Zagier's conjecture on $L(E,2)$

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1 Introduction

In this paper we introduce an elliptic analog of the Bloch-Suslin complex and prove that it (essentially) computes the weight two parts of the groups $K_2(E)$ and $K_1(E)$ for an elliptic curve E over an arbitrary field k. Combining this with the results of Bloch and Beilinson we proved Zagier's conjecture on $L(E,2)$ for modular elliptic curves over $\mathbb{Q}$.

1. The elliptic dilogarithm. The dilogarithm is the following multi-valued analytic function of on $\mathbb{C}P^1\backslash\{0,1,\infty\}$):

$$Li_2(z) = -\int_0^z \log(1-t)\frac{dt}{t}$$

It has a single-valued version, the Bloch-Wigner function:

$$\mathcal{L}_2(z) := ImLi_2(z) + \arg(1-z)\cdot\log|z|$$

The elliptic analog of the dilogarithm was defined and studied by Spencer Bloch in his seminal paper [Bl1].

The story goes as follows. Let $E(\mathbb{C}) = \mathbb{C}^*/q^{\mathbb{Z}}$ be the complex points of an elliptic curve E. Here $q := exp(2\pi i\tau), Im\tau > 0$.

The function $\mathcal{L}_2(z)$ has a singularity of type $|z|\log|z|$ near $z=0$. It satisfies the relation $\mathcal{L}_2(z) = -\mathcal{L}_2(z^{-1})$. So averaging $\mathcal{L}_2(z)$ over the action of the group $\mathbb{Z}$ on $\mathbb{C}^*$ generated by $z \longmapsto qz$ we get the convergent series:

$$\mathcal{L}_{2,q}(z) := \sum_{n\in\mathbb{Z}} \mathcal{L}_2(q^n z), \qquad \mathcal{L}_{2,q}(z^{-1}) = -\mathcal{L}_{2,q}(z)$$

This function can be extended by linearity to the set of all divisors on $E(\mathbb{C})$ setting $\mathcal{L}_{2,q}(P) := \sum_i n_i \mathcal{L}_{2,q}(P_i)$ for a divisor $P = \sum n_i(P_i)$.

2. The results on $L(E,2)$. Let $L(E,s) = L(h^1(E),s)$ be the Hasse-Weil L-function of an elliptic curve E over $\mathbb{Q}$. We will always suppose that an elliptic curve E has at least one point over $\mathbb{Q}$: zero for the addition law.

Let v be a valuation of a number field K, and h_v the corresponding canonical local height on $E(K)$. As usual $x \sim_{\mathbb{Q}^*} y$ means that $x = qy$ for a certain $q \in \mathbb{Q}^*$. Let $J = J(E)$ be the Jacobian of E.

Theorem 1.1 *Let E be a modular elliptic curve over $\mathbb{Q}$. Then there exists a $\mathbb{Q}$-rational divisor $P = \sum n_j(P_j)$ over $\bar{\mathbb{Q}}$ which satisfy the conditions a)-c) listed below and such that*

$$L(E,2) \sim_{\mathbb{Q}^*} \pi \cdot \mathcal{L}_{2,q}(P) \tag{1}$$

The conditions on divisor P:

a)
$$\sum n_j P_j \otimes P_j \otimes P_j = 0 \quad in \quad S^3 J(\bar{\mathbb{Q}}) \tag{2}$$

b) For any valuation v of the field $\mathbb{Q}(P)$ generated by the coordinates of the points P_j

$$\sum n_j h_v(P_j) \cdot P_j = 0 \quad in \quad J(\bar{\mathbb{Q}}) \otimes \mathbb{R} \tag{3}$$

c) For every prime p where E has a split multiplicative reduction one has an integrality condition on P, see (4) below.

The integrality condition . Suppose E has a split multiplicative reduction at p with N-gon as a special fibre. Let L be a finite extention of $\mathbb{Q}_p$ of degree $n = ef$ and $\mathcal{O}_L$ the ring of integers in L. Let E^0 be the connected component of the Néron model of E over $\mathcal{O}_L$. Let us fix an isomorphism $E^0_{F_{p^f}} = \mathbb{G}_m/F_{p^f}$. It provides a bijection between $\mathbb{Z}/eN\mathbb{Z}$ and the components of $E_{F_{p^f}}$. For a divisor P such that all its points are defined over L denote by $d(P;\nu)$ the degree of the restriction of the flat extension of a divisor P to the ν'th component of the (eN)-gon.

Let $B_3(x) := x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$ be the third Bernoulli polynomial.

The integrality condition at p is the following condition on a divisor P, provided by the work of Schappaher and Scholl ([SS]). For a certain (and hence for any, see s. 3.3) extention L of $\mathbb{Q}_p$ such that all points of the divisor P are defined over L one has ($[L : \mathbb{Q}_p] = ef$):

$$\sum_{\nu \in \mathbb{Z}/(eN)\mathbb{Z}} d(P;\nu) B_3(\frac{\nu}{eN}) = 0 \tag{4}$$

Remarks. 1. For a p-adic valuation v of the field $K(P)$ one has $(\log p)^{-1} h_v(P_j) \in \mathbb{Q}$. So the condition b) in this case looks as follows

$$(\log p)^{-1} \sum n_j h_v(P_j) \cdot P_j = 0 \quad \text{in} \quad J(K(P)) \otimes \mathbb{Q} \tag{5}$$

In particular the right hand side is a finite dimensional $\mathbb{Q}$-vector space.

2. Lemma 1.5 below shows that, assuming (2), if the condition (3) is valid for all archimedean valuations but one then it is valid for all of them. In particular if $P \in \mathbb{Z}[E(\mathbb{Q})]$ we can omit (3) for the archimedean valuation.

The proof of theorem (1.1) is based on the results of S. Bloch [Bl1] on regulators on elliptic curves, a "weak" version of Beilinson's conjecture for modular curves proved by A.A. Beilinson in [B2] and the results presented in s.2-3 below.

To prove the theorem we introduce for an elliptic curve E over an *arbitrary* field k a new complex (the elliptic motivic complex $B(E;3)$) and prove that its cohomology essentially computes the weight 2 parts of $K_2(E)$ and $K_1(E)$ (see theorems (1.5) and (3.8)). This complex mirrors the properties of the elliptic dilogarithm. It is an elliptic deformation of the famous Bloch-Suslin complex which computes $K_3^{ind}(F) \otimes \mathbb{Q}$ and $K_2(F)$ for an arbitrary field F (see [DS], [S] and s.1.6).

In particular we replace the "arithmetical" condition b) by its refined "geometrical" version (see s. 1.4), which is equivalent to the condition b) for curves over number fields.

Our results imply

Theorem 1.2 *Let E be an elliptic curve over $\mathbb{Q}$. Then*

i) For any element $\gamma \in K_2(E)$ there exists a $\mathbb{Q}$-rational divisor P on E satisfying the conditions a), b) from theorem (1.1) such that the value of the Bloch-Beilinson regulator map $r_2 : K_2(E) \longrightarrow \mathbb{R}$ on γ is $\sim_{\mathbb{Q}^*} \mathcal{L}_{2,q}(P)$

ii) For any $\mathbb{Q}$-rational divisor P on E satisfying the conditions a), b) there exists an element $\gamma \in K_2(E) \otimes \mathbb{Q}$ such that $r_2(\gamma) \sim_{\mathbb{Q}^*} \mathcal{L}_{2,q}(P)$.

Theorem (1.2ii) implies immediately

Corollary 1.3 *Let E be an elliptic curve over $\mathbb{Q}$. Let us assume that the image of $K_2(E)_{\mathbb{Z}} \otimes \mathbb{Q}$ under the regulator map is $L(E,2) \cdot \mathbb{Q}$. (This is a part of the Bloch-Beilinson conjecture).*

Then for any $\mathbb{Q}$-rational divisor P on $E(\bar{\mathbb{Q}})$ satisfying the conditions a) - c) of theorem (1.1) one has

$$q \cdot L(E,2) = \pi \cdot \mathcal{L}_{2,q}(P)$$

where q is a rational number, perhaps equal to 0.

Remark. Corollary (1.3) has an analog for an elliptic curve over any number field. Its formulation is an easy exercise to the reader.

Unlike in Zagier's conjecture on ζ-functions of number fields one can not expect $P_i \in E(\mathbb{Q})$: the Mordell-Weil group of an elliptic curve over $\mathbb{Q}$ could be trivial.

The conditions a)-b) are obviously satisfied if P is (a multiple of) a torsion divisor. Moreover, if E is a curve with complex multiplication then $L(E,2)$ is the value of the elliptic dilogarithm on a torsion divisor ([Bl1]). However if E is not a CM curve this should not be true in general. Thus one has to consider the non-torsion divisors, and so it is necessary to use the conditions a)-b) in full strength.

The conditions a) and b) were guessed by D. Zagier several years ago after studying the results of the computer experiments with $\mathbb{Q}$-rational points on some elliptic curves, which he did with H. Cohen.

3. A numerical example. E is given by equation $y^2 - y = x^3 - x$. The discriminant Δ = conductor = 37. So E has split multiplicative reduction at $p = 37$ with one irreducible component of the fiber of the Néron model. Therefore the integrality condition is empty.

Local nonarchimedean heights on E. Let $P = [a/p^{2\delta}, b/p^{3\delta}] \in E(\mathbb{Q})$ where a, b are prime to p. If p is prime to Δ then $h_p(P) = 0$ if $\delta \leq 0$ and $h_p(P) = \delta \cdot \log p$ if $\delta > 0$. The local height at $p = 37$ is given by $h_{37}(P) = -1/6 + 2\delta$ (see the formula for the local height in s. 4.3 of or [Sil]).

The Mordell-Weil group has rank one and is generated by the point $P = [0,0]$. Consider the following integral points on E:

$$P = [0,0], \quad 2P = [1,0], \quad 3P = [1,1], \quad 4P = [2,3], \quad 6P = [6,-14]$$

and also

$$5P = [\frac{1}{4}, \frac{5}{8}], \quad 10P = [\frac{161}{16}, \frac{2065}{64}]$$

There are no height conditions at $p \neq 37$ for the integral points and there is just one at $p = 2$ for the points $5P$ and $10P$.

Consider the divisor $\sum n_k(kP)$. Notice that $S^3J(\mathbb{Q}) = \mathbb{Z}$ and the condition a) is $\sum n_k \cdot k^3 = 0$. The height condition at $p = 37$ gives $\sum n_k \cdot k = 0$ provided that the coordinates of (kP) are prime to 37.

The divisor

$$P_k = (kP) - k(P) - \frac{k^3 - k}{6}((2P) - 2(P))$$

satisfies the conditions $\sum n_k \cdot k = \sum n_k \cdot k^3 = 0$. Also $P_{10} - 4 \cdot P_5$ satisfies the height condition at $p = 2$.

