CANONICAL DEGREE OF CURVES ON A MINIMAL SURFACE

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In memory of
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Abstract. Given a smooth, projective, minimal complex surface X and an irreducible curve C on it, we prove that the canonical degree CK_X is bounded from above by an explicit function in the Chern numbers of X and the geometric genus of C (Theorem 1.1). As an immediate consequence, it follows that the number of rational or elliptic curves on a surface of general type is universally bounded if the oriented topological type of the ambient surface is fixed.

1. Introduction

Let X be a smooth, projective, complex algebraic variety. An *entire curve* on X is, by definition, an image of the complex line $\mathbb{C}$ via a non-constant holomorphic mapping. A theorem of R. Brody [3] says that X is hyperbolic if and only if X does not contain any entire curve. Inspired by this result and by the measure hyperbolicity of varieties of general type (proved by K. Kodaira and S. Kobayashi-T. Ochiai [7]), Mark L. Green suggested that a surface of general type might be hyperbolic modulo rational and elliptic curves therein (see [6], (7.4.13)). Serge Lang proposed a generalization to arbitrary dimension of Green's suggestion ([8], Chapter IV, Section 5), and conjectured that the union of all the entire curves on X could not be Zariski dense as long as X was of general type.

If an entire curve is algebraic, i.e. if its Zariski closure is a closed curve C in X, C is either a rational curve or an elliptic curve. Conversely a rational or elliptic curve is obviously the closure of an entire curve. In this sense, we may call rational and elliptic curves on X *algebraic entire curves*.

When X is a surface, Lang's conjecture clearly asserts the algebraicity and finiteness of the entire curves on a given X of general type. Yet the conjecture yields a far stronger result for the algebraic entire curves on surfaces, should it be true in arbitrary dimension. Namely we can derive

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from it the *uniform boundedness* of the numbers of algebraic entire curves on a minimal surface of general type X, meaning that the number should be bounded by a constant depending only on the topological invariants c_1^2 and c_2. (For an arithmetic analogue, see [4]).

Notwithstanding its importance, very few supporting evidences for two-dimensional Lang's conjecture or Green's conjecture have been known so far. The known results are essentially classified into two types.

The first type of resuts gets finiteness of algebraic entire curves C under certain condition on the cotangent bundle of the ambient surface. As an extreme case, assume that Ω^1_X is ample. Fix an arbitrary ample divisor H on X. Then we can find a small positive rational number t such that $\Omega^1_X(-tH)$ is still ample. By looking at the natural homomorphism $\Omega^1_X \to \omega_{\tilde{C}}$, we see that $\omega_{\tilde{C}}(-tH)$ is also ample on C so that $2g-2 > tCH > 0$. This proves (1) the non-existence of non-hyperbolic curves and gives (2) the upper bound of CH in terms of the genus[1] g plus a geometric quantity $t = t(X, H)$. In 1977, F. Bogomolov [1] generalized this simple observation to the case where Ω^1_X is *big*. (Ω^1_X is said to big if the dimension of $\mathrm{H}^0(X, \mathrm{Sym}^n \Omega^1_X)$ grows cubically in n as n tends to infinity.) In particular, if $c_1(X)^2 > c_2(X)$, there are only finitely many algebraic entire curves on X, a surface of general type.

Unfortunately, Bogomolov's result does not apply to some important classes of surfaces including hypersurfaces in $\mathbb{P}^3$. If it does, his method, though ingeneous, is not effective and does not provide us a uniform bound of the number of non-hyperbolic curves.

The second type of results, on the other hand, proves a uniform bound of the number of algebraic entire curves that are subject to some constraint. For instance, let C be a smooth irreducible curve of genus g on a minimal surface of general type X. Then the Bogomolov-Miyaoka-Yau inequality for open surfaces (explicitly stated first in [12]) yields the estimate

$$CK_X \le 3c_2(X) - K_X^2 + 4g - 4,$$

implying the uniform boundedness of the number of *smooth* rational/elliptic curves on X. A modification of the inequality gives a bound of CK_X, the *canonical degree* (perhaps "*canonical height*" sounds better), for such C that has only a bounded number of ordinary double points and ordinary triple points [9]. This kind of condition on the number of certain singularities is of course artificial and even absurd, but the method is anyhow capable to give an effective result.

In the present paper, we give an effective, uniform bound of the canonical degree of an irreducible curve of given genus on a minimal surface of positive Kodaira dimension, without any codition on the ambient space or on the singularities of curves. Our main result is the following

[1]Throughout the paper, the genus of a complete irreducible curve will always mean the genus of its normalization.

Theorem 1.1. *Let X be a smooth projective complex surface and C an irreducible curve of geometric genus g on X (with arbitrary singularities). Assume that the canonical divisor K_X is nef, i.e. X is minimal and that $CK_X > \max\{2g-2, 3g-3\}$. Then one of the following four inequalities holds:*

$$CK_X \leq 6c_2(X) - 2c_1(X)^2 + 6g - 6, \tag{1}$$

$$\frac{(1-4(g-1)/CK_X)^3}{(1-3(g-1)/CK_X)^2} CK_X \leq 8c_2(X) - 2c_1(X)^2, \tag{2}$$

$$\frac{(1-2(g-1)/CK_X)^3}{(1-3(g-1)/CK_X)^2} CK_X \leq 8c_2(X) - 2c_1(X)^2, \tag{3}$$

$$CK_X \leq 14^3(c_2(X) - \frac{1}{4}c_1(X)^2). \tag{4}$$

Immediate consequences of the theorem are listed below.

Corollary 1.2. *If C is an algebraic entire curve, i.e. if $g(C) \leq 1$, then*

$$CK_X \leq 14^3(c_2(X) - (1/4)c_1(X)^2) = 2744c_2(X) - 686c_1(X)^2.$$

Corollary 1.3. *If $g(C) \geq 2$, then CK_X is bounded from above by the maximum of $5g(C) - 5$, $32c_2 - 8c_1^2 + 4g - 4$ and $14^3(c_2 - (1/4)c_1^2)$.*

When the ambient surface X is of general type, theory of Chow varieties tells us that the uniform boundedness of the canonical degrees implies the uniform boundedness of the number of algebraic entire curves thereon, which is thought of as a supporting evidence for Lang's conjecture.

Corollary 1.4. *If X is a minimal surface of general type, then the number of rational/elliptic curves on X is bounded by a constant which depends only on the oriented topological type of X.*

However, the real importance of our result is exhibited by the following

Corollary 1.5. *Given the equations of a pluricanonically embedded surface $X \subset \mathbb{P}^N$, we can find all the rational curves on X by solving a system of algebraic equations in a bounded number of variables.*

Indeed, a rational curve on $\mathbb{P}^N$ of degree d is given by a system of equations

$$X_i(t) = \sum_{j=0}^{d} a_{i,j} t^j, \quad i = 0, 1, \ldots, N$$

in a parametric representation in terms of the linear coordinate t of $\mathbb{P}^1$. This curve lies on X if and only if $g(X(t)) \equiv 0$ for every defining equation g, which is an algebraic relation between the $a_{i,j}$'s.

In [14], T. Shioda observed that the Mordell-Weil rank of a certain elliptic surface does not increase much by cyclic base changes $\mathbb{P}^1 \to \mathbb{P}^1$. The following corollary provides us a natural explanation to his observation.

Corollary 1.6. *Let X be a minimal elliptic surface of Kodaira dimension one, with canonical fibre structure $\pi : X \to B$. Given a multi-section $C \subset X$ of genus g, the mapping degree of C over B is effectively bounded in terms of the genus of C.* (This statement is trivial unless $B = \mathbb{P}^1$.) *In particular, the Mordell-Weil rank of X remains universally bounded after any base change $B' \to B$ as long as the genus of B' is bounded.*

Our universal bound also gives a characterization of surfaces of general type in terms of algebraic entire curves.

Corollary 1.7. *Let X be a projective complex surface. Then*

1) *The number of algebraic entire curves on X is finite if and only if X is either of general type or a simple abelian surface.*

2) *Let $\mathcal{C}(X)$ be the set of irreducible curves on a polarized surface (X, H). Define a function $\lambda_X : \mathcal{C}(X) \to \mathbb{R}$ by*

$$\lambda_X(C) = \frac{2g(C) - 2}{(C, H)}.$$

Then X is of general type if and only if $0 \in \mathbb{R}$ is away from the set of accumulation values of the function λ_X. (This property of surfaces of general type can be viewed as an algebraic analogue of measure hyperbolicity. [2])

It is evident that there are uncountably many algebraic entire curves on a rational, ruled or elliptic surface. By [11], an algebraic K3 surface carries a one-parameter family of (singular) elliptic curves, whence follows (1). The proof of (2) thus reduces to the statement that

$$\inf_{C \in \mathcal{C}(X)} \lambda_X(C) = 0$$

for a simple abelian surface X. Any simple abelian surface X is an étale quotient of a principally polarized abelian surface A with theta divisor Θ of genus two. Consider a general translation Θ_a by an element $a \in A$ and look at the image of Θ_a via the composite of the multiplication $\mathbf{m}_N : A \to A$ by a positive integer N and the étale morphism $A \to X$. This image has fixed genus two and arbitrarily large degree if N is chosen sufficiently large accordingly.

This paper is organized as follows.

In Section 2, we introduce certain subsheaves $\Omega^{(n)}\langle iC\rangle$ of $\operatorname{Sym}^n \Omega^1_X(iC)$ and their natural filtrations. These filtrations essentially come from the parabolic structure of Ω^1_X attached to the natural homomorphism $\Omega^1_X|_C \to \nu_*\omega_{\tilde{C}}$, where $\nu : \tilde{C} \to C$ is the normalization.

