Max-Planck-Institut für Mathematik
Bonn

Twisted correlation functions on self-sewn Riemann surfaces via generalized vertex algebra of intertwiners

by

Alexander Zuevsky

Max-Planck-Institut für Mathematik
Preprint Series 2012 (63)

Twisted correlation functions on self-sewn Riemann surfaces via generalized vertex algebra of intertwiners

Alexander Zuevsky

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

MPIM 12-63

TWISTED CORRELATION FUNCTIONS ON SELF-SEWN RIEMANN SURFACES VIA GENERALIZED VERTEX ALGEBRA OF INTERTWINERS

A. ZUEVSKY

Abstract. We review (based on the talk at the Conference "Conformal Field Theory, Automorphic Forms and Related Topics", Heidelberg Universität, Heidelberg, Germany, 2011) our recent results on computation of the partition and n-point "intertwined" functions for modules of vertex operator superalgebras with formal parameter associated to local parameters on Riemann surfaces obtained by self-sewing of a lower genus Riemann surface. We introduce the torus intertwined n-point functions containing two intertwining operators in the supertrace. Then we define the partition and n-point correlation functions for a vertex operator superalgebra on a genus two Riemann surface formed by self-sewing of the torus. For the free fermion vertex operator superalgebra we present a closed formula for the genus two continuous orbifold partition function in terms of an infinite dimensional determinant with entries arising from the original torus Szegő kernel. This partition function is holomorphic in the sewing parameters on a given suitable domain and possess natural modular properties. We describe modularity of the generating function for all n-point correlation functions in terms of a genus two Szegő kernel determinant.

1991 *Mathematics Subject Classification.* 17B69, 30F10, 32A25, 11F03.

Key words and phrases. Vertex operator superalgebras, intertwining operators, Riemann surfaces, Szegő kernel, modular forms.

The author was supported by Max–Planck Institut für Mathematik, Bonn, Germany.

1. Introduction

In this paper (based on the talk at the Conference "Conformal Field Theory, Automorphic Forms and Related Topics", Heidelberg Universität, Heidelberg, Germany, 2011) we review our recent result on construction and computation of correlation functions of vertex operator superalgebras with a formal parameter associated to local coordinates on a self-sewn Riemann surface of genus g which forms a genus $g+1$ surface. In particular, we review result presented in the papers [TZ1]- [TZ5] accomplished in collaboration with M. P. Tuite (National University of Ireland, Galway).

1.1. **Vertex operator super algebras.** A Vertex Operator Superalgebra (VOSA) [B, DL, Ka, FHL, FLM] is a quadruple $(V, Y, \mathbf{1}, \omega)$: $V = V_{\bar{0}} \oplus V_{\bar{1}} = \bigoplus_{r \in \frac{1}{2}\mathbb{Z}} V_r$, $\dim V_r < \infty$, is a superspace, Y is a linear map $Y : V \to (\mathrm{End} V)[[z, z^{-1}]]$: so that for any vector (state) $u \in V$ we have $u(k)\mathbf{1} = \delta_{k,-1} u$, $k \geq -1$,

$$Y(u, z) = \sum_{n \in \mathbb{Z}} u(n) z^{-n-1},$$

$u(n) V_\alpha \subset V_{\alpha + p(u)}$, $p(u)$-parity. The linear operators (modes) $u(n) : V \to V$ satisfy creativity

$$Y(u, z)\mathbf{1} = u + O(z),$$

and lower truncation

$$u(n)v = 0,$$

conditions for u, $v \in V$ and $n \gg 0$.

These axioms identity impy locality, associativity, commutation and skew-symmetry:

$$(z_1 - z_2)^m Y(u, z_1) Y(v, z_2) = (-1)^{p(u,v)} (z_1 - z_2)^m Y(v, z_2) Y(u, z_1),$$

$$(z_0 + z_2)^n Y(u, z_0 + z_2) Y(v, z_2) w = (z_0 + z_2)^n Y(Y(u, z_0)v, z_2) w,$$

$$u(k) Y(v, z) - (-1)^{p(u,v)} Y(v, z) u(k) = \sum_{j \geq 0} \binom{k}{j} Y(u(j)v, z) z^{k-j},$$

$$Y(u, z)v = (-1)^{p(u,v)} e^{zL(-1)} Y(v, -z) u,$$

for u, v, $w \in V$ and integers m, $n \gg 0$, $p(u, v) = p(u)p(v)$.

The vacuum vector $\mathbf{1} \in V_{\bar{0},0}$ is such that, $Y(\mathbf{1}, z) = Id_V$, and $\omega \in V_{\bar{0},2}$ the conformal vector satisfies

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2},$$

where $L(n)$ form a Virasoro algebra for a central charge C

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{C}{12}(m^3 - m)\delta_{m,-n}.$$

$L(-1)$ satisfies the translation property

$$Y(L(-1)u, z) = \partial_z Y(u, z).$$

$L(0)$ describes a grading with $L(0)u = wt(u)u$, and $V_r = \{u \in V | wt(u) = r\}$.

1.2. **VOSA modules.**

Definition 1. A V-module for a VOSA V is a pair (W, Y_W), W is a $\mathbb{C}$-graded vector space $W = \bigoplus_{r \in \mathbb{C}} W_r$, $\dim W_r < \infty$, $W_{r+n} = 0$ for all r and $n \ll 0$. $Y_W : V \to \mathrm{End}(W)[[z, z^{-1}]]$

$$Y_W(u, z) = \sum_{n \in \mathbb{Z}} u_W(n) z^{-n-1},$$

for each $u \in V$ $u_W : W \to W$. $Y_W(\mathbf{1}, z) = \mathrm{Id}_W$, and for the conformal vector

$$Y_W(\omega, z) = \sum_{n \in \mathbb{Z}} L_W(n) z^{-n-2},$$

where $L_W(0)w = rw$, $w \in W_r$. The module vertex operators satisfy the Jacobi identity:

$$\begin{aligned} z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) & Y_W(u, z_1) Y_W(v, z_2) \\ -(-1)^{p(u,v)}\delta\left(\frac{z_2 - z_1}{-z_0}\right) & Y_W(v, z_2) Y_W(u, z_1) \\ = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) & Y_W\left(Y(u, z_0)v, z_2\right). \end{aligned}$$

Recall that

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^n.$$

The above axioms imply that $L_W(n)$ satisfies the Virasoro algebra for the same central charge C and that the translation property

$$Y_W(L(-1)u, z) = \partial_z Y_W(u, z).$$

1.3. **Twisted modules.** We next define the notion of a twisted V-module [FHL, DLM2]. Let g be a V-automorphism g, i.e., a linear map preserving $\mathbf{1}$ and ω such that

$$gY(v,z)g^{-1}=Y(gv,z),$$

for all $v\in V$. We assume that V can be decomposed into g-eigenspaces

$$V=\oplus_{\rho\in\mathbb{C}}V^{\rho},$$

where V^{ρ} denotes the eigenspace of g with eigenvalue $e^{2\pi i\rho}$.

