Scalar-Flat Closed Manifolds not Admitting Positive Scalar Curvature Metrics

Akito Futaki

Tokyo Insitute of Technology
Department of Mathematics
Oh-okayama
Meguro
Tokyo

Japan

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

MPI / 92-57

SCALAR-FLAT CLOSED MANIFOLDS NOT ADMITTING POSITIVE SCALAR CURVATURE METRICS

AKITO FUTAKI

Max-Planck-Institut für Mathematik
Gottfried-Claren-Strasse 26, D-5300 Bonn 3, Germany

Tokyo Institute of Technology
Department of Mathematics, Oh-okayama, Meguro, Tokyo, Japan

Dedicated to Professor Shoshichi Kobayashi on his sixtieth birthday

1. INTRODUCTION

All the closed connected manifolds are divided into the following three classes:

(P) manifolds admitting a Riemannian metric of positive scalar curvature;
(Z) manifolds admitting a metric of non-negative scalar curvature, but not admitting a metric of positive scalar curvature;
(N) manifolds not admitting a metric of non-negative scalar curvature.

If a closed manifold M has a metric of non-negative scalar curvature which is not identically zero, then by a conformal change of the metric we get a metric of positive scalar curvature [KW]. Therefore any metric of non-negative scalar curvature on a manifold in the class (N) is in fact scalar-flat.

It is desirable to have a characterization of these three classes. A characterization of the class (P) has been given by Gromov-Lawson [GL] and Stolz [S1], [S2]: if M is simply connected and $\dim M \geq 5$, then M belongs to the class (P) if and only if M is either non-spin or spin with vanishing Lichnerowicz-Hitchin obstruction $\alpha(M)$. In this paper we remark that their results also imply the following.

Theorem 1. *Let M be a closed simply connected manifold with $\dim M \geq 5$. Then M belongs to the class (Z) if and only if M is the product of manifolds $M_1 \times \cdots \times M_l$ such that*

(1) *$\pm M_i$ admits a Ricci-flat Kähler metric or a Riemannian metric with $Spin(7)$ holonomy (in both cases M_i is necessarily spin);*
(2) *$\alpha(M) \neq 0$.*

Note that $Spin(7)$ holonomy occurs only in dimension 8, and the metric is Ricci-flat. So far no compact example of a Riemannian manifold with $Spin(7)$ holonomy has been known.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

The condition $\alpha(M) \neq 0$ in Theorem 1 can not be omitted. For example a non-singular hypersurface M of degree $4k+5$ in the $(4k+4)$-dimensional complex projective space is a simply connected Calabi-Yau manifold, but $\alpha(M) = 0$ since $KO^{-(8k+6)}(pt) = 0$. Thus M admits a metric of positive scalar curvature by the theorem of Stolz.

Corollary 2. *Let M^n be connected closed manifold with finite fundamental group such that its universal cover is spin. If $\sharp\pi_1(M) \cdot |\hat{A}(M)| > 2^{n/4}$, then M does not admit a metric of non-negative scalar curvature.*

In the proof we use a standard fact that if a spin manifold has a parallel spinor then M has special holonomy (c.f. [H],[F], [W]). We will review this fact for the sake of completeness.

Remark. In the case of infinite fundamental groups, it is known that if M is a closed spin manifold with $\hat{A}(M) \neq 0$ then M belongs to the class (N) (c.f. [O], [M]).

I am grateful to Stephan Stolz for explaining his results to me and for his interest in this work, and to Kaoru Ono for helpful discussions.