The computer calculation (using PARI) shows

$$\frac{8\pi \cdot \mathcal{L}_{2,q}(P_3)}{37 \cdot L(E,2)} = -8.0000..., \qquad \frac{8\pi \cdot \mathcal{L}_{2,q}(P_4)}{37 \cdot L(E,2)} = -26.0000...,$$

$$\frac{8\pi \cdot \mathcal{L}_{2,q}(P_6)}{37 \cdot L(E,2)} = -90.0000..., \qquad \frac{8\pi \cdot \mathcal{L}_{2,q}(P_{10} - 4 \cdot P_5)}{37 \cdot L(E,2)} = -248.0000...$$

4. The group $B_2(E)$ and a refined version of conditions a) - b). Let E be an elliptic curve over an arbitrary field k and $J := J(k)$ be the group of k-points of the Jacobian of E. Let $\mathbb{Z}[X]$ be the free abelian group generated by a set X. We will define in s. 2.1 a group $B_2(E/k) = B_2(E)$ such that

a) one has an exact sequence

$$0 \longrightarrow k^* \longrightarrow B_2(E/k) \xrightarrow{p} S^2J(k) \longrightarrow 0 \tag{6}$$

b) one has a canonical (up to a choice of a sixth root of unity) surjective homomorphism

$$h : \mathbb{Z}[E(k)\backslash 0] \longrightarrow B_2(E/k) \tag{7}$$

whose projection to $S^2J(k)$ is given by the formula $\{a\} \longmapsto a \cdot a$.

c) if K is a local field there is a canonical homomorphism

$$H : B_2(E/K) \longrightarrow \mathbb{R}$$

whose restriction to the subgroup $K^* \subset B_2(E/K))$ is given by $x \longmapsto \log|x|$, (see s. 2.3). Moreover the canonical local height h_K is given by the composition

$$\mathbb{Z}[E(K)\backslash 0] \xrightarrow{h} B_2(E/K) \xrightarrow{H} \mathbb{R}$$

The group $B_2(E)$ appears naturally as a version of the theory of biextensions. It is a "motivic" version of theta functions. Set $\{a\}_2 := h(\{a\}) \in B_2(E)$.

The conditions a)-b) on a divisor $\sum_j n_j(P_j)$ are equivalent to the following single one:

$$\sum n_j\{P_j\}_2 \otimes P_j = 0 \quad \text{in} \quad B_2(E(\bar{\mathbb{Q}})) \otimes J(\bar{\mathbb{Q}}) \tag{8}$$

More precisely,

Lemma 1.4 *Let K be a number field and $P_j \in E(K)$. Then*

$$\sum n_j\{P_j\}_2 \otimes P_j = 0 \quad \textit{in} \quad B_2(E(K)) \otimes J(K) \otimes \mathbb{Q}$$

if and only if the following two conditions hold:

$$\sum n_j P_j \otimes P_j \otimes P_j = 0 \quad \textit{in} \quad S^3 J(K) \otimes \mathbb{Q}$$

and for any valuation v of the field K

$$\sum n_j h_v(P_j) \cdot P_j = 0 \quad \textit{in} \quad J(K) \otimes_{\mathbb{Q}} \mathbb{R}$$

Proof. Multiplying the exact sequence (6) by $J(K) \otimes \mathbb{Q}$ we get

$$0 \longrightarrow K^* \otimes J(K) \otimes \mathbb{Q} \longrightarrow B_2(E(K)) \otimes J(K) \otimes \mathbb{Q} \stackrel{p\otimes id}{\longrightarrow} S^2 J(K) \otimes J(K) \otimes \mathbb{Q} \longrightarrow 0$$

and use the fact that the local norms $|\cdot|_v$ separate all the elements in $K^* \otimes \mathbb{Q}$.

5. The elliptic motivic complex. Let us suppose first that k is an algebraically closed field. In chapter 3 we define a subgroup $R_3(E) \subset \mathbb{Z}[E(k)]$. When $k = \mathbb{C}$ it is a subgroup of all functional equations for the elliptic dilogarithm. In particular the homomorphism

$$\mathcal{L}_{2,q} : \mathbb{Z}[E(\mathbb{C})] \longrightarrow \mathbb{R}, \quad \{a\} \longmapsto \mathcal{L}_{2,q}(a)$$

annihilates the subgroup $R_3(E/\mathbb{C})$. Consider the homomorphism ($J := J(k)$)

$$\delta_3 : \mathbb{Z}[E(k)] \longrightarrow B_2(E) \otimes J, \quad \{a\} \longrightarrow -\frac{1}{2}\{a\}_2 \otimes a$$

An important result (theorem (3.3)) is that $\delta_3(R_3(E)) = 0$. Setting

$$B_3(E) := \frac{\mathbb{Z}[E(k)]}{R_3(E)}$$

we get a homomorphism $\delta_3 : B_3(E) \longrightarrow B_2(E) \otimes J$. Let us consider the following complex

$$B(E;3): \qquad B_3(E) \xrightarrow{\delta_3} B_2(E) \otimes J \longrightarrow J \otimes \Lambda^2 J \longrightarrow \Lambda^3 J \tag{9}$$

Here the middle arrow is $\{a\}_2 \otimes b \longmapsto a \otimes a \wedge b$ and the last one is the canonical projection. The complex is placed in degrees $[1, 4]$. It is acyclic in the last two terms. This is our elliptic motivic complex.

Let I_E be the augmentaion ideal of the group algebra $\mathbb{Z}[E]$, and I_E^4 its fourth pour. Let $B_3^*(E)$ be the quotient of I_E^4 by the subgroup generated by the elements $(f) * (1-f)^-$, where $*$ is the convolution in the group algebra $\mathbb{Z}[E]$, $f \in k(E)^*$, and $g^-(t) := g(-t)$. Then there is a homomorphism

$$\delta_3 : B_3^*(E) \longrightarrow k^* \otimes J \tag{10}$$

which fits the following commutative diagram

$$\begin{array}{ccccccc}
0 & & 0 & & & & \\
\downarrow & & \downarrow & & & & \\
B_3^*(E) & \longrightarrow & k^* \otimes J & & & & \\
\downarrow & & \downarrow & & & & \\
B_3(E) & \xrightarrow{\delta_3} & B_2(E) \otimes J & \longrightarrow & J \otimes \Lambda^2 J & \longrightarrow & \Lambda^3 J \\
\downarrow & & \downarrow & & \downarrow = & & \downarrow = \\
S^3 J & \longrightarrow & S^2 J \otimes J & \longrightarrow & J \otimes \Lambda^2 J & \longrightarrow & \Lambda^3 J \\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}$$

where the vertical sequences are exact, and the bottom one is the Koszul complex, and thus also exact. Let us denote by $B^*(E;3)$ the complex (10). It is canonically quasiisomorphic to the complex $B(E;3)$.

The complex $B^*(E;3)$ looks simpler then $B(E;3)$. However a definition of the differential in $B^*(E;3)$ which does not use the embedding to $B(E;3)$ is rather awkward, see s. 4.6.

If k is not algebraically closed we postulate the Galois descent property:

$$B(E/k;3) := B(E/\bar{k};3)^{Gal(\bar{k}/k)}; \qquad B^*(E/k,3) := B^*(E/\bar{k};3)^{Gal(\bar{k}/k)}$$

6. Relation with algebraic K-theory. Let k be an arbitrary field. Let $\mathcal{K}_2$ be the sheaf of K_2 groups in the Zariski topology on E. One has canonical inclusion $K_2(k) \hookrightarrow H^0(E, \mathcal{K}_2)$ and surjective projection $H^1(E, \mathcal{K}_2) \to k^*$.

Theorem 1.5 *Let $k = \bar{k}$. Then there is a sequence*

$$Tor(k^*, J) \hookrightarrow \frac{H^0(E, \mathcal{K}_2)}{K_2(k)} \longrightarrow B_3^*(E) \longrightarrow k^* \otimes J \longrightarrow Ker(H^1(E, \mathcal{K}_2) \to k^*) \to 0$$

It is exact in the term $k^* \otimes J$ and exact modulo 2-torsion in the other terms.

For an abelian group $A(E)$ depending functorially on E let $A(E)^-$ be the subgroup of skewinvariants under the involution $x \to -x$ of E. Recall that one has the γ-filtration on the Quillen K-groups. One can show that modulo 2-torsion

$$\frac{H^0(E, \mathcal{K}_2)}{K_2(k)} = gr_2^\gamma K_2(E)^-, \qquad Ker(H^1(E, \mathcal{K}_2) \to k^*) = gr_2^\gamma K_1(E)^-$$

Recall that the Bloch-Suslin complex for an arbitrary field k is defined as follows:

$$B_2(k) \xrightarrow{\delta} \Lambda^2 k^*; \qquad B_2(k) := \frac{\mathbb{Z}[k^*]}{R_2(F)}; \quad \delta : \{x\} \longmapsto (1-x) \wedge x$$

Here $R_2(k)$ is the subgroup generated by the elements $\sum_i (-1)^i \{r(x_1, ..., \hat{x}_i, ..., x_5)\}$, where x_i runs through all 5-tuples of distinct points over k on the projective line and r is the cross ratio.

One should compare theorem (1.5) with the following exact sequence provided by Suslin's theorem on $K_3^{ind}(k)$ ([S]) and Matsumoto's theorem on $K_2(k)$ ([M]) (see also a closely related results by Dupont and Sah [DS]):

$$0 \longrightarrow \tilde{T}or(k^*, k^*) \longrightarrow K_3^{ind}(k) \longrightarrow B_2(k) \xrightarrow{\delta} \Lambda^2 k^* \longrightarrow K_2(k) \longrightarrow 0$$

Here $\tilde{T}or(k^*, k^*)$ is a nontrivial extension of $Tor(k^*, k^*)$ by $\mathbb{Z}/2\mathbb{Z}$.

Remark. To guess an elliptic analog of the Steinberg relation $(1-x) \otimes x \in \mathbb{C}^* \otimes \mathbb{C}^*$ we might argue as follows. $E(\mathbb{C}) = \mathbb{C}^*/q^{\mathbb{Z}}$, so let us try to make sence out of $\sum_{n \in \mathbb{Z}} (1 - q^n x) \otimes q^n x$. Let $p : \mathbb{C}^* \to E(\mathbb{C})$. Projecting $\mathbb{C}^* \otimes \mathbb{C}^*$ to $\mathbb{C}^* \otimes J(\mathbb{C})$ we get $\prod_{n \in \mathbb{Z}} (1 - q^n x) \otimes x$, $x \in E(\mathbb{C})$. Regularizing the infinite product we obtain $\theta(x) \otimes x$ where

$$\theta(x) := q^{1/12} z^{-\frac{1}{2}} \prod_{j \geq 0} (1 - q^j z) \prod_{j > 0} (1 - q^j z^{-1})$$

Unfortunately this seems to make no sence: $\theta(x)$ is not a function on $E(\mathbb{C})$ with values in $\mathbb{C}^*$, but a section of a line bundle. Only introducing the *group*

$B_2(E/\mathbb{C})$ and realizing that $\theta(x)$ *is a function on* $E(\mathbb{C})$, *but with values in the group* $B_2(E/\mathbb{C})$, we find the elliptic analog of the Steinberg relation: $\theta(x)\otimes x \in B_2(E/\mathbb{C})\otimes J(\mathbb{C})$ (compare with s. 4.1, 4.2, 4.6). For an arbitrary base field $\{x\}_2 \in B_2(E)$ replaces $\theta(x)$.

