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JORDAN GROUPS AND ALGEBRAIC SURFACES

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Abstract. We prove that an analogue of Jordan's theorem on finite subgroups of general linear groups holds for the groups of biregular automorphisms of quasi-projective algebraic surfaces. This gives a positive answer to a question of Vladimir L. Popov.

1. Introduction

Throughout this paper, k is an algebraically closed field of characteristic zero and $\mathbb{P}^1$ is the projective line over k. If U is an irreducible algebraic variety over k then $U(k), k(U)$, $\mathrm{Aut}(U)$ and $\mathrm{Bir}(U)$ stand for its set of k-points, the field of rational functions, the group of biregular k-automorphisms and the group of birational k-automorphisms respectively. Unless otherwise stated, by a point of U we mean a k-point. By an elliptic curve we mean an irreducible smooth projective curve of genus 1. It is well known (e.g., see [16]) that if X is an elliptic curve and $\mathcal{T} \subset X(k)$ is a *nonempty* finite set of points on X then the (sub)group

$$\mathrm{Aut}(X, \mathcal{T}) = \{u \in \mathrm{Aut}(X) \mid u(\mathcal{T}) = \mathcal{T}\} \subset \mathrm{Aut}(X)$$

is finite. If $\mathcal{S}$ is a smooth irreducible projective surface over k then an irreducible closed curve C in $\mathcal{S}$ is called a (-1)-curve if it is smooth rational and its self-intersection index is -1.

The following definition was inspired by the classical theorem of Jordan [2, Sect. 36] about finite subgroups of general linear groups (over fields of characteristic zero).

Definition 1.1 (Definition 2.1 of [8]). A group B is called a *Jordan group* if there exists a positive integer J_B such that every finite subgroup B_1 of B contains a normal commutative subgroup, whose index in B_1 is at most J_B.

Remark 1.2. Clearly, a subgroup of a Jordan group is also Jordan. If a Jordan group G_1 is a subgroup of *finite* index in a group G then G is also Jordan.

V. L. Popov ([8, Sect. 2], see also [9]) posed a question whether $\mathrm{Aut}(S)$ is a Jordan group when S is an irreducible algebraic surface over k. He obtained a positive answer to his question for almost all surfaces. (The case of rational surfaces was treated earlier by J.-P. Serre [12, Sect. 5.4].) The only remaining case is when S is birationally (but not biregularly) isomorphic to a product $X \times \mathbb{P}^1$ of an elliptic curve X and the projective line. In [16] the second named author proved that

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$\mathrm{Aut}(S)$ is a Jordan group if S is a *projective* surface. The aim of this paper is to extend this result to the case of *quasi-projective* surfaces. Our main result is the following statement, which gives a positive answer to Popov's question.

Theorem 1.3. *If X is an elliptic curve over k and S is an irreducible normal quasi-projective algebraic surface that is birationally isomorphic to $X \times \mathbb{P}^1$ then $\mathrm{Aut}(S)$ is a Jordan group.*

Remark 1.4. The group $\mathrm{Bir}(X \times \mathbb{P}^1)$ is *not* Jordan [15].

Remark 1.5. Suppose that S is a *non*-smooth irreducible normal surface. Since it is normal, there are only finitely many singular points on S. Then, by [9, Sect. 2, Cor. 8], $\mathrm{Aut}(S)$ is Jordan. This implies that in the course of the proof of Theorem 1.3 we may assume that S is smooth.