2. Holonomy groups

Let P be a principal bundle with a connection over a manifold M with structure group G. The fiber over $x \in M$ will be denoted by P_x, and the Lie algebra of G will be denoted by $\mathfrak{g}$. A connection on P is a smooth splitting of TP into the tangent bundle T_f along the fibers and a right invariant horizontal distribution. A piecewise C^1 curve $\gamma : [0,1] \to M$ is called a loop with base point x if $\gamma(0) = \gamma(1) = x$. Let us fix $p \in P_x$, and consider the horizontal lift $\tilde{\gamma}_p : [0,1] \to P$ of γ such that $\tilde{\gamma}_p(0) = p$. Define $h_{\gamma,p} \in G$ by $\tilde{\gamma}_p(1) = \tilde{\gamma}_p(0)h_{\gamma,p}$. The set of $h_{\gamma,p}$ for all possible loops γ with base point x consists a subgroup of G, which we denote by $Hol(P)_p$. Let us choose another $q = pk \in P_x$ with $k \in G$. Then by the right invariance of the horizontal distribution, we have $\tilde{\gamma}_q = \tilde{\gamma}_p k$ and

$$\tilde{\gamma}_p(0)kh_{\gamma,q} = \tilde{\gamma}_q(0)h_{\gamma,q} = \tilde{\gamma}_q(1) = \tilde{\gamma}_p(1)k = \tilde{\gamma}_p(0)h_{\gamma,p}k.$$

Thus $h_{\gamma,q} = Ad(k^{-1})h_{\gamma,p}$, and $h_{\gamma,p}$ and $h_{\gamma,q}$ define the same element h_γ in the fiber $Ad(P)_x$ of the adjoint bundle $Ad(P) := P \times_{Ad} G$. Hence $Hol(P)_p$ and $Hol(P)_q$ define the same subgroup $Hol(P)_x$ in $Ad(P)_x$, which we call the holonomy group of P at $x \in M$. If we choose another base point $y \in M$, then $Hol(P)_y$ is isomorphic to $Hol(P)_x$; this isomorphism class is called the holonomy group of P.

Let $\rho : G \to GL(V)$ be a representation of G in a real vector space V. Let $E = P \times_\rho V$ be the associated vector bundle. Then $Ad(P)_x$, and hence $Hol(P)_x$, act on E_x in the canonical way.

The connection on P defines uniquely a covariant derivative in E. Let $e_1, \ldots, e_r$ be a local frame field of E over an open set $U \subset M$. Let σ be a local section of P over U. Then the covariant derivative ∇ of E is defined by

$$\nabla e_j = \sum_{i=1}^{r} \rho_*(\sigma^*\omega)^i_j e_i,$$

where ω is the connection form on P, i.e. the projection $TP_p \to T_f \cong \mathfrak{g}$ along the horizontal distribution.

Let $\gamma(t)$ be a path in M. A section s of E over γ is said to be parallel if $\nabla_{\frac{\partial}{\partial t}} s = 0$.

Lemma 2.1. *Let s be a parallel section of E over γ. Then there exists a vector $v \in V$ and a horizontal lift $\tilde{\gamma}$ of γ such that $s(t) = (\tilde{\gamma}(t), v) \in P \times_\rho V = E$.*

Proof. Let $e_1(t), \ldots, e_r(t)$ be a local frame field of E along γ. The parallel section $s(t) = \sum_i u^i(t)e_i(t)$ is uniquely obtained by solving the ordinary differential equation

$$\frac{\partial u^i(t)}{\partial t} + \sum_j \rho_*(\omega(\sigma_*(\frac{\partial}{\partial t})))^i_j u^j(t) = 0, \qquad 1 \le i \le r$$

with the initial value $s(0) = \sum_i u^i(0)s_i(0)$. On the other hand, if $s(0) = (p, v) \in P \times_\rho V = E$, and if we take the horizontal lift $\tilde{\gamma}$ of γ with $\tilde{\gamma}(0) = p$, then $s_1(t) = (\tilde{\gamma}(t), v)$ is a parallel section by the definition of ∇. From the uniquenessof s we have $s = s_1$. This completes the proof.

