GENERALIZED HIRZEBERUCH CONJECTURE FOR HILBERT–PICARD MODULAR CUSPS

by

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Period relations for twisted Legendre equations

by William L. Hoyt

1. Introduction

Fix a square free polynomial $T \in \mathbb{C}[t]$ and let $L = L_T$ and $q = q_T$ be the parabolic cohomology group and the quadratic form which are associated as in §§ 3,6 below with the twisted Legendre equation over $\mathbb{C}(t)$

(i) $$y^2 = Tx(x-1)(x-t) \ .$$

In § 3 it is shown that L has rank $2d+e$ with $2d = \deg(T)$ if $\deg(T)$ is even and $2d = \deg(T)-1$ if $\deg(T)$ is odd and $e =$ the number of $a \neq 0,1$ such that $T(a) = 0$. The main purpose of this paper is to prove that there is a bijective isomorphism $\psi : \mathbb{Z}^{2d+e} \xrightarrow{\sim} L$ such that

(ii) $$q(\psi(x_1,\dots,x_{2d+e})) = \tfrac{1}{2}(x_1^2 + \dots + x_{2d}^2 - x_{2d+1}^2 - \dots - x_{2d+e}^2) \ .$$

The proof of (ii), which is completed in § 6, is based on general results of Endo [3] which imply that all elements of $L \otimes \mathbb{C}$ can be represented by periods $p(G)$ of suitable vector valued integrals of the second kind $G = \int dG$, that q can be defined by an integral $q(p(G)) = \int {}^tGPdG$, and that this integral for $q(p(G))$ has a $\mathbb{Z}$-bilinear expansion in terms of suitable values of G. Proofs of the results of [3] for the special case considered here are sketched in § 4 for the convenience of the reader; and explicit expansions for the integral for $q(p(G))$ are derived in §§ 6,7. In addition it is shown in § 5 that $d =$ the geometric genus of an associated elliptic surface $X_T \longrightarrow \mathbb{P}_1$, that the holomorphic

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Hirzebruch [**H**] conjectured that the signature defects of Hilbert modular cusps should be given by the values at $s = 1$ of the corresponding Shimizu L-functions, which was proved by Atiyah, Donnelly and Singer [**ADS1-2**] and by Müller [**Mu1**]. The conjecture was reformulated in [**SO**] to the case of general isolated cusp singularities, and the generalized conjecture was partially solved by Müller [**Mu2**]. In this paper we show that the signature defects of Hilbert-Picard modular cusps are still given by special values of Shimizu L-functions. One finds the precise statement in Section 4. This case is not treated in [**SO**]. There is another case which is acceptable to the definition of signature defects by Hirzebruch, that is, Picard modular cusps. We calculated in [**O**] the signature defect of Picard modular cusps.

Let F be a totally real number field of degree $n(> 1)$, K a totally imaginary quadratic extension of F and $\mathcal{O}_K$ the ring of integers in K. Let $SU(m+1,1) := \{g \in SL(m+2, \mathbf{C}); {}^*gI_{m+1,1}g = I_{m+1,1}\}$ ($m \geq 1$), where $I_{m+1,1} = \begin{pmatrix} I_{m+1} & 0 \\ 0 & -1 \end{pmatrix}$. The group $SU(m+1,1)$ acts on the complex unit ball B_{m+1} in $\mathbf{C}^{m+1}$ as linear fractional transformations. $SU(m+1,1;\mathcal{O}_K) := SU(m+1,1) \cap SL(m+2,\mathcal{O}_K)$ is the *Hilbert-Picard modular group*. $SU(m+1,1;\mathcal{O}_K)$ acts on the product $(B_{m+1})^n$ of n copies of the complex unit ball through n embeddings of K in $\mathbf{C}$ which are not complex conjugate each other. The quotient space $SU(m+1,1;\mathcal{O}_K) \setminus (B_{m+1})^n$ is the *Hilbert-Picard modular variety*, and is compactified to a normal complex space by addition of finite points called *Hilbert-Picard modular cusps*.

In order to look one cusp it is available to realize $(B_{m+1})^n$ as a Siegel domain of second kind: By a holomorphic mapping

$$B_{m+1} \ni (z_1, \ldots, z_{m+1}) \mapsto (\sqrt{-1}(1+z_1), \sqrt{2}z_2, \ldots, \sqrt{2}z_{m+1})/(1-z_1) \in D_0,$$

the complex unit ball B_{m+1} is biholomorphic to

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$D_0 := \{(z, u_1, \ldots, u_m) \in \mathbf{C}^{m+1}; 2\mathrm{Im}z - \sum_{i=1}^m |u_i|^2 > 0\}$. Hence $(B_{m+1})^n$ is biholomorphic to $D = (D_0)^n$. And the group $SU(m+1,1)$ is transformed into the group $G_0 = \{g \in SL(m+2, \mathbf{C}); {}^*gH_{m+1,1}g = H_{m+1,1}\}$ by conjugation, where

$$H_{m+1,1} = \begin{pmatrix} 0 & 0 & -\sqrt{-1} \\ 0 & I_m & 0 \\ \sqrt{-1} & 0 & 0 \end{pmatrix}.$$

Let $G := (G_0)^n$ and $\Gamma \subset G$ the discrete subgroup corresponding to the Hilbert-Picard modular group. The isotropy subgroup G_∞ of the point at infinity of D is a parabolic subgroup P. The group P splits into $P = UAM$, where

$$A = \{ \begin{pmatrix} \delta & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & \delta^{-1} \end{pmatrix}^n \in G; \delta > 0\},$$

$$M = \{ \begin{pmatrix} y\beta & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & y^{-1}\beta \end{pmatrix} \in G; \begin{array}{ll} B = (B_1, \ldots, B_n), & B_i \in U(m) \\ y\beta = (y_1\beta_1, \ldots, y_n\beta_n), & y_i > 0 \\ y_1 \cdots y_n = 1, & \det B_i = \beta_i^{-2} \end{array} \}$$

and

$$U = \{[a, r] := \begin{pmatrix} 1 & \sqrt{-1}\,{}^t\bar{a} & \sqrt{-1}|a|^2/2 + r \\ 0 & I_m & a \\ 0 & 0 & 1 \end{pmatrix}; \begin{array}{l} a = (a_1, \ldots, a_n) \in (\mathbf{C}^m)^n \\ r = (r_1, \ldots, r_n) \in \mathbf{R}^n \end{array} \}.$$

According to the theory of toroidal embedding [**AMRT**] we see that a desingularization of the Hilbert-Picard modular cusp $(P \cap \Gamma \setminus D \cup \{\infty\}, \infty)$ is given by replacing the singularity ∞ by a toric bundle divisor over an abelian variety of dimension nm in the sense of Satake[**S**]. Further we can easily see that the boundary manifold X of a suitable compact neighborhood of the cusp obtained by slicing along the cusp is paralellizable. Hence we can define the signature defect $\sigma(X, f)$ by giving a framing f on X.