Corollary 1.6. *Suppose that V is an irreducible normal quasi- projective algebraic variety over k. If $\dim(V) \le 2$ then $\mathrm{Aut}(V)$ is Jordan.*

Proof of Corollary 1.6. We have $\mathrm{Aut}(V) \subset \mathrm{Bir}(V)$. If V is *not* birationally isomorphic to a product of the projective line and an elliptic curve then $\mathrm{Bir}(V)$ is Jordan ([8, Th. 2.32]) and therefore its subgroup $\mathrm{Aut}(V)$ is also Jordan. If V is birationally isomorphic to a product of the projective line and an elliptic curve then $\dim(V) = 2$ and Theorem 1.3 implies that $\mathrm{Aut}(V)$ is Jordan. □

Theorem 1.7. *Let V be an irreducible quasi-projective algebraic variety over k. If $\dim(V) \le 2$ then $\mathrm{Aut}(V)$ is Jordan.*

Proof of Theorem 1.7. One may view V as a Zariski-open subset of an irreducible *projective* variety $\bar{V}$. Let $\bar{\nu} : \bar{V}^{\nu} \to \bar{V}$ be the *normalization* of $\bar{V}$ [14, Ch. II, Sect. 5], [7, Ch. III, Sect. 8], [5, Ch. 2, Sect. 2.14])). Here $\bar{\nu}$ is a birational (surjective) regular map and V^{ν} is an irreducible *normal projective* variety of the same dimension (as V) over k [7, Th. 4 on p. 203]. Let us put $V^{\nu} = \bar{\nu}^{-1}(V) \subset \bar{V}$ and define $\nu : V^{\nu} \to V$ as the restriction of $\bar{\nu}$ to V^{ν}. Then V^{ν} is an open (dense) subset in projective $\bar{V}$ and therefore is normal quasi-projective (irreducible) while ν is a birational regular map that is the the normalization map for V. The *universality property* of the normalization ([14, Ch. II, Sect. 5, Th. 5], [4, Ch. III, Sect. 3, Ex. 3.8]) implies that every biregular automorphism of V lifts uniquely to a biregular automorphism of V^{ν} [5, Ch. 2, Sect. 2.14, Th. 2.25 on p, 141]. This give rise to the *embedding* of groups

$$\mathrm{Aut}(V) \hookrightarrow \mathrm{Aut}(V^{\nu}).$$

By Corollary 1.6, the group $\mathrm{Aut}(V^{\nu})$ is Jordan. Since $\mathrm{Aut}(V)$ is isomorphic to a subgroup of Jordan $\mathrm{Aut}(V^{\nu})$, it is also Jordan.

□

Corollary 1.8. *Let V be a quasi-projective algebraic variety over k. If $\dim(V) \le 2$ then $\mathrm{Aut}(V)$ is Jordan.*

Proof. Let $V_1, \dots, V_r$ be all the *irreducible* components of V. Clearly, all V_i are irreducible projective varieties with $\dim(V_i) \le \dim(V) \le 2$. By Theorem 1.7, all $\mathrm{Aut}(V_i)$ are Jordan. Now Lemma 1 in Section 2.2 of [9] implies that $\mathrm{Aut}(V)$ is also Jordan. □

Remark 1.9. Suppose that k is the field $\mathbb{C}$ of complex numbers and X is a smooth irreducible quasi-projective non-projective surface. Then $X(\mathbb{C})$ carries the natural structure of a connected oriented smooth *real noncompact* fourfold and the group $\mathrm{Aut}(X)$ embeds naturally in the group of the diffeomorphisms of the fourfold $X(\mathbb{C})$. While $\mathrm{Aut}(X)$ is always Jordan, there are examples of connected oriented smooth *noncompact* real fourfolds, whose group of diffeomorphisms is *not* Jordan [10].

The paper is organized as follows. In Section 2 we discuss *minimal closures* of surfaces. In Section 3 we prove Theorem 1.3.

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2. Minimal closures

2.1. Let X be an elliptic curve over k and S be a *smooth* irreducible surface over k that is birationally isomorphic to $X \times \mathbb{P}^1$. There exists an irreducible smooth projective surface $\bar{S}$ such that its certain Zariski-open subset is biregularly isomorphic to S (further we identify S with this open subset). Clearly, the inclusion map $S \subset \bar{S}$ is a birational morphism. This implies that

$$\mathrm{Aut}(S) \subset \mathrm{Bir}(S) = \mathrm{Bir}(\bar{S})$$

and therefore one may view $\mathrm{Aut}(S)$ as a subgroup of $\mathrm{Bir}(\bar{S})$. Since $\bar{S}$ is birationally isomorphic to S, it also birationally isomorphic to $X \times \mathbb{P}^1$.