[2]The function λ_X depends on the choice of the ample divisor H. However, if one replaces H by another ample divisor H' to get a new function λ'_X, one can find positive constants $a = a(H, H')$, $b = b(H, H')$, such that $a\lambda_X(C) \le \lambda'_X(C) \le b\lambda_X(C)$ regardless which $C \in \mathcal{C}(X)$ one picks up. In this sense, qualitative behaviour of λ_X does not depend on the choice of H.

In Section 3, we give estimates for the dimension of the global sections of $\Omega^{(n)}\langle iC\rangle(-tnK_X)$. For instance, if C is an algebraic entire curve with only ordinary singularities, the dimension turns out to be positive whenever $i \geq (1-t)n > 0$ and $u = 1-t$ satisfies

$$u^3 > \frac{c_2(X) - \frac{1}{4}K_X^2}{(C, K_X)} + O\Big(\frac{1}{n}\Big).$$

Furthermore, a non-zero global section of this sheaf must have large *slope function*, defined in Section 4.

In Section 4, we study the interplay between the slope function and the cohomology class of an effective divisor on $\mathbb{P}(\mathcal{E})$, the projective bundle attached with a 2-vector bundle $\mathcal{E}$ with parabolic structure.

The proof of Theorem 1.1 will be completed in Section 5, by combining the results in the previous part of the paper.[3]

In this article, everything is over the complex number field, and we always work in the category of analytic spaces. Open neighbourhoods, local coordinates, local equations, etc. will be understood in analytic topology.

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2. Canonical filtration of the symmetric tensors associated with a curve

Let C be a reduced (but possibly reducible) curve on a smooth surface X. For simplicity of the notation, we denote by Ω the sheaf of holomorphic 1-forms Ω^1_X and by $\Omega^{(n)}$ the sheaf of the n-th symmetric tensors $\operatorname{Sym}^n \Omega$. Given a local equation f of C, df/f is a meromorphic 1-form with logarithmic pole along (the smooth locus of) C. Replacing f by a second equation gf, with g invertible, the 1-form $d(gf)/(gf)$ differs from the original one by $d\log g$, a holomorphic differential. Thus the sheaf

$$\Omega\langle C\rangle = \frac{df}{f}\mathcal{O}_X + \Omega \subset \Omega(C)$$

is globally well-defined. $\Omega\langle C\rangle/\Omega$ is a trivial $\mathcal{O}_C$-module with a standard basis $df/f \mod \Omega$, where f is any local equation of C.

[3]It might be worth pointing out that our proof is quite elementary. Apart from fundamentals of algebraic geometry and undergraduate mathematics, what we need are: (1) Semistable reduction theorem for Proposition 2.3; (2) K_X-semistability and Bogomolov semistability of Ω^1_X for Lemma 3.4; and (3) Bogomolov-Miyaoka-Yau-Sakai inequality for Proposition 5.1.

If C is smooth, this sheaf is the usual sheaf of logarithmic 1-forms $\Omega(\log C)$. If C has singularities (including the case where C has only nodes), however, it is not locally free any more.

Products of n elements of $\Omega\langle C\rangle$ generate a subsheaf

$$\Omega^{(n)}\langle nC\rangle = \frac{(df)^n}{f^n}\mathcal{O}_X + \frac{(df)^{n-1}}{f^{n-1}}\Omega + \cdots + \Omega^{(n)} \subset \Omega^{(n)}(nC).$$

By construction, $\Omega^{(n)}\langle nC\rangle$ has a natural increasing filtration given by the subsheaves

$$\begin{aligned}\Omega^{(n)}\langle jC\rangle &= \Omega^{(j)}\langle jC\rangle \cdot \Omega^{(n-j)} = \Omega^{(n)}\langle nC\rangle \cap \Omega^{(n)}(jC)\\ &= \frac{(df)^j}{f^j}\Omega^{(n-j)} + \frac{(df)^{j-1}}{f^{j-1}}\Omega^{(n-j+1)} + \cdots + \Omega^{(n)}\\ &\subset \Omega^{(n)}(jC), \qquad\qquad j = 0, 1, \ldots, n.\end{aligned}$$

The associated graded module is naturally isomorphic to

$$\bigoplus_{i=0,\ldots,n} \mathcal{O}_C \otimes \Omega^{(i)}.$$

Since $\Omega^{(n)}\langle jC\rangle$ is a subsheaf of $\Omega^{(n)}(jC)$, we obtain a series of subsheaves $\Omega^{(n)}\langle jC\rangle(-jC)$ of $\Omega^{(n)}$, which are locally of the form

$$(df)^j\Omega^{(n-j)} + f(df)^{j-1}\Omega^{(n-j+1)} + \cdots + f^j\Omega^{(n)}.$$

Thus we have the inclusion relation

$$\Omega^{(n)}\langle nC\rangle(-nC) \subset \Omega^{(n)}\langle (n-1)C\rangle(-(n-1)C) \subset \cdots \subset \Omega^{(n)}\langle 0C\rangle = \Omega^{(n)}.$$

Because they are not necessarily locally free, these subsheaves do not define a nice filtration of $\Omega^{(n)}$. However, by taking the double duals

$$\Omega^{(n)}\langle -jC\rangle = (\Omega^{(n)}\langle jC\rangle)^{**}(-jC),$$

we get a nicer decreasing filtration

$$\Omega^{(n)} = \Omega^{(n)}\langle -0C\rangle \supset \cdots \supset \Omega^{(n)}\langle -nC\rangle.$$

Along the smooth locus of C, we have of course $\Omega^{(n)}\langle -jC\rangle = \Omega^{(n)}\langle jC\rangle(-jC)$ and, if C is defined by one of the local coordinates x, it is simply the locally free $\mathcal{O}_X$-module generated by:

$$dx^n, dx^{n-1}dy, \ldots, dx^j dy^{n-j}, x dx^{j-1}dy^{n-j+1}, \ldots, x^j dy^n,$$

or, more graphically, by

$$\begin{array}{lllll} & & & & x^j dy^n, \\ & & & \cdots & \\ & & x dx^{j-1}dy^{n-j+1}, & & \\ dx^n, \quad \ldots, \quad dx^j dy^{n-j}. & & & & \end{array}$$

Let $\nu : \tilde{C} \to X$ denote the normalization of $C \subset X$. There is a natural $\mathcal{O}_X$-homomorphism from Ω to $\nu_*\omega_{\tilde{C}}$, which induces a generically surjective $\mathcal{O}_{\tilde{C}}$-homomorphism $\nu^*\Omega \to \omega_{\tilde{C}}$. Let $\mathfrak{k}$ and $\mathfrak{q}$ denote the kernel and the image of this $\mathcal{O}_{\tilde{C}}$-homomorphism, respectively. On the smooth locus of C, we can identify $\tilde{C}$ with C, and then $\mathfrak{k}$ is the conormal bundle $\mathcal{O}_C(-C)$ whereas $\mathfrak{q}$ is the canonical bundle $\omega_{\tilde{C}}$. At each point point of C, we have

$$\begin{aligned} \mathfrak{k} \otimes_{\mathcal{O}_{\tilde{C}}} \mathfrak{q} &\simeq \nu^* \mathcal{O}_X(K_X), \\ \mathfrak{q} &\subset \omega_{\tilde{C}}. \end{aligned}$$

The equality $\mathfrak{q} = \omega_{\tilde{C}}$ globally holds if and only if $\tilde{C} \to X$ is an immersion or, equivalently, for each singularity of C, the local analytic branches of C are all smooth. When C is a compact irreducible curve of geometric genus g, we have thus obvious relations between the degrees of $\mathfrak{k}$ and $\mathfrak{q}$:

$$\begin{aligned} \deg \mathfrak{k} + \deg \mathfrak{q} &= CK_X, \\ \deg \mathfrak{q} &\leq 2g-2, \\ \deg \mathfrak{k} &\geq CK_X - 2g + 2. \end{aligned}$$

In particular, if C is compact, irreducible and $CK_X > 4g-4$, the vector bundle $\nu^*\Omega$ is unstable and $0 \subset \mathfrak{k} \subset \nu^*\Omega$ is the *Harder-Narsimhan filtration* introduced by [5].

The subsheaf $\mathfrak{k} \subset \nu^*\Omega$ defines a natural filtration also on $\nu^*\Omega^{(n)}$:

$$\textbf{(HN)} \qquad 0 \subset \mathfrak{k}^n \subset \nu^*\mathfrak{k}^{n-1}\Omega \subset \cdots \subset \mathfrak{k}\nu^*\Omega^{(n-1)} \subset \nu^*\Omega^{(n)},$$

which is also the Harder-Narasimhan filtration of $\nu^*\Omega^{(n)}$ provided $CK_X > 4g-4$ (plus compactness and irreducibility of C).

Notice that the natural homomorphism $\Omega|_C \to \nu_*\omega_{\tilde{C}}$ defines a (simplest) *parabolic structure* on Ω in the sense of C.S. Seshadri (cf. [13] for vector bundles over punctured curves; for the generalization to higher dimension, see e.g. [10]). Our filtration $\Omega^{(n)}\langle -jC\rangle$ is of course understood in terms of this parabolic structure.

The filtration (**HN**) on $\nu^*\Omega^{(n)}$ and the filtration $\{\Omega^{(n)}\langle -jC\rangle\}$ are apparently related to each other. More precisely, we have

Lemma 2.1. *There is a natural inclusion*

$$\Omega^{(n)}\langle -jC\rangle / \big(\Omega^{(n)}\langle -jC\rangle \cap \Omega^{(n)}(-C)\big) \hookrightarrow \mathfrak{k}^j \nu^*\Omega^{(n-j)},$$

which is an isomorphism along the smooth locus of C.

The proof is easy and left to the reader.