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§1 Hilbert-Picard Modular Cusps. Let F be a totally real number field of degree $n(>1)$ with the ring of integers $\mathcal{O}_F$ in F, K a totally imaginary quadratic extension of F with integers $\mathcal{O}_K$ and $\{\varphi_1, \bar{\varphi}_1, \ldots, \varphi_n, \bar{\varphi}_n\}$ the set of embeddings of K into $\mathbf{C}$. Let $T \in K$ with $\bar{T} = -T$ and $\sqrt{-1}T < 0$. Then we define an alternating form $E : K^m \times K^m \to F$ by

$$E(u, v) = \mathrm{trace}_{K/F}({}^t uT\bar{v}),$$

where $u, v \in K^m$ are regarded as column vectors. Let N be a complete lattice in F, M a free $\mathbf{Z}$-module of rank $2mn$ in K^m satisfying the condition that for $l_1, l_2 \in M$, $E(l_1, l_2) \in N$, and let $\bar{\Gamma} \subset \mathcal{O}_K^\times$ be a finite index free subgroup preserving M and N, where the action of $\bar{\Gamma}$ on M is componentwise multiplication and that of $\bar{\Gamma}$ on N is multiplication through the relative norm of K to F. Set $V := \mathrm{Norm}_{K/F}(\bar{\Gamma})$. Then $V \subset \mathcal{O}_F^\times$ is a finite index free subgroup of the group of totally positive units in $\mathcal{O}_F$. We may consider that V acts on M through $\bar{\Gamma}$. From this 4-tuple (T, M, N, V) we construct a normal isolated singularity.

By the embedding $\varphi = (\varphi_1, \ldots, \varphi_n) : K \to \mathbf{C}^n$, we regard M as a $\mathbf{Z}$-lattice in $\mathbf{C}^{mn}$ and N in $\mathbf{R}^n$. Then $M_\mathbf{R} = M \otimes_\mathbf{Z} \mathbf{R} \simeq \mathbf{C}^{mn}$ and $N_\mathbf{R} = N \otimes_\mathbf{Z} \mathbf{R} \simeq \mathbf{R}^n$. We identify $M_\mathbf{R}$ with $\mathbf{C}^{mn}$ through this isomorphism, and we also denote $E \otimes_\mathbf{Q} \mathbf{R}$ simply by E. We define a Hermitian form $H : M_\mathbf{R} \times M_\mathbf{R} \to N_\mathbf{C}$ by

$$H(l_1, l_2) = E(l_1, \sqrt{-1}l_2) + \sqrt{-1}E(l_1, l_2) \quad \text{for} \quad l_1, l_2 \in M_\mathbf{R}.$$

We set $D := \{(z, u) \in N_\mathbf{C} \times M_\mathbf{R}; 2\mathrm{Im} z - H(u, u) \in C\}$, where $C := (\mathbf{R}_{>0})^n$ the first quadrant cone in $N_\mathbf{R}$. The domain D is biholomorphic to $(B_{m+1})^n$. Let $\mathcal{N} := \mathcal{N}(N_\mathbf{R}, M_\mathbf{R}) = \{[a, r]; a \in M_\mathbf{R}, r \in N_\mathbf{R}\}$ be a group with the multiplication law $[a, r][b, s] = [a + b, r + s - \frac{1}{2}E(a, b)]$. The group $\mathcal{N}$ acts on D as

$$[a, r].(z, u) = (z + r + \sqrt{-1}H(u, a) + \frac{\sqrt{-1}}{2}H(u, u), u + a).$$

The group V also acts on $N_\mathbf{C} \times M_\mathbf{R}$. Hence the semi-direct product $S(N, M, V) := \mathcal{N}(N, M) \rtimes V$ acts on D. The point $p = (\sqrt{-1}\infty, 0)$ at infinity of D gives the *Hilbert-Picard modular cusp singularity* $(S(N, M, V) \setminus D \cup \{p\}, p)$ associated to (T, M, N, V).

§2 Framed Manifold (X, f). We define a level set $C_t (t \in \mathbf{R})$ in $C = (\mathbf{R}_{>0})^n$ by

$$C_t := \{(y_1, \ldots, y_n) \in (\mathbf{R}_{>0})^n; y_1 \ldots y_n = e^{2nt}\},$$

and also define D_t in D by

$$D_t := \{(z, u) \in N_{\mathbf{C}} \times M_{\mathbf{R}}; 2\mathrm{Im}z - H(u, u) \in C_t\}.$$

Then the action of $S(N, M, V)$ preserves D_t. Let $X_t := S(N, M, V) \backslash D_t$. Set $X = X_0$. By composition $V \overset{\varphi}{\to} (\mathbf{R}_{>0})^n \overset{log}{\to} \mathbf{R}^n$, we identify V as a $\mathbf{Z}$-lattice in $\mathbf{R}^{n-1}$. Then X is just the solvmanifold $S(N, M, V) \setminus S(N_{\mathbf{R}}, M_{\mathbf{R}}, V_{\mathbf{R}})$ and inherits a natural framing f on its tangent bundle induced by the left invariant framing on the solvable Lie group $S(N_{\mathbf{R}}, M_{\mathbf{R}}, V_{\mathbf{R}})$.