Let us fix a birational isomorphism between $\bar{S}$ and $X \times \mathbb{P}^1$. The projection map $X \times \mathbb{P}^1 \to X$ gives rise to a rational map $\bar{\pi} : \bar{S} \to X$ with dense image. Since $\bar{S}$ is smooth and X becomes abelian variety (after a choice of a base point), it follows from a theorem of Weil [1, Sect. 4.4] that $\bar{\pi}$ is regular. Since $\bar{S}$ is projective, $\bar{\pi} : \bar{S} \to X$ is surjective, because its image is closed.

For each $x \in X(k)$ we write $\bar{F}_x$ for the effective divisor $\bar{\pi}^*(x)$ on $\bar{S}$ that is the pullback (under $\bar{\pi}$) of the divisor (x) on $\bar{S}$. Clearly, the support of $\bar{F}_x$ coincides with the curve $\bar{\pi}^{-1}(x)$ on $\bar{S}$. One say that the fiber of $\bar{\pi}$ over x is *reduced* if all irreducible components of the divisor $\bar{F}_x$ have multiplicity 1. We say that the fiber of $\bar{\pi}$ over x is *irreducible* if the curve $\bar{\pi}^{-1}(x)$ is irreducible; if this is the case then its multiplicity in $\bar{F}_x$ is 1 [6, Ch. 3, Sect. 1.4, Lemma 1.4.1(1) on p. 195].

It is known [13, Ch. IV] that for all but finitely many $x \in X(k)$ the fiber of $\bar{\pi}$ over x is irreducible and reduced, and the curve $\bar{\pi}^{-1}(x)$ is smooth (and irreducible). We call such fibers nonsingular and other fibers *singular*.

If C is a rational curve on $\bar{S}$ then the restriction of $\bar{\pi}$ to C must be a constant map, because every map from a rational curve to an elliptic curve is constant. This implies that C lies in a fiber of $\bar{\pi}$. In particular, every (-1)-curve on $\bar{S}$ lies in a fiber of $\bar{\pi}$.

However, if $x \in X(k)$ and the fiber $\bar{\pi}^{-1}(x)$ is singular then the corresponding divisor $\bar{F}_x$ enjoys the following properties [6, Ch. I, Sect. 2.12; Ch. 3, Sect. 1.4, Lemma 1.4.1 on p. 195]] (see also [3]).

(i) Each irreducible component of $\bar{F}_x$ is a smooth rational curve (and the corresponding graph is a tree) ([3, Sect. 3], [6, Ch. 3, Sect. 1.4, Lemma 1.4.1(2) on p. 195].

(ii) At least, one of the irreducible components of $\bar{F}_x$ is a (-1)-curve ([3, Sect. 4.2], [6, Ch. 3, Sect. 1.4, Lemma 1.4.1(5) on p. 195]).

(iii) If one of the irreducible components of $\bar{F}_x$ is a (-1)-curve of multiplicity 1 then there is another irreducible (-1)-component of $\bar{F}_x$ ([3, Sect. 4.2], [6, Ch. 3, Sect. 1.4, Lemma 1.4.1(6) on p. 195]).

2.2. If $\sigma \in \mathrm{Bir}(\bar{S})$ then there is a unique *biregular* automorphism $f(\sigma) : X \to X$ such that the composition $\bar{\pi}\sigma$ is a *regular* map that coincides with the composition

$$f(\sigma)\bar{\pi} : S \xrightarrow{\bar{\pi}} X \xrightarrow{f(\sigma)} X$$

[11, Lecture V, Sect. 1.4, p. 99]. Clearly, σ sends the fiber $\bar{\pi}^{-1}(x)$ to the fiber $\bar{\pi}^{-1}(f(\sigma)(x))$ for all $x \in X(k)$. We get a surjective group homomorphism