Let X° denote the open subset $X \setminus \mathrm{Sing}(C)$. The natural filtration

$$\Omega^{(n)} \supset \cdots \supset \Omega^{(n)}(-jC)$$

on X° induces a finer filtration on $\Omega^{(n)}\langle -jC\rangle$. (In other words, we are considering a double filtration on $\Omega^{(n)}$.) Namely, we define:

$$\Omega^{(n)}\langle -jC\rangle_i = \big\{\eta \in \Omega^{(n)}\langle -jC\rangle \big|\ \eta|_{X^\circ} \in \Omega^{(n)}(-iC) + \Omega^{(n)}\langle -(j+1)C\rangle|_{X^\circ}\big\}.$$

Thus the quotient $\Omega^{(n)}\langle -jC\rangle/\Omega^{(n)}\langle -(j+1)C\rangle$, viewed as a torsion free $\mathcal{O}_C$-module, naturally has successive quotients $\mathfrak{Q}_{j,i}$, $i=0,\dots,j$. On the smooth locus of C,

$$\mathfrak{Q}_{j,i} \simeq f^i(df)^{j-i}\omega_C^{n-j+i} \simeq \omega_C^{n-j+i}(-jC).$$

Let $\tilde{\mathfrak{Q}}_{j,i}$ denote $\nu^*\mathfrak{Q}_{j,i}/(\text{torsion})$.

Lemma 2.2. *If C has only ordinary double points and if $j \le n/2$, then there is a natural isomorphism*

$$\tilde{\mathfrak{Q}}_{i,j} \simeq \omega_{\tilde{C}}^{\otimes(n-2j+i)}(j\nu^*K_X).$$

Proof. If C is smooth at p, then ω_C, which is the quotient of Ω by $df = f\frac{df}{f}$, is canonically isomorphic to

$$\det\Omega\otimes\left(f\frac{df}{f}\right)^{-1}\mathcal{O}_C \simeq f^{-1}\det\Omega\otimes\mathcal{O}_C \simeq \mathcal{O}_C(K_X+C).$$

This isomorphism (*adjunction formula*) shows the assertion at a smooth point of C.

Assume that p is a node of C, and let (x,y) be a local coordinate at $p\in C$. We may assume that $f=xy$. Under the condition that $j\le n/2$, direct calculation shows that $\Omega^{(n)}\langle -jC\rangle$ is generated by:

$$\begin{array}{ccccccc}
y^j\,dx^n & & & & & & x^j\,dy^n \\
 & \cdots & & & & \cdots & \\
 & & dx^{n-j}dy^j & \cdots & dx^j\,dy^{n-j} & &
\end{array}$$

so that

$$(*)\qquad \begin{aligned}
&\Omega^{(n)}\langle -jC\rangle/\Omega^{(n)}\langle -(j+1)C\rangle \\
=&y^j\,dx^n(\mathcal{O}/y\mathcal{O})\oplus x^j(dy^n)(\mathcal{O}/x\mathcal{O}) \\
&\oplus y^{j-1}(dxdy)dx^{n-2}(\mathcal{O}/y\mathcal{O})\oplus x^{j-1}(dxdy)dy^{n-2}(\mathcal{O}/x\mathcal{O}) \\
&\oplus\cdots \\
&\oplus (dxdy)^j\,dx^{n-2j}(\mathcal{O}/y\mathcal{O})\oplus(dxdy)^j(dy^{n-2j})(\mathcal{O}/x\mathcal{O}).
\end{aligned}$$

The $\mathcal{O}_C$-submodule $\Omega^{(n)}\langle -jC\rangle_i/\Omega^{(n)}\langle -(j+1)C\rangle$ is generated by the first $j-i+1$ lines of the above equality.

Note that $dx(\mathcal{O}/y\mathcal{O})\oplus dy(\mathcal{O}/x\mathcal{O})$ is naturally identified with $\omega_{\tilde{C}}$. Meanwhile, on the local branch $x=0$ of C,

$$\begin{aligned}
x(\mathcal{O}/x\mathcal{O}) &= \frac{dx\wedge dy}{d(xy)/(xy)\wedge dy}(\mathcal{O}/x\mathcal{O}) \\
&\simeq (dy)^{-1}(dxdy)(\mathcal{O}/x\mathcal{O})\simeq\omega_{\tilde{C}}^{-1}(\nu^*K_X).
\end{aligned}$$

Thus each line of the equality (*) is of the form

$$(dx\wedge dy)^j(dx)^{n-2(j-i)-i}\mathcal{O}/x\mathcal{O}\oplus(dx\wedge dy)^j(dy)^{n-2(j-i)-i}\mathcal{O}/y\mathcal{O}.$$

Now the assertion is clear. $\square$

Thanks to the semistable reduction theorem, this result carries over to general cases.

Proposition 2.3. *For general (X, C), there exists an inclusion:*

$$\tilde{\mathfrak{Q}}_{i,j} \hookrightarrow \omega_{\tilde{C}}^{\otimes(n-2j+i)}(j\nu^* K_X),$$

for $j \leq n/2$.

Proof. We have never assumed that X or C is compact. Hence, by replacing X by a small Stein open subset, we only have to prove that

$$\mathrm{H}^0(\tilde{C}, \tilde{\mathfrak{Q}}_{i,j}) \hookrightarrow \mathrm{H}^0\big(\tilde{C}, \omega_{\tilde{C}}^{\otimes(n-2j+i)}(j\nu^* K_X)\big).$$

We view C (locally) as the fibre over the origin of the fibration $\pi : X \to \Delta$, where $\Delta \subset C$ is a small disc with local coordinate t. Then $\Omega^{(n)}\langle -jC\rangle$ is the saturation in $\Omega^{(n)}$ of

$$(\pi^*\omega_\Delta)^n + \pi^* t(\pi^*\omega_\Delta)^{n-1}\Omega + \cdots + (\pi^* t^j)\Omega^{(n)}.$$

By the semistable reduction theorem, after a suitable cyclic base change, desingularization and blow-down, we get a new fibration $g : Y \to \Delta'$ such that the singular fibre $D = g^{-1}(0)$ is a reduced divisor with normal crossings on a smooth surface Y. Thus we get a commutative diagram

$$\begin{array}{ccc} \overline{X} & \overset{\gamma}{\leftarrow} & \overline{Y} \\ \downarrow & & \downarrow \\ X & & Y \\ \downarrow & & \downarrow \\ \Delta & \leftarrow & \Delta'. \end{array}$$

Here $\overline{X}$ and $\overline{Y}$ stand for blown-up surfaces of X and Y, respectively. The top line and the botom line are finite cyclic Galois covers, the latter being given simply by $\tau \mapsto t = \tau^r$.

From the construction follow the facts

(1) the projection $\overline{X} \to \Delta$ is smooth at a general point of the strict transform $\overline{C}$ of C on $\overline{X}$;

(2) and hence $\gamma^{-1}(\overline{C})$ is set-theoretically isomorphic to $\overline{C}$;

(3) $\gamma^{-1}(\overline{C})$ does not collapse to a point on Y;

(4) the Galois group $G = \mathrm{Gal}(Y/X) = \mathrm{Gal}(\Delta'/\Delta) \simeq \mathbb{Z}/(r)$ naturally acts on the pair $(\overline{Y}, \overline{D})$, which descends to the action on (Y, D).

The filtration $\Omega^{(n)}\langle -jC\rangle_i$ does not behave nicely under base change simply because the defining equation f of C changes to g^r after base change. Nevertheless $\Omega^{(n)}\langle -jC\rangle_i(jC)$ behaves far better. In fact, $\Omega^{(n)}\langle -jC\rangle(jC)$ is, by definition, the double dual of

$$\big(\frac{df}{f}\big)^j\Omega^{(n)} + \cdots + \Omega^{(n)},$$

while

$$\frac{df}{f} = \frac{dt}{t} = r\frac{d\tau}{\tau} = r\frac{dg}{g}.$$

The submodule $\Omega^{(n)}\langle -jC\rangle_i(jC)$ is the double dual of

$$f\big(\frac{df}{f}\big)^j\Omega^{(n-j)} + \cdots + f\big(\frac{df}{f}\big)^{j-i+1}\Omega^{(n-j+i-1)} + \big(\frac{df}{f}\big)^{j-i}\Omega^{(n-j+i)} + \cdots + \Omega^{(n)},$$

which is "almost" compatible with the base change.

In general, given torsion free sheaves $\mathcal{E}_1 \subset \mathcal{E}_2$, the quotient $\mathcal{E}_2/\mathcal{E}_1$ will inflate to a bigger sheaf if we replace $\mathcal{E}_i$'s by their double duals. From this viewpoint, Lemma 2.2 is interpreted as an upper bound of the inflation:

$$\begin{aligned}&\Omega_Y^{(n)}\langle -jD\rangle_i(jD)/\Omega_Y^{(n)}\langle -jD\rangle_{i+1}(jD)\\ &\qquad\simeq \omega_D^i((j-i)(K+D)) \otimes \omega_{\tilde{D}}^{n-2j+i}.\end{aligned}$$

This sheaf is a quotient of

$$\sum_{i=0}^{j}\Big(\frac{(dx\wedge dy)/g}{dg/g}\Big)^i \cdot (dx\wedge dy/g)^{j-i}\Omega_Y^{(n-2j+i)} \subset \Omega^{(n)}(jD),$$

$dx \wedge dy$ being a local generator of $\det \Omega_Y$. Here we used two identifications

$$(dx\wedge dy/g)\mathcal{O} \simeq \det\Big((dg/g)\mathcal{O} + \Omega_Y)^{**}\Big),$$

$$\big((dx\wedge dy/g)/(dg/g)\big)\mathcal{O} \simeq \Big(\big((dg/g)\mathcal{O}+\Omega\big)^{**}/(dg/g)\mathcal{O}\Big)^{**}.$$

By construction, there are natural isomorphisms

$$\gamma^*\mu_X^*\omega_C \simeq \mu_Y^*\omega_D,$$
$$\gamma^*\mu_X^*(K_X + C) \simeq \mu_Y^*(K_Y + D),$$

where $\gamma : \overline{Y} \to \overline{X}$ is the cyclic Galois cover and μ stands for the blowing up. These isomorphisms send the local generators $(ds\wedge dt/f)/(df/f)$ and $(ds\wedge dt)/f$ to $(dx\wedge dy/g)/(dg/g)$ and $r(dx\wedge dy)/g$, respectively.