In [W] J.Wildeshaus, assuming standard conjectures about mixed motives, gave a conjectural inductive definition of groups similar to $B_n(E)$ and discuss an elliptic analog of weak version of Zagier's conjecture.

Acknowledgement. This paper was essentially written when the first author enjoyed the hospitality of MPI(Bonn) and IHES during the Summer of 1995, and completed when the authors met in 1996 at the MPI (Bonn). The generous support of both institutions gratefully acknowledged.

A.G. was partially supported by the NSF Grant DMS-9500010 and A.L. by the Soros International Scientific Foundation.

It is our pleasure to thank Don Zagier and Maxim Kontsevich for interesting discussions. We are very grateful to the referee who made a lot of useful remarks and pointed out some errors.

2 The group $B_2(E)$

1. A construction of the group $B_2(X)$. Let k be an arbitrary field.

For any two degree zero line bundles L_1 and L_2 on a regular curve X over k let us define, following Deligne ([De]), a k^*-torsor $[L_1, L_2]$.

Motivation. Let s_i be a section of a line bundle L_i. If $div(s_2) = \sum m_i P_i$ then $< s_1, s_2 >$ is $s_1(div(s_2)) \in \otimes L_1^{m_i}|_{p_i} = [L_1, L_2]$. Such a tensor product turns out to be symmetric and does not depend on the choice of s_2. If $L_1 = \mathcal{O}$ then $[L_1, L_2] = k^*$.

For $f \in k(X)^*$ and a closed point $x \in X_1$ set $\bar{f}(x) := Nm_{k(x)/k} f(x)$. We extend $\bar{f}$ by linearity to the group of k-rational divisors on E.

Definition. The elements of $[L_1, L_2]$ are pairs $< s_1, s_2 >$, where s_i is a section of the line bundle L_i and the divisors $div(s_1)$ and $div(s_2)$ are disjoint. For a rational function f, g such that $div(f)$ is disjoint from $div(s_2)$ and $div(g)$ is disjoint from $div(s_1)$, one has $((s) := div(s))$:

$$< f\cdot s_1, s_2 >= \bar{f}((s_2)) < s_1, s_2 >, \quad < s_1, g\cdot s_2 >= \bar{g}((s_1)) < s_1, s_2 >$$

There are two *a priory* different expansions

$$< f\cdot s_1, g\cdot s_2 >= \bar{f}((g))\bar{f}(s_2) < s_1, g\cdot s_2 >= \bar{f}((g))\bar{f}((s_2))\bar{g}((s_1)) < s_1, s_2 > \tag{11}$$

and

$$< f\cdot s_1, g\cdot s_2 >= \bar{g}((f))\bar{g}((s_1)) < f\cdot s_1, s_2 >= \bar{g}((f))\bar{g}((s_1))\bar{f}((s_2)) < s_1, s_2 > \tag{12}$$

The right hand sides coincide thanks to the Weil reciprocity $\bar{f}((g)) = \bar{g}((f))$. So the k^*-torsor $[L_1, L_2]$ is well defined.

There is a canonical isomorphism of k^*-torsors

$$[L_1 \otimes L_2, M] \longrightarrow [L_1, M] \otimes_{k^*} [L_2, M]$$

and a similar additivity isomorphism for the second divisor. Further, the two possible natural isomorphisms

$$[L_1 \otimes L_2, M_1 \otimes M_2] \longrightarrow \otimes_{1\leq i,j\leq 2}[L_i, M_j]$$

coincide.

Therefore for any element $s \in S^2 J_X$ one gets a k^*-torsor $[s]$ defined up to an isomorphism. Moreover $[s_1 + s_2] = [s_1] \otimes_{k^*} [s_2]$.

The above facts just mean that the collection of k^*-torsors $[s]$ defines an extension of type (6). This is the definition of the group $B_2(X)$.

To construct the homomorphism (7) we want to make sense of elements $< s_1, s_2 >$ where the divisors of the sections s_i *may not be disjoint*. The definition is suggested by the motivation given above. Namely

$$< s_1, s_2 > \quad \in \quad [L_1, L_2] \otimes_{k^*} V(div(s_1), div(s_2)) \tag{13}$$

where $V(L_1, L_2)$ is a k^*-torsor defined as follows. Recall that if $A \to B$ is a homomorphism of abelian groups and T is an A-torsor, we can define a B-torsor $f_*T := T \otimes_A B$ where A acts on B via the homomorphism f. Thus for a $k(x)^*$-torsor T_x^*X we define a k^*-torsor $N(T_x^*X)$ using the norm homomorphism $Nm : k(x)^* \to k^*$.

For two arbitrary divisors l_1 and l_2 on X consider the following k^*-torsor

$$V(l_1, l_2) := \otimes_{x\in X_1} N(T_x^*X)^{\otimes ord_x l_1 \cdot ord_x l_2}$$

Here $ord_x l$ is the multiplicity of the divisor l at the point x. One has a canonical isomorphism $V(l_1 + l_2, m) \longrightarrow V(l_1, m) \otimes V(l_2, m)$. If the divisors l_1 and l_2 are disjoint then $V(l_1, l_2) = k^*$.

Any rational function f provides a canonical isomorphism

$$\varphi_f : V(l_1, l_2) \longrightarrow V((f) + l_1, l_2) \qquad v \longmapsto \tilde{f}(l_2) \cdot v$$

In this formula $\tilde{f}(x) \in N(T_x^*X)^{\otimes ord_x(f)}$ is the "leading term" of the function f at x. It is defined as follows. Choose a local parameter t at the point x. If $f(t) = at^k +$ higher order terms then $\tilde{f}(x) = a(dt)^k \otimes 1 \in N(T_x^*X)^k = (T_x^*X)^k \otimes k^*$. So $\tilde{f}(x)^{ord_x l_2} \in N(T_x^*X)^{\otimes ord_x(f) \cdot ord_x l_2}$ and the formula above makes sense.

Let us recall the full version of the Weil reciprocity law. Let

$$\partial_x : \{f, g\} \longmapsto (-1)^{ord_x(f) \cdot ord_x(g)} \frac{f(x)^{ord_x(g)}}{g(x)^{ord_x(f)}} \tag{14}$$

be the tame symbol. Then for *any* two rational functions f, g on a curve over an algebraically closed field k one has $\prod_{x \in X} \partial_x(f, g) = 1$.

There exists a canonical isomorphism of k^*-torsors

$$S : [L_1, L_2] \otimes V((s_1), (s_2)) \longrightarrow [L_2, L_1] \otimes V((s_2), (s_1))$$

given on generators by the formula

$$S : \quad < s_1, s_2 > \quad \longmapsto \quad (-1)^{degL_1 \cdot degL_2 + \sum_{x \in X_1} ord_x s_1 \cdot ord_x s_2} < s_2, s_1 >$$

The defining properties of the torsor $[L_1, L_2]$ look as follows:

$$< f \cdot s_1, s_2 >= \bar{f}((s_2)) < s_1, s_2 >, \quad < s_1, g \cdot s_2 >= S(< s_1, (g) >) < s_1, s_2 >$$

The formula

$$S < (f), (g) >=< (f), (g) > \tag{15}$$

is equivalent to the Weil reciprocity. Similar to (11), (12) and using the formula (15) we see that S is well defined.

2. The homomorphism (7). From now on $X = E$ is an elliptic curve over k, so there is canonical isomorphism $T_x^*E = T_0^*E$. For $c \in k^*$ one obviously has $< c \cdot s_1, s_2 >=< s_1, s_2 >$. Therefore we may consider $< s_1, s_2 >$ when s_1, s_2 are divisors on E. Consider a map

$$\{a\} \longmapsto < (a) - (0), (a) - (0) > \in (T_0^*E)^{\otimes 2} \otimes_{k^*} [a \cdot a]$$

There is an almost canonical (up to a sixth root of unity) choice of an element in $(T_0^*E)^{\otimes 2}$. Namely, the quotient of E by the involution $x \longmapsto -x$ is isomorphic to P^1. The image of 0 on E is the point ∞ on P^1. The images of the three nonzero 2-torsion points on E gives 3 distinguished points on P^1. Their ordering = choice of a level 2 structure on E. A choice of ordering gives a *canonical* coordinate t on $\mathbb{A}^1 := P^1 \backslash \infty$ for which $t = 0$ is the first point and $t = 1$ is the second. This coordinate provides a vector in $T^*_\infty P^1$ and so a vector in $(T^*_0 E)^2$. Therefore we have six different trivializations of $(T^*_0 E)^2$ and thus a canonical one of $(T^*_0 E)^{12}$: their product. The sixth root of ($16\times$ this trivialization) is the (almost) canonical element in $(T^*_0 E)^2$ we need.

Using this element in $(T^*_0 E)^2$ we get a map (7). The composition $\mathbb{Z}[E(k)\backslash 0] \longrightarrow B_2(E) \longrightarrow S^2 J$ is obviously given by $\{a\} \longmapsto a \cdot a$.

If E is written in the Weierstrass form $y^2 = (x - e_1)(x - e_2)(x - e_3)$ then e_1, e_2, e_3 are the coordinates of the distinguished points and $\frac{x - e_i}{e_j - e_k}$ is the canonical coordinate corresponding to the ordering e_i, e_j, e_k. Let Δ be the discriminant of E. Then $\Delta = 16 \cdot \prod_{i<j}(e_i - e_j)^2$. The canonical trivialization is $\Delta^{1/6}(dx/2y)^2$.

3. The canonical height. In this section we recall the construction of the canonical local heights via the biextension (compare with [Za] and [Bl2]). The canonical local height gives a homomorphism $B_2(K) \longrightarrow \mathbb{Q}$ (resp. to $\mathbb{R}$) when K is a nonarchimedean (resp. archimedean) local field.

The construction of the group $B_2(E)$ provides us with a collection of k^*-torsors $T_{(x,y)}$ where (x, y) is a point of $J \times J$. The torsors $T_{(0,y)}$ and $T_{(x,0)}$ are trivialized: we have a distinguished element $< \emptyset, (y) - (0) > \in T_{(0,y)}$ and a similar one in $T_{(x,0)}$.

