assigned A. P., $\le x$ and prime to primes greater than x^c, *Proc. Amer. Math. Soc.* **2** (1951), 318–319.

[31] R.A. Rankin, The difference between consecutive prime numbers, *J. London Math. Soc.* **13** (1938), 242–247.

[32] K. Soundararajan, An asymptotic expression related to the Dickman function, *Ramanujan J.* **29** (2012), 25–30.

[33] W. Specht, Zahlenfolgen mit endliche viele Primteilern, *Bayer. Akad. Wiss. Math.-Natur. Abt. S.-B.* **1948** (1949), 149–169.

[34] B. Sury and R. Thangadurai, S Chowla and S S Pillai, *Resonance* **17**, no. 9, (2012), 855–883.

[35] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Cambridge Studies in Advanced Mathematics **46**, Cambridge University Press, Cambridge, 1995.

[36] I.M. Vinogradov, On a bound for the least nth power non-residue, *Trans. Amer. Math. Soc.* **29** (1927), 218–226.

Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany
E-mail address: moree@mpim-bonn.mpg.de