Definition 2. A g-twisted V-module for a VOSA V is a pair (W^g, Y_g), $W^g=\bigoplus_{r\in\mathbb{C}} W^g_r$, $\dim W^g_r<\infty$, $W^g_{r+n}=0$ for all r and $n\ll 0$. $Y_g : V\to \mathrm{End}\, W^g\{z\}$, the vector space of End W^g-valued formal series in z with arbitrary complex powers of z. For $v\in V^{\rho}$

$$Y_g(v,z)=\sum_{n\in\rho+\mathbb{Z}} v_g(n)z^{-n-1},$$

with $v_g(\rho+l)w=0$, $w\in W^g$, $l\in\mathbb{Z}$ sufficiently large. $Y_g(\mathbf{1},z)=\mathrm{Id}_{W^g}$,

$$Y_g(\omega,z)=\sum_{n\in\mathbb{Z}} L_g(n)z^{-n-2},$$

where $L_g(0)w=rw$, $w\in W^g_r$. The g-twisted vertex operators satisfy the twisted Jacobi identity:

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_g(u,z_1)Y_g(v,z_2)\\
&\quad -(-1)^{p(u,v)}z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y_g(v,z_2)Y_g(u,z_1)\\
&\quad = z_2^{-1}\left(\frac{z_1-z_0}{-z_2}\right)^{-\rho}\delta\left(\frac{z_1-z_0}{-z_2}\right)Y_g(Y(u,z_0)v,z_2),
\end{aligned}$$

for $u\in V^{\rho}$.

1.4. **Creative intertwining operators.** We define the notion of creative intertwining operators in [TZ3]. Suppose we have a VOA V with a V-module (W, Y_W).

Definition 3. A *Creative Intertwining Vertex Operator* $\mathbb{Y}$ for a VOA V-module (W, Y_W) is defined by a linear map

$$\mathbb{Y}(w,z)=\sum_{n\in\mathbb{Z}} w(n)z^{-n-1},$$

for $w\in W$ with modes $w(n) : V\to W$; satisfies creativity

$$\mathbb{Y}(w,z)\mathbf{1}=w+O(z),$$

for $w \in W$ and lower truncation

$$w(n)v = 0,$$

for $v \in V$, $w \in W$ and $n \gg 0$. The intertwining vertex operators satisfy the Jacobi identity:

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) Y_W(u,z_1)\mathbb{Y}(w,z_2) \\ &-z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right) \mathbb{Y}(w,z_2)Y(u,z_1) \\ &= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) \mathbb{Y}\left(Y_W(u,z_0)w,z_2\right), \end{aligned}$$

for all $u \in V$ and $w \in W$.

These axioms imply that the intertwining vertex operators satisfy translation, locality, associativity, commutativity and skew-symmetry:

$$\begin{aligned} \mathbb{Y}(L_W(-1)w,z) &= \partial_z \mathbb{Y}(w,z), \\ (z_1-z_2)^m Y_W(u,z_1)\mathbb{Y}(w,z_2) &= (z_1-z_2)^m \mathbb{Y}(w,z_2)Y(u,z_1), \\ (z_0+z_2)^n Y_W(u,z_0+z_2)\mathbb{Y}(w,z_2)v &= (z_0+z_2)^n \mathbb{Y}(Y_W(u,z_0)w,z_2)v, \\ u_W(k)\mathbb{Y}(w,z) - \mathbb{Y}(w,z)u(k) &= \sum_{j\geq 0}\binom{k}{j}\mathbb{Y}(u_W(j)w,z)z^{k-j}, \\ \mathbb{Y}(w,z)v &= e^{zL_W(-1)}Y_W(v,-z)w, \end{aligned}$$

for u, $v \in V$, $w \in W$ and integers m, $n \gg 0$.

1.5. **Example: Heisenberg intertwiners.** Consider the Heisenberg vertex operator algebra M, [Ka] generated by weight one normalized Heisenberg vector a with modes obeying

$$[a(n), a(m)] = n\delta_{n,-m},$$

n, $m \in \mathbb{Z}$. In [TZ3] we consider an extension $\mathbb{M} = \cup_{\alpha\in\mathbb{C}} M_\alpha$ of M by its irreducible modules M_α generated by a $\mathbb{C}$-valued continuous parameter α automorphism $g = e^{2\pi i \alpha a(0)}$.

We introduce an extra operator q which is canonically conjugate to the zero mode $a(0)$, i.e.,

$$[a(n), q] = \delta_{n,0}.$$

The state $\mathbf{1}\otimes e^{\alpha}\in\mathbb{M}$ is created by the action of $e^{\alpha q}$ on the state $\mathbf{1}\otimes e^{0}$. Using q-conjugation and associativity properties, we explicitly construct in [TZ3] the creative intertwining operators $\mathbb{Y}(u,z):M\to M_{\alpha}$. We then prove

Theorem 1 (Tuite-Z). *The creative intertwining operators $\mathbb{Y}$ for $\mathbb{M}$ are generated by q-conjugation of vertex operators of M. For a Heisenberg state u,*

$$\begin{aligned}\mathbb{Y}(u\otimes e^{\alpha},z) &= e^{\alpha q}\, Y_{-}(e^{\alpha},z)\, Y(u\otimes e^{0})\, Y_{+}(e^{\alpha},z)\, z^{\alpha\, a(0)},\\ Y_{\pm}(e^{\alpha},z) &\equiv \exp\left(\mp\alpha\sum_{n>0} a(\pm n)\,\frac{z^{\mp n}}{n}\right).\end{aligned}$$

The operators $\mathbb{Y}$ with some extra cocycle structure satisfy a natural extension from rational to complex parameters of the notion of a *Generalized VOA* as described by Dong and Lepowsky [DL, DLM3]. We then prove in [TZ3]:

Theorem 2 (Tuite-Z). $\mathbb{Y}(u\otimes e^{\alpha},z)$ *satisfy the generalized Jacobi identity*

$$\begin{aligned}& z_0^{-1}\left(\frac{z_1-z_2}{z_0}\right)^{-\alpha\beta}\delta\left(\frac{z_1-z_2}{z_0}\right)\,\mathbb{Y}(u\otimes e^{\alpha},z_1)\,\mathbb{Y}(v\otimes e^{\beta},z_2)\\ &\quad -C(\alpha,\beta)z_0^{-1}\left(\frac{z_2-z_1}{z_0}\right)^{-\alpha\beta}\delta\left(\frac{z_2-z_1}{-z_0}\right)\\ &\qquad \mathbb{Y}(v\otimes e^{\beta},z_2)\,\mathbb{Y}(u\otimes e^{\alpha},z_1)\\ &\quad = z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\mathbb{Y}(\mathbb{Y}(u\otimes e^{\alpha},z_0)(v\otimes e^{\beta}),z_2)\left(\frac{z_1-z_0}{z_2}\right)^{\alpha a(0)},\end{aligned}$$

for all $u\otimes e^{\alpha}, v\otimes e^{\beta}\in\mathbb{M}$.