The collection of torsors $T_{(x,y)}$ glue to a k^*-bundle on $J \times J$. It is isomorphic to the (rigidified) Poincaré line bundle minus zero section.

Let L be a degree zero line bundle on an elliptic curve over k. Let us denote by L^* the complement of the zero section in L. It is a principal homogeneous space over a certain commutative algebraic group $A(L)$ over k which is an extension

$$0 \longrightarrow \mathbb{G}_m \longrightarrow A(L) \longrightarrow J \longrightarrow 0$$

The group $A(L)$ is described as follows. For any $a \in E$ let $t_a : x \longmapsto a + x$ be the shift by a. Then $t_a^* L$ is isomorphic to L (because L is of degree zero). The set of isomorphisms from L to $t_a^* L$ form a k^*-torsor. These torsors glue together to a k^*-torsor over E which is isomorphic to L^*. On the other hand the collection of isomorphisms $L \longrightarrow t_a^* L$ form a group. This is the group $A(L)$. It is commutative: the commutator provides a morphism from $E \times E$ to $\mathbb{G}_m$, which has to be a constant map.

Now let K be a local field and E be an elliptic curve over K. We get a group extension

$$0 \longrightarrow K^* \longrightarrow A(L)(K) \longrightarrow J(K) \longrightarrow 0$$

Let $U(G)$ be the maximal compact subgroup of a locally compact commutative group G. One can show (use lemma 6.1, ch. 11 in [La]) that $U\Big(A(L)(K)\Big)$ projects surjectively onto $J(K)$ if K is archimedean or E has a good reduction over K. In the case of bad reduction the image is a subgroup of finite index.

There is canonical homomorphism

$$A(L)(K) \longrightarrow A(L)(K)/U\Big(A(L)(K)\Big) =: H$$

The quotient H is isomorphic to $\mathbb{Z}$ (resp. to a subgroup in $\mathbb{Q}$ which is an extension of $\mathbb{Z}$ by a finite group) when K is nonarchimedean and E has good reduction (resp. bad reduction), and to $\mathbb{R}$ if $K = \mathbb{C}, \mathbb{R}$.

Therefore we get a homomorphism $A(L)(K) \longrightarrow \mathbb{Z}(\text{resp} \quad \mathbb{Q})$ for the nonarchimedean case and $A(L)(K) \longrightarrow \mathbb{R}$ for the archimedean one.

For a given x the torsors $T_{(x,y)}$ form a group $T_{(x,\cdot)}$ which is isomorphic to the group $A(L|_{x\times J})$, and there is a similar statement for the torsors $T_{(x,y)}$ for a given y. Applying the homomorphism $A(L|_{x\times J}) \longrightarrow H$ we will get a homomorphism of the group $T_{(x,\cdot)}$ to H. Similarly we have a homomorphism of the group $T_{(\cdot,y)}$ to H.

Consider the map

$$U(T_{(x_1,\cdot)}) \times U(T_{(x_2,\cdot)}) \longrightarrow T_{(x_1+x_2,\cdot)}$$

induced by the multiplication $T_{(x_1,y)} \times T_{(x_2,y)} \longrightarrow T_{(x_1+x_2,y)}$. Its image is a subgroup. This follows from the commutativity of the diagram

$$\begin{array}{ccc}
T_{(x_1,y_1)} \times T_{(x_2,y_1)} \times T_{(x_1,y_2)} \times T_{(x_2,y_2)} & \longrightarrow & T_{(x_1,y_1+y_2)} \times T_{(x_2,y_1+y_2)} \\
\downarrow & & \downarrow \\
T_{(x_1+x_2,y_1)} \times T_{(x_1+x_2,y_2)} & \longrightarrow & T_{(x_1+x_2,y_1+y_2)}
\end{array}$$

It is therefore a compact subgroup, and so a maximal compact subgroup in $T_{(x_1+x_2,\cdot)}$. In particular the restrictions of the homomorphisms $T_{(x,\cdot)} \longrightarrow H$ and $T_{(\cdot,y)} \longrightarrow H$ to $T_{(x,y)}$ coincide.

So we get a well defined homomorphism $B_2(E(K)) \longrightarrow H$. Now the composition

$$\mathbb{Z}[E(K)\backslash 0] \xrightarrow{h} B_2(E(K)) \longrightarrow H \tag{16}$$

is the canonical Néron height on $E(K)$. The restriction of the homomorphism (16) to the subgroup $K^* \subset B_2(E(K))$ coincides with the logarithm of the norm homomorphism. In the nonarchimedean case H is a subgroup of $\log p \cdot \mathbb{Q} \subset \mathbb{R}$.

Remark. The homomorphism $h : \mathbb{Z}[E(K)\backslash 0] \longrightarrow B_2(E(K))$ was defined up to a sixth root of unity. This does not affect the definition of the height because the norm vanishes on roots of unity.

3 An elliptic analog of the Bloch-Suslin complex

1. The group $B_3(E)$ and complex $B(E;3)$. We will assume in s.3.1 - 3.3 that $k = \bar{k}$. We will always use notation $J := J(k)$. Set $g^-(t) := g(-t)$. Denote by $*$ the convolution in the group algebra $\mathbb{Z}[E(k)]$.

Definition 3.1 *$R_3(E)$ is the subgroup of $\mathbb{Z}[E(k)]$ generated by the elements $(f) * (1-f)^-$, $\quad f \in k(E)^*$, $\{0\}$, and the "distribution relations"*

$$m \cdot (\{a\} - m \cdot \sum_{mb=a} \{b\}), \quad a \in E(k), \quad m = -1, 2$$

Remarks. a) For $m = -1$ we get the elements $\{a\} + \{-a\} \in R_3(E)$. If we remove them from the definition of $R_3(E)$, we get the same group.

b) It would be more natural to add to the subgroup $R_3(E)$ the distribution relations for all $m \in \mathbb{Z}\backslash 0$: we should get the same group (compare with lemma (3.20)). But we will not need this.

Consider the homomorphism

$$\beta : \otimes^2 k(E)^* \longrightarrow \mathbb{Z}[E(k)] \qquad \beta : f \otimes g \longmapsto f * g^- := \sum n_i m_j \{a_i - b_j\}$$

(the Bloch map), where $(f) = \sum n_i(a_i)$ and $(g) = \sum m_j(b_j)$.

Recall that

$$\delta_3 : \mathbb{Z}[E(k)] \longrightarrow B_2(E) \otimes J, \qquad \{a\} \longmapsto -\frac{1}{2}\{a\}_2 \otimes a$$

and $i : k^* \hookrightarrow B_2(E)$ is the canonical embedding (see (6)). Let I_E be the augmentation ideal of the group algebra $\mathbb{Z}[E(k)]$ and $p : I_E \to J$ the canonical projection.

Recall that if $k = \bar{k}$ the tame symbol provides a homomorphism

$$\otimes^2 k(E)^* \xrightarrow{\partial} k^* \otimes \mathbb{Z}[E]$$

The Weil reciprocity law shows that its image belong to $k^* \otimes I_E$.

Theorem 3.2 *The following diagram is commutative*

$$\begin{array}{ccc} \otimes^2 k(E)^* & \xrightarrow{\partial} & k^* \otimes I_E \\ \downarrow \beta & & \downarrow i \otimes p \\ \mathbb{Z}[E(k)] & \xrightarrow{\delta_3} & B_2(E) \otimes J \end{array}$$

Proof. Let $(f) = \sum n_i(a_i), \quad (g) = \sum m_j(b_j)$. Then

$$\delta_3 \circ \beta(f \wedge g) = -\frac{1}{2} \cdot \sum_{i,j} m_i n_j < a_i - b_j, a_i - b_j > \otimes (a_i - b_j)$$

The term $\sum_{i,j} m_i n_j < (a_i - b_j), (a_i - b_j) > \otimes a_i$ equals to

$$\sum_{i,j} m_i n_j < (a_i) - (0), (a_i) - (0) > \otimes a_i + \sum_{i,j} m_i n_j < (b_j) - (0), (b_j) - (0) > \otimes a_i -$$

$$2 \cdot \sum_{i,j} m_i n_j < (a_i) - (0), (b_j) - (0) > \otimes a_i$$

The first term here is zero because $\sum_j m_j = 0$. The second is zero because $\sum_i n_i a_i = 0$ in J. The last one can be written as

$$-2 \cdot \sum_i m_i < (a_i) - (0), (g) > \otimes a_i$$

So the theorem follows from the definition of the tame symbol.

Theorem 3.3 $\delta_3(R_3(E)) = 0$.

Proof. We will denote by $\{a\}_3$ projection of the generator $\{a\}$ onto the quotient $B_3(E)$. The map δ_3 kills the distribution relations:

$$\delta_3\Big(m(\{a\}_3 - m \sum_{mb=a} \{b\}_3))\Big) = m(\{a\}_2 \otimes a - m \sum_{mb=a} \{b\}_2 \otimes b) =$$

$$m(\{a\}_2 - \sum_{mb=a} \{b\}_2) \otimes a = 0$$

The last equality is provided by corollary (4.3), which will be proved later, in s. 4.4. (The proof does not depend on any results or constructions in chapter 3).

The fact that $\delta_3(f*(1-f))=0$ lies deeper and follows from the theorem (3.2). Theorem (3.3) is proved.

Set $B_3(E):=\mathbb{Z}[E(k)]/R_3(E)$. We get a complex

$$B(E,3):= \qquad B_3(E)\longrightarrow B_2(E)\otimes J\longrightarrow J\otimes\Lambda^2 J\longrightarrow\Lambda^3 J$$

Let I_E^k be k-th power of the augmentation ideal.

Lemma 3.4 *$\beta(f\otimes g)\in I_E^4$. Moreover, β is surjective onto I_E^4.*

Proof. A divisor $\sum n_i(a_i)$ is principal if and only if $\sum n_i=0$ and $\sum n_i a_i=0$ in $J(E)$. So I_E^2 coincides with the subgroup of $\mathbb{Z}[E]$ given by the divisors of functions. So the convolution of two principal divisors belongs to I_E^4 and, moreover, generate it.

Let $B_3^*(E)$ be the quotient of I_E^4 by the subgroup generated by the elements $(f)*(1-f)^-$.

Lemma 3.5 $\delta_3(I_E^4)\in k^*\otimes J\subset B_2(E)\otimes J$

Proof. It is easy to see that $\delta_3(I_E^4)\subset h(I_E^3)\otimes J$. Further, $h(I_E^3)\subset k^*$ because $p\circ h(I_E^3)=0$ (see the properties of the group $B_2(E)$ listed in s. 1.4). The lemma follows.