$$f : \mathrm{Bir}(\bar{S}) \to \mathrm{Aut}(X),\ \sigma \mapsto f(\sigma)$$

that fits into a short exact sequence

$$\{1\} \to \mathrm{Bir}_X(\bar{S}) \subset \mathrm{Bir}(\bar{S}) \xrightarrow{f} \mathrm{Aut}(X) \to \{1\}$$

where the subgroup $\mathrm{Bir}_X(\bar{S})$ consists of all birational automorphisms $\sigma \in \mathrm{Bir}(\bar{S})$ such that $\bar{\pi}\sigma = \bar{\pi}$ (i.e. σ leaves invariant every fiber of $\bar{\pi}$). In addition, $\mathrm{Bir}_X(\bar{S})$ is isomorphic to the projective linear group $\mathrm{PGL}(2, k(X))$ over the field $k(X)$ of rational functions on X [11, Lecture V, Sect. 1.4, p. 99].

2.3. We write π for the composition

$$S \subset \bar{S} \xrightarrow{\bar{\pi}} X,$$

i.e., for the restriction of π to S. Recall that $\mathrm{Aut}(S) \subset \mathrm{Bir}(\bar{S})$. Since S is a surface, it is not contained in a union of finitely many fibers of π in $\bar{S}$. This implies that $\pi(S)$ is infinite and therefore is everywhere dense in X. It follows from [14, vol. 1, Ch. 1, Sect. 5, Th. 6] that either $\pi(S) = X$ or the complement $T_0 := X(k) \setminus \pi(S(k))$ is a finite set and

$$S \subset \pi^{-1}(X \setminus T_0) \subset \bar{S}.$$

If we write $\mathrm{Aut}_X(S)$ for the intersection (in $\mathrm{Bir}(\bar{S})$) of $\mathrm{Aut}(S)$ and $\mathrm{Bir}_X(\bar{S})$ then we get a short exact sequence

$$\{1\} \to \mathrm{Aut}_X(\bar{S}) \subset \mathrm{Aut}(S) \xrightarrow{f} f(\mathrm{Aut}(S)) \to \{1\}$$

where

$$\mathrm{Aut}_X(\bar{S}) \subset \mathrm{Bir}_X(\bar{S}),\ f(\mathrm{Aut}(X)) \subset \mathrm{Aut}(X).$$

Similarly to the case of projective surfaces, if $x \in X(k)$ then we write F_x for the effective divisor $\pi^*(x)$ on S that is the pullback(under π) of the divisor (x) on S. Clearly, the support of F_x coincides with the curve $\pi^{-1}(x)$ on S. It is also clear that the divisor F_x on S is the pullback of the divisor $\bar{F}_x$ on $\bar{S}$ under the (open) inclusion map $S \subset \bar{S}$. One says that the fiber of π over x is *reduced* if all irreducible components of the divisor F_x have multiplicity 1. We say that the fiber of π over x is irreducible if it is a multiple of a *simple* divisor, i.e., the curve $\bar{\pi}^{-1}(x)$ is irreducible. Clearly, if the fiber of $\bar{\pi}$ over x is irreducible (resp. reduced, resp. smooth) then the fiber of π over x is irreducible (resp. reduced, resp. smooth). On the other hand, if $\bar{F}_x$ has an irreducible component, say, $\bar{C}$ that appears in $\bar{F}_x$ with multiplicity $m > 1$ and, in addition, $\bar{C}$ meets S then $C := \bar{C} \bigcap S$ is an irreducible curve in S that is a component of F_x that appears in F_x with the same multiplicity m; in particular, the fiber of π over x is *not* reduced. Notice also that if $\bar{C}_1$ and

$\bar{C}_2$ are distinct irreducible components of $\bar{F}_x$ then $C_1 := \bar{C}_1 \bigcap S$ and $C_2 := \bar{C}_2 \bigcap S$ are *distinct* irreducible components of F_x; in particular, the fiber of π over x is *not* irreducible.