1.6. **Invariant form for extended Heisenberg algebra.** The definitions of invariant forms [FHL, L] for a VOSA and its g-twisted modules were given by Scheithauer [S] and in [TZ2] correspondingly. A bilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{M}$ is said to be invariant if for all $u\otimes e^{\alpha}$, $v\otimes e^{\beta}$, $w\otimes e^{\gamma}\in\mathbb{M}$ we have

$$\langle\mathbb{Y}(u\otimes e^{\alpha},z)v\otimes e^{\beta},w\otimes e^{\gamma}\rangle=e^{i\pi\alpha\beta}\langle v\otimes e^{\beta},\mathbb{Y}^{\dagger}(u\otimes e^{\alpha},z)w\otimes e^{\gamma}\rangle,$$

$$\mathbb{Y}^{\dagger}\left(u\otimes e^{\alpha},z\right) = \mathbb{Y}\left(e^{-z\lambda^{-2}L(1)}\left(-\frac{\lambda}{z}\right)^{2L(0)}(u\otimes e^{\alpha}),-\frac{\lambda^2}{z}\right).$$

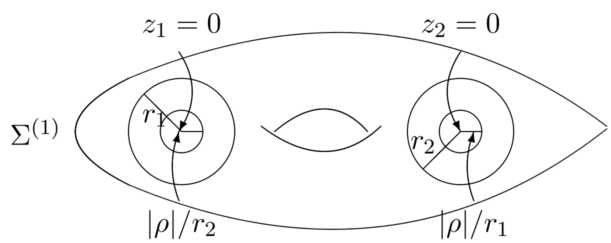
We are interested in the Möbius map $z\mapsto w=\frac{\rho}{z}$ associated with the sewing condition so that $\lambda=-\xi\,\rho^{\frac{1}{2}}$, with $\xi\in\{\pm\sqrt{-1}\}$. We prove in [TZ3]

Theorem 3 (Tuite-Z). *The invariant form $\langle .,.\rangle$ on $\mathbb{M}$ is symmetric, unique and invertible with*

$$\langle v\otimes e^{\alpha},w\otimes e^{\beta}\rangle=\lambda^{-\alpha^2}\delta_{\alpha,-\beta}\langle v\otimes e^{0},w\otimes e^{0}\rangle.$$

2. The Szegő kernel

2.1. **Torus self-sewing to form a genus two Riemann surface.** In [TZ1] we describe procedures of sewing Riemann surfaces [G, FK]. Consider a self-sewing of the oriented torus $\Sigma^{(1)} = \mathbb{C}/\Lambda$, $\Lambda = 2\pi i(\mathbb{Z}\tau \oplus \mathbb{Z})$, $\tau \in \mathbb{H}_1$.



Define annuli $\mathcal{A}_a$, $a = 1, 2$ centered at $z = 0$ and $z = w$ of $\Sigma^{(1)}$ with local coordinates $z_1 = z$ and $z_2 = z - w$ respectively. We use the convention $\overline{1} = 2$, $\overline{2} = 1$. Take the outer radius of $\mathcal{A}_a$ to be $r_a < \frac{1}{2}D(q) = \min_{\lambda\in\Lambda,\lambda\neq 0} |\lambda|$. Introduce a complex parameter ρ, $|\rho| \leq r_1r_2$. Take inner radius to be $|\rho|/r_{\overline{a}}$, with $|\rho| \leq r_1r_2$. r_1, r_2 must be sufficiently small to ensure that the disks do not intersect. Excise the disks

$$\{z_a, |z_a| < |\rho|r_{\overline{a}}^{-1}\} \subset \Sigma^{(1)},$$

to form a twice-punctured surface

$$\widehat{\Sigma}^{(1)} = \Sigma^{(1)} \backslash \bigcup_{a=1,2} \{z_a, |z_a| < |\rho|r_{\overline{a}}^{-1}\}.$$

Identify annular regions $\mathcal{A}_a \subset \widehat{\Sigma}^{(1)}$, $\mathcal{A}_a = \{z_a, |\rho|r_{\overline{a}}^{-1} \leq |z_a| \leq r_a\}$ as a single region $\mathcal{A} = \mathcal{A}_1 \simeq \mathcal{A}_2$ via the sewing relation

$$z_1z_2 = \rho,$$

to form a compact genus two Riemann surface $\Sigma^{(2)} = \widehat{\Sigma}^{(1)} \backslash \{\mathcal{A}_1 \cup \mathcal{A}_2\} \cup \mathcal{A}$, parameterized by

$$\mathcal{D}^\rho = \{(\tau, w, \rho) \in \mathbb{H}_1 \times \mathbb{C} \times \mathbb{C}\ , |w - \lambda| > 2|\rho|^{\frac{1}{2}} > 0,\ \lambda \in \Lambda\}.$$

2.2. **The Prime form.** Recall the prime form $E^{(g)}(z, z')$ [M, F1, F2]

$$E^{(g)}(z, z') = \frac{\vartheta\left[\begin{smallmatrix}\gamma\\ \delta\end{smallmatrix}\right]\left(\int_{z'}^{z} \nu|\Omega^{(g)}\right)}{\zeta(z)^{\frac{1}{2}}\zeta(z')^{\frac{1}{2}}} \sim (z - z')\, dz^{-\frac{1}{2}}\, dz'^{-\frac{1}{2}} \quad \text{for } z \sim z',$$

is a holomorphic differential form of weight $(-\frac{1}{2}, -\frac{1}{2})$ on $\widetilde{\Sigma}^{(g)} \times \widetilde{\Sigma}^{(g)}$,

$$E^{(g)}(z, z') = -E^{(g)}(z', z),$$

and has multipliers 1 and $e^{-i\pi\Omega^{(g)}_{jj}-\int_{z'}^{z}\nu_j}$ along the a_i and b_j cycles in z. Here

$$\zeta(z)=\sum_{i=1}^{g}\partial_{z_i}\vartheta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}(0|\Omega^{(g)})\nu_i(z),$$

(a holomorphic 1-form, and let $\zeta(z)^{\frac{1}{2}}$ denote the form of weight $\frac{1}{2}$ on the double cover $\widetilde{\Sigma}^{(g)}$ of $\Sigma^{(g)}$).

In particular, the prime form on the torus is [M]

$$E^{(1)}(z,z')=K^{(1)}(z-z',\tau)\;dz^{-\frac{1}{2}}\;dz'^{-\frac{1}{2}},$$

$$K^{(1)}(z,\tau)=\frac{\vartheta_1(z,\tau)}{\partial_z\vartheta_1(0,\tau)},$$

for $z\in\mathbb{C}$ and $\tau\in\mathbb{H}_1$ and where $\vartheta_1(z,\tau)=\vartheta\begin{bmatrix}1/2\\ 1/2\end{bmatrix}(z,\tau)$.