So we get a complex

$$B^*(E;3): \quad B_3^*(E)\xrightarrow{\delta_3}k^*\otimes J \tag{17}$$

2. Relation with algebraic K-theory. Let us remind the long exact sequence of localization

$$K_3(k(E))\xrightarrow{\partial_3}\oplus_{x\in E_1}K_2(k(x))\longrightarrow K_2(E)\longrightarrow$$

$$K_2(k(E))\xrightarrow{\partial_2}\oplus_{x\in E_1}k(x)^*\longrightarrow K_1(E)\longrightarrow k(E)^*\xrightarrow{\partial_1}\oplus_{x\in E_1}\mathbb{Z}$$

The group $K_1(E)$ has a subgroup k^* which comes from the base. One can show that

$$H^0(E,\mathcal{K}_2)=gr_2^{\gamma}K_2(E), \qquad H^1(E,\mathcal{K}_2)=gr_2^{\gamma}K_1(E)=K_1(E)/k^*$$

Lemma 3.6 *Modulo 2-torsion one has*

$$H^0(E,\mathcal{K}_2)=K_2(k)\oplus H^0(E,\mathcal{K}_2)^-$$

$$Ker\Big(H^1(E,\mathcal{K}_2)\longrightarrow k^*\Big)=H^1(E,\mathcal{K}_2)^-=K_1(E)^-$$

Proof. It follows easily using the transfer related to the projection $E \to P^1$ given by factorization along the involution $x \to -x$.

Recall that $H^3(B(E,3)) = H^4(B(E,3)) = 0$.

Theorem 3.7 *Let $k = \bar{k}$. Then the commutative diagram from theorem (3.2) provides a morphism of complexes*

$$\begin{array}{ccc} \frac{K_2(k(E))}{k^*\cdot k(E)^*} & \xrightarrow{\partial} & k^* \otimes J \\ \tilde{\beta} \downarrow & & \downarrow id \\ B_3^*(E) & \xrightarrow{\delta_3} & k^* \otimes J \end{array}$$

where $\tilde{\beta}$ is surjective and $Ker\tilde{\beta} = Tor(k^*, J)$ modulo 2-torsion.

This theorem and lemma (3.6) implies immediately

Theorem 3.8 *Let $k = \bar{k}$. Then there are an embedding*

$$i : Tor(k^*, J(k)) \hookrightarrow \frac{H^0(E, \mathcal{K}_2)}{K_2(k)}$$

and canonical isomorphisms

$$\frac{H^0(E, \mathcal{K}_2)}{Tor(k^*, J(k)) + K_2(k)} = H^1B^*(E;3) \quad \textit{modulo 2-torsion} \qquad (18)$$

$$Ker\Big(H^1(E, \mathcal{K}_2) \longrightarrow k^*\Big) = H^2B^*(E;3) \qquad (19)$$

For an arbitrary field k (18) and (19) are isomorphisms modulo torsion.

Proof of theorem (3.7). We will first prove that we have a morphism of complexes and $Tor(k^*, J) \subset Ker\tilde{\beta}$, and then that $Tor(k^*, J) = Ker\tilde{\beta}$ modulo 2-torsion.

Recall that the tame symbol homomorphism ∂ maps $K_2(k(E))$ to $k^* \otimes I_E$. Therefore the complex computing the groups $H^0(E, \mathcal{K}_2)$ and $Ker\Big(H^1(E, \mathcal{K}_2) \longrightarrow k^*\Big)$ looks as follows

$$K_2(k(E)) \xrightarrow{\partial} k^* \otimes I_E$$

Notice that $\partial(\{c, f\}) = c \otimes (f)$, so it has a subcomplex

$$k^* \cdot k(E)^* \xrightarrow{\partial} k^* \otimes I_E^2 \qquad (20)$$

where $\cdot$ is the product in K-theory. One obviously has $Ker\partial = K_2(k)$, and factorising by $K_2(k)$ we get the identity map $k^* \otimes I_E^2 \to k^* \otimes I_E^2$.

The map $\beta : \otimes^2 k(E)^* \to I_E^4$ followed by the natural projection $I_E^4 \to B_3^*(E)$ leads to surjective map

$$\tilde{\beta} : \otimes^2 k(E)^* \longrightarrow B_3^*(E)$$

Notice that $\beta(k^* \otimes k(E)^*) = 0$ and $\beta(f \otimes (1-f)) = 0$ by the definition of the group $B_3^*(E)$. So we get the desired morphism of complexes.

Tensoring the exact sequence $0 \to I_E^2 \to I_E \to J \to 0$ by k^* and using the fact that I_E is a free abelian group, we get an exact sequence

$$0 \longrightarrow Tor(k^*, J) \longrightarrow k^* \otimes I_E^2 \longrightarrow k^* \otimes I_E \xrightarrow{id \otimes p} k^* \otimes J \longrightarrow 0 \qquad (21)$$

So we get the following commutative diagram, where the vertical sequences are complexes, the complex on the right is the exact sequence (21), and α is injective:

$$\begin{tikzcd}
 & 0 \arrow[d] \\
 & Tor(k^*, J) \arrow[dl] \arrow[d] \\
k^* \otimes I_E^2 \arrow[r, "="] \arrow[d, "\alpha"'] & k^* \otimes I_E^2 \arrow[d] \\
\frac{K_2(k(E))}{K_2(k)} \arrow[r, "\partial"] \arrow[d, "\tilde{\beta}"'] & k^* \otimes I_E \arrow[d] \\
B_3^*(E) \arrow[r, "\delta_3"] & k^* \otimes J \arrow[d] \\
 & 0
\end{tikzcd}$$

From this diagram we see that $Tor(k^*, J) \subset Ker\tilde{\beta}$. Notice that

$$Coker\Big(k^* \otimes I_E^2 \xrightarrow{\alpha} \frac{K_2(k(E))}{K_2(k)}\Big) = \frac{\otimes^2 I_E^2}{\{(1-f) * (f)^-\}}$$

Theorem 3.9 *The map* $\bar{\beta} : \frac{\otimes^2 I_E^2}{\{(1-f)*(f)^-\}} \longrightarrow B_3^*(E)$ *is an isomorphism modulo* 2*-torsion.*

Proof of theorem (3.9). It consists of three steps of quite different nature.

Step 1. Set $(t_a f)(x) := f(x-a)$.

Proposition 3.10 *Let f and g be rational functions on E. Then*

$$\{f,g\} - \{t_a f, t_a g\} = 0 \quad \text{in} \quad \frac{K_2(k(E))}{k^* \cdot (k(E))^*}$$

Let L/K be an extension of fields. Then one has a natural map $p^* : K_2(K) \to K_2(L)$ and the transfer map $p_* : K_2(L) \to K_2(K)$. We need the following result ([BT]).

Lemma 3.11 *Let L/K be a degree 2 extension of fields. Then $K_2(L)$ is generated by symbols $\{k,l\}$ with $k \in K, l \in L$ and $p_*(\{k,l\}) = \{k, Nm_{L/K} l\}$.*

In particular $p^* p_* = Id + \sigma$ where σ is a nontrivial element in $Gal(L/K)$, and thus modulo 2-torsion any Galois invariant element in $K_2(L)$ belongs to $p^* K_2(K)$.

Lemma 3.12 *Assume $k = \bar{k}$. Then any rational function f on an elliptic curve E over k can be decomposed into a product of functions with divisors of the following kind: $(a) - 2(b) + (-a + 2b)$.*

Proof of the lemma 3.12. Any function f can be decomposed into a product of the functions with divisors of the following kind: $(a) - (b) - (c) + (-a + b + c)$.

Indeed, let $(f) = \sum n_a(a)$. We will use induction on $\sum |n_a|$. Since $\sum n_a = 0$, replacing if needed f by f^{-1} we can find points a, b, c such that $n_a > 0$, $n_b < 0$, $n_c < 0$. Then $(f) - [(a) - (b) - (c) + (-a + b + c)]$ is a principal divisor with smaller $\sum |n_a|$.

Further $(a) - (b) - (c) + (-a+b+c) = [(a) - 2(d) + (-a+b+c)] - [(b) - 2(d) + (c)$ for $2d = b + c$.

Proof of the proposition (3.10). According to the lemma we can assume that $(f) = (b) - 2(0) + (-b)$, $(g) = (c) - 2(d) + (-c + 2d)$. Therefore $(t_a f) = (b + a) - 2(a) + (-b + a)$, $(t_a g) = (c + a) - 2(d + a) + (-c + 2d + a)$.

The quotient of E under the involution $\sigma_a : x \to a - x$ is isomorphic to $\mathbb{P}^1$. The symbol $\{t_a f, t_a g\} + \{\sigma_a t_a f, \sigma_a t_a g\}$ is σ_a-invariant, so it comes from $K_2(k(\mathbb{P}^1))$. It is known that $K_2(k(\mathbb{P}^1))$ is generated by $k(\mathbb{P}^1)^* \otimes k^*$. So

$$\{f,g\} - \{t_a f, t_a g\} \sim \{f,g\} + \{\sigma_a t_a f, \sigma_a t_a g\} = \{f, g \cdot \sigma_0 g\} \sim 0$$

because $\sigma_a t_a = \sigma_0$, $\sigma_0 f = f$, and the symbol $\{f, g \cdot \sigma_0 g\}$ is σ_0-invariant.

Step 2. Let A be an abelian group. Let $S_k(A) \subset S^k I_A$ be the subgroup generated by the elements

$$((x_1)*X_1)\circ...\circ((x_k)*X_k)-X_1\circ...\circ X_k, \quad x_1+...+x_k=0, \quad x_i \in A, \quad X_i \in I_A$$

This subgroup clearly belongs to the kernel of the convolution map

$$S^k I_A \to I_A^k, \quad X_1 \circ ... \circ X_k \longmapsto X_1 * ... * X_k$$

So we get a homomorphism $\alpha_k : S^k I_A / S_k(A) \longrightarrow I_A^k$.

Proposition 3.13 *For any abelian group A the homomorphism α_k is injective.*

Proof. We may assume that A is finitely generated. Therefore the group ring $\mathbb{Z}[A]$ looks as follows:

$$\mathbb{Z}[A] = \mathbb{Z}[t_1, ..., t_a, t_1^{-1}, ..., t_a^{-1}] \times \prod_{a+1 \leq j \leq m} \frac{\mathbb{Z}[t_j]}{(t_j^{N_j}-1)} \tag{22}$$

Under this isomorphism the augmentation ideal I_A goes to the maximal ideal $(t_1-1, ..., t_m-1)$. The subgroup $S_k(A)$ is generated by the elements

$$(t_{i_1}-1)f_1 \circ ... \circ (t_{i_{k-1}}-1)f_{k-1} \circ (t_{i_k}-1)f_k \quad - \tag{23}$$
$$(t_{i_1}-1) \circ ... \circ (t_{i_{k-1}}-1) \circ (t_{i_k}-1)f_1...f_k$$

So any element of the quotient $S^k I_A / S_k(A)$ can be written as

$$(t_{i_1}-1) \circ ... \circ (t_{i_{k-1}}-1) \circ (t_{i_k}-1)f \tag{24}$$

The homomorphism α_k sends it to $(t_{i_1}-1)...(t_{i_{k-1}}-1)(t_{i_k}-1)f$.