§3 Representations of $S(N_{\mathbf{R}}, M_{\mathbf{R}}, V_{\mathbf{R}})$. Let $\mathcal{G}$ be the Lie algebra of $S(N_{\mathbf{R}}, M_{\mathbf{R}}, V_{\mathbf{R}})$. Denote by $Y_1, \ldots, Y_{n-1} \in \mathcal{G}$ the elements corresponding to a basis of $\mathrm{Lie}(V_{\mathbf{R}}) \simeq \mathbf{R}^{n-1}$, and by $X_1, \ldots, X_n \in \mathcal{G}$ the elements corresponding to a basis of $\mathrm{Lie}(N_{\mathbf{R}}) \simeq \mathbf{R}^n$. Let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbf{C}^m$. Then we denote by U_{ij} and $V_{ij} \in \mathcal{G}$ $(i = 1, \ldots, n; j = 1, \ldots, m)$ the elements corresponding to e_j and $\sqrt{-1}e_j$ in $i-$th component of $\mathrm{Lie}(M_{\mathbf{R}}) \simeq (\mathbf{C}^m)^n$ respectively. Then we have relations:

$$\begin{aligned}
&[Y_i, X_i] = 2X_i, \quad [Y_i, X_n] = -2X_n, \quad && (i = 1, \ldots, n-1, \\
&[Y_i, U_{ik}] = U_{ik}, \quad [Y_i, V_{ik}] = V_{ik}, && j = 1, \ldots, m, \\
&[Y_i, U_{nk}] = -U_{nk}, \quad [Y_i, V_{nk}] = -V_{nk}, && k = 1, \ldots, m) \\
&[U_{jk}, V_{jk}] = -d_j X_j,
\end{aligned}$$

where $d_j = -\sqrt{-1}\varphi_j(T) > 0$. The set $\{Y_i, X_j, U_{jk}, V_{jk}; i = 1, \ldots, n-1, j = 1, \ldots, n \text{ and } k = 1, \ldots, m\}$ forms a basis of $\mathcal{G}$ over $\mathbf{R}$, and hence induces the framing f.

Set $S_{\mathbf{Z}} := S(N, M, V)$ and $S_{\mathbf{R}} := S(N_{\mathbf{R}}, M_{\mathbf{R}}, V_{\mathbf{R}})$. We consider the right quesi-regular representation of $S_{\mathbf{Z}}$ on $L^2(S_{\mathbf{Z}} \setminus S_{\mathbf{R}})$. Let $f \in C^\infty(S_{\mathbf{Z}} \setminus S_{\mathbf{R}})$. We may consider f as a function on $S_{\mathbf{R}}$ invatiant under the left action of $S_{\mathbf{Z}}$. For a fixed $v \in V_{\mathbf{R}}$ the function $f(\cdot, v) : \mathcal{N} \to \mathbf{C}$ is invariant under the action of $\mathcal{N}(N, M)$, hence it belongs to $L^2(\mathcal{N}(N, M) \setminus \mathcal{N})$. The right quasi-regular representation $R_{\mathcal{N}}$ of $\mathcal{N}$ on $L^2(\mathcal{N}(N, M) \setminus \mathcal{N})$ decomposes discretely into orthogonal direct sum $R_{\mathcal{N}} = \oplus M(\pi)\pi$ ($\pi \in \hat{\mathcal{N}}$) of irreducible representations, each occurring with finite multiplicity $m(\pi)$. Note that $\mathcal{N}$ is the direct product of copies of a Heisenberg group of dimension $2m+1$. We know unitary irreducible representation of Heisenberg groups (see, for example, [**Mo**]):

Let $\mathcal{G}^*$ be the dual vector space to $\mathcal{G} = N_{\mathbf{R}} \times M_{\mathbf{R}}$. On $N_{\mathbf{R}}$ we have an inner product $< \cdot, \cdot >_N$ defined by $\mathbf{R}$-linear extention of the rational bilinear form $\mathrm{trace}_{F/\mathbf{Q}}(xy)$ for $x, y \in F$. Let $E_0 := \mathrm{trace}_{F/\mathbf{Q}}(E|_{M \times M})$. On $M_{\mathbf{R}}$ then we have a non-degenerate Hermitian form H_0 so that its imaginary part coincides with $E_0 \otimes_{\mathbf{Z}} \mathbf{R}$. For $\tau \in M_{\mathbf{R}} \subset \mathcal{G}^*$ we define the one dimensional representation π_τ by

$$\pi_\tau([a,r]) = \exp(2\pi\sqrt{-1}H_0(\tau, a)).$$

For $\nu \in N_{\mathbf{R}} \subset \mathcal{G}^*$ we define the representation π_ν on $\mathcal{H}(\pi_\nu) = L^2(W_2) := L^2(\sum t_{jk}V_{jk} \in \mathcal{N})$ by

$$(\pi_\nu([a,r])f)(v_2) = \exp(2\pi\sqrt{-1} < \nu, r - E(w_1, v_2) + \frac{1}{2}E(w_1, w_2) >_N) \times f(v_2 - w_2),$$

where $W_1 := \{\sum t_{jk}U_{jk} \in \mathcal{N}\} \simeq \mathbf{R}^{nm}$ and $a = w_1 + w_2 \in W_1 \oplus W_2 = M_{\mathbf{R}}$.

Let N^* and M^* be the dual $\mathbf{Z}$-modules to N and M with respect to $< \cdot, \cdot >_N$ and H_0, respectively.

LEMMA 3.1. *The representation $R_{\mathcal{N}}$ of $\mathcal{N}$ on $L^2(\mathcal{N}(N, M)\backslash\mathcal{N})$ decomposes as*

$$R_{\mathcal{N}} = \bigoplus_{\tau \in M^*} \pi_\tau \oplus \bigoplus_{\nu \in N^* - 0} \mathrm{m}(\pi_\nu)\pi_\nu,$$

where $\mathrm{m}(\pi_\nu) = \sqrt{\det E_0}|\mathrm{Norm}_{F\backslash\mathbf{Q}}(\nu)|^m$.

For the proof see Theorem 37 of [**Mo**].

Since $S_{\mathbf{R}}$ is the semi-direct product of $\mathcal{N}$ and $V_{\mathbf{R}}$, we have the following lemma.

LEMMA 3.2. *We have the decomposition*

$$L^2(S_{\mathbf{Z}} \setminus S_{\mathbf{R}}) = L^2(V \setminus V_{\mathbf{R}}) \oplus \bigoplus_{\tau \in V\backslash(M^* - 0)} L^2(V_{\mathbf{R}}) \oplus \bigoplus_{\nu \in V\backslash(N^* - 0)} m(\pi_\nu)L^2(V_{\mathbf{R}}) \otimes \mathcal{H}(\pi_\nu).$$

§4 Signature defects. Assume $n(m+1)=2k$. Let (X,f) be the framed manifold defined in Section 2. Then there exists a compact oriented manifold W with $\partial W = X$. Since X is framed, we can define the Pontrjagin classes of W as relative classes $p_j \in H^{4j}(W,\partial W)$. Let $L_k(p_1,\dots,p_k) \in H^{4k}(W,\partial W)$ be the Hirzebruch L-polynomial. The signature defect is defined as

$$\sigma(X,f) = L_k(p_1,\dots,p_k)[W,\partial W] - \operatorname{sign}(W,\partial W),$$

where $[W,\partial W] \in H_{4k}(W,\partial W)$ is the fundamental class and $\operatorname{sign}(W,\partial W)$ is the signature of the bilinear form on $H^{2k}(W,\partial W)$ defined by cup product. The signature defect is independent of the choice of a bounding manifold W (cf. **[H]**).