2.3. **The Szegő kernel.** The Szegő Kernel [M, F1, F2] is defined by

$$S^{(g)}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(z,z'|\Omega)=\frac{\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\int_{z'}^{z}\nu\right)}{\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0)E^{(g)}(z,z')}\sim\frac{dz^{\frac{1}{2}}dz'^{\frac{1}{2}}}{z-z'}\quad\text{for } z\sim z',$$

with $\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0)\neq 0$,

$$\theta_j=-e^{-2\pi i\beta_j},\quad \phi_j=-e^{2\pi i\alpha_j},\quad j=1,\ldots,g,$$

where $E^{(g)}(z_1,z_2)$ is the genus g prime form. The Szegő kernel has multipliers along the a_i and b_j cycles in z given by $-\phi_i$ and $-\theta_j$ respectively and is a meromorphic $(\frac{1}{2},\frac{1}{2})$-form on $\widetilde{\Sigma}^{(g)}\times\widetilde{\Sigma}^{(g)}$.

$$S^{(g)}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(z,z')=-S^{(g)}\begin{bmatrix}\theta^{-1}\\ \phi^{-1}\end{bmatrix}(z',z),$$

where $\theta^{-1}=(\theta_i^{-1})$ and $\phi^{-1}=(\phi_i^{-1})$.

Finally, we describe the modular invariance of the Szegő kernel under the symplectic group $Sp(2g,\mathbb{Z})$ where we find [Fay]

$$S^{(g)}\begin{bmatrix}\tilde\theta\\ \tilde\phi\end{bmatrix}(z,z'|\tilde\Omega^{(g)})=S^{(g)}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(z,z'|\Omega^{(g)}),$$

with $\tilde\theta_j=-e^{-2\pi i\tilde\beta_j}$, $\tilde\phi_j=-e^{2\pi i\tilde\alpha_j}$,

$$\begin{pmatrix}-\tilde\beta\\ \tilde\alpha\end{pmatrix}=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}-\beta\\ \alpha\end{pmatrix}+\frac{1}{2}\begin{pmatrix}-\mathrm{diag}(AB^T)\\ \mathrm{diag}(CD^T)\end{pmatrix},$$

$$\tilde\Omega=(A\Omega+B)\,(C\Omega+D)^{-1},$$

where $\mathrm{diag}(M)$ denotes the diagonal elements of a matrix M.

On the torus $\Sigma^{(1)}$ the Szegő kernel for $(\theta,\phi)\neq(1,1)$ is

$$S^{(1)}\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z,z'|\tau)=P_1\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z-z',\tau)\,dz^{\frac{1}{2}}\,dz'^{\frac{1}{2}},$$

where

$$\begin{aligned}P_1\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z,\tau) &= \frac{\vartheta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](z,\tau)}{\vartheta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](0,\tau)}\frac{\partial_z\vartheta_1(0,\tau)}{\vartheta_1(z,\tau)}\\ &= -\sum_{k\in\mathbb{Z}}\frac{q_z^{k+\lambda}}{1-\theta^{-1}q^{k+\lambda}},\end{aligned}$$

for $\vartheta_1(z,\tau)=\vartheta\left[\begin{array}{c}\frac{1}{2}\\ \frac{1}{2}\end{array}\right](z,\tau)$, $q_z=e^z$, and $\phi=\exp(2\pi i\lambda)$ for $0\leq\lambda<1$.

2.4. **Genus two Szegő kernel in torus self-sewing (ρ-formalism).** It is convenient to define $\kappa\in\left[-\frac{1}{2},\frac{1}{2}\right)$ by $\phi_2=-e^{2\pi i\kappa}$. Then we prove [TZ1] the following

Theorem 4 (Tuite-Z). *$S^{(2)}$ is holomorphic in ρ for $|\rho|<r_1r_2$ with*

$$S^{(2)}(x,y)=S^{(1)}_\kappa(x,y)+O(\rho),$$

for x, $y\in\widehat{\Sigma}^{(1)}$ where $S^{(1)}_\kappa(x,y)$ is defined for $\kappa\neq-\frac{1}{2}$, by

$$\begin{aligned}S^{(1)}_\kappa\left[\begin{array}{c}\theta_1\\ \phi_1\end{array}\right](x,y|\tau,w) &= \left(\frac{\vartheta_1(x-w,\tau)\vartheta_1(y,\tau)}{\vartheta_1(x,\tau)\vartheta_1(y-w,\tau)}\right)^\kappa\\ &\quad\cdot\frac{\vartheta^{(1)}\left[\begin{array}{c}\alpha_1\\ \beta_1\end{array}\right](x-y+\kappa w,\tau)}{\vartheta^{(1)}\left[\begin{array}{c}\alpha_1\\ \beta_1\end{array}\right](\kappa w,\tau)\,K^{(1)}(x-y,\tau)}dx^{\frac{1}{2}}dy^{\frac{1}{2}},\end{aligned}$$

with similar expression for $S^{(1)}_{-\frac{1}{2}}(x,y)$ for $\kappa=-\frac{1}{2}$.

Let $k_a=k+(-1)^{\overline{a}}\kappa$, for $a=1$, 2 and integer $k\geq1$. We introduce the moments for $S^{(1)}_\kappa(x,y)$:

$$\begin{aligned}&G_{ab}(k,l)=G_{ab}\left[\begin{array}{c}\theta^{(1)}\\ \phi^{(1)}\end{array}\right](\kappa;k,l)\\ &=\frac{\rho^{\frac{1}{2}(k_a+l_b-1)}}{(2\pi i)^2}\oint_{\mathcal{C}_{\overline{a}}(x_{\overline{a}})}\oint_{\mathcal{C}_b(y_b)}(x_{\overline{a}})^{-k_a}(y_b)^{-l_b}S^{(1)}_\kappa(x_{\overline{a}},y_b)\,dx_{\overline{a}}^{\frac{1}{2}}\,dy_b^{\frac{1}{2}},\end{aligned}$$

with associated infinite matrix $G = (G_{ab}(k,l))$. We define also half-order differentials

$$h_a(k,x) = h_a \begin{bmatrix} \theta^{(1)} \\ \phi^{(1)} \end{bmatrix} (\kappa; k, x) = \frac{\rho^{\frac{1}{2}(k_a - \frac{1}{2})}}{2\pi i} \oint_{\mathcal{C}_a(y_a)} y_a^{-k_a} S_\kappa^{(1)}(x, y_a)\, dy_a^{\frac{1}{2}},$$
$$\overline{h}_a(k,y) = \overline{h}_a \begin{bmatrix} \theta^{(1)} \\ \phi^{(1)} \end{bmatrix} (\kappa; k, y) = \frac{\rho^{\frac{1}{2}(k_a - \frac{1}{2})}}{2\pi i} \oint_{\mathcal{C}_{\overline{a}}(x_{\overline{a}})} x_{\overline{a}}^{-k_a} S_\kappa^{(1)}(x_{\overline{a}}, y)\, dx_{\overline{a}}^{\frac{1}{2}},$$

and let $h(x) = (h_a(k,x))$ and $\overline{h}(y) = (\overline{h}_a(k,y))$ denote the infinite row vectors indexed by a, k. From the sewing relation $z_1 z_2 = \rho$ we have

$$dz_a^{\frac{1}{2}} = (-1)^{\overline{a}}\, \xi\, \rho^{\frac{1}{2}}\, \frac{dz_{\overline{a}}^{\frac{1}{2}}}{z_{\overline{a}}},$$

for $\xi \in \{\pm\sqrt{-1}\}$, depending on the branch of the double cover of $\Sigma^{(1)}$ chosen. It is convenient to define

$$T = \xi G D^\theta,$$

with an infinite diagonal matrix

$$D^\theta(k,l) = \begin{bmatrix} \theta^{-1} & 0 \\ 0 & -\theta \end{bmatrix} \delta(k,l).$$