Lemma 2.2. *Let s be a global parallel section of E. Then $Hol(P)_x \subset Ad(P)_x$ is contained in the isotropy subgroup $Ad(P)^{s(x)}$ of $s(x)$.*

Proof. Let $(p, h_\gamma) \in Hol(P)_x$ correspond to a loop γ with base point x. By Lemma 2.1, $s|_\gamma$ can be written as $s|_{\gamma(t)} = (\tilde{\gamma}(t), v)$ for some $v \in V$. Then, since $s|_{\gamma(0)} = s|_{\gamma(1)}$, we have

$$(\tilde{\gamma}(0), v) = (\tilde{\gamma}(1), v) = (\tilde{\gamma}(0)h_\gamma, v) = (\tilde{\gamma}(0), h_\gamma v) = (p, h_\gamma)(\tilde{\gamma}(0), v).$$

Thus $(p, h_\gamma) \in Ad(P)^{s(x)}$, completing the proof.

Let $Hol^0(P)_x$ be the subgroup of $Hol(P)_x$ consisting of all holonomies for null-homotopic loops with base point x. The group $Hol^0(P)$ is called the restricted holonomy group of P. Obviously $Hol^0(P)_x$ is connected and is contained in the identity component $Ad^0(P)_x$ of $Ad(P)_x$. Before we state the following proposition, we note that the representation of $Ad(P)_x$ in E_x is equivalent to ρ.

Proposition 2.3. *Suppose that the identity component G^0 of G is compact, so that the restriction $\rho|_{G^0}$ of ρ to G^0 splits into irreducible orthogonal representations. Suppose further that each irreducible component of $\rho|_{G^0}$ has dimention ≥ 2. If E admits a parallel section, then $Hol^0(P)$ is strictly smaller than G^0.*

proof. Let $F_x \subset E_x$ be an irreducible component of $\rho|_{G^0}$, and $\pi : E_x \to F_x$ be the projection. By Lemma 2.2, $Hol^0(P)_x \subset Ad(P)^{s(x)} \cap Ad^0(P)_x \subset Ad(P)^{\pi s(x)} \cap Ad^0(P)_x$. If $Hol^0(P)_x = Ad^0(P)_x$, then $Ad^0(P)_x \subset Ad(P)^{\pi s(x)}$. Thus $Ad^0(P)_x$ leaves $\pi s(x)$ fixed. But then F_x is reducible since $\dim F_x \ge 2$. This is a contradiction and completes the proof.

Let G' be a covering group of G with covering map λ. We say that a principal bundle P' with structure group G' covers P if there is a covering $\mu : P' \to P$ such that $\mu(pg) = \mu(p)\lambda(g)$ for $p \in P'$ and $g \in G'$. In this situation the right invariant distribution in P naturally lifts to P' to define a connection in P'.

Lemma 2.4. *There are covering maps of $Hol(P')$ onto $Hol(P)$, and $Hol^0(P')$ onto $Hol^0(P)$.*

proof. Let γ_1 and γ_2 be loops in M with base point x, and pick $p \in P_x$ and $p' \in P'_x$. It suffices to show that if $h_{\gamma_1,p'} = h_{\gamma_2,p'}$ in $Hol(P')_{p'}$ then $h_{\gamma_1,p} = h_{\gamma_2,p}$ in $Hol(P)_p$. But this is true because a closed path in P' is mapped under the covering map onto a closed path. This completes the proof.

In the rest of this section we will consider the Riemannian case, for which the reader is referred to [B], chapters 10 and 14, and [Sa]. Let M be an oriented Riemannian manifold of dimension n. The Levi-Civita connection defines a parallelism on the tangent bundle of M. Thus the parallel transport along a loop γ with base point x determines an element $f_\gamma \in SO(T_xM)$. Let P_{SO} be the principal bundle associated to the tangent bundle. Then a point $p \in (P_{SO})_x$ stands for an oriented orthonormal basis $p = (e_1, \ldots, e_n)$ of T_xM, and we can express f_γ in terms of a special orthogonal matrix $h_{\gamma,p}$ by

$$f_\gamma(p) = (f_\gamma(e_1), \ldots, f_\gamma(e_n)) = (e_1, \ldots, e_n)h_{\gamma,p} = ph_{\gamma,p}.$$

This h_γ determines an element of the adjoint bundle, which is nothing more than f_γ.