Thus

$$\begin{aligned}&\mathrm{H}^0\big(X, \Omega^{(n)}\langle -jC\rangle_i(jC)\big) \simeq \mathrm{H}^0\big(\overline{X}, \mu_X^*\Omega^{(n)}\langle -jC\rangle_i(jC)\big)\\ &\quad\subset \mathrm{H}^0\big(\overline{Y}, \mu_Y^*\Omega^{(n)}\langle -jD\rangle_i(jD)\big) \simeq \mathrm{H}^0\big(Y, \Omega^{(n)}\langle -jD\rangle_i(jD)\big)\end{aligned}$$

must be contained in

$$\begin{aligned}&\sum_{i=0}^{j}\Big(\frac{(dx\wedge dy)/g}{dg/g}\Big)^i \cdot (dx\wedge dy/g)^{j-i}\Omega_Y^{(n-2j+i)}\\ &\qquad= \sum_{i=0}^{j}\Big(\frac{(ds\wedge dt)/f}{df/f}\Big)^i \cdot (ds\wedge dt/f)^{j-i}\Omega_Y^{(n-2j+i)}.\end{aligned}$$

Since $\hat{C}$, the image of $\gamma^{-1}(\overline{C})$ on Y, is bimeromorphic to $\tilde{C}$, the restriction of this sheaf to $\hat{C}$ maps naturally to $\sum \omega_C^i((j-i)(K+C)) \otimes \omega_{\tilde{C}}^{n-2j+i}$. In particular $\mathrm{H}^0\big(X, \Omega^{(n)}\langle -jC\rangle_i(jC)\big) / \mathrm{H}^0\big(X, \Omega^{(n)}\langle -jC\rangle_{i+1}(jC)\big)$ must be contained in the space of global sections of

$$\omega_C^i((j-i)(K+C)) \otimes \omega_{\tilde{C}}^{n-2j+i}.$$

Noting that ω_C is canonically isomorphic to $\mathcal{O}_C(K_X + C)$ via the multiplication by df/f, we have the assertion. $\square$

Remark. There should be an elementary proof of Proposition 2.3 without resorting to the semistable reduction theorem. It is indeed fairly easy to directly show the assertion in case C is an immersed curve (i.e. $\nu : \tilde{C} \to X$ is an immersion).

3. Global dimension count

Let X be a compact complex surface and C a reduced, irreducible curve on X.

We use the same notation as in the previous section. Thus $\Omega^{(n)}\langle nC\rangle$ is an enlargement of $\Omega^{(n)} = \mathrm{Sym}^n\,\Omega$ by adjoining polynomials of df/f of degree $\leq n$.

First we calculate the Euler characteristic of $\Omega^{(n)}\langle nC\rangle$:

Proposition 3.1. *If $n \gg 0$ and $CK_X > c_1(X)^2 - c_2(X)$, then the Euler characteristic $\chi(X, \Omega^{(n)}\langle nC\rangle)$ is large:*

$$\chi(X, \Omega^{(n)}\langle nC\rangle) = \frac{n^3}{6}\big(c_1(X)^2 - c_2(X) + CK_X\big) + O(n^2).$$

Proof. By Riemann-Roch, $\chi(X, \Omega^{(n)}) = \frac{n^3}{6}(c_1^2 - c_2)$ up to quadratic terms in n. Then look at the increasing filtration $\Omega^{(n)}\langle jC\rangle$ on $\Omega^{(n)}\langle nC\rangle$ with successive quotients $\mathcal{O}_C \otimes \Omega^{(n-j)}$, which has Euler characteristic

$$(n-j)(n-j+1)CK_X + (n-j+1)\chi(C, \mathcal{O}_C).$$

Adding up these terms, we get the equality as above. (Note that the contribution involving $\chi(C, \mathcal{O}_C)$ is only up to quadratic order in n.) $\square$

The same proof shows the following:

Corollary 3.2. *We have the following asymptotic equality:*

$$\chi(X, \Omega^{(n)}\langle nC\rangle(-tnK_X)) = \frac{n^3}{6}\big((1-3t+3t^2)c_1(X) - c_2(X) + (1-3t)CK_X + O(n^2).$$

The corollary guarantees the non-vanishing of $\mathrm{H}^0(X,\Omega^{(n)}\langle nC\rangle(-tnK_X))$ for fairly large value t, roughly up to $1/3$ when CK_X is large. However, it is not optimal by any means. We will below improve the estimate of the maximum value of rational numbers t such that $\mathrm{H}^0(X,\Omega^{(n)}\langle nC\rangle(-tnK_X))$ does not vanish.

Fix an interger $j = vn$ and consider the subsheaf

$$\begin{aligned}\Omega^{(n)}\langle nC\rangle_{[j]} =&\Omega^{(n)} + \frac{df}{f}\Omega^{(n-1)}\langle -(n-j)C\rangle + \big(\frac{df}{f}\big)^2\Omega^{(n-2)}\langle -(n-j)C\rangle \\ &+\cdots \\ &+\big(\frac{df}{f}\big)^j\Omega^{(n-j+1)}\langle -(n-j)C\rangle + \big(\frac{df}{f}\big)^j\Omega^{(n-j)}\langle -(n-j)C\rangle\end{aligned}$$

of $\Omega^{(n)}\langle jC\rangle \subset \Omega^{(n)}\langle nC\rangle$. This is globally well-defined, thanks to the natural inclusion $df\Omega^{(k)}\langle -jC\rangle \subset \Omega^{(k+1)}\langle -jC\rangle$. If C is locally smooth and defined by $x = 0$, this sheaf is locally identical with $\Omega^{(n)}\langle -nC\rangle\big((n-j)C\big) + \Omega^{(n)}$ spanned by the local basis

$$\begin{array}{cccc} & & & dx^{n-j}dy^j, \quad \ldots \quad dy^n, \\ & & x^{-1}dx^{n-j+1}dy^{j-1}, & \\ & \cdots & & \\ x^{-j}dx^n & & & \end{array}$$

whereas $\Omega^{(n)}\langle jC\rangle$ is generated by

$$\begin{array}{cccc} & & & dy^n, \\ & & x^{-1}dxdy^{n-1}, & \\ & \cdots & & \\ x^{-j}dx^n, \quad \ldots, \quad x^{-j}dx^{n-j}dy^j & & & \end{array}$$

Let $\nu : \tilde{C} \to C \hookrightarrow X$ be the normalization map. Assume that $CK_X > \max\{4g(C)-4, 0\}$, so that $\nu^*\Omega$ is unstable. Let $\mathfrak{k} \subset \nu^*\Omega$ denote the maximal destabilizing subsheaf, to get the exact sequence

$$0 \to \mathfrak{k} \to \nu^*\Omega \to \mathfrak{q} \to 0.$$

Let $d = d(X,C)$ be the rational number (*defect coefficient*) defined by

$$(2d+1)CK_X = \deg\mathfrak{k}.$$

It is obvious that $\deg\mathfrak{q} = -2dCK_X$ and that $d \geq (1-g)/CK_X$.

With these new symbols, we get:

Proposition 3.3. (1) *We have the following asymptotic inequality*

$$\begin{aligned}&\chi(X,\Omega^{(n)}\langle nC\rangle_{[vn]}(-tnK_X)) \\ &\qquad \geq \chi(X,\Omega^{(n)}(-tnK_X)) \\ &\qquad\qquad + n^3CK_X\int_0^v x\big((1-t+2d)-(1+3d)x\big)dx + O(n^2).\end{aligned}$$

(2) *If we vary v, the integral above, viewed as a function in v, takes at*

$$v_0 = \frac{(1-t)+2d}{1+3d}$$

the maximum value

$$\frac{n^3(1-t+2d)^3 CK_X}{6(1+3d)^2}.$$

In particular,

$$\begin{aligned}&\chi(X,\Omega^{(n)}\langle nC\rangle_{[v_0 n]}(-tnK_X))\\&\quad\geq \frac{n^3}{6}\left\{\big((1-3t+3t^2)c_1^2-c_2\big)+\frac{(1-t+2d)^3}{(1+3d)^2}CK_X\right\}+O(n^2),\end{aligned}$$

Proof. There is an increasing filtration

$$\big(\Omega^{(n)}\langle nC\rangle_{[j]}+\Omega^{(n)}\langle kC\rangle\big)/\Omega^{(n)}\langle nC\rangle_{[j]}^{(n)},\quad k=0,\dots,n$$

of $\Omega^{(n)}\langle nC\rangle/\Omega^{(n)}\langle nC\rangle_{[j]}$, with successive quotients being an $\mathcal{O}_C$-submodule of the quotients

$$\begin{cases}\nu^*\Omega^{(n-k)}/\mathfrak{k}^{n-j}\nu^*\Omega^{(j-k)} & \text{if } k\leq j;\\ \nu^*\Omega^{(n-k)} & \text{if } j<k\leq n.\end{cases}$$

The Euler characteristic of these $\mathcal{O}_{\tilde{C}}$-modules are:

$$\binom{n-k+1}{2}CK+\binom{j-k+1}{2}CK-2\binom{j-k+1}{2}\deg\mathfrak{k}+(n-j)\chi(\tilde{C},\mathcal{O})$$

or

$$\binom{n-k+1}{2}CK+(n+k+1)\chi(\tilde{C},\mathcal{O})$$

according as $k\leq j$ or $k>j$. If $j=vn$, these numbers add up to

$$\frac{n^3}{6}\big((1-v^3)CK-3v^2(1-v)\deg\mathfrak{k}\big)+O(n^2).$$

Hence we deduce the inequality

$$\begin{aligned}&\chi(X,\Omega^{(n)}\langle nC\rangle_{[vn]})\\&\quad\geq\frac{n^3}{6}\big(c_1^2-c_2+CK-(1-v^3)CK+3v^2(1-v)\deg\mathfrak{k}\big)+O(n^2)\\&\quad=\frac{n^3}{6}(c_1^2-c_2+v^3CK+3v^2(1-v)\deg\mathfrak{k})+O(n^2).\end{aligned}$$