Let us suppose first that $a > 0$, i.e. A is an infinite group. We will use the induction on both m and k. The case $m = 1$ is trivial: if $(t-1)^k f = 0$ then $(t-1) \circ ... \circ (t-1)f = 0$, even in the case $f(t) \in \frac{\mathbb{Z}[t_j]}{(t_j^{N_j}-1)}$.

Consider an element

$$P = \sum_j (t_{i_1(j)}-1) \circ ... \circ (t_{i_k(j)}-1)f_j \in Ker\alpha_k \tag{25}$$

Any element f of the right hand side of (22) can be written as $f'(t_2, ..., t_m) + (t_1-1)f''(t_1, ..., t_m)$, So writing $f_j = f'_j + (t_1-1)f''_j$ and setting $P' := \sum_j (t_{i_1(j)}-1) \circ ... \circ (t_{i_k(j)}-1)f'_j$ we get $P = P' + (t_1-1) \circ Q''$.

Further, let P'_I be the sum of all the terms of P' where non of the indices $i_l(j)$ equal to 1. Then $P' = P'_I + (t_1 - 1) \circ Q'_{II}$. The restriction of P'_I to the divisor $t_1 = 1$ in $Spec\, \mathbb{Z}[A]$ coincides with the restriction of P. Therefore it belongs to $Ker \alpha_k$ and thus is zero by the induction asumption for $(k, m-1)$. Since by the definition P'_I does not depend on t_1, this implies $P'_I = 0$.

Thus $P = (t_1 - 1) \circ Q$ for some $Q \in S^{k-1} I_A / S_{k-1}(A)$. Therefore $\alpha_{k-1}(Q) = 0$ since $t_1 - 1$ is not a divisor of zero (we have assumed $a > 0$). Thus $Q = 0$ by the induction assumption for $(k-1, m)$.

Now let A be a finite group. Decomposing $f(t)$ in (24) into a sum of monomials $(t_1 - 1)^{a_1} ... (t_q - 1)^{a_q}$ we can write an element (24) as a sum

$$P = \sum (t_1 - 1)^{b_1} \circ ... \circ (t_k - 1)^{b_k} (t_{k+1} - 1)^{b_{k+1}} ... (t_{k+l} - 1)^{b_{k+l}} \qquad (26)$$

We will treat for a moment this sum as element of $S^k \mathbb{Z}[t_1, ..., t_m]$, not $S^k I_A / S_k(A)$. Let

$$\tilde{\alpha}_k(P) := \sum (t_1 - 1)^{b_1} ... \cdot (t_{k+l} - 1)^{b_{k+l}} \in \mathbb{Z}[t_1, ..., t_m]$$

be its image in $\mathbb{Z}[t_1, ..., t_m]$ under the product map. If $\tilde{\alpha}_k(P) = 0$ then clearly $P = 0$ modulo the relations (23) with $f_i \in \mathbb{Z}[t_1, ..., t_m]$.

Since $\alpha_k(P) = 0$, $\tilde{\alpha}_k(P)$ is a linear combination of monomials of type $(t_i^{N_i} - 1) \cdot Q(t)$. We are going to show that one can find another presentation P' for the element in $S^k I_A / S_k(A)$ given by P such that $\tilde{\alpha}_k(P') = \tilde{\alpha}_k(P) - (t_i^{N_i} - 1) \cdot Q(t)$.

Let us factorize $t_i^{N_i} - 1 = (t_i - 1) \cdot p_i(t_i - 1)$, where $p_i(u) = \sum c_j u^j$ is a polynomial in one variable. Notice that $c_0 \neq 0$. We must have a monomial in the sum P which goes under the map $\tilde{\alpha}_k$ to $c_0 \cdot (t_i - 1) \cdot Q(t)$. It can be written (modulo $S_k(A)$) as $c_0 \cdot (t_i - 1) \circ R$. Therefore the terms in (26) which are maped by $\tilde{\alpha}_k$ to $c_{j-1}(t_i - 1)^j \cdot Q(t)$, $j > 1$, can be written as $c_{j-1}(t_i - 1)^j \circ R$. Therefore the sum of these terms in (26) can be written as $(t_i^{N_i} - 1) \circ R$ and thus represent a zero element in $S^k I_A / S_k(A)$. The proposition is proved.

Step 3. Let us write β as a composition $\beta = i \circ \alpha$:

$$\otimes^2 I_E^2 \xrightarrow{i} \otimes^2 I_E^2 \xrightarrow{\alpha} I_E^4$$

$$i : A \otimes B \longmapsto A \otimes B^-; \quad \alpha : A \otimes B \longmapsto A * B$$

Let $S_E \subset \otimes^2 I_E^2$ be the subgroup generated by the elements $(f) \otimes (1 - f)^-$, $f \in k(E)^*$. One has

$$\otimes^2 I_E^2 = \frac{\otimes^2 k(E)^*}{k^* \otimes k(E)^* + k(E)^* \otimes k^*}, \qquad \frac{\otimes^2 I_E^2}{S_E} = \frac{K_2(k(E))}{k^* \cdot k(E)^*}$$

Let A_+ (resp A_-) be the coinvariants of the involution $x \to -x$ on E acting on a group A functorially depending on E (resp $A \otimes_{\mathbb{Z}} \lambda$, where λ is the standard $\mathbb{Z}$-line where the involution acts by inverting the sign).

Lemma 3.14 $(Ker\alpha)_+ \subset S_E$.

Proof. Let $x \in (Ker\alpha)_+$. Then $\partial(x) = 0$, so x defines an element of $(H^0(E, \mathcal{K}_2)/K_2(k))_+$. But this group is zero modulo 2-torsion by lemma (3.6).

Lemma 3.15 $\Lambda^2 I_E^2 \subset S_E$.

Proof. $A \otimes B + B \otimes A$ belongs to the subgroup generated by the Steinberg relations $(f) \otimes (1-f)$. Thus $A \otimes B^- + B \otimes A^- \in S_E$. The lemma above states that $B \otimes A^- + B^- \otimes A \in S_E$. So $A \otimes B^- - B^- \otimes A \in S_E$.

Let $\alpha' : S^2 I_E^2 \to I_E^4, \quad A \circ B \longmapsto A * B$. To prove the theorem we need to show that $(Ker\alpha')_- \subset S_E$.

Let $A_i \in I_E$. The element

$$< A_1, A_2, A_3, A_4 >:= (A_1 * A_2) \circ (A_3 * A_4) - (A_1 * A_3) \circ (A_2 * A_4)$$

clearly belongs to $Ker\alpha$, and thus to $Ker\partial$.

Proposition 3.16 $< A_1, A_2, A_3, A_4 > \in S_E$.

To prove this we need the following lemma.

Lemma 3.17 *If A_i is a principal divisor for some $1 \leq i \leq 4$, then $< A_1, A_2, A_3, A_4 > \in S_E$. In particular we get a well defined homomorphism*

$$< \cdot, \cdot, \cdot, \cdot >: \otimes^4 J \to S^2 I_E^2 / S_E \tag{27}$$

Proof. Let us show that if $B_0, B_1 \in I_E$, then $< B_0 * B_1, A_2, A_3, A_4 > \in S_E$ (the other cases are similar). We will write $A \stackrel{S_E}{=} B$ if $A - B \in S_E$. It follows from the proposition (3.10) that for $A, B \in I_E^2$ and $X \in \mathbb{Z}[E(k)]$ one has $(X * A) \circ B \stackrel{S_E}{=} A \circ (X * B)$. So

$$(B_0 * B_1 * A_2) \circ (A_3 * A_4) \stackrel{S_E}{=} (B_0 * B_1) \circ (A_2 * A_3 * A_4) \stackrel{S_E}{=} (B_0 * B_1 * A_3) \circ (A_2 * A_4)$$

Since $(a) - (0) + (b) - (0) - ((a+b) - (0)) \in I_E^2$, we get (27). The lemma is proved.

Proof of the proposition (3.16). Let (A) be the image of $A \in I_E$ in the Jacobian. Using $(A^-) = -(A)$ and the previous lemma we get

$$< A_1, A_2, A_3, A_4 > - < A_1^-, A_2^-, A_3^-, A_4^- > \in S_E$$

On the other hand according to the lemma (3.6)

$$< A_1, A_2, A_3, A_4 > + < A_1^-, A_2^-, A_3^-, A_4^- > \in S_E$$

So $2 < A_1, A_2, A_3, A_4 > \in S_E$. Using lemma (3.17) and the divisibility (by 2) of $J(k)$ we conclude that $< A_1, A_2, A_3, A_4 > \in S_E$.

Proposition 3.18 *$Ker(S^2 I_E^2 \xrightarrow{\alpha'} I_E^4)$ is generated by the elements*

$$((x) * A) \circ ((-x) * B) - A \circ B, \quad A, B \in I_E^2$$

and $< A_1, A_2, A_3, A_4 >$, $A_i \in I_E$.

Proof. Let $T \subset I_E^2$ be the subgroup generated by the elements $< A_1, A_2, A_3, A_4 >$. There is a surjective homomorphism

$$S^4 I_E \longrightarrow S^2(I_E^2)/T, \quad A_1 \otimes ... \otimes A_4 \longmapsto (A_1 * A_2) \circ (A_3 * A_4)$$

The proposition follows immediately from the proposition (3.13).

Theorem (3.9) follows from proposition (3.18) (3.13) and (3.17).

Proposition 3.19 *Assume that $k = \bar{k}$. Then*

a) There is an injective homomorphism of complexes $B^*(E;3) \to B(E;3)$.

b)The quotient $B(E;3)/B^*(E;3)$ is isomorphic to the Koszul complex

$$S^3 J \longrightarrow S^2 J \otimes J \longrightarrow J \otimes \Lambda^2 J \longrightarrow \Lambda^3 J \tag{28}$$

In particular the complexes $B(E;3)$ and $B^*(E;3)$ are quasiisomorphic.

We will need

Lemma 3.20 *Suppose that*

$$D := m \sum_i (\{a_i\} - m \sum_{mb_i = a_i} \{b_i\}) \in I_E^4$$

Then D belongs to the subgroup generated by the elements $(f) * (1 - f)^-$.

Let $[m] : E \to E$ be the isogeny of multiplication by m.