In this Riemannian situation we denote by $Hol(M)_x$ the holonomy group at x, and by $Hol^0(M)_x$ the subgroup consisting of elements h_γ for null-homotopic loops γ. $Hol^0(M)$ is called the restricted holonomy group of M. The following facts are well known: $Hol^0(M)$ is a connected closed subgroup of $SO(n)$; If $\widetilde{M}$ is the universal cover then $Hol^0(M) \cong Hol(\widetilde{M})$; If $M = M_1 \times M_2$ is a Riemannian product, then $Hol^0(M) = Hol^0(M_1) \times Hol^0(M_2)$, and conversely if the restricted holonomy group splits as a non-trivial product, then $\widetilde{M}$ splits as a Riemannian product (known as the de Rham decomposition). We will say that M is irreducible if its holonomy group does not split non-trivially. For irreducible Riemannian manifolds Berger-Simons theorem says that, if M is not locally symmetric, $Hol^0(M)$ is one of the following:

(1) $SO(n)$;
(2) $U(m)$, where $n = 2m$, in which case M is Kähler but not Ricci-flat;
(3) $SU(m)$, where $n = 2m$, in which case M is Ricci-flat Kähler;
(4) $Sp(k) \cdot Sp(1) := Sp(k) \times Sp(1)/\{\pm 1\}$ where $n = 4k$, in which case M is called a Quaternionic Kähler manifold, and is an Einstein manifold but neither Ricci-flat nor Kähler;
(5) $Sp(k)$, where $n = 4k$, in which case M is called a hyperkähler manifold, and is Ricci-flat Kähler;
(6) G_2, in which case $n = 7$ and M is Ricci-flat;
(7) $Spin(7)$, in which case $n = 8$ and M is Ricci-flat.

Remark 2.5. A locally symmetric space is Einstein with non-zero scalar curvature. Thus for an irreducible Riemannian manifold M, if $Hol^0(M) \neq SO(n)$ and M is Ricci-flat, then $Hol^0(M)$ is either $SU(m)$, $Sp(k)$, G_2 or $Spin(7)$.

Remark 2.6. The reduction of the holonomy group defines naturally a reduction of the structure group of P_{SO} to $Hol(M)$. If the holonomy group reduces to G_2 or $Spin(7)$, then M is spin since $\pi_1(G_2) = 1 = \pi_1(Spin(7))$. We have used this fact in the statement of Theorem 1.

3. Proofs

The group $Spin(n)$ is generated by the elements in the real Clifford algera Cl_n of even degree and with unit length. Let M be a spin manifold of dimension n, and $P_{Spin} \to P_{SO}$ be a spin structure. By a real (resp. complex) spin-module we mean a $Spin(n)$ module obtained by restriction to $Spin(n)$ of a module of Cl_n (resp. the complexified Clifford algebra $\mathbb{C}l_n$). We call a real (resp. complex) spinor bundle the vector bundle which associates to P_{Spin} via a real (resp. complex) spin-module. Given a spinor bundle S we can define the Dirac operator D by $Ds = \sum_i e_i \nabla_{e_i} s$, where s is a section of the spinor bundle and $e_1, \ldots, e_n$ are a local oriented orthonormal frame of M. We have the following Lichnerowicz formula:

$$D^2 = \nabla^* \nabla + \frac{1}{4}\kappa$$

where κ denotes the scalar curvature of M. This formula says that if M admits a positive scalar curvature then there is no harmonic spinor, and that if the scalar curvature is identically zero then a harmonic spinor is parallel.