THEOREM.

$$\sigma(X,f) = 2^{nm}\sqrt{\det E_0}\,L(N^*,V;-m),$$

where $L(N^*,V;s)$ is the Shimizu L-function defined by

$$L(N^*,V;s) = \sum_{\nu\in V\backslash(N^*-0)} \frac{\operatorname{sign}(\operatorname{Norm}_{F/\mathbf{Q}}(\nu))}{|\operatorname{Norm}_{F/\mathbf{Q}}(\nu)|^s} \quad (\operatorname{Re}s>1).$$

COROLLARY. *When n or m is odd, $\sigma(X,f)$ vanishes.*

PROOF: When n is odd, we have $L(N^*,V;s)=0$ by definition. When m is odd, it follows from the zeros and poles of Γ-function appearing in a functional equation of L-function.

§5 Eta invariants. Let X be a $(4k-1)$-dimensional compact oriented manifold without boundary. The tangential signature operator on X is a first order elliptic differential operator acting on square integrable differential forms of even degree defined on $2p$-forms by $(-1)^{k+p+1}(*d-d*)$, where d is the exterior differential and $*$ is the Hodge star operator (see[**APS1**]).

In this section we define the operator A on the manifold $X = S(N,M,V)\backslash S(N_{\mathbf{R}},M_{\mathbf{R}},V_{\mathbf{R}})$ by slightely modifying the tangential signature operator as in [**ADS1**]. We define A on $2p$-forms by

$$(-1)^{k+p+1}(*d^\nabla - d^\nabla *),$$

where d^∇ is the covariant differential of the flat connection ∇ defined by the framing f. The space of square integrable forms of even degree on X is identified with $L^2(S_{\mathbf{Z}} \setminus S_{\mathbf{R}}) \otimes_{\mathbf{C}} (\wedge^{ev}\mathcal{G}^* \otimes \mathbf{C})$, where $\wedge^{ev}\mathcal{G}^* := \oplus_{p=0}^{2k-1} \wedge^{2p} \mathcal{G}^*$ is the set of even degree alternating forms on $\mathcal{G}$ with values in $\mathbf{R}$. Put $\mathcal{M} := \wedge^{ev}\mathcal{G} \otimes_{\mathbf{R}} \mathbf{C}$, which is identified with the space of constant forms of even degree on X.

PROPOSITION 5.1.

$$\eta(A,0) = 2^{nm}\sqrt{\det E_0}L(N^*, V; -m).$$

For the proof of the proposition we devote the rest of this section and the next section.

LEMMA 5.2. *On $L^2(S_{\mathbf{Z}} \setminus S_{\mathbf{R}}) \otimes_{\mathbf{C}} \mathcal{M}$, the operator A is written as*

$$A = \sum_{i=1}^{n-1} Y_i \otimes F_i + \sum_{j=1}^{n}\{\sum_{k=1}^{m}(U_{jk} \otimes E_{jk}^{(1)} + V_{jk} \otimes E_{jk}^{(2)}) - \sqrt{-1}X_j \otimes E_j\},$$

where $F_i, E_{jk}^{(l)}, E_j \in \mathrm{End}(\mathcal{M}), F_i$ and $E_{jk}^{(l)}$ are skew Hermitian and E_j are Hermitian. Moreover $F_j^2 = (E_{jk}^{(l)})^2 = -1, E_j^2 = 1$ and distinct pairs anti-commute.

PROOF: It follows from that A is self-adjoint and that A^2 has the same leading symbol as that of the Laplace-Beltrami operator on forms.

Now consider the diffeomorphism $\psi_t : D_t \to D_0$ defined by $\psi_t(z,u) = (e^{-2t}z, e^{-t}u)$. By the diffeomorphism ψ_t we identify $X_t = S_{\mathbf{Z}} \setminus D_t$ with $S(e^{-2t}N, e^{-t}M, V) \setminus S_{\mathbf{R}}$. Set $S_{\mathbf{Z}}(t) := S(e^{-2t}N, e^{-t}M, V)$. On the compact solvmanifold $S_{\mathbf{Z}}(t) \setminus S_{\mathbf{R}}$ the framing f defines a metric, g_t, and the operator A, we denote it by $A(t)$. We define a diffeomorphism $\varphi_t : S_{\mathbf{Z}}(t) \setminus S_{\mathbf{R}} \to S_{\mathbf{Z}} \setminus S_{\mathbf{R}}$ by $\varphi_t(S_{\mathbf{Z}}(z,u,v)) = S_{\mathbf{Z}}(e^{2t}z, e^t u, v)$. Transform the operator $A(t)$ on $S_{\mathbf{Z}}(t) \setminus S_{\mathbf{R}}$ to an operator B on $S_{\mathbf{Z}} \setminus S_{\mathbf{R}}$: For $\Phi \in L^2(S_{\mathbf{Z}} \setminus S_{\mathbf{R}}) \otimes \mathcal{M}$,

$$B(\Phi)(S_{\mathbf{Z}}g) := A(t)(\Phi \circ \varphi_t)(\varphi_t^{-1}(S_{\mathbf{Z}}g)).$$

Then we have for $f \otimes \omega \in L^2(S_{\mathbf{Z}} \setminus S_{\mathbf{R}}) \otimes \mathcal{M}$

$$\begin{aligned} B(f \otimes \omega) &= \sum_{i=1}^{n-1} Y_i f \otimes F_i\omega \\ &+ \sum_{j=1}^{m}\{e^t \sum_{k=1}^{n}(U_{jk}f \otimes E_{jk}^{(1)}\omega + V_{jk}f \otimes E_{jk}^{(2)}\omega) - e^{2t}\sqrt{-1}X_j f \otimes E_j\omega\}. \end{aligned}$$

Since B is an $S_{\mathbf{R}}-$invariant operator, we can decompose B into the sum of the operators on the representation spaces of $S_{\mathbf{R}}$ on $L^2(S_{\mathbf{Z}} \setminus S_{\mathbf{R}})$. We can decompose $B = B_0 + \sum_{\tau \in V \setminus (M^* - 0)} B_\tau + \sum_{\nu \in V \setminus (N^* - 0)} B_\nu$ according to the decomposition obtained in Lemma 3.2.