Defining $\det(I-T)$ by the formal power series in ρ

$$\log\det(I-T) = Tr\log(I-T) = -\sum_{n\geq 1} \frac{1}{n} Tr(T^n),$$

we prove in [TZ1]

Theorem 5 (Tuite-Z)**.**

a.) $(I-T)^{-1} = \sum_{n\geq 0} T^n$ *is convergent for* $|\rho| < r_1 r_2$*,*

b.) $\det(I-T)$ *is non-vanishing and holomorphic in* ρ *on* $\mathbb{D}^\rho$*.*

Theorem 6 (Tuite-Z)**.** $S^{(2)}(x,y)$ *is given by*

$$S^{(2)}(x,y) = S_\kappa^{(1)}(x,y) + \xi h(x) D^\theta (I-T)^{-1} \overline{h}^T(y).$$

3. Intertwined n-point functions

As in ordinary (non-intertwined) case [DLM1,H,MN,MT1,MT3,MT4,MTZ, TZ2,Z1] we construct in [TZ4] the partition and n-point functions [DVFHLS, EO,FS,GKV,GV,KNTY,Pe,R,TUY,U] for vertex operator algebra modules.

3.1. **Torus intertwined n-point functions.** Let g_i, f_i, $i=1,2$ be VOSA V automorphisms commuting with $\sigma v = (-1)^{p(v)} v$. For $u \in V_{\sigma g_2}$ and the states $v_1, \ldots, v_n \in V$ we define the *intertwined* n-point function [TZ4] on the torus by

$$\begin{aligned} & Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (u, z_2;\ v_1, x_1; \ldots; v_n, x_n; \overline{u}, z_1;\ \tau) \\ & \equiv \mathrm{STr}_{V_{\sigma g_1}} \Big(f_1 \ \mathbb{Y}\left(q_{z_2}^{L_{\sigma g_2}(0)} u, q_{z_2}\right) \ \ Y(q_1^{L(0)} v_1, q_1) \\ & \qquad \ldots Y(q_n^{L(0)} v_n, q_n) \ \mathbb{Y}\left(q_{z_1}^{L_{\sigma g_2^{-1}}(0)} \overline{u}, q_{z_1}\right) \ q^{L_{\sigma g_1}(0) - c/24} \Big), \end{aligned}$$

where $q = \exp(2\pi i \tau)$, $q_k = \exp(x_k)$, $q_{z_j} = \exp(z_j)$, $j = 1,\ 2;\ 1 \le k \le n$, for variables $x_1, \ldots, x_n$ associated to the local coordinates on the torus, and $\overline{u}$ is dual for u with respect to the invariant form on $V_{\sigma g_2}$. The supertrace over a V-module N is defined by

$$\mathrm{STr}_N(X) = Tr_N(\sigma X).$$

For an element $u \in V_{\sigma g_2}$ of a VOSA g-twisted V-module we introduce also the differential form

$$\begin{aligned} & \mathbb{F}^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (u, z_2;\ v_1, x_1;\ \ldots;\ v_n, x_n;\ \overline{u}, z_1;\ \tau) \\ & \equiv Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (u, z_2;\ v_1, x_1; \ldots; v_n, x_n; \overline{u}, z_1;\ \tau) \\ & \qquad \cdot dz_2^{wt[u]} \ dz_1^{wt[\bar{u}]} \ \prod_{i=1}^{n} dx_i^{wt\ [v_i]}, \end{aligned}$$

associated to the torus intertwined n-point function.

3.2. **Genus two partition and n-point functions in ρ-formalism.** Let f_i, $i = 1, 2$ be automorphisms, and $V_{\sigma g_j}$ be twisted V-modules of a vertex operator superalgebra V. For $x_1, \ldots, x_n \in \Sigma^{(1)}$ with $|x_k| \ge |\rho|/r_2$ and $|x_k - w| \ge |\rho|/r_1$, $k = 1, \ldots, n$, we define the genus two n-point function [TZ4] in the ρ-formalism by

$$\begin{aligned} & Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (v_1, x_1; \ldots; v_n, x_n; \tau, w, \rho) \\ & = \sum_{k \ge 0} \sum_{u \in V_{\sigma g_2}[k]} \rho^k Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (u, w + z_2; v_1, x_1; \ldots; v_n, x_n; f_2 \overline{u}, z_1; \tau), \end{aligned}$$

where $(f, g) = ((f_i), (g_i))$, where f (respectively g) denotes the pair f_1, f_2 (respectively g_1, g_2). The sum is taken over any $V_{\sigma g_2}$-basis.

In particular, introduce the genus two partition function

$$Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}(\tau, w, \rho) \quad = \quad \sum_{u \in V_{\sigma g_2}} Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(u, w; f_2\, \overline{u}, 0;\ \tau)\,,$$

where $Z^{(1)}\left[\begin{smallmatrix} f_1 \\ g_1 \end{smallmatrix}\right](u, w; f_2\, \overline{u}, 0; \tau)$ is the genus one intertwined two point function.

Remark 1. We can generalize the genus two n-point function by introducing and computing the differential form associated to the torus n-point function containing several intertwining operators in the supertrace as well as corresponding genus two n-point functions.

Similar to the ordinary genus two case [TZ2], we define the differential form [TZ4] associated to the n-point function on a sewn genus two Riemann surface for $v_i \in V$ and $x_i \in \Sigma^{(2)}$, $i = 1, \ldots, n$ with $|x_i| \geq |\rho|/r_2$, $|x_i - w| \geq |\rho|/r_1$,

$$\begin{aligned} &\mathbb{F}^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}(v_1, \ldots, v_n; \tau, w, \rho) \\ &\qquad \equiv Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}(v_1, x_1; \ldots; v_n, x_n; \tau, w, \rho)\ \prod_{i=1}^{n}\ dx_i^{wt[v_i]}. \end{aligned}$$

4. Free fermion VOSA

4.1. **Torus intertwined two-point function.** The rank two free fermionic VOSA $V(H, \mathbb{Z} + \frac{1}{2})^{\otimes 2}$, [Ka] is generated by $\psi^{\pm}$ with

$$[\psi^{+}(m), \psi^{-}(n)] = \delta_{m,-n-1},\ [\psi^{+}(m), \psi^{+}(n)] = 0,\ [\psi^{-}(m), \psi^{-}(n)] = 0,$$

The rank two free fermion VOSA intertwined torus n-point function is parameterized by $\theta_1 = -e^{-2\pi i \beta_1}$, $\phi_1 = -e^{2\pi i \alpha_1}$, and $\phi_2 = -e^{-2\pi i \kappa}$, [TZ2, TZ4] where

$$\sigma f_1 = e^{2\pi i \beta_1 a(0)}, \quad \sigma g_1 = e^{-2\pi i \alpha_1 a(0)}, \quad \sigma g_2 = e^{2\pi i \kappa a(0)},$$

for real valued α_1, β_1, κ, $(\theta_1, \phi_1) \neq (1, 1)$.