Let $l : Spin(n) \to Hom(Cl_n, Cl_n)$ be the left multiplication of $Spin(n)$ on Cl_n. The bundle $P_{Spin} \times_l Cl_n$ admits an action of Cl_n by the right multiplication. Thus the kernel $\operatorname{Ker} \mathfrak{D}$ of the Dirac operator $\mathfrak{D}$ is a $\mathbb{Z}_2$-graded Cl_n-module.

Let $\widehat{\mathfrak{M}}_n$ be the Grothendieck group of $\mathbb{Z}_2$-graded Cl_n-modules. Let $i : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be the inclusion. Then i induces $i_* : Cl_n \to Cl_{n+1}$ and $i^* : \widehat{\mathfrak{M}}_{n+1} \to \widehat{\mathfrak{M}}_n$. Through the isomorphism $\widehat{\mathfrak{M}}_n / i^* \widehat{\mathfrak{M}}_{n+1} \cong KO^{-n}(pt) := KO(D^n, S^{n-1})$, $\operatorname{Ker} \mathfrak{D}$ defines an element of $KO^{-n}(pt)$ which we define to be $\alpha(M)$. Moreover α induces a ring homomorphism $\alpha_* : \Omega_*^{Spin} \to KO^{-*}$ where Ω_*^{Spin} denotes the spin cobordism ring. By the Lichnerowicz formula, if M admits a metric of positive scalar curvature, then $\alpha(M) = 0$. Conversely, by the theorem of Stolz, if M is a closed simply-connected spin manifold of dimension ≥ 5 such that $\alpha(M) = 0$, then M admits a metric of positive scalar curvature.

Recall that $KO^{-n}(pt)$ is isomorphic to $\mathbb{Z}_2$ for $n = 8k+1$ and $8k+2$, to $\mathbb{Z}$ for $n = 8k$ and $8k+4$, and to 0 for other dimensions. For actual purposes $\operatorname{ind} \mathfrak{D}$ are computed by $\dim_{\mathbb{R}} \operatorname{Ker} D^0 \mod 2$ for $n = 8k+1$, $\dim_{\mathbb{C}} \operatorname{Ker} D^0 \mod 2$ for $n = 8k+2$, $\hat{A}(M)/2$ for $n = 8k+4$, $\hat{A}(M)$ for $n = 8k$, and by 0 for other dimensions, where $D = D^0 + D^1$ denotes the Dirac operator for the real spinor bundle. In any event if $\operatorname{ind} \mathfrak{D} \neq 0$, there exists a harmonic spinor for the real spinor bundle. This last fact can be seen also from the fact that $P_{Spin} \times_l Cl_n$ is a direct sum of irreducible real spinor bundles, each component having vanishing second fundamental form.

Lemma 3.1. *Let $M = M_1 \times \cdots \times M_k$ be a Riemannian product of closed spin manifolds. Then a spinor bundle on M has a harmonic (resp. parallel) spinor if and only if a spinor bundle of each M_i has a harmonic (resp. parallel) spinor.*

Proof. We describe the case of real spinor bundles. It suffices to show in the case of $k = 2$. Suppose that $\dim M_i = n_i$ for $i = 1, 2$, and that $n = n_1 + n_2$. We denote by $P_{Spin(n)}$ (resp. $P_{Spin(n_i)}$) the spin structure of M (resp. M_i). We may assume that a harmonic spinor exists in an irreducible spinor bundle on M. Since an irreducible spin-module is imbedded into Cl_n which is considered as a left Cl_n module, we may further assume that $S_M := P_{Spin(n)} \times_l Cl_n$ has a harmonic spinor. Set $S_{M_i} = P_{Spin(n_i)} \times_l Cl_{n_i}$. Then since $Cl_{p+q} \cong Cl_p \hat{\otimes} Cl_q$, we have $S_M = \pi_1^* S_{M_1} \otimes \pi_2^* S_{M_2}$ where $\pi_i : M \to M_i$ denotes the projection. Denote by $\mathfrak{D}$ and $\mathfrak{D}_i$ the Dirac operators of S_M and S_{M_i}. Let $\eta = \phi \otimes \psi$ be a local section of S_M. Since $\mathfrak{D}^2 = \nabla^* \nabla + \frac{1}{4}\kappa_M$, we have

$$\mathfrak{D}^2 \eta = (\mathfrak{D}_1^2 \phi) \otimes \psi + \phi \otimes (\mathfrak{D}_2^2 \psi).$$