Lemma 3.21 *Let $f, g \in k(E)^*$, $k = \bar{k}$. Then*

$$[m]^*\{f,g\} = m\{f,g\} \quad in \quad \frac{K_2(k(E)}{k^* \cdot k(E)}$$

Proof of the lemma (3.21). One has an exact sequence

$$0 \longrightarrow \frac{H^0(E, \mathcal{K}_2)}{K_2(k) + Tor(k^*, J)} \longrightarrow \frac{K_2(k(E))}{k^* \cdot k(E)} \longrightarrow k^* \otimes J \longrightarrow 0$$

The operator $[m]_*$ acts on the left and right (different from 0) groups here by multiplication by m (see [BL], ch. 5,6 where this was proved rationally; that proof works integrally). Further, $[m]_*$ has no Jordan blocks since $[m]_*[m]^* = m^2$. The lemma follows.

Proof of the lemma (3.20). There is an isomorphism

$$j : \mathbb{Z}[E(k)]/I_E^4 = \mathbb{Z} \oplus J \oplus S^2J \oplus S^3J, \qquad \{a\} \longmapsto (1, a, a \cdot a, a \cdot a \cdot a)$$

One has

$$j(m(\{a\} - m\sum_{mb=a}\{b\})) = m((1-m^3), (1-m^2)a, (1-m)a \cdot a, 0) \qquad (29)$$

Using this we see that $\sum_i \{a_i\} \in I_E^4$. Thus $\sum_i \{a_i\} = \sum (f_j) * (g_j)^-$. It is easy to see that $\beta\Big([m]^* \sum_i \{f_i, g_i\} - m\{f_i, g_i\}\Big) = D$. The lemma is proved.

Proof of the proposition (3.19). a) We need only to show that $B_3^*(E)$ injects to $B_3(E)$. This boils down to the lemma (3.20) above, since in our definition of $R_3(E)$ we used only the distribution relations for $m = -1, 2$.

b) By definition $B_3(E)/B_3^*(E)$ is isomorphic to the quotient of $\mathbb{Z}[E(k)]/I_E^4$ modulo (the image of) the distribution relations.

Using the computaion (29) and divisibility of $J(k)$ we see that the map j maps the subgroup generated by the distribution relations for any given $|m| > 1$ surjectivly onto $2\mathbb{Z} \oplus J \oplus J^2$. Therefore $B_3(E)/B_3^*(E) = S^3J$. Let us recall that $B_2(E)/k^* = S^2J$. So the terms of the quotient $B(E;3)/B^*(E;3)$ are the same as in (28). It is easy to see that the differentials coincide. The proposition is proved.

3. Zagier's conjecture on $L(E,2)$ for modular elliptic curves over $\mathbb{Q}$. Let us recall that for a curve X over $\mathbb{R}$ one has $H^2_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(2)) =$

$H^1(X/\mathbb{R}, \mathbb{R}(1))$. Let $\bar{X} := X \otimes \mathbb{C}$. The cup product with $\omega \in \Omega^1(\bar{X})$ provides an isomorphism of vector spaces over $\mathbb{R}$:

$$H^1(X/\mathbb{R}, \mathbb{R}(1)) \longrightarrow H^0(\bar{X}, \Omega^1)^{\vee}$$

So we will present elements of $H^2_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(2))$ as functionals on $H^0(\bar{X}, \Omega^1)$.

Bloch constructed the regulator map

$$r_{\mathcal{D}} : K_2(E) \longrightarrow H^2_{\mathcal{D}}(E, \mathbb{R}(2))$$

If we represent an element of $K_2(E)$ as $\sum_i \{f_i, g_i\}$ (with all the tame symbols vanish) then Beilinson's construction of the regulator looks as follows:

$$< r_{\mathcal{D}} \sum_i \{f_i, g_i\}, \omega >= \frac{1}{2\pi i} \sum_i \int_{E(\mathbb{C})} \log |f_i| d \arg(g_i) \wedge \omega$$

Let f and g be rational functions on E such that

$$(f) = \sum n_i(a_i), \quad (g) = \sum m_j(b_j)$$

Let $\Gamma = H_1(E(\mathbb{C}), \mathbb{Z})$. We may assume that $\Gamma = \{\mathbb{Z} \oplus \mathbb{Z} \cdot \tau\} \subset \mathbb{C}$ and z is the coordinate in $\mathbb{C}$. Let us briefly recall how the regulator integral $< r_{\mathcal{D}}\{f, g\}, dz >$ is computed by means of the elliptic dilogarithm ([Bl1], see also [RSS]). The intersection form on Γ provides a pairing

$$(\cdot, \cdot) : E(\mathbb{C}) \times \Gamma \longrightarrow S^1; \qquad (z, \gamma) := exp(\frac{2\pi i(z\bar{\gamma} - \bar{z}\gamma)}{\tau - \bar{\tau}}) \tag{30}$$

Let

$$K_{2,1}(z; \tau) := \frac{(Im\tau)^2}{\pi} \sum_{\gamma \in \Gamma \setminus 0} \frac{(u, \gamma)}{\gamma^2 \bar{\gamma}}, \qquad z = exp(2\pi i u)$$

Then one has

$$\frac{1}{2\pi i} \int_{E(\mathbb{C})} \log |f| d \arg g \wedge dz = \frac{1}{i\pi} \sum_{a,b \in E(\mathbb{C})} v_a(f) v_b(g) K_{2,1}(a - b; \tau)$$

To prove this one may use that

$$\int_{E(\mathbb{C})} \log |f| di \arg g \wedge dz = - \int_{E(\mathbb{C})} \log |f| d \log |g| \wedge dz$$

together with the following lemma and the fact that the Fourier transform sends the convolution to the product.

Lemma 3.22

$$\log|f(z)| = -\frac{Im\tau}{2\pi}\sum_{\gamma\in\Gamma\backslash 0} v_a(f)\frac{(z-a,\gamma)}{|\gamma|^2} + C_f \tag{31}$$

where C_f is a certain constant.

Proof. One can get a proof applying $\partial\bar{\partial}$ to the both parts of (31). The constant C_f can be computed from the decomposition of f on the product of theta functions using the formula in s. 18 ch. VIII of [We]. It does not play any role in our considerations since $\int_{E(\mathbb{C})} C_f \cdot d\log|g| \wedge \omega = 0$ by the Stokes formula.

The relation between the Eisenstein-Kronecker series $K_{2,1}(z)$ and the elliptic dilogarithm is the following ([Bl1], [Z]): $K_{2,1}(z;\tau) = \mathcal{L}_{2,q}(z) - iJ_q(z)$, where the function $J_q(z)$ is defined as follows. Let us average the function $J(z) := \log|z|\log|1-z|$ over the action of the group $\mathbb{Z}$ generated by the shift $z \longmapsto qz$ regularizing divergencies. We will get

$$J_q(z) := \sum_{n=0}^{\infty} J(q^n z) - \sum_{n=1}^{\infty} J(q^n z^{-1}) + \frac{1}{3}(\log|q|)^2 \cdot B_3(\frac{\log|z|}{\log|q|})$$

Here $B_3(x)$ is the third Bernoulli polynomial. The function $J_q(z)$ is invariant under the shift $z \longmapsto qz$ and satisfies $J_q(z) = -J_q(z^{-1})$.

It follows from the main result of Beilinson in [B2], see also [SS2], that for a modular elliptic curve E over $\mathbb{Q}$ there always exists an element in $K_2(E)_{\mathbb{Z}}$ whose regulator gives (up to a standard nonzero factor) $L(E,2)$. So we get the formula

$$L(E,2) \sim_{\mathbb{Q}^*} \pi \cdot \sum_i \sum_{a,b\in E(\mathbb{C})} v_a(f^{(i)}) v_b(g^{(i)}) \mathcal{L}_{2,q}(a-b)$$

Finally, the results of the present section implies that the element $\sum_i \sum_{a,b\in E(\mathbb{C})} v_a(f^{(i)}) v_b(g^{(i)})\{a-b\}_3 \in I^4_{E(\bar{\mathbb{Q}})}$ must satisfy all the conditions of the theorem(1.1).

Theorem (1.2) follows from the surjectivity of the map β and the arguments above.

The integrality condition ([BG], [SS]). Let E be an elliptic curve over $\mathbb{Q}$. Choose a minimal regular model $E_{\mathbb{Z}}$ of E over $\mathbb{Z}$. One has the exact sequence

$$K_2(E_{\mathbb{Z}}) \longrightarrow K_2(E_{\mathbb{Q}}) \xrightarrow{\partial} \oplus_p K_1'(E_p) \tag{32}$$

The group $K_1'(E_p) \otimes \mathbb{Q}$ is not zero if and only if E_p has a split multiplicative reduction with special fibre a Néron N-gon. In this case $K_1'(E_p) \otimes \mathbb{Q} = \mathbb{Q}$.

Consider an element $\sum_i \{f_i, g_i\} \in K_2(\mathbb{Q}(E))$ which has zero tame symbol at all points. It defines an element of $H^0(E, \mathcal{K}_2)$. Suppose first that the closure of the support of the divisors f_i, g_i is contained on the smooth part of $E_{\mathbb{Z}}$. Then Schappaher and Scholl ([SS]) proved that the image of this element under the map ∂ is computed by the following formula:

$$\partial(\sum_i \{f_i, g_i\} = \pm \frac{1}{3N} \sum_{\nu \in \mathbb{Z}/N\mathbb{Z}} d((f) * (g^-); \nu) B_3(\frac{\nu}{N}) \cdot \Phi$$

here Φ is a generator in $K_1'(E_p) \otimes \mathbb{Q}$.

In general one should extend $\mathbb{Q}$ to $\mathbb{Q}((f_i),(g_i))$, which is the field of the definition of the divisors $(f_i),(g_i)$. After this we get precisely the condition (4).

Let us explane why the expression $\sum_{\nu \in \mathbb{Z}/(eN)\mathbb{Z}} d(P;\nu) B_3(\frac{\nu}{eN})$ does not depent on the field L. Let L and L' are two extentions of $\mathbb{Q}_p$ such that all the points are defined over them. We can assume that $L \subset L'$ (by taking the composit). Denote by $n_1 = e_1 f_1$ degree of this extension. Looking at the Tate uniformization we see that points which intersect ν-th component of special fiber corresponding to the field L intersect $e_1\nu$-th component of special fiber corresponding to the field L'.

4. Main results from the motivic point of view. Let $\mathcal{MM}_X$ be the (hypothetical) abelian category of all mixed motivic sheaves over a regular scheme X over a field k. Let $\mathbb{Q}(-1) := h^2(P^1)$, $\mathbb{Q}(n) := \mathbb{Q}(1)^{\otimes n}$ and $\mathcal{H} := h^1(E)(1)$.