For $u = \mathbf{1} \otimes e^{\kappa} \equiv e^{\kappa} \in V_{\sigma g_2}$ and $v_i = \mathbf{1}$, $i = 1, \ldots, n$ we obtain [TZ4] the basic intertwined two-point function on the torus

$$\begin{aligned} &Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(e^{\kappa}, z_2;\ e^{-\kappa}, z_1;\ \tau) \\ &\qquad \equiv \mathrm{STr}_{V_{\sigma g_1}}\left(f_1 \mathbb{Y}\left(q_{z_2}^{L(0)} e^{\kappa}, q_{z_2}\right) \mathbb{Y}\left(q_{z_1}^{L(0)} e^{-\kappa}, q_{z_1}\right) q^{L_{\sigma g_1}(0) - c/24}\right). \end{aligned}$$

We then consider the differential form

$$\begin{aligned}&\mathbb{G}_n^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(x_1, y_1, \ldots, x_n, y_n)\\ &\quad \equiv \mathbb{F}^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(e^{\kappa}, w; \psi^+, x_1; \psi^-, y_1; \ldots; \psi^+, x_n; \psi^-, y_n; e^{-\kappa}, 0; \tau),\end{aligned}$$

associated to the torus intertwined $2n$-point function

$$Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(\ e^{\kappa}, w;\ \psi^+, x_1; \psi^-, y_1; \ldots; \psi^+, x_n\ ; \psi^-, y_n; e^{-\kappa}, 0;\ \tau),$$

with alternatively inserted n states ψ^+ and n states ψ^- distributed on the resulting genus two Riemann surface $\Sigma^{(2)}$ at points x_i, $y_i \in \Sigma^{(2)}$, $i = 1, \ldots, n$.

We then prove in [TZ4]

Theorem 7 (Tuite-Z)**.** *For the rank two free fermion vertex operator superalgebra V and for $(\theta, \phi) \neq (1, 1)$ the generating form is given by*

$$\begin{aligned}&\mathbb{G}_n^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(x_1, y_1, \ldots, x_n, y_n)\\ &\quad = Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}\left(e^{\kappa}, w;\ e^{-\kappa}, 0;\ \tau\right)\ \det S_{\kappa}^{(1)},\end{aligned}$$

$$Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}\left(e^{\kappa}, w;\ e^{-\kappa}, 0;\ \tau\right) = \frac{1}{\eta(\tau)} \frac{\vartheta^{(1)}\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}(\kappa w, \tau)}{K^{(1)}(w, \tau)^{\kappa^2}},$$

is the basic intertwined two-point function on the torus, and $n \times n$-matrix $S_{\kappa}^{(1)} = \left[S_{\kappa}^{(1)}\begin{bmatrix} \theta_1 \\ \phi_1 \end{bmatrix}(x_i, y_j \mid \tau, w)\right]$, i, $j = 1\ldots, n$, with elements given by parts of the Szegő kernel.

4.2. **Genus two partition function.** In [TZ4] we then prove:

Theorem 8 (Tuite-Z)**.** *Let $V_{\sigma g_i}$, $i = 1$, 2 be σg_i-twisted V-modules for the rank two free fermion vertex operator superalgebra V. Let $(\theta, \phi) \neq (1, 1)$. Then the partition function on a genus two Riemann surface obtained in the ρ-self-sewing formalism of the torus is a non-vanishing holomorphic function on $\mathbb{D}^{\rho}$ given by*

$$Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}(\tau, w, \rho) = Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}\left(e^{\kappa}, w;\ e^{-\kappa}, 0;\ \tau\right)\ \det\left(1 - T\right),$$

where $Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(e^{\kappa}, w; e^{-\kappa}, 0; \tau)$ is the intertwined V-module $V_{\sigma g_1}$ torus basic two-point function.

We may similarly compute the genus two partition function in the ρ-formalism for the original rank one fermion VOSA $V\left(H, \mathbb{Z}+\frac{1}{2}\right)$ in which case we can only construct a σ-twisted module. Then we have [TZ4] the following

Corollary 1 (Tuite-Z)**.** *Let V be the rank one free fermion vertex operator superalgebra and $f_1,\ g_1 \in \{\sigma, 1\}$, $a=1,\ 2$ be automorphisms. Then the partition function for V-module $V_{\sigma g_1}$ on a genus two Riemann surface obtained from the ρ-formalism of a self-sewn torus $\Sigma^{(1)}$ is given by*

$$Z^{(2)}_{\mathrm{rank}1}\begin{bmatrix} f \\ g \end{bmatrix}(\tau, w, \rho) = Z^{(1)}_{\mathrm{rank}1}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(e^{\kappa}, w; e^{-\kappa}, 0; \tau) \det\left(I-T\right)^{1/2},$$

where $Z^{(1)}_{\mathrm{rank}\ 1}\left[\begin{smallmatrix} f_1 \\ g_1 \end{smallmatrix}\right](e^{\kappa}, w;\ e^{-\kappa}, 0;\ \tau)$ is the rank one fermion intertwined partition function on the original torus.

4.3. **Genus two generating form.** In [TZ4] we define matrices

$$S^{(2)} = \left(S^{(2)}(x_i, y_j)\right), \quad S^{(1)}_{\kappa} = \left(S^{(1)}_{\kappa}(x_i, y_j)\right),$$
$$H^{+} = \left(\left(h(x_i)\right)(k, a)\right), \quad H^{-} = \left(\left(\overline{h}(y_i)\right)(l, b)\right)^{T}.$$

$S^{(2)}$ and $S^{(1)}_{\kappa}$ are finite matrices indexed by $x_i,\ y_j$ for $i,\ j=1,\ldots,n$; H^{+} is semi-infinite with n rows indexed by x_i and columns indexed by $k \geq 1$ and $a=1,\ 2$ and H^{-} is semi-infinite with rows indexed by $l \geq 1$ and $b=1, 2$ and with n columns indexed by y_j. We then prove

Lemma 1 (Tuite-Z)**.**

$$\det\begin{bmatrix} S^{(1)}_{\kappa} & \xi\, H^{+}\, D^{\theta_2} \\ H^{-} & I-T \end{bmatrix} = \det S^{(2)}\ \det(I-T),$$

with T, D^{θ_2}.