Let $\{\lambda_i\}_{i=1}^{\infty}$ and $\{\mu_j\}_{j=1}^{\infty}$ be the eigenvalues of $\mathfrak{D}_1^2$ and $\mathfrak{D}_2^2$. Since the Dirac operators are self-adjoint, λ_i and μ_j are all non-negative. Let $\{\phi_i\}_{i=1}^{\infty}$ and $\{\psi_j\}_{j=1}^{\infty}$ be the corresponding eigensections. Then they respectively form L^2-bases of $C^\infty(S_{M_1})$ and $C^\infty(S_{M_2})$. Thus $\{\phi_i \otimes \psi_j\}_{i,j=1}^{\infty}$ are eigensections of S_M which form an L^2-basis of $C^\infty(S_M)$, and the corresponding eigenvalues are $\{\lambda_i + \mu_j\}_{i,j=1}^{\infty}$. Clearly $\lambda_1 + \mu_1 = 0$ if and only if $\lambda_1 = \mu_1 = 0$. This proves the case of harmonic spinors. For parallel spinors, we have only to replace $\mathfrak{D}^2$ by $\nabla^* \nabla$. This completes the proof.

Proposition 3.2 ([F]). *Let S be a spinor bundle on a Riemannian spin manifold M. If S has a non-zero parallel spinor ψ, then M is Ricci-flat.*

Proof. Using the first Bianchi identity one sees

$$\begin{aligned} 0 &= \sum_j e_j (\nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i} - \nabla_{[e_i, e_j]}) \psi \\ &= -\frac{1}{4} \sum_{j,k,l} e_j R_{ijkl} e_k e_l \psi = -\frac{1}{2} \sum_l Ric(e_i, e_l) e_l \psi \end{aligned}$$

for any i. Thus $Ric = 0$, as desired.

Proposition 3.3 ([H]). *Let M be a closed Riemannian spin manifold of dimension n. If a spinor bundle S on M has a parallel spinor, then M is Ricci-flat and the resricted holonomy group $Hol^0(M)$ of M reduces to a product whose irreducible components are $SO(1)$, $SU(m)$, $Sp(k)$, G_2, or $Spin(7)$.*

Proof. First of all M is Ricci-flat by Proposition 3.2. Passing to the universal cover$\widetilde{M}$ we may assume that there exists a parallel spinor of a spinor bundle. By the de Rham decomposition $\widetilde{M}$ splits into a Riemannian product in such a way that the holonomy group of each irreducible component M_i is irreducible. By Lemma 3.1 M_i has a parallel spinor. If $n_i := \dim M_i = 1$, then $Hol(M_i) = SO(1)$. If $n_i \geq 2$, then an irreducible spin representation has dimension ≥ 2. Then by Proposition 2.3, the dimension of the holonomy group of the principal bundle associated to the spinor bundle is strictly smaller than $\dim Spin(n_i) = \dim SO(n_i)$. Thus by Lemma 2.4 the holonomy group $Hol(M_i)$ is strictly smaller than $SO(n_i)$. Since M_i is Ricci-flat, possible holonomy groups are $SU(m)$, $Sp(k)$, G_2 or $Spin(7)$. Since $Hol^0(M) = Hol(\widetilde{M})$, we are done.