By tensoring $\mathcal{O}(-tnK_X)$, everything must be suitably adjusted to yield

$$\begin{aligned}&\chi(X,\Omega^{(n)}\langle nC\rangle_{[vn]}(-tnK_X))\\&\quad\geq\frac{n^3}{6}\big((1-3t+3t^2)c_1^2-c_2+(v^3-3v^2t)CK+3v^2(1-v)\deg\mathfrak{k}\big)\end{aligned}$$

up to an error term of quadratic order in n. Now the statement (1) is a tautological restatement of this, while the statement (2) is an immediate corollary to (1). $\square$

Lemma 3.4. *Let the notation be as above. If $u = 1 - t < \frac{1}{2}$, then*

$$\begin{aligned}\dim \mathrm{H}^2(X, &\Omega^{(n)}\langle nC\rangle_{[vn]}(-tnK_X)) \\ &\leq \dim \mathrm{H}^2(X, \Omega^{(n)}(-tnK_X)) \\ &= \dim \mathrm{H}^0(X, \Omega^{(n)}(-unK_X)) + O(n^2) \\ &\leq \frac{n^3(1-2u)^2}{8} c_1(X)^2 + O(n^2).\end{aligned}$$

Proof. The first inequality is trivial if one notes that $\Omega^{(n)}\langle nC\rangle_{[vn]}$ contains $\Omega^{(n)}$. The second equality is a direct consequence of the Serre duality and the natural isomorphism $\Omega^* \simeq \Omega(-K_X)$. Let us prove the third inequality.

If the Kodaira dimension $\kappa(X)$ of X is one or less, it is more or less trivial to show that $\mathrm{h}^0(X, \Omega^{(n)}(-unK_X))$ is at most quadratic order in n. Therefore we may assume that X is of general type. In this case, it is known that the cotangent sheaf Ω is K_X-semistable. (It is a corollary of S.-T. Yau's famous theorem [16] on the existence of Kähler Einstein metric when K_X is ample. Also in case K_X is nef and big, semistability follows from a theorem of H. Tsuji on the existence of degenerate Kähler Einstein metric [15].) In particular, if Γ is a general member of $|(\frac{1}{2} - u)nK_X|$ and n sufficiently large, $\Omega|_\Gamma$ is semistable. Hence so is $\Omega^{(n)}(-unK_X)|_\Gamma$, with rank $n+1$ and with average degree $n^2(1-2u)^2K_X^2/4$. Noting that the genus of Γ is $n^2(1-2u)^2/8$ modulo a linear correction term, we get

$$\dim \mathrm{H}^0(\Gamma, \Omega^{(n)}(-unK_X)) \leq \frac{n^3(1-2u)^2}{8} + O(n^2).$$

Now look at the exact sequence

$$0 \to \Omega^{(n)}(-\frac{n}{2}K_X) \to \Omega^{(n)}(-unK_X) \to \Omega^{(n)}(-unK_X)|_\Gamma \to 0.$$

The dimesion of the global sections of the third term is bounded as above, whilst the first term has, by Bogomolov semi-stability, global sections of dimension at most of linear order in n. □

Corollary 3.5. *Assume that $u = 1 - t < 1/2$. Then*

$$\begin{aligned}&\dim \mathrm{H}^0(X, \Omega^{(n)}\langle nC\rangle_{[v_0 n]}(-tnK_X)) \\ &\quad \geq \frac{n^3}{6}\left\{\left((1-u)^3 + u^3\right)c_1^2 - c_2 + \frac{(u+2d)^3}{(1+3d)^2}CK - \frac{3(1-2u)^2}{4}K^2\right\} + O(n^2) \\ &\quad = \frac{n^3}{6}\left\{\frac{1}{4}c_1^2 - c_2 + \frac{(u+2d)^3}{(1+3d)^2}CK_X\right\} + O(n^2).\end{aligned}$$

Here v_0 is a rational number defined by

$$v_0 = \frac{u+2d}{1+3d}$$

as in Lemma 3.3.

From this corollary immediately follows:

Corollary 3.6. *Let u be the minimum rational number $\in (1/n)\mathbb{Z}$ such that*

$$\mathrm{H}^0(X, \Omega^{(n)}\langle nC\rangle(-(1-u)nK_X)) \neq 0.$$

Then the inequality

$$\frac{(u+2d)^3}{(1+3d)^2} CK_X + \frac{1}{4}c_1^2 - c_2 \leq O\big(\frac{1}{n}\big)$$

holds.

Finally, we note that, if the hypothesis of the above lemma is satisfied, then the global sections are necessarily contained in a small subsheaf:

Lemma 3.7. *Put*

$$j = j(u,n) = \left\lceil \frac{u+2d}{1+2d} n \right\rceil,$$

where the symbol $\lceil * \rceil$ denotes the round-up of $*$, i.e. the smallest integer larger than or equal to $*$. Then:

$$\begin{aligned}\mathrm{H}^0(X, \Omega^{(n)}\langle nC\rangle(-(1-u)nK_X)) &= \mathrm{H}^0(X, \Omega^{(n)}\langle jC\rangle(-(1-u)nK_X)) \\ &= \mathrm{H}^0(X, \Omega^{(n)}\langle nC\rangle_{[j]}(-(1-u)nK_X)).\end{aligned}$$

If $1-u > (1/2) + \max\{0, g-1\}$, then, in view of Proposition 2.3, we have a stronger result:

Lemma 3.8. *Assume that $1-u > (1/2) + \max\{0, g-1\}$. Then*

$$\begin{aligned}&\mathrm{H}^0(X,\Omega^{(n)}\langle nC\rangle(-(1-u)nK_X)) \\ &\quad = \mathrm{H}^0\Big(X, \big(\Omega^{(n)}\langle nC\rangle_{[j]} \cap \Omega^{(n)}\langle -(j+\frac{n}{2})C\rangle(jC)\big)(-(1-u)nK_X)\Big)\end{aligned}$$

The proofs of Lemmas 3.7 and 3.8 are similar as that of Lemma 3.2.

4. Parabolic structure, slopes and Newton polygon

Let X be a smooth surface and $\mathcal{E}$ a vector bundle of rank two on X. Let $H = H_{\mathcal{E}}$ denote the tautological line bundle on the projective bundle $\pi : \mathbb{P}(\mathcal{E}) \to X$. Given an effective divisor D on $\mathbb{P}(\mathcal{E})$ such that $\pi(D) = X$, there exist a positive integer n and a $\mathbb{Q}$-divisor Δ on X such that D is linearly equivalent to $n(H + \pi^*\Delta)$; n is of course the mapping degree of the projection from D to X.

Let $C \subset X$ be a compact, irreducible, reduced curve (note that we did not assume that X is compact). A *parabolic structure* on $\mathcal{E}$ with respect to C is, by definition, an invertible subsheaf $\mathcal{L}_C \subset \mathcal{E}|_C$ of rank one such that $\mathcal{L}_C$ locally extends to a saturated invertible subsheaf of $\mathcal{E}$ at each point $p \in C$. (By a *saturated subsheaf*, we mean an subsheaf $\mathcal{L}$ such that $\mathcal{E}/\mathcal{L}$ is a torsion free $\mathcal{O}_X$-module.)

Let $\mathcal{L}_C$ be a parabolic structure of $\mathcal{E}$. Then, locally around a smooth point of C, $\mathcal{L}_C$ is a line subbundle of of $\mathcal{E}|_C$, and we can locally extend $\mathcal{L}_C \subset \mathcal{E}|_C$ to a subbundle $\mathcal{L}$. Let $\mathcal{L}'$ be the quotient $\mathcal{E}/\mathcal{L}$:

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}' \to 0.$$

Thus $\mathcal{E}$ is locally a direct sum $s_0\mathcal{O} \oplus s_1\mathcal{O}$, s_0 and s_1 being local bases of $\mathcal{L}$ and of $\mathcal{L}'$, respectively. We regard (s_0, s_1) as a homogeneous coordinate of the fibre $\simeq \mathbb{P}^1$ over $p \in X$.

Recall that a geometric realization of $\mathbb{P}(\mathcal{E})$ is the quotient of $\mathcal{E}^*\backslash$(zero-section) via the natural $\mathbb{C}^*$-action (fibrewise scalar multiplication). The subbundle $\mathcal{L}$, which is naturally identified with the annihilator of $\mathcal{L}'^* \subset \mathcal{E}^*$, defines a relative hyperplane divisor $\sim H - \pi^*L_1$, locally given by a single equation $s_0 = 0$. The extension $\mathcal{L}$ is not unique, so that the s_0 involves some indeterminacy apart from non-vanishing holomorphic multiplicative factor. However, such additional indeterminacy vanishes modulo the ideal of C.

An effective divisor $D \sim n(H + \pi^*\Delta)$ on $\mathbb{P}(\mathcal{E})$ is, locally around a given smooth point $p \in C$, defined by a homogeneous polynomial $\sum a_i s_0^{n-i} s_1^i$ in (s_0, s_1) of degree n with coefficients in $\mathcal{O}_{X,p}$. Though the choice of s_0 is fairly rigid, that of s_1 is almost arbitrary. One can easily check that, after a suitable linear transformation by an upper triangular matrix with entries in the holomorphic functions on X, the coefficient a_0 of s_0^n can be chosen to be one, unless D contains the fibre $\pi^{-1}(p)$.

In what follows, we always fix an irreducible curve C on X together with a parabolic structure $\mathcal{L}_C \subset \mathcal{E}|_C$. Given an effective divisor $D \subset \mathbb{P}(\mathcal{E})$, we assume that D does not contain any vertical component and normalize the first coefficient a_0 to 1 (at a general point of C).