Introduce the differential form

$$\mathbb{G}^{(2)}_{n}\begin{bmatrix} f \\ g \end{bmatrix}(x_1, y_1, \ldots, x_n, y_n)$$
$$= \mathbb{F}^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}(\psi^{+}, \psi^{-}, \ldots, \psi^{+}, \psi^{-}; \tau, w, \rho),$$

associated to the rank two free fermion VOSA genus two $2n$-point function

$$Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}(\psi^{+}, x_1; \psi^{-}, y_1; \ldots; \psi^{+}, x_n\ ; \psi^{-}, y_n; \tau, w, \rho),$$

with alternatively inserted n states ψ^{+} and n states ψ^{-}. The states are distributed on the genus two Riemann surface $\Sigma^{(2)}$ at points $x_i,\ y_i \in \Sigma^{(2)}$, $i=1,\ldots,n$. Then we have

Theorem 9 (Tuite-Z). *All n-point functions for rank two free fermion VOSA twisted modules $V_{\sigma g}$ on self-sewn torus are generated by the differential form*

$$\mathbb{G}_n^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}(x_1, y_1, \ldots, x_n, y_n) = Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}(\tau, w, \rho)\ \det S^{(2)},$$

where the elements of the matrix $S^{(2)} = \left[S^{(2)}\begin{bmatrix} \theta \\ \phi \end{bmatrix}(x_i, y_j \mid \tau, w)\right]$, $i, j = 1, \ldots, n$ and $Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}(\tau, w, \rho)$ is the genus two partition function.

5. Modular invariance properties

Following the ordinary case [DLM1, MT3, MT5] we would like to describe modular properties of genus two "intertwined" partition and n-point generating functions. As in [MT3], consider $\hat{H} \subset Sp(4, \mathbb{Z})$ with elements

$$\mu(a, b, c) = \begin{pmatrix} 1 & 0 & 0 & b \\ a & 1 & b & c \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\hat{H}$ is generated by $A = \mu(1, 0, 0)$, $B = \mu(0, 1, 0)$ and $C = \mu(0, 0, 1)$ with relations $[A, B]C^{-2} = [A, C] = [B, C] = 1$. We also define $\Gamma_1 \subset Sp(4, \mathbb{Z})$ where $\Gamma_1 \cong SL(2, \mathbb{Z})$ with elements

$$\gamma_1 = \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & 1 & 0 & 0 \\ c_1 & 0 & d_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad a_1 d_1 - b_1 c_1 = 1.$$

Together these groups generate $L = \hat{H} \rtimes \Gamma_1 \subset Sp(4, \mathbb{Z})$. From [MT3] we find that L acts on the domain $\mathcal{D}^\rho$ of as follows:

$$\begin{aligned} \mu(a, b, c).(\tau, w, \rho) &= (\tau, w + 2\pi i a\tau + 2\pi i b, \rho), \\ \gamma_1.(\tau, w, \rho) &= \left(\frac{a_1\tau + b_1}{c_1\tau + d_1}, \frac{w}{c_1\tau + d_1}, \frac{\rho}{(c_1\tau + d_1)^2}\right). \end{aligned}$$

We then define [TZ4] a group action of $\gamma_1 \in SL(2, \mathbb{Z})$ on the torus intertwined two-point function $Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(u, w;\ v, 0; \tau)$ for u, $v \in V_{\sigma g}$:

$$Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}\Bigg|\gamma_1\ (u, w;\ v, 0;\ \tau) = Z^{(1)}\left(\gamma_1.\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}\right)(u, \gamma_1.w;\ v, 0;\ \gamma_1.\tau),$$

with the standard action $\gamma_1.\tau$ and $\gamma_1.w$, and $\gamma_1.\begin{bmatrix} f_1 \\ g_1 \end{bmatrix} = \begin{bmatrix} f_1^{a_1} g_1^{b_1} \\ f_1^{c_1} g_1^{d_1} \end{bmatrix}$, and the torus multiplier $e^{(1)}_{\gamma_1}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix} \in U(1)$, [MTZ], [TZ1]. Then we have [TZ4]

Theorem 10 (Tuite-Z)**.** *The torus intertwined two-point function for the rank two free fermion VOSA is a modular form (up to multiplier) with respect to L*

$$Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}\Bigg| \gamma_1 \ (u, w;\ v, 0;\ \tau)$$
$$= e^{(1)}_{\gamma_1}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(c_1\tau + d_1)^{wtu+wtv+\kappa^2} Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(u, w; v, 0; \tau),$$

where $u,\ v \in V_{\sigma g}$*.*

The action of the generators A, B and C is given by [TZ1]

$$A\begin{bmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_1\ f_2\ \sigma \\ g_1\ g_2^{-1}\ \sigma \\ g_2 \end{bmatrix}, \quad B\begin{bmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} f_1\ g_2\ \sigma \\ f_2\ g_1\ \sigma \\ g_1 \\ g_2 \end{bmatrix}, \quad C\begin{bmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2\ g_2\ \sigma \\ g_1 \\ g_2 \end{bmatrix}.$$

In a similar way we may introduce the action of $\gamma \in L$ on the genus two partition function [TZ4]

$$Z^{(2)}\left[\begin{array}{c} f \\ g \end{array}\right]\Bigg| \gamma\ (\tau, w, \rho) = Z^{(2)}\left(\gamma.\left[\begin{array}{c} f \\ g \end{array}\right]\right)\gamma.\left(\tau, w, \rho\right),$$

$$\gamma_1.\begin{bmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} f_1^{a_1} g_1^{b_1} \\ f_2 \\ f_1^{c_1} g_1^{d_1} \\ g_2 \end{bmatrix}.$$

We may now describe the modular invariance of the genus two partition function for the rank two free fermion VOSA under the action of L. Define a genus two multiplier $e^{(2)}_{\gamma}\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in U(1)$ for $\gamma \in L$ in terms of the genus one multiplier as follows

$$e^{(2)}_{\gamma}\begin{bmatrix} f \\ g \end{bmatrix} = e^{(1)}_{\gamma_1}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix},$$

for the generator $\gamma_1 \in \Gamma_1$. We then find [TZ4]

Theorem 11 (Tuite-Z)**.** *The genus two partition function for the rank two VOSA is modular invariant with respect to L with the multiplier system, i.e.,*

$$Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}\Bigg| \gamma\ (\tau, w, \rho) = e^{(2)}_{\gamma}\begin{bmatrix} f \\ g \end{bmatrix}\ Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}(\tau, w, \rho).$$

Finally, we can also obtain modular invariance for the generating form

$$\mathbb{G}^{(2)}_n\begin{bmatrix} f \\ g \end{bmatrix}(x_1, y_1, \ldots, x_n, y_n),$$

for all genus two n-point functions [TZ4].

Theorem 12. $\mathbb{G}_n^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}(x_1, y_1, \ldots, x_n, y_n)$ *is modular invariant with respect to L with a multiplier.*

6. Acknowledgement

The author would like to express his deep gratitude to the organizers of the Conference "Conformal Field Theory, Automorphic Forms and Related Topics", Heidelberg Universität, Heidelberg, Germany, 2011. Some parts of the papers described were finished at Max–Planck Institut für Mathematik in Bonn, and we thank the Institute for a fruitful scientific environment, and a possibility to continue this research program.