LEMMA 5.3. *We have*

$$B_0 = \sum_{i=1}^{n-1} \frac{\partial}{\partial y_i} \otimes F_i,$$

$$B_\tau = \sum_{i=1}^{n-1} \frac{\partial}{\partial y_i} \otimes F_i$$
$$+ 2\pi\sqrt{-1} \sum_{j=1}^{n} e^{t+y_j} \sum_{k=1}^{m} \{H_0(\tau, U_{jk}) \otimes E_{jk}^{(1)} + H_0(\tau, V_{jk}) \otimes E_{jk}^{(2)}\}$$

and

$$B_\nu = \sum_{i=1}^{n-1} \frac{\partial}{\partial y_i} \otimes F_i - \sum_{j=1}^{n} (e^{t+y_j} \sum_{k=1}^{m} \{2\pi\sqrt{-1} < \nu, X_j >_N d_j t_{jk} \otimes E_{jk}^{(1)}$$
$$+ \frac{\partial}{\partial t_{jk}} \otimes E_{jk}^{(2)}\} - 2\pi\sqrt{-1} < \nu, X_j >_N e^{2t+2y_j} \otimes E_j).$$

PROOF: It follows from Lemmas 3.1 and 3.2.

LEMMA 5.4. *We have*

$$\eta(B_0, s) = 0 \quad \text{and} \quad \eta(B_\tau, s) = 0 \quad \textit{for all} \quad \tau \in M^*.$$

PROOF: From Lemma 5.2, E_j are Hermitian and unitary matrices. Since $E_j {B_0}^* E_j = -B_0$ and $E_j {B_\tau}^* E_j = -B_\tau$, we have $\eta(B_0, s) = -\eta(B_0, s)$ and $\eta(B_\tau, s) = -\eta(B_\tau, s)$.

LEMMA 5.5. *For sufficiently large* Res, *we have*

$$\eta(B, s) = \sqrt{\det E_0} \sum_{\nu \in V \setminus (N^* - 0)} |\mathrm{Norm}_{F/\mathbf{Q}}(\nu)|^m \eta(B_\nu, s).$$

PROOF: It follows from Lemmas 3.2 and 5.4.

§6 Proof of Proposition 5.1. Let

$$B_{\nu,j} = -e^{t+y_j} \sum_{k=1}^{m} (2\pi\sqrt{-1} < \nu, X_j > d_j t_{jk} \otimes E_{jk}^{(1)} + \frac{\partial}{\partial t_{jk}} \otimes E_{jk}^{(2)})$$
$$+ 2\pi < \nu, X_j > e^{2t+2y_j} \otimes E_j.$$

Then we have

$$(B_{\nu,j})^2 = e^{2t+2y_j} \sum_{k=1}^{m} \{-(\frac{\partial}{\partial t_{jk}})^2 + (2\pi < \nu, X_j > d_j t_{jk})^2\} \otimes \mathrm{id}$$
$$+ e^{4t+4y_j} (2\pi\sqrt{-1} < \nu, X_j >)^2 \otimes \mathrm{id}$$
$$- e^{2t+2y_j} 2\pi\sqrt{-1} d_j < \nu, X_j > \otimes \sum_{k=1}^{m} E_{jk}^{(1)} E_{jk}^{(2)}.$$

Put

$$\Delta_j := \sum_{k=1}^{m} \{-(\frac{\partial}{\partial t_{jk}})^2 + (2\pi < \nu, X_j > d_j t_{jk})^2\}.$$

LEMMA 6.1. *Δ_j on $L^2(\mathbf{R}^m)$ has eigenvalues*

$$\{2\pi d_j | < \nu, X_j > | \sum_{k=1}^{m} (2 l_k^{(j)} + 1); l^{(j)} = (l_1^{(j)}, \ldots, l_m^{(j)}) \in (\mathbf{Z}_{\geq 0})^m\}.$$

PROOF: Let $h_m(x) := (-1)^m e^{x^2} (\frac{d}{dx})^m e^{-x^2}$ be the Hermite polynomial for nonnegative integer m, which satisfies the Hermite differential equation:

$$(\frac{d}{dx})^2 h_m(x) - 2x \frac{d}{dx} h_m(x) + 2m h_m(x) = 0.$$

Set $f_m(x) := e^{-x^2/2} h_m(x)$. Then $\{f_m(x)\}_{m=0}^{\infty}$ forms a complete orthogonal basis of $L^2(\mathbf{R})$. Set $g_m(y) := f_m(\sqrt{2\pi d_j | < \nu, X_j > |} y)$. Then $g_m(y)$ satisfies the differential equation

$$\{(2\pi d_j | < \nu, X_j > |)^2 y^2 - (\frac{d}{dx})^2\} g_m(y) = 2\pi d_j | < \nu, X_j > | (2m+1) g_m(y).$$

Hence if we put

$$\Phi_l(t) := \prod_{k=1}^{m} g_{l_k}(t_k)$$

for $l = (l_1, \ldots, l_m) \in (\mathbf{Z}_{\geq 0})^m$, then $\{\Phi_l\}$ forms a complete orthogonal basis of $L^2(W_2)$ and satisfies

$$\sum_{k=1}^{m}\{-(\frac{\partial}{\partial t_k})^2 + (2\pi < \nu, X_j > d_j t_k)^2\}\Phi_l = 2\pi d_j| < \nu, X_j > |\sum_{k=1}^{m}(2l_k + 1)\Phi_l.$$

Next we simultaneously diagonalize the operators $\sqrt{-1}E_{jk}^{(1)}E_{jk}^{(2)}$. Since $(E_{jk}^{(1)}E_{jk}^{(2)})^2 = -1$ and since distinct pair among $\{\sqrt{-1}E_{jk}^{(1)}E_{jk}^{(2)}; 1 \leq j \leq n, 1 \leq k \leq m\}$ commute, we can decompose $\mathcal{M}$ into the direct sum of $V_\varepsilon := \{v \in \mathcal{M}; \sqrt{-1}E_{jk}^{(1)}E_{jk}^{(2)}v = \varepsilon_{jk}v \quad \text{for all } j \text{ and } k\}$, where $\varepsilon = (\varepsilon_{jk}) \in \{+1, -1\}^{nm}$. Put $\varepsilon_0 \in \{+1, -1\}^{nm}$ with all (j, k)-components equal to $+1$. Then for any $\varepsilon \in \{+1, -1\}^{nm}$ the mapping

$$\prod_{(j,k):\varepsilon_{jk}=-1} \sqrt{-1}E_{jk}^{(2)}E_{jk}^{(1)} : V_{\varepsilon_0} \to V_\varepsilon$$

is bijective. Hence $\dim V_\varepsilon = \dim\mathcal{M}/2^{nm} = 2^{n(m+2)-2}$.