Pick up a general point $p \in C$ and let (x, y) be a local coordinate system of X at p such that $x = 0$ defines C. The *minimum slope*, of an effective divisor D at p (with respective to the pair $(C, \mathcal{L}_C)$) is defined as the non-negative rational number

$$\begin{aligned}\sigma_{\min}(D) &= \sigma_{\min}(D; C, \mathcal{L}_C)\\ &= \min\{1, \min_i\{\frac{\operatorname{ord}_C a_i}{i}\}\},\end{aligned}$$

where D is given by the homogeneous equation $\sum a_i(x,y) s_0^{n-i} s_1^i$ and the symbol $\operatorname{ord}_C(*)$ is defined to be the integer α such that x^α exactly divides $*$ (i.e. the vanishing order of $*$ along C). The positive integer n is, in the meantime, called the *covering degree* of D. (The choice of local extensions $\mathcal{L}$ and its bases s_0 leaves the indeterminacy of $\operatorname{ord} a_i$ up to order i, so that $\sigma_{\min}(D)$ makes sense only up to one. In case the subsheaf $\mathcal{L}_C$ is globally given in the r-th infinitesimal neighbourhood of C, the minimal slope will be well-defined to be a rational number less than or equal to r.)

The zero-orders of the coefficients a_i also determine the *Newton polygon* $\operatorname{Newt}_{(C,\mathcal{L}_C;p)}(D)$ of the effective divisor $D \sim n(H + \pi^*\Delta)$ at p (with respect to the pair $(C, \mathcal{L}_C)$), which is, by definition, the convex closure of the union of the $n+1$ half lines

$$\{(T,S)\in\mathbb{R}^2; T=i, S\geq\min\{\operatorname{ord}a_i, i\}\},\quad i=0,1,\dots,n$$

in the (T,S)-plane $\simeq\mathbb{R}^2$.

One easily checks that the minimum slope, or the Newton polygon, of D is well-defined by the parabolic structure of $\mathcal{E}$, i.e. independent of the choice of $\mathcal{L}$, s_i, etc. Furthermore, they are independent also of the choice of a general point $p\in C$ (recall that C is irreducible).

The ring of holomorphic functions on X at a general point of $C\subset X$ is identified with the ring $\mathcal{O}_C\{\{x\}\}$ of the convergent power series in x with coefficients in $\mathcal{O}_C$. The field of meromorphic functions in a small neighbourhood of C is $\mathbb{C}(C)((x))$, the field of convergent Laurent series in x with coefficients in the function field of C. Its algebraic closure is the field of the *Puiseux series* $\overline{\mathbb{C}(C)}((x))[x^{1/\infty}]$ obtained by adjoining all the roots of x to the field of the Laurent series with coefficients in the algebraic closure of $\mathbb{C}(C)$. The ring of integers in this field is $\overline{\mathcal{O}_C}\{\{x\}\}[x^{1/\infty}]$, $\overline{\mathcal{O}_C}$ being the integral closure of $\mathcal{O}_C$ in $\overline{\mathbb{C}(C)}$.

Any analytically locally irreducible divisor $D\subset\mathbb{P}(\mathcal{E})$ over a small neighbourhood of $p\in C$ defines a finite extension of $\mathcal{O}_C\{\{x\}\}$. Therefore D can be written in the form

$$s_0^n+x^bs_2^n=0$$

or

$$x^bs_0^n+(1-x^b)(s_0-s_1)^n=0$$

after a suitable local analytic coordinate change at a general point $p\in C$. Here b is a positive integer with $(b,n)=1$. In this case, the *slope* of this locally irreducible divisor D at p is defined to be $\min\{1,b/n\}$ in the former case and 0 in the latter case and is denoted by $\sigma(D)$ (of course it coincides with $\sigma_{\min}(D)$ in this situation). The Newton polygon of D is the region given by $0\leq T\leq n, S\geq\sigma(D)T$ or $0\leq T\leq n, S\geq 0$, accordingly.

Given a general point p of C, the locally irreducible divisor D intersects $\pi^{-1}(p)$ at a single point. $\sigma(D)$ is positive if and only if the intersection point coincides $\pi^{-1}(p)\cap\{s_0=0\}$ at p or, equivalently, D passes through the curve $\pi^{-1}(C)\cap\{s_0=0\}$. The intersection multiplicity $D\cap\{s_1=0\}$ along this curve is at least $\min\{n\sigma(D),1\}$ (exactly $n\sigma(D)$ if $n\sigma(D)<1$).

Let D be an arbitrary effective divisor on $\mathbb{P}(\mathcal{E})$. Choose a general point $p\in C$ and decompose D into the sum of analytically locally irreducible components D_α near p. Let n_α and σ_α denote the covering degree and the slope of D_α. Choose the indices α such that σ_α is increasing.

Then we define the *slope function* $\sigma=\sigma_D:[0,n]\to\mathbb{R}$ to be the monotone increasing step function $\sigma(T)=\sigma(D_\alpha)$, if $n_1+\dots+n_{\alpha-1}<T\leq n_1+\dots+n_\alpha$.

Lemma 4.1. *Assume that X is projective. Let D be an irreducible divisor $\sim n(H+\pi^*\Delta)$ with covering degree $n\geq 2$. Let A be a general hyperplane* *section on X. Then*

$$\begin{aligned} nA(2\Delta + \det \mathcal{E}) &\geq \frac{1}{n-1} \sum_{i=1}^{n} \sigma_D(i)(I(i)-1)AC \\ &> \frac{1}{n} \sum_{i=1}^{n} \sigma(i)(I(i)-1)AC \\ &\geq AC\left(\frac{1}{n}\int_0^n \sigma(T)(n-T)dT - \varepsilon(D)\right), \end{aligned}$$

where $I(i)$, called the counting function, is the number of the integers j such that $\sigma(j) \geq \sigma(i)$, and $\varepsilon(D)$ is the correction term:

$$\varepsilon(D) = \begin{cases} 1 & \text{if } \sigma_D(n-1) < 1 \text{ and } \sigma_D(n) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For simplicity, we assume that $\det \mathcal{E}$ is numerically trivial. Take a suitable ramified Galois cover $Y \to X$ such that the inverse image of D decomposes into a union of sections S_i. Fix one irreducible component S_i once for all. Choose A generally so that A meets C and the branch locus of f transversally at smooth points. Let $q \in Y$ be an arbitrary point of the inverse image of $A \cdot C$. A small neighbourhood of S_i of q is mapped to a unique local irreducible component $D_{\alpha(i)}$. Hence S_i meets the other components S_j at $\{q\} \times \{s_0 = 0\}$ with intersection multiplicity $\geq \min\{N\sigma(D_{\alpha(i)}), N\sigma(D_{\alpha(j)}\}$, where N is the branch index of f along C. Hence if we write $S_i \sim H + \Delta_i$, we have

$$S_i(\sum_{j\neq i} S_j)\pi^* f^* A = (\sum_{j \neq i} \Delta_j + (n-1)\Delta_i)) f^* A \geq \sum_{i=1}^{m} \sigma(i)(I(i)-1) f^* A f^* C,$$

On the other hand, one can easily check that

(1) $\mathrm{NS}(Y) \otimes \mathbb{Q}$ is a direct sum of $\mathrm{NS}(X) \otimes \mathbb{Q}$ and its orthogonal complement, and that
(2) $\Delta_i - \Delta_j$ sits in the orthogonal compliment (here we use the fact that D is irreducible).

Now the assertion is streightforward, if one notices that the slope function σ_D takes value a/b at b or more integer points (here the integers a, b are of course assumed to have no common factor). □

Call D a *special irreducible divisor* if the correction term $\varepsilon(D)$ is non-zero.

Corollary 4.2. *Let $D \sim n(H + \pi^*\Delta)$ be an irreducible divisor of covering degree ≥ 2. Let $\overline{\sigma} = \overline{\sigma}(D)$ denote the mean slope:*

$$\overline{\sigma}(D) = \frac{1}{n}\int_0^n \sigma(T)dT.$$

Assume that the slope function $\sigma(T) = \sigma_D(T)$ is bounded from below by $\sigma_{\min} < 1$. Then the inequality

$$nA(2\Delta + \det \mathcal{E}) \geq \frac{n}{2} \cdot \frac{(\overline{\sigma} - \sigma_{\min})^2 + \sigma_{\min}(1-\sigma_{\min})}{1-\sigma_{\min}} AC + \frac{\varepsilon(D)}{2} AC$$

holds.

Proof. The slope function $\sigma(T)$ is monotone increasing with values between $\sigma_{\min}$ and 1, while the weight function $n - T$ is monotone decreasing. If the mean value $\overline{\sigma}$ is given, the integral

$$\int_0^n \sigma(T)(n-T)dT$$

takes the minimum value when $\sigma(T)$ is the two-step function

$$\sigma(T) = \begin{cases} \sigma_{\min}, & 0 \leq T \leq T_0, \\ 1, & T_0 < T \leq n. \end{cases}$$

Here T_0 is of course the solution of $\sigma_{\min} T_0 + (n - T_0) = n\overline{\sigma}$. By computing the integral of this step function, we get the estimate. □

Corollary 4.3. *Let D be an effective divisor on $\mathbb{P}(\mathcal{E})$ of which each irreducible component has covering degree ≥ 2. If $D \sim nH + \pi^*\Delta$ of average slope $\overline{\sigma}$ and minimum slope $\sigma_{\min} < 1$, then*

$$nA(2\Delta + \det \mathcal{E}) \geq \left[\frac{n}{2} \frac{(\overline{\sigma} - \sigma_{\min})^2 + \sigma_{\min}(1-\sigma_{\min})}{1-\sigma_{\min}} + \frac{\nu(D)}{2} \right] AC$$

where $\nu(D)$ denotes the number of special irreducible components contained in D.