References

[B] Borcherds, R.E.: Vertex algebras, Kac-Moody algebras and the Monster, Proc. Nat. Acad. Sc. **83** (1986) 3068–3071.

[DL] Dong, C. and Lepowsky, J.: *Generalized Vertex Algebras and Relative Vertex Operators*, Progress in Mathematics **112** Birkhäuser (Boston, MA, 1993).

[DLM1] Dong, C., Li, H. and Mason, G.: Modular-invariance of trace functions in orbifold theory and generalized moonshine, Comm. Math. Phys. **214** (2000) 1–56.

[DLM2] Dong, C., Li, H. and Mason, G.: Twisted representation of vertex operator algebras, Math. Ann. **310** (1998) 571–600.

[DLM3] Dong, C., Li, H. and Mason, G.: Simple currents and extensions of vertex operator algebras. Comm. Math. Phys. 180 (1996) 671–707.

[DVFHLS] di Vecchia, P., Hornfeck, K., Frau, M., Lerda, A. and Sciuto, S.: N-string, g-loop vertex for the fermionic string, Phys. Lett. **B211** (1988) 301–307.

[EO] Eguchi, T. and Ooguri, H.: Chiral bosonization on a Riemann surface. Phys. Lett. **B187** (1987) 127–134.

[F1] Fay J. D.: *Theta functions on Riemann Surfaces*, Lecture Notes in Mathematics, Springer-Verlag, (Berlin-New York, 1973).

[F2] Fay J. D.: Kernel functions, analytic torsion and moduli spaces, Mem. Amer. Math. Soc. **96** (1992).

[FHL] Frenkel, I., Huang, Y. and Lepowsky, J.: On axiomatic approaches to vertex operator algebras and modules, Mem. Amer. Math. Soc. **104** (1993).

[FK] Farkas H. M. and Kra I, *Theta Constants, Riemann surfaces and the Modular Group*, Graduate Studies in Mathematics, **37**, AMS (Providence, 2001).

[FLM] Frenkel I., Lepowsky J. and Meurman A., *Vertex Operator Algebras and the Monster*, Academic Press, (New York, 1988).

[FS] Freidan, D. and Shenker, S.: The analytic geometry of two dimensional conformal field theory, Nucl. Phys. **B281** (1987) 509–545.

[GKV] Gaberdiel M.R., Keller, Ch.A. and Volpato R. Genus two partition functions of chiral conformal field theories, arXiv:1002.3371, (2010).

[GV] Gaberdiel, M.R. and Volpato, R.: Higher genus partition functions of meromorphic conformal field theories, J. High Energy Phys. (2009) **6**, 048.

[G] Gunning, R.C.: *Lectures on Riemann Surfaces*, Princeton Univ. Press, (Princeton, 1966).

[H] Huang, Y.: *Two-Dimensional Conformal Geometry and Vertex Operator Algebras*, Progress in Mathematics, **148** Birkhäuser, (Boston, 1997).

[Ka] Kac, V.: *Vertex Operator Algebras for Beginners*, University Lecture Series **10**, AMS (Providence 1998).

[KNTY] Kawamoto, N., Namikawa, Y., Tsuchiya, A. and Yamada, Y.: Geometric realization of conformal field theory on Riemann surfaces, Commun. Math. Phys. **116**(1988), 247–308.

[L] Li, H.: Symmetric invariant bilinear forms on vertex operator algebras, J. Pure. Appl. Alg. **96** (1994) 279–297.

[MN] Matsuo, A. and Nagatomo, K,: Axioms for a vertex algebra and the locality of quantum fields, Math. Soc. Jap. Mem. **4** (1999).

[MT1] Mason, G. and Tuite, M.P.: Free bosonic vertex operator algebras on genus two Riemann surfaces I, Commun. Math. Phys. **300** (2010) 673–713.

[MT2] Mason, G. and Tuite, M.P.: On genus two Riemann surfaces formed from sewn tori, Commun. Math. Phys. **270** (2007) 587–634.

[MT3] Mason, G. and Tuite, M.P.: Free bosonic vertex operator algebras on genus two Riemann surfaces II. arXiv:1111.2264v1.

[MT4] Mason, G. and Tuite, M.P.: Chiral n-point functions for free boson and lattice vertex operator algebras, Commun. Math. Phys. **235** (2003) 47–68.

[MT5] Mason, G. and Tuite, M.P.: Vertex operators and modular forms, *A Window into Zeta and Modular Physics* eds. K. Kirsten and F. Williams, MSRI Publications **57** 183–278, Cambridge University Press, (Cambridge, 2010).

[MTZ] Mason, G., Tuite, M. P. and Zuevsky, A.: Torus n-point functions for $\mathbb{R}$-graded vertex operator superalgebras and continuous fermion orbifolds, Commun. Math. Phys. **283** (2008) 305–342.

[M] Mumford, D.: *Tata lectures on Theta I and II*, Birkhäuser, (Boston, 1983).

[Pe] Pezzella, F.: g-loop vertices for free fermions and bosons, Phys. Lett. **B220** (1989) 544–550.

[R] Raina, A. K.: Fay's trisecant identity and conformal field theory, Comm. Math. Phys. **122**, 625–641 (1989).

[S] Scheithauer, N.: Vertex algebras, Lie algebras and superstrings, J. Alg. **200** (1998) 363–403.

[TUY] Tsuchiya, A, Ueno, K. and Yamada, Y.: Conformal field theory on universal family of stable curves with gauge symmetries, Adv. Stud. Pure. Math. **19** (1989) 459–566.

[TZ1] Tuite, M. P. and Zuevsky A.: The Szegő Kernel on a Sewn Riemann Surface, Commun. Math. Phys., **306**, n.3, 617–645 (2011).

[TZ2] Tuite, M. P. and Zuevsky, A.: Genus Two Partition and Correlation Functions for Fermionic Vertex Operator Superalgebras I, Comm. Math. Phys., **306**, n. 2, 419–447 (2011).

[TZ3] Tuite, M. P. and Zuevsky, A.: A Generalized Vertex Operator Algebra for Heisenberg Intertwiners. J. Pure and Applied Alg., v. 216, n. 6, p. 1253-1492, 2012.

[TZ4] Tuite, M. P. and Zuevsky, A.: Genus Two Partition and Correlation Functions for Fermionic Vertex Operator Superalgebras II. To appear, 2012.

[TZ5] Tuite, M. P. and Zuevsky, A.: The Bosonic Vertex Operator Algebra on a Genus g Riemann Surface. RIMS Kokyuroko 1756 81-93, 2011.

[U] Ueno, K.: Introduction to conformal field theory with gauge symmetries, *Geometry and Physics - Proceedings of the conference at Aarhus Univeristy, Aaarhus, Denmark*, Marcel Dekker, (New York 1997).

[Z1] Zhu, Y.: Modular invariance of characters of vertex operator algebras, J. Amer. Math. Soc. **9** (1996) 237–302.

Max–Planck Institut für Mathematik, Vivatsgasse 7, 53111, Bonn, Germany
E-mail address: zuevsky@mpim-bonn.mpg.de