It follows from the results about the fibers of $\bar{\pi}$ mentioned in Sect. 2.1 (see also theorems of Bertini [14, vol. 1, Ch. 2, Sect. 6.1 and 6.2] that either all the fibers of π are smooth irreducible reduced or the set T_1 of points $x \in \pi(S(k)) \subset X(k)$ such that, at least, one of these properties does not hold, is finite. Clearly,

$$f(\mathrm{Aut}(S)) \subset \mathrm{Aut}(X, T_0), \; f(\mathrm{Aut}(S)) \subset \mathrm{Aut}(X, T_1).$$

This implies that if either T_0 or T_1 is *non*-empty then $f(\mathrm{Aut}(S))$ is a *finite* group and $\mathrm{Aut}_X(\bar{S})$ is a subgroup of *finite index* in $\mathrm{Aut}(S)$.

2.4. It follows from the theorem of Jordan that the projective linear group $\mathrm{PGL}(2, k(X))$ is Jordan [8, 16]. Since $\mathrm{Bir}_X(\bar{S})$ is isomorphic to $\mathrm{PGL}(2, k(X))$ (see Sect. 2.2), it is also a Jordan group. This implies in turn that its subgroup $\mathrm{Aut}_X(S)$ is also Jordan. It follows that if either T_0 or T_1 is *non*-empty then $\mathrm{Aut}(S)$ contains the Jordan subgroup $\mathrm{Aut}_X(S)$ of finite index and therefore is Jordan itself.

Definition 2.5. The projective surface $\bar{S}$ is called a (relative) *minimal closure* of S if every (-1)-curve on $\bar{S}$ meets S. See [3, Sect. 4.9]. A minimal closure of S always exists [3, Prop. 4.10]. (Warning: if $\bar{S}$ is a minimal closure then the complement of S in $\bar{S}$ does *not* have to be a divisor!)

Lemma 2.6 (Lemma 4.12 of [3])**.** *Assume that $\pi(S) = X$ and all the fibers of π are smooth irreducible and reduced.*

If $\bar{S}$ is a minimal closure of S then all the fibers of $\bar{\pi} : \bar{S} \to X$ are irreducible.

Proof. Suppose that there exists $x \in X(k)$ such that the fiber of $\bar{\pi}$ over x is not irreducible and therefore is singular. Then $\bar{F}_x$ contains as an irreducible component a (-1)-curve, say $\bar{C}_1$ with multiplicity $m \geq 1$ (Sect. 2.1). The minimality of $\bar{S}$ implies that $C_1 = \bar{C}_1 \bigcap S$ is non-empty and therefore is an irreducible component of F_x with the same multiplicity m (Sect. 2.3). Since the fiber of π over x is reduced, $m = 1$. This implies that $\bar{F}_x$ contains another irreducible component $\bar{C}_2$ that is also a (-1)-curve. Again $C_2 = \bar{C}_2 \bigcap S$ is an irreducible component of F_x that does not coincide with C_1. This implies that the fiber of π over x is *not* irreducible, which is not the case. □

Theorem 2.7. *Assume that $\pi(S) = X$ and all the fibers of π are smooth irreducible and reduced. Let $\bar{S}$ be a minimal closure of S Then every biregular automorphism of S extends uniquely to a biregular automorphism of $\bar{S}$. In other words,*

$$\mathrm{Aut}(S) \subset \mathrm{Aut}(\bar{S}) \subset \mathrm{Bir}(\bar{S}).$$

Proof. By Lemma 2.6, every fiber $\bar{F}_x$ is an irreducible curve isomorphic to $\mathbb{P}^1$ ([3], Lemma 4.12).