Proof. Decompose D into irreducible components D_α, with covering degrees n_α, average slopes $\overline{\sigma}_\alpha$. Then we have

$$\begin{aligned} nA(2\Delta + \det \mathcal{E}) &= \sum_\alpha n_\alpha A(2\Delta_\alpha + \det \mathcal{E}) \\ &\geq \sum_\alpha \left[\frac{n_\alpha}{2} \frac{(\overline{\sigma}_\alpha - \sigma_{\min})^2 + \sigma_{\min}(1-\sigma_{\min})}{1-\sigma_{\min}} + \frac{\varepsilon(D_\alpha)}{2} \right] AC. \end{aligned}$$

The right hand side is bounded from below by

$$\left[\frac{n}{2} \frac{(\overline{\sigma} - \sigma_{\min})^2 + \sigma_{\min}(1-\sigma_{\min})}{1-\sigma_{\min}} + \frac{\nu(D)}{2} \right] AC,$$

by the Schwarz inequality. □

Note that the inequality above remains true after replacing A by an arbitrary ample or, more generally, nef, divisor. In particular, if $C^2 \geq 0$ (which implies that C is nef) and if $\deg \mathcal{L}_C$ is bounded from above, we have:

Corollary 4.4. *Assume that* $C^2 \geq 0$ *and that*

$$-2\deg \mathcal{L}_C + C\det \mathcal{E} \geq \frac{C^2}{2}.$$

Let $D \sim n(H + \pi^*\Delta)$ *be an arbitrary effective divisor, with slope function* $\sigma(x)$ *bounded from below by* $\sigma_{\min} > 0$*. Then the inequality*

$$nC(2\Delta + \det \mathcal{E}) \geq \frac{nC^2}{2} \cdot \frac{(\overline{\sigma} - \sigma_{\min})^2 + \sigma_{\min}(1 - \sigma_{\min})}{1 - \sigma_{\min}} + \frac{\nu(D)C^2}{2}$$

holds. Here $\overline{\sigma}$ *stands for the average slope of* D*.*

Proof. What is not covered by Corollary 4.3 is the contribution from the irreducible components of covering degree one, which is taken care of by the degree bound of $\mathcal{L}_C$. □

5. Proof of Theorem 1.1

Let X be a minimal complex surface and C an irreducible curve of genus g on it. Let n be a sufficiently large integer and let u denote the minimal rational number such that $\mathrm{H}^0(X, \Omega^{(n)}\langle nC\rangle(-(1-u)nK_X)$ stays non-vanishing.

We prove the theorem in three separate cases:

Case 0. $KC \leq \max\{2g-2, 3g-3\}$.
Case 1. $C^2 \leq 0$.
Case 2. $C^2 > 0$, $u \geq (1/2) - (g-1)/CK_X$ or $u \geq 1/2$.
Case 3. $C^2 > 0$, $u < (1/2) - (g-1)/CK_X$, $u < 1/2$.

Case 0 is a trivial case, and there is nothing to prove.

In what follows, we always assume that $KC > \max\{2g-2, 3g-3\}$.

Case 1 is taken care of by the following

Proposition 5.1. *If* $C^2 \leq 0$*, then the inequality*

$$CK_X \leq 6c_2(X) - 2c_1(X)^2 + 6g - 6$$

holds.

Proof. Take a minimal embedded resolution $\mu : \tilde{X} \to X$ such that the total transform of C is supported by a divisor with simple normal crossings $\overline{C} = \tilde{C} + F$. Here $\tilde{C}$ is the smooth strict transform of C and F is the union of the exceptional curves.

Apply the Bogomolov-Miyaoka-Yau-Sakai inequality to the pair $(\tilde{X}, \overline{C})$:

$$\begin{aligned}(K_{\tilde{X}} + \tilde{C} + F)^2 &\leq 3c_2(\tilde{X}) - 3e(\overline{C}) \\ &= 3c_2(X) - 3e(C).\end{aligned}$$

The topological Euler number $e(C)$ turns out to be equal to $2 - 2g - F\tilde{C} + \#\,\mathrm{Sing}(C)$ by easy calculation. In the meantime, the left-hand side is the sum of three terms: Namely, $\tilde{C}(\tilde{C} + K_{\tilde{X}})$, $F(F + K_{\tilde{X}}) + 2F\tilde{C}$ and $K_{\tilde{X}}(\overline{C} + K_{\tilde{X}})$.

The first term is nothing but $2g-2$.

Each connected component of F is a tree of $\mathbb{P}^1$ of arithmetic genus 0, so that $F(F+K_{\tilde{X}}) = -2\#\operatorname{Sing}(C)$, meaning that the second term is $-2e(C)-4g+4$.

Since our resolution μ is minimal, the third term is bounded from below by $CK_X + K_X^2$.

Thus the left-hand side is bounded from below by $CK_X + K_X^2 - 2g + 2 - 2e(C)$, and hence we get

$$CK_X \le 3c_2(X) - c_1(X)^2 - e(C) + 2g - 2.$$

It is easy to show the inequality

$$-e(X) \le \frac{1}{2}(C^2 + CK_X) + g - 1.$$

(The equality is attained if and only if C has only ordinary double points.) Noticing the hypothesis $C^2 \le 0$, we eventually get

$$CK_X \le 3c_2(X) - c_1(X)^2 + \frac{CK_X}{2} + 3g - 3,$$

whence the assertion follows. $\square$

In Case 2, we have the following

Proposition 5.2. (1) *If $C^2 > 0$ and $u \ge 1/2$, then*

$$\frac{(1 - \frac{4(g-1)}{CK_X})^3}{8(1 - \frac{3(g-1)}{CK_X})^2} CK_X \le c_2 - \frac{1}{4}c_1^2.$$

(2) *If $C^2 > 0$ and $u \ge (1/2) + g - 1$, then*

$$\frac{(1 - \frac{2(g-1)}{CK_X})^3}{8(1 - \frac{3(g-1)}{CK_X})^2} CK_X \le c_2 - \frac{1}{4}c_1^2.$$

Proof. $d \ge (1-g)/CK_X$ being the deficiency number, Corollary 3.6 tells us that

$$\frac{(u+2d)^3}{(1+3d)^2} CK_X \le c_2 - \frac{1}{4}c_1^2.$$

Therefore, if $u \ge 1/2$, then

$$\frac{(1+4d)^3}{8(1+3d)^2} CK_X \le c_2 - \frac{1}{4}c_1^2.$$

Putting $x = (1+3d)$, we rewrite the coefficient of CK_X into

$$\frac{8}{27} \cdot \frac{(x - \frac{1}{4})^3}{x^2},$$

which is, viewed as a function defined on the interval $(0, \infty)$, monotone increasing. However, since $d \ge (1-g)/CK_X$, the minimum possible value of x is $1-\big(3(g-1)/CK_X\big)$, and we obtain

$$\frac{(1-\frac{4(g-1)}{CK_X})^3}{8(1-\frac{3(g-1)}{CK_X})^2}CK_X \leq c_2 - \frac{1}{4}c_1^2.$$

Similarly, if $u \geq (1/2)+g-1$, we get

$$\frac{(1-\frac{2(g-1)}{CK_X})^3}{8(1-\frac{3(g-1)}{CK_X})^2}CK_X \leq c_2 - \frac{1}{4}c_1^2.$$

□

What is left is Case 3. We will prove

Proposition 5.3. *In Case 3, the inequality*

$$CK_X \leq 14^3(c_2 - \frac{1}{4}c_1^2)$$

holds.

Assume hereafter that $C^2 > 0$, $CK_X > \max\{2g-2, 3g-3\}$, $u < \min\{1/2, 1/2-(g-1)/CK\}$. The last condition is of course equivalent to $t = (1-u) > \max\{1/2, 1/2+(g-1)\}$.

These conditions imply that a non-zero global section of $\Omega^{(n)}\langle nC\rangle(-unK_X)$ is contained in a much smaller subsheaf. For instance, it is contained in

$$\Omega^{(n)}\langle -\frac{n}{2}C\rangle + (df/f)\Omega^{(n-1)}\langle -\frac{n}{2}C\rangle \\ + \cdots + (df/f)^{n/2}\Omega^{(n/2)}\langle -\frac{n}{2}C\rangle$$

in view of Proposition 2.3.

The subsheaf

$$\mathcal{L}_C = \mathrm{Ker}(\Omega \to \nu_*\omega_{\tilde{C}}) = (df\mathcal{O}_X + f\Omega)/f\Omega$$

defines a natural parabolic structure on Ω. When we speak about a parabolic structure below, we always mean this structure on Ω.

Lemma 5.4. *Let $D \subset \mathbb{P}(\Omega)$ be an irreducible effective divisor $\sim H + \pi^*\Delta$, i.e. D is a rational section of π. If the slope of D with respect to the parabolic structure of Ω is one, then $C\Delta \geq C^2$.*

Proof. $D \sim H + \pi^*\Delta$ is effective and irreducible if and only if $\mathcal{O}_X(-\Delta)$ is a saturated subsheaf of Ω. $\sigma(D) = 1$ if and only if the restriction of $\mathcal{O}_X(-\Delta)$ to C sits in the subsheaf $\mathcal{L}_C \simeq \mathcal{O}_C(-C)$. Now the assertion is trivial. □

Lemma 5.5. *Let $D \subset \mathbb{P}(\Omega)$ be an irreducible effective divisor $\sim n(H + \pi^*\Delta)$, $n \geq 2$. If D is a special irreducible divisor, then*

$$\mathcal{O}_X(-n\Delta) \hookrightarrow df\Omega^{(n-1)} + f\Omega^{(n)}.$$

Proof. By definition, at each smooth point p of C, there is a unique analytically local irreducible component D_0 which has slope one. This component is independent of the choice of p, so that we infer that D_0 defines an irreducible component of the restriction of D to a small analytic neighbourhood of C. This effective divisor D_0 clearly defines an invertible subsheaf of $df\Omega + f\Omega$ near C, so that D gives a subsheaf of $(df\Omega + f\Omega)\Omega^{(n-1)}$ near C. This implies the assertion. □

By dividing out the both sides by the defining equation f, we rephrase the lemma above in terms of the filtration introduced in Section 1.