LEMMA 6.2. *$(B_{\nu,j})^2$ restricted to $L^2(\mathbf{R}^m) \otimes V_\varepsilon$ has eigenvalues*

$$e^{2t+2y_j}2\pi d_j| < \nu, X_j > |\sum_{k=1}^{m}(2l_k^{(j)} + 1) + e^{4t+4y_j}(2\pi < \nu, X_j >)^2$$
$$- e^{2t+2y_j}(2\pi d_j < \nu, X_j >)\sum_{k=1}^{m}\varepsilon_{jk}.$$

Consider the operator $(\sum_{j=1}^{n} B_{\nu,j})^2 = \sum_{j=1}^{n}(B_{\nu,j})^2$ on $L^2(W_2) \otimes \mathcal{M} = (\otimes^n L^2(\mathbf{R}^m)) \otimes \mathcal{M}$. Let $\mathcal{E}$ be any eigenspace of $\sum_{j=1}^{n}(B_{\nu,j})^2$. Then there exists an integer a such that

$$\mathrm{Tr}(B_{\nu,j}|\varepsilon) = a\{e^{2t+2y_j}2\pi d_j| < \nu, X_j > |\sum_{k=1}^{m}(2l_k^{(j)} + 1)$$
$$+ e^{4t+4y_j}(2\pi < \nu, X_j >)^2 - e^{2t+2y_j}(2\pi d_j < \nu, X_j >)\sum_{k=1}^{m}\varepsilon_{jk}\}^{1/2}.$$

On the other hand we have the following.

LEMMA 6.3.. *For any eigenspace $\mathcal{E}$ of the operator $\sum_{j=1}^{n}(B_{\nu,j})^2$ we have*

$$\mathrm{Tr}(B_{\nu,j}|_{\mathcal{E}}) \in e^{2t+2y_j}(2\pi < \nu, X_j >)\mathbf{Z}.$$

PROOF: Any eigenspace $\mathcal{E}$ has the form $\sum_i V_{\varepsilon(i)}$. No $E_{jk}^{(l)}$ preserves V_ε. But only E_j preserves it. Hence $\mathrm{Tr}(B_{\nu,j}|_{\mathcal{E}}) = e^{2t+2y_j}(2\pi < \nu, X_j >)\mathrm{Tr}(E_j|_{\mathcal{E}})$. Since $E_j|_{\mathcal{E}}) = \pm 1$, we have the lemma.

We assume $\mathrm{Tr}(B_{\nu,j}|_{\mathcal{E}}) \neq 0$, then we have

$$l_k^{(j)} = 0 \quad \text{and} \quad \varepsilon_{jk} = 1 \quad \text{for all } k \text{ if} < \nu, X_j >> 0,$$

or

$$l_k^{(j)} = 0 \quad \text{and} \quad \varepsilon_{jk} = -1 \quad \text{for all } k \text{ if} < \nu,_j >< 0.$$

For $\nu \in N^* - \{0\}$ we define $\varepsilon(\nu) = (\varepsilon_{jk}(\nu)) \in \{+1, -1\}^{nm}$ such that $\varepsilon_{jk}(\nu) = 1$ if $< \nu, X_j >> 0$ and that $\varepsilon_{jk}(\nu) = -1$ if $< \nu, X_j >< 0$. We denote by $B_{\nu,0}$ the operator B_ν restricted to $L^2(V_{\mathbf{R}}) \otimes V_{\varepsilon(\nu)}$.

LEMMA 6.4.

$$\eta(B_\nu, s) = \eta(B_{\nu,0}, s).$$

PROOF: From Lemma 6.2 we have a series of finite dimensional vector bundles on $V_{\mathbf{R}}$. Except for the bundle associated to $V_{\varepsilon(\nu)}$, any bundle associated to $\mathcal{E} \subset L^2(W_2)$ splits into the direct sum of the bundles with the same rank with respect to eigenvalues of $B_{\nu,j}$ with $\mathrm{Tr}(B_{\nu,j}|_{\mathcal{E}}) = 0$. Since F_i and $B_{\nu,j'}$ $(j' \neq j)$ anticommute with $B_{\nu,j}$, we have $\mathrm{Tr}(B_{\nu,j}) = 0$, hence $\eta(B_{\nu,j}|_{\mathcal{E}}, s) = 0$.

$B_{\nu,0}$ has the form

$$\sum_{i=1}^{n-1} \frac{\partial}{\partial y_i} \otimes F_i + \sum_{j=1}^{n} 2\pi < \nu, X_j > e^{2t+2y_j} \otimes E_j,$$

where F_i and E_j are regarded to be restricted to $V_{\varepsilon(\nu)}$. Fortunately this operator $B_{\nu,0}$ has the same form as that of the operator A_μ in [**ADS1**]. In fact we have

$$B_{\nu,0} = \mathrm{sign}(\mathrm{Norm}(\nu))|\mathrm{Norm}(\nu)|^{1/n}(e^{2t}/2)B_h,$$

where $B_h = h\sum_{i=1}^{n-1} \partial/\partial(2y_i) \otimes F_i + \sum_{j=1}^{n} e^{2y_j} E_j$ and $h = |\mathrm{Norm}(\nu)|^{-1/n}$. If we restrict $t \in \mathbf{R}$ to $t \geq 0$, we can apply the method in Sections 5 to 12 in [**ADS1**] to get Proposition 5.1 for the operator $A(t)$ $(t \geq 0)$. The only difference is the normalization constant because of $\dim V_{\varepsilon(\nu)} = 2^{n(m+2)-2}$. Note that $A(0) = A$ and that $\eta(A(t), 0)$ is independent of $t \geq 0$.

§7 Proof of Theorem. In this section we connect with the eta invariant of A and the signature defect.

Let W be an oriented compact Riemannian manifold of dimension $4k = 2n(m+1)$ with $\partial W = S_{\mathbf{Z}} \setminus S_{\mathbf{R}}$. Asssume that the metric on W is the product metric in a neighborhood of ∂W. From Theorem 13.1 in [**ADS1**] we may write

$$\eta(A,0) = \int_W \mathcal{D}_0 - l_0,$$

where l_0 is an integer and $\mathcal{D}_0$ is invariant under scaling of the metric on W.