Let $g : S \to S$ be a biregular automorphism of S. Let us extend g to a birational map

$$\bar{g} : \bar{S} \to \bar{S}.$$

Assume that $\bar{g}$ is *not* a regular map. Let S' be a *resolution of the singularities* of $\bar{g}$, i.e. a smooth irreducible surface included into the following commutative digram.

$$\begin{array}{ccc} S' & & \\ u\downarrow & \searrow g' & \\ \bar{S} & \overset{\bar{g}}{\dashrightarrow} \bar{S} & \\ \cup & \cup & \\ S & \xrightarrow{g} S & \\ \pi\downarrow & \downarrow & \pi \\ X & \overset{h}{\longleftrightarrow} X & \end{array} ,$$

where u is a birational morphism that is a composition of finitely many blow ups and induces a biregular isomorphism between $u^{-1}(S)$ and S (such an u exists, because g is defined on S), g' and $\bar{\pi}' = \bar{\pi} \circ u$ are morphisms, and $h = f(g) \in \mathrm{Aut}(X)$ is a biregular automorphism of X. Let $D' \subset S'$ be the union of all exceptional curves for g' and let $D = g'(D') \subset \bar{S}$, which is a finite set. Every point z of $\bar{S}$ that does *not* lie on D has only one preimage ${g'}^{-1}(z) \in S'$. Let B' be the union of exceptional curves for u. Clearly,

$$B' \subset S' \setminus u^{-1}(S).$$

This implies that

$$u(B') \bigcap S = \emptyset.$$

We want to show that $B \subset D'$, because then one may contract all components of B' and $\bar{g}$ would appear to be a morphism.

Let C' be an irreducible component of B'. The *point* $u(C')$ lies in $u(B')$ and therefore does *not* belong to S.

Since X is an elliptic curve, and C' is rational, $\bar{\pi}(g'(C'))$ is a point $x \in X(k)$. Thus, since all the fibers of $\bar{\pi}$ are irreducible (thanks to Lemma 2.6), either

Case 1. $g'(C')$ is a point and therefore $C' \subset D'$;

or

Case 2. $g'(C') = \bar{F}_x = \bar{\pi}^{-1}(x) \subset \bar{S}$ for a point $x = h(x_1) \in X(k)$ with $x_1 := h^{-1}(x) \in X(k)$. Let $s \in F_x \setminus (F_x \cap D) \subset S$ be a point of the fiber F_x, which is not in the image of D'. Then, since $g'(C') = \bar{F}_x$, there is a point $c \in C'$ such that $g'(c) = s$. We have

$$c \in C' \subset B' \subset S' \setminus u^{-1}(S).$$

In particular,

$$c \notin u^{-1}(S).$$

On the other hand, since $g \in \mathrm{Aut}(S)$ and u^{-1} is a biregular isomorphism between S and $u^{-1}(S) \subset S'$, there is a point

$$s_1 \in F_{x_1} \subset S$$

such that

$$g'(u^{-1}(s_1)) = g(s_1) = s.$$

It follows that the preimage ${g'}^{-1}(s)$ is a *finite* set that contains (at least) *two distinct points* $c \notin u^{-1}(S)$ and $u^{-1}(s_1) \in u^{-1}(S)$, which is impossible for a birational morphism g' [14, Ch. 2, Sect. 4, Th. 2]. This contradiction shows that the Case 2 does not occur.

This proves that every $g \in \operatorname{Aut}(S)$ extends to a regular birational map $\bar{g} : \bar{S} \to \bar{S}$. Since the same is true for $g^{-1} \in \operatorname{Aut}(S)$, the map $\bar{g}$ is a biregular automorphism of $\bar{S}$.

□

3. Proof of Theorem 1.3

In light of results of Section 2.4, we may and will assume that every fiber of π is smooth irreducible and reduced, and $\pi(S) = X$. Let $\bar{S}$ be a minimal closure of S. By Theorem 2.7, $\operatorname{Aut}(S)$ is a subgroup of $\operatorname{Aut}(\bar{S})$. Since $\bar{S}$ is projective, the results of [16] imply that the group $\operatorname{Aut}(\bar{S})$ is Jordan and therefore its every subgroup is Jordan. This implies that $\operatorname{Aut}(S)$ is Jordan.

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