Proof of Theorem 1. Let M be a simply connected closed manifold of dimension ≥ 5 which belongs to the class (Z). By the theorems of Gromov-Lawson and Stolz that M is spin and $\alpha(M) \neq 0$. Thus M has a harmonic spinor, but as M admits a scalar-flat metric, we may assume that the harmonic spinor is parallel. Then by Proposition 3.3 the holonomy group of M is a product of $SO(1)$, $SU(m)$, $Sp(k)$, G_2 and $Spin(7)$. However $SO(1)$ is ruled out because M is simply connected, and G_2 is also ruled out because if M contains a 7-dimensional irreducible factor then we have $\alpha(M) = 0$ by the multiplicative property of α. The converse is obvious. This completes the proof.

Proof of Corollary 2. Suppose that the universal cover $\widetilde{M}$ belongs to the class (Z). Let M_i be an irreducible component of $\widetilde{M}$ as in Theorem 1. By [W], $|\hat{A}(M_i)| = 1$ if $Hol(M_i) = Spin(7)$, $\hat{A}(M_i) = 2$ if $Hol(M_i) = SU(m_i)$ and m_i is even, and $\hat{A}(M_i) = k_i + 1$ if $Hol(M_i) = Sp(k_i)$. Hence we have $|\hat{A}(\widetilde{M})| \leq 2^{\frac{n}{4}}$ where the equality holds when $\widetilde{M}$ is the product of $K3$-surfaces, from which the corollary follows.

Remark 3.4. In the case where M is spin and with finite fundamental group, there are existence results of positive scalar curvature metrics for certain fundamental groups (e.g. $\mathbb{Z}/2$) under the assumption that all index obstructions coming from flat bundles vanish (c.f.[RS],[R],[KS]). These results also imply the characterization of the class (Z). For example one can prove that, if M is spin with $\pi_1(M) = \mathbb{Z}/2$, then M belongs to the class (Z) if and only if there exists a metric on M such that the Riemannian covering $\widetilde{M}$ is a product of Ricci-flat Kähler manifolds and manifolds with $Spin(7)$ and G_2 holonomy and if M has non-vanishing index obstruction. In this case it is not easy to rule out the G_2 holonomy since it is not clear if the positive scalar curvature on $\widetilde{M}$ is $\mathbb{Z}/2$-invariant to descend to M.

References

[B] A.L.Besse: Einstein manifolds, Springer-Verlag, Berlin and New York, 1986

[F] Th.Friedrich: Zur Existenz paralleler Spinorfelder über Riemannschen Mannigfaltigkeiten, Colloq. Math. vol.XLIV(1981), 277–290

[GL] M.Gromov and H.B.Lawson: The classification of simply connected manifolds of positive scalar curvature, Ann. of Math. **111** (1980), 423–434

[H] N.Hitchin: Harmonic spinors, Adv. in Math. **14** (1974), 1-55

[KW] J.L.Kazdan and F.W.Warner: Scalar curvature and conformal deformation of Riemannian structure, Journ. Diff. Geom. **10**(1975), 113–134

[KS] S.Kwasik and R.Schultz: Positive scalar curvature and periodic fundamental group, Comment. Helv. Math. **65** (1990), 271–286

[M] V.Mathai: Non-negative scalar curvature, preprint, Univ. of Adelaide, 1990

[O] K.Ono: The scalar curvature and the spectrum of the Laplacian of spin manifolds, Math. Ann. **281** (1988), 163–168

[R] J.Rosenberg: The KO-assembly map and positive scalar curvature, Lecture Notes in Math. **1474** (1991), Springer-Verlag, Berlin and New York

[RS] J.Rosenberg and S.Stolz: Manifolds of positive scalar curvature, preprint, 1990

[Sa] S.Salamon: Riemannian geometry and holonomy groups, Pitman Research Notes in Math. **201** (1989), Longman Scientific & Technical, Essex

[S1] S.Stolz: Simply connected manifolds of positive scalar curvature, Bull. Amer. Math. Soc. **23** (1990), 427–432

[S2] S.Stolz: Simply connected manifolds of positive scalar curvature, to appear in Ann. of Math.

[W] M.Wang: Parallel spinors and parallel forms, Ann. Global Anal. Geom. **7** (1989), 59–68