Let $H := 1 \otimes \mathcal{M} \subset L^2(S_{\mathbf{Z}} \setminus S_{\mathbf{R}}) \otimes \mathcal{M}$ be the space of constant forms and $H^{\perp}$ the orthogonal complement.

LEMMA 7.1. *Given positive constant c, we may assume*

$$\mathrm{Ker} B(t) = H \quad \text{and} \quad B(t) - c \geq 0 \quad \text{on} \quad H^{\perp}$$

for sufficiently large t.

PROOF: It follows from Lemma 5.3.

Now let $C(t)$ be the tangential signature operator on the compact solvmanifold $S_{\mathbf{Z}}(t) \setminus S_{\mathbf{R}}$. Then for $f \otimes \omega \in L^2(S_{\mathbf{Z}}(t) \setminus S_{\mathbf{R}}) \otimes \mathcal{M}$,

$$C(t)(f \otimes \omega) = A(t)(f \otimes \omega) + f \otimes C_0\omega,$$

where C_0 is the restriction of $C(t)$ to H. Note that $C_0 \in \mathrm{End}(\mathcal{M})$. Let us deform linearly from $A(t)$ to $C(t)$. Set

$$A_\lambda := (1-\lambda)A(t) + \lambda C(t) = A(t) + \lambda C_0 \quad \text{for} \quad 0 \leq \lambda \leq 1.$$

LEMMA 7.2. *For sufficiently large t we have*

$$\mathrm{Ker} A_\lambda(t) = \begin{cases} H & \text{for} \quad \lambda = 0, \\ \mathrm{Ker} C_0 & \text{for} \quad \lambda > 0. \end{cases}$$

PROOF: We can choose a positive constant c in Lemma 7.1 so that $c > \|C_0\|^2$.

Fix t sufficiently large so that Lemma 7.2 holds and denote $A_\lambda(t)$ by simply A_λ. Then

$$\eta(A_\lambda, 0) = \eta(A_\lambda|_{H^\perp}, 0) + \eta(A_\lambda|_H, 0).$$

LEMMA 7.3. *The eta invariant $\eta(A_\lambda|_{H^\perp}, 0)$ is continuous in λ.*

PROOF: From Lemma 7.2 we see that zero is not eigenvalues of $A_\lambda|_{H^\perp}$. Since the discontinuities are produced by the zero-eigenvalues (see [**APS2**]), the result follows.

In the following we simply denote by the same symbol $A_\lambda(t)$ the operator on $S_{\mathbf{Z}} \setminus S_{\mathbf{R}}$ transformed by the diffeomorphism $\varphi_t : S_{\mathbf{Z}}(t) \setminus S_{\mathbf{R}} \to S_{\mathbf{Z}} \setminus S_{\mathbf{R}}$ defined in Section 5. We may regard this as changing the metric g_0 on $S_{\mathbf{Z}} \setminus S_{\mathbf{R}}$ by $(\varphi_t^{-1})^* g_t$.

From Theorem 4.2 in [**APS1**] we have

$$l_\lambda = \int_W \mathcal{D}_\lambda - \eta(A_\lambda, 0),$$

where l_λ is integer and $\mathcal{D}_\lambda$ is continuous in λ.

LEMMA 7.4. *For all $0 \leq \lambda \leq 1$ we have*

$$\eta(A_\lambda, 0) = 0.$$

PROOF: From the argument in pp. 67-68 of [**APS1**] it is sufficient to consider $*d$ on the space of constant $(2k-1)$-forms $\Omega_0^{2k-1} := 1 \otimes \wedge^{2k-1}\mathcal{G}^* \otimes \mathbf{C}$.

We denote by y_i, x_j, u_{jk} and $v_{jk} \in \mathcal{G}^*$ the dual elements to Y_i, X_j, U_{jk} and V_{jk}, respectively. Then the volume form

$$\bigwedge_{i=1}^{n-1} (x_i \wedge y_i) \wedge x_n \wedge \bigwedge_{1 \leq j \leq n, 1 \leq k \leq m} (u_{jk} \wedge v_{jk})$$

defines the Hodge star operator $*$. And the exterior differential d acts as

$$dy_i = 0, \quad dx_i = -2y_i \wedge x_i + d_i \sum_{k=1}^{m} u_{ik} \wedge v_{ik},$$

$$dx_n = 2\sum_{i=1}^{n-1} y_i \wedge x_n + d_n \sum_{k=1}^{m} u_{nk} \wedge v_{nk},$$

$$du_{ik} = -y_i \wedge u_{ik}, \quad du_{nk} = \sum_{j=1}^{n-1} y_j \wedge u_{nk},$$

$$dv_{ik} = -y_i \wedge v_{ik}, \quad dv_{nk} = \sum_{j=1}^{n-1} y_j \wedge v_{nk},$$

where $i = 1, \dots, n-1$ and $k = 1, \dots, m$.

For $\theta \in \Omega^{2k-1}$ we may write

$$\theta = x_{I_1} \wedge y_{I_2} \wedge u_{J_1K_1} \wedge v_{J_2K_2}.$$

Set $F := \{\theta \in \Omega_{2k-1}; J_1 = J_2 \quad \text{and} \quad K_1 = K_2\}$.

CLAIM 1. $\operatorname{sign}(*d|_{\Omega^{2k-1}}) = \operatorname{sign}(*d|_F)$.

PROOF OF CLAIM 1: Let $\theta' = x_{I_1} \wedge y_{I_2} \wedge u_{J_1K_1} \wedge v_{J_2K_2} \in \Omega^{2k-2}$. Choose $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$ so that $j \notin J_1 \cup J_2$ or $k \notin K_1 \cup K_2$. Set $\theta^- = \theta' \wedge u_{jk}$ and $\theta^+ = \theta' \wedge v_{jk}$. Then we have $*d\theta^+ = - * d\theta^-$.

We denote simply by $w_{jk} = u_{jk} \wedge v_{jk}$. Then we may write $\theta = x_{I_1} \wedge y_{I_2} \wedge w_{JK} \in F$. Next we define the weight $w(\theta) = (l_1, \dots, l_n)$ of $\theta \in F$ by

$$l_i = \delta(I_1, i) + {}^\sharp\{(j,k) \in JK; j = i\}, \quad \delta(I_1, i) = \begin{cases} 1 & \text{if} \quad i \in I_1, \\ 0 & \text{otherwise.} \end{cases}$$

Set $F_0 := \{\theta \in F; w(\theta) = w(*d\theta)\}$.