Corollary 5.6. *Let $D \sim n(H + \pi^*\Delta)$ be an effective divisor. If $\nu = \nu(D)$ is the number of the special irreducible components of D, then the vector space $\mathrm{H}^0\big(X, \Omega^{(n)}\langle \nu C\rangle(-\nu C - n\Delta)\big)$ does not vanish.*

Fix a sufficiently large integer n. Let u be a positive rational number such that

1) un is an integer;
2) $\mathrm{H}^0\big(X, \Omega^{(n)}\langle nC\rangle(-n(1-u)K)\big) \neq 0$;
3) $\mathrm{H}^0\big(X, \Omega^{(n)}\langle nC\rangle(-(n(1-u)+a)K)\big) = 0$.

Here $a = a(X)$ is a fixed positive integer such that $|aK| \neq \emptyset$ (we may assume that $a = 1$ unless $p_g = 0$).

Let v be the smallest non-negative rational number such that

$$\mathrm{H}^0(X, \Omega^{(n)}\langle nC\rangle(-n(1-u)K)) = \mathrm{H}^0(X, \Omega^{(n)}\langle vnC\rangle(-n(1-u)K)).$$

Any non-zero global section of $\Omega^{(n)}\langle nvC\rangle(-n(1-u)K)$ defines a non-zero section of $\Omega^{(n)}(-(1-u)nK + vnC)$ or, equivalently, an effective divisor $D \sim n(H + \pi^*(vC - (1-u)K))$.

Lemma 5.7. *Assume that $C^2 > 0$. In the notation above, we have an estimate of the number of the special irreducible components of D:*

$$\nu(D) \leq nv + O(1).$$

Proof. Put $\nu = \nu(D)$. Then by Lemma 5.6, we get

$$\mathrm{H}^0(X, \Omega^{(n)}\langle \nu C\rangle(-\nu C - n(1-u)K + nvC)) \neq 0.$$

Since $C^2 > 0$, we can find a positive integer $c = c(X, C)$, independent of n, such that $|cC - K| \neq \emptyset$. Therefore, by replacing cC by K, we get

$$\mathrm{H}^0\big(X, \Omega^{(n)}\langle \nu C\rangle((nv - \nu + ac)C - (n(1-u) + a)K)\big) \neq 0.$$

This is possible only when $nv - \nu + ac \geq 0$, i.e. $\nu \leq nv + ac$. □

Proof of Proposition 5.3.

Let us recall results obtained in Sections 2 and 3 once again. We defined a rational number d by

$$\deg \mathfrak{k} = (1 + 2d)KC,$$

or, equivalently, $-2d$ is the degree of the image of the natural map $\nu^*\Omega \to \omega_{\tilde{C}}$. Then

(A) By Corollary 3.6, u, being defined as above, satisfies:

$$\frac{(u+2d)^3}{(1+3d)^2}KC + \frac{1}{4}K_X^2 - c_2(X) \le 0.$$

(B) By Lemma 3.8, v satisfies

$$v \le \frac{u+2d}{1+2d}.$$

Pick up a non-zero element from $\mathrm{H}^0(X, \Omega^{(n)}\langle vnC\rangle(-(1-u)nK))$. We can choose an element which is not contained in $\mathrm{H}^0(X, \Omega^{(n)}\langle (vn-1)C\rangle(-(1-u)nK))$. This non-zero element determines an effective divisor $D \simeq n(H + vC - (1-u)K)$ on $\mathbb{P}(\mathcal{E})$.

The Newton polygon of the divisor $D \subset \mathbb{P}(\Omega)$ is contained in the region given by

$$S \ge \begin{cases} T, & 0 \le T \le nv \\ vn, & nv \le T \le n/2 \\ T - \frac{n}{2} + vn, & n/2 \le T \le n. \end{cases}$$

This region is not convex. Its convex hull is given by

$$S \ge \begin{cases} 2vT, & 0 \le T \le n/2 \\ T - \frac{n}{2} + vn, & n/2 \le T \le n. \end{cases}$$

Hence the minimum slope $\sigma_{\min}$ and the average slope $\overline{\sigma}$ of D satisfy

$$\begin{aligned} \sigma_{\min} &\ge 2v \\ \overline{\sigma} &\ge v + \frac{1}{2}. \end{aligned}$$

By applying Corollary 4.4 and Lemma 5.7 to D, we get

$$\begin{aligned} 2vnC^2 &\ge 2(vnC - n(\frac{1}{2} - u)K)C \\ &\ge \frac{C^2}{n}\int_0^{n/2} \sigma_{\min}(n-x)dx + \frac{C^2}{n}\int_{n/2}^{n}(n-x)dx - \frac{\nu(D)}{2}C^2 \\ &\ge \frac{3vnC^2}{4} + \frac{n}{8}C^2 - \frac{vn}{2}C^2 + O(1) \\ &= \left(\frac{n}{4}v + \frac{n}{8}\right)C^2 + O(1). \end{aligned}$$

Thus we get

$$v \ge \frac{1}{14} + O(\frac{1}{n}),$$

provided $u \le \frac{1}{2}$. Hence by (B),

$$\frac{u+2d}{1+2d} \ge \frac{1}{14} + O(\frac{1}{n}),$$

or, equivalently,

$$u + 2d \geq \frac{1+2d}{14} + O(\frac{1}{n}).$$

Then by the property (A) above,

$$\frac{(1+2d)^3}{14^3(1+3d)^2} CK + \frac{1}{4}K^2 - c_2 \leq O(\frac{1}{n}).$$

The rational number d takes value in the interval $[\frac{1-g}{CK}, \infty)$. It is easy to see that the function $(1+2d)^3/(1+3d)^2$ defined on $(-1/3, \infty)$ is monotone decreasing on $(-1/3, 0]$ and monotone increasing on $[0, \infty)$, and its minimum value is 1. Thus we eventually arrive at the inequlity as stated.

6. Concluding remarks

Our result above is obtained by crude estimates and by no means optimal. The coefficient $14^3 = 2744$ in our bound for the canonical degree of rational/elliptic curves is indeed fairly large, but it is not absurdly large.

Example. T. Shioda showed that an elliptic surface birational to the affine surface

$$y^2 = x^3 + at^n x + bt^m, \ \ 2m \neq 3m, \ ab \neq 0$$

has Mordell-Weil rank at most 56 and the maximum 56 is attained if and only if

$$\begin{aligned} m &\equiv 0 \quad \mod 2 \\ 2m - 3n &\equiv 0 \quad \mod 2^3 \cdot 3^2 \cdot 5 \cdot 7 \end{aligned}$$

(see [14], Corollary 9). If we take $m = 17, n = 7$ and general coefficients $a, b \in \mathbb{C}$, the minimal model X of this surface is an elliptic surface of Kodaira dimension one and has numerical invariants $c_1^2 = 0, c_2 = 36$, with canonical divisor linearly equivalent to a single closed fibre of the elliptic fibration.

By a cyclic base change $\mathbb{P}^1 \to \mathbb{P}^1$, $u \mapsto t = u^r$, the Mordell-Weil rank increases, and attains the maximum if and only if r is divisible by $2^3 \cdot 3^2 \cdot 5 \cdot 7$. Hence if one takes a "general" section of the fibre product $\mathbb{P}^1 \times_{\mathbb{P}^1} X$, its image C on the original surface X is a rational curve which is a $2^3 \cdot 3^2 \cdot 5 \cdot 7$-sheeted cover of the base curve $\mathbb{P}^1$ or, equivalently,

$$CK_X = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 70 c_2(X).$$

Thus the coefficient of c_2 must be large in a general estimate of canonical degree of curves. However, once one imposes suitable constraints on the singularity of C, then one gets a much better estimate.

For instance,

Proposition 6.1. *Let X be a minimal surface and C an irreducible curve with only ordinary double points as its singularities. Suppose that $CK_X > 0$ (i.e. C is not a (-2)-curve.) Then*

$$CK_X - 6(g-1) - 8c_2(X) + 2c_1(X)^2$$

is bounded from above by a term of of order $O\big(\frac{|g-1|}{CK_X}\big)$. More precisely, we have:

(1) *If $g(C) = 0$, then $x = CK_X$ satisfies*

$$x + O(1/x) = \frac{(1+(2/x))^3}{(1+(3/x))^2}x \leq 8c_2(X) - 2c_1(X)^2.$$

(2) *If $g(C) = 1$, then*

$$CK_X \leq 8c_2(X) - 2c_1(X)^2.$$

(3) *If $g(C) = g \geq 2$, $x = CK_X$ satisfies one of the following two equations*

$$x \leq 6c_2(X) - 2c_1(X)^2 + 6g - 6,$$

$$x - 6(g-1) + O\left(\frac{g-1}{x}\right) = \frac{(1-4(g-1)/x)^3}{(1-3(g-1)/x)^2}x \leq 8c_2 - 2c_1^2.$$

To prove this statement, it suffices to show that Case 3 in the beginning of Section 5 does not occur. By our assumption that C has only nodes, one checks that

$$\Omega^{(n)}\langle -nC\rangle \simeq \operatorname{Sym}^n \Omega(\log C)(-nC).$$

If Case 3 occurs, then it follows that

$$\begin{aligned} &\mathrm{h}^0\big(X, \Omega^{(n)}\langle -nC\rangle(\frac{n}{2}(C-K))\big) \\ &\qquad = \mathrm{h}^0\big(X, \operatorname{Sym}^n(\Omega(\log C))(-\frac{n}{2}(C+K))\big) > O(n^2), \end{aligned}$$

which contradicts the Bogomolov-semistability of the sheaf of the logarithmic 1-forms [2].

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