CLAIM 2. $\operatorname{sign}(*d|_F) = \operatorname{sign}(*d|_{F_0})$.

The claim follows from the fact that $(*d)^2$ preserves weights.

If we set $w(\theta) = (l_1, \dots, l_n)$ for $\theta \in F_0$, then $w(*d\theta) = (m+1-l_1, \dots, m+1-l_n)$. When m is even, hence $F_0 = 0$. We assume m is odd in the followings. We can easily see that d restricted to F_0 coincides with the operator $\sum_{j=1}^{n}\sum_{k=1}^{m}\operatorname{ext}(w_{jk})\operatorname{int}(X_j)$. Hence $(*d)^2$ does not affect on the factor y_{I_2}. But $*d$ transforms y_{I_2} to $y_{I_2'}$ up to sign, where $I_2' = \{1,\dots,n-1\} - I_2$. Thus we conclude that $\operatorname{sign}(*d|_{F_0}) = 0$.

Remark. When m is even, we can also prove Lemma 7.4 by employing the same argument of the proof of Lemma 14.8 of [**ADS1**].

LEMMA 7.5. $l_0 = \operatorname{sign}(W, \partial W)$.

PROOF: From Theorem 4.14 of [**APS1**] we have $l_1 = \operatorname{sign}(W, \partial W)$. From Lemmas 7.3 and 7.4 we see that $\eta(A_\lambda, 0)$ is continuous in λ. Since an integer-valued function $l_\lambda = \int_W \mathcal{D}_\lambda - \eta(A_\lambda, 0)$ is continuous, l_λ is constant.

Next we must identify the integral $\int_W \mathcal{D}_0$. Let h be a nonnegative C^∞-function on $I = [0,1]$ satisfying

$$0 \le h \le 1, \quad h([0,1/4]) = 1 \quad \text{and} \quad h([3/4,1]) = 0.$$

As in Section 13 in [**ADS1**] we extend the flat connection ∇ on $S_{\mathbf{Z}} \setminus S_{\mathbf{R}}$ to a metric connection ϕ on W so that its torsion tensor is $hT(\nabla)$ on $\partial W \times I$ and vanishes on the rest. We denote by $p_j(\phi)$ the j-th Pontrjagin form defined from the curvature form of ϕ by means of the Chern-Weil theory. Then

$$L_k(p_1, \dots, p_k)[W, \partial W] = \int_W L_k(p_1(\phi), \dots, p_k(\phi)),$$

where $p_j \in H^{4j}(W, \partial W; \mathbf{Z})$ are the relative Pontrjagin classes associated to the framing f. We put $\Omega(\phi) := L_k(p_1(\phi), \dots, p_k(\phi))$ for simplicity. The signature defect is

$$\begin{aligned}\sigma(S_{\mathbf{Z}}(t) \setminus S_{\mathbf{R}}) &= \int_W \Omega(\phi) - \operatorname{sign}(W, \partial W) \\ &= \int_W \Omega(\phi) - \int_W \mathcal{D}_0 + \eta(A(t), 0).\end{aligned}$$

We may choose the connection ϕ so that it defines the integrand $\mathcal{D}_0$ as in Theorem 13.2 of [**ADS1**]. Since the integrands $\Omega(\phi)$ and $\mathcal{D}_0$ restricted to $W - \partial W \times I$ coincide, the integrals turn out

$$\int_W (\Omega(\phi) - \mathcal{D}_0) = \int_{\partial W \times I} (\Omega(\phi) - \mathcal{D}_0).$$

Up to now we have seen that

$$\int_{\partial W\times I}(\Omega(\phi)-\mathcal{D}_0)=\sigma(S_{\mathbf{Z}}(t)\setminus S_{\mathbf{R}},f)-\eta(A(t),0)$$
$$=\sigma(X,f)-\eta(A,0). \tag{7.6}$$

The first equality holds for sufficiently large t and the second one follows from the result of Section 6 and the invariance of the signature defects under diffeomorphism.

We will consider the behavior of the integral in (7.6) under changing t of the metric $(\varphi_t^{-1})^* g_t$ on $S_{\mathbf{Z}}\setminus S_{\mathbf{R}}$. The integrand is a $O(4k)$-invariant $4k$-form and has weight zero under scaling the metric $g\to\mu^2 g$. Moreover we have

LEMMA 7.7. *On $\partial W\times I$ we have*

$$\Omega(\phi)-\mathcal{D}_0=\sum a_i(h)P_i(T(\nabla)),$$

where $a_i(h)$ is apolynomial in h and in the derivatives of h with values in 1-forms on I, and $P_i(T(\nabla))$ is an $O(4k-1)$-invariant $4k$-form valued polynomial in the components of $T(\nabla)$ and in its covariant derivatives with respect to the flat connection ∇. Moreover each P_i has nonnegative weight.

For the proof see Proposition 13.5 of [**ADS1**].

Every invariant polynomial is a finite linear combination of elementary monomials $m(T(\nabla))$ in the torsion tensor $T(\nabla)$ with values in q-forms defined in [**ABP**].

LEMMA 7.8. *If we change the metric g_0 on $S_{\mathbf{Z}}\setminus S_{\mathbf{R}}$ by $(\varphi_t^{-1})^* g_t$, then elementary monomials $m(T(\nabla))$ with $(4k-1)$-forms change as multiplication by $e^{2n(m+1)t}$.*

PROOF: Recall [**ABP**] that an elementary monomial $m(T)$ with values in q-forms is given by

$$m(T)=\sum_q^{*} T_{\alpha^1}\dots T_{\alpha^r},$$

where $T_\alpha=T_{\alpha_1\alpha_2\alpha_3,\alpha_4\dots\alpha_l}$, the indices $\alpha_4,\dots,\alpha_l$ refer to covariant derivatives. Alternation runs over q indices and the remaining indices are contracted. Since $T=T(\nabla)$ is parallel, the indices α^i have the length three.

If all indices α in T_α are contracted, it is not affected by the change of the metric. Hence $q = 4k - 1$ indices of alternation change $m(T)$ totally by $e^{2n(m+1)t}$.

From the equation (7.6) and Lemmas 7.7 and 7.8 we see that

$$\sigma(X, f) = \eta(A, 0).$$

Combining this equality with Proposion 5.1 we complete the proof of Theorem.

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