The motivic refinement of our results is the following

Conjecture 3.23 *There exists a canonical quasiisomorphism in the derived category*

$$B(E,3) \otimes \mathbb{Q} = RHom_{\mathcal{MM}_k}(\mathbb{Q}(0), \mathcal{H}(1))$$

Let us explain the relationship of this conjecture with our results.

Let $\pi : E \longrightarrow Spec(k)$ be the structure morphism. There are the Tate sheaves $\mathbb{Q}(n)_E := \pi^*\mathbb{Q}(n)$. Beilinson's description of Ext groups between the Tate sheaves over E gives us

Conjecture 3.24

$$Ext^i_{\mathcal{MM}(E)}(\mathbb{Q}(0)_E, \mathbb{Q}(2)_E) = gr_2^{\gamma} K_{4-i}(E) \otimes \mathbb{Q}$$

Lemma 3.25

$$RHom_{\mathcal{MM}_k}(\mathbb{Q}(0), \mathcal{H}(1)) = RHom_{\mathcal{MM}_E}(\mathbb{Q}(0), \mathbb{Q}(2))^- \qquad (33)$$

Indeed, let $p : E \longrightarrow Spec(k)$ be the canonical projection. Then we should have the motivic Leray spectral sequence

$$E_2^{p,q} = Ext^p_{\mathcal{MM}_k}\Big(\mathbb{Q}(0), R^q p_* \mathbb{Q}(2)\Big)$$

degenerating at E_2 and abutting to $Ext^{p+q}_{\mathcal{MM}_E}\Big(\mathbb{Q}(0), \mathbb{Q}(2)\Big)$. Noting that

$$h^0(E)^- = h^2(E)^- = 0; \qquad h^1(E)^- = h^1(E)$$

we get (33). Conjecture (3.24) together with this lemma tell us that the cohomology of the elliptic motivic complexes are given by the formula

$$R^i Hom_{\mathcal{MM}_k}(\mathbb{Q}(0), \mathcal{H}(1)) \otimes \mathbb{Q} = gr_2^\gamma K_{3-i}(E)^- \otimes \mathbb{Q} \qquad (34)$$

Conjecture (3.23) follows from theorem (3.7) and conjecture (3.24), see [G2]. A very interesting and important is the following:

Problem. To construct explicitely the general elliptic motivic complexes

$$RHom_{\mathcal{MM}_k}(\mathbb{Q}(0), Sym^n \mathcal{H}(m))$$

For $m = 1$ it is considered in [G2].

5. Degeneration to the nodal curve (compare with [Bl1], [DS]). Let k be an algebraicly closed field. Denote by $\mathcal{I}$ the group of rational functions such that $f(0) = f(\infty) = 0$. Let I_{k^*} be the augmentation ideal of $\mathbb{Z}[k^*]$.

For an element $f \otimes g \in (1 + \mathcal{I}) \otimes k(t)^*$ consider the Bloch map

$$\beta(f \otimes g) := \sum_{x,y \in k^*} v_x(f) v_y(g) \{y/x\} + v_\infty(g) \cdot ((f) + (f^-)) \in I^2_{k^*}$$

Set

$$\mathbb{Z}[k^*] \xrightarrow{\delta} k^* \otimes k^*, \qquad \{x\} \longmapsto (1-x) \otimes x, \quad \{1\} \longmapsto 0 \qquad (35)$$

Let $p : I_{k^*} \to k^*$ be the natural projection $\{x\} \to x$.

Theorem 3.26 *The following diagram is commutative:*

$$\begin{array}{ccc} (1+\mathcal{I}) \otimes k(t)^* & \xrightarrow{\partial} & I_{k^*} \otimes k^* \\ \beta \downarrow & & \downarrow p \otimes id \\ I^2_{k^*} & \xrightarrow{\delta} & k^* \otimes k^* \end{array}$$

The proof is a direct calculation similar to (but simpler then) the proof of theorem (3.2).

Let $S(k)$ be the subgroup generated by the elements $(1-f)\otimes f$ where $f\in\mathcal{I}$ and the subgroup $(1+\mathcal{I})\otimes k^*$. Set $B_2^*(k) := I^2_{k^*}/\beta(S(k))$.

Lemma 3.27 *$Ker\beta \subset S(k)$*

Proof. An easy analog of theorem (3.10) for the nodal curve claims that the homomorphism

$$q: k(t)^* \longrightarrow (1+\mathcal{I})\otimes k(t)^*, \quad f(t)\longmapsto f(x)\otimes(x-1)$$

is surjective. It is clear that $\beta\circ q$ is injective. The lemma is proved.

This lemma implies that $\bar\beta : (1+\mathcal{I})\otimes k(t)^* \to B_2^*(k)$ is an isomorphism. So we get a morphism of complexes

$$\begin{array}{ccc} \frac{(1+\mathcal{I})\otimes k(t)^*}{S(k)} & \xrightarrow{\partial} & k^*\otimes k^* \\ \bar\beta\downarrow & & \downarrow id \\ B_2^*(k) & \xrightarrow{\delta} & k^*\otimes k^* \end{array}$$

Such that $\bar\beta$ is surjective and $Ker\bar\beta = Tor(k^*,k^*)$, similar to the theorem (3.8). So we see that when E degenerates to a nodal curve the complex $B^*(E/k,3)$ degenerates to the complex $B_2^*(k)\to k^*\otimes k^*$.

Let $S_1(k)$ be the subgroup generated by the elements $(1-f)\otimes f$ where $f\in\mathcal{I}$.

Theorem 3.28 *Let $k=\bar k$. The identity map on $\mathbb{Z}[k^*]$ provides a homomorphism of groups $B_2^*(k)\to B_2(k)$. Its kernel is isomorphic to S^2k^*. This map provides a quasiisomorphism of complexes*

$$\begin{array}{ccc} B_2^*(k) & \longrightarrow & k^*\otimes k^* \\ \downarrow & & \downarrow \\ B_2(k) & \longrightarrow & k^*\wedge k^* \end{array}$$

One can show that the group $B_2^*(k)$ is isomorphic to a group defined by S. Lichtenbaum in [Li].

It is known ([Lev]) that

$$K_2(\mathbb{P}^1_{\{0,\infty\}},\{0,\infty\}) = \frac{(1+\mathcal{I})\otimes k(t)^*}{S_1(k)}$$

Using this and the results of this subsection we can get a proof of Suslin's theorem for $k = \bar{k}$.

5. On an elliptic analog of the 5-term relation for the elliptic dilogarithm. Notice that

$$B_2(F) = Coker(\mathbb{Z}[M_{0,5}(k)] \xrightarrow{\partial} \mathbb{Z}[\mathbb{G}_m(k)])$$

Here $M_{0,5}$ is the configuration space of 5 distict points on the projective line. So one may ask how to present the group $B_3(E)$ in a similar form:

$$B_3(E) \stackrel{?}{=} Coker(\mathbb{Z}[X(k)] \xrightarrow{\partial} \mathbb{Z}[E(k)])$$

where X is a *finite dimensional* variety. In the our definition we have an infinite dimensional X (more precisely it is an inductive limit of finite dimensional varieties).

Here is a guess. Let us realize E as a cubic in P^2. Let p be a point in P^2 and l_1, l_2, l_3 any three lines through this point. Set $A_i := l_i \cap E$. Let $\tilde{R}_3^*(E) \subset I_E^4$ be the subgroup generated by the elements

$$\{p; l_1, l_2, l_3\} := A_1 * A_2^- + A_2 * A_3^- + A_3 * A_1^- \qquad (36)$$

and those linear combinations of the elements $\{a\}_3 + \{-a\}_3$ which lie in I_E^4. (Probably they belong to the subgroup generated (36)).

Lemma 3.29 $\tilde{R}_3^*(E) \subset R_3^*(E)$.

Proof. Let f_i be a linear homogeneous equation of the line l_i. Since the lines l_1, l_2, l_3 intersect in a point, these equations are linearly dependent, so we may choose them in such a way that $f_1 + f_2 = f_3$. Thus $f_1/f_3 \wedge f_2/f_3 = f_1/f_3 \wedge (1 - (f_1/f_3))$. Applying the map β to this Steinberg relation and using the relations $\{a\}_3 + \{-a\}_3 = 0$ we get the element $\{p; l_1, l_2, l_3\}$.

One obviously has $\sum_{j=1}^4 (-1)^j \{p; l_1, ...\hat{l}_j, ..., l_4\} = 0$, so we can assume that the line l_3 is, say, a vertical line.

Conjecture 3.30 $\tilde{R}_3^*(E) = R_3^*(E)$.

4 $B_2(E)$, θ-functions and action of isogenies

1. Elliptic curves over $\mathbb{C}$. Let us represent E as a quotient $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$. Below ξ denotes the coordinate on $\mathbb{C}$. The canonical trivialization of $(T_0^* E)^{12}$

is $-16\prod(e_j - e_i)(d\xi)^{12}$, which is equal to $\Delta(\tau)(d\xi)^{12}$, where

$$\Delta(\tau) = (2\pi i)^{12}\eta^{24}(\tau), \quad \eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) \quad (q = \exp(2\pi i\tau))$$

So the trivialization of $(T_0^*E)^2$ is $(2\pi i)^2\eta^4(\tau)(d\xi)^2$.

Now we will give the analytic description of the Deligne pairing $[*,*]$. Let L_a be the line bundle corresponding to the divisor $(a)-(0)$. Choose a representative $\alpha \in \mathbb{C}$ of a (α is defined up to $\mathbb{Z}+\mathbb{Z}\tau$). Let us define L_α on E as the quotient of the trivial line bundle on $\mathbb{C}$ under the action of $\mathbb{Z}+\mathbb{Z}\tau : 1$ acts trivially and τ acts by multiplication by $\exp(2\pi i\alpha)$. We identify $L_{\alpha+1}$ with L_α trivially and $L_{\alpha+\tau}$ with L_α by multiplication by $\exp(2\pi i\xi)$.

The fiber of the Poincare line bundle $[L_\alpha, L_\beta]$ on $J\times J$ over the point (α,β) is equal to $L_\alpha|_\beta \otimes L_\alpha^{-1}|_0$. This line bundle is described as a quotient

$$\frac{\mathbb{C}\times\mathbb{C}\times\mathbb{C}}{(\mathbb{Z}\oplus\mathbb{Z}\tau)\oplus(\mathbb{Z}\oplus\mathbb{Z}\tau)}$$

$$(\alpha,\beta,\lambda) \longmapsto (\alpha+m+n\tau, \beta+m'+n'\tau, \lambda\cdot exp(2\pi i(n\beta+n'\alpha+nn'\tau)))$$

Let us show that $L_\alpha = L_a$. Consider a slight modification of the Jacobi θ-function ($z = exp(2\pi i\xi)$):

$$\theta(\xi) = \theta(\xi;\tau) = q^{1/12}z^{-\frac{1}{2}}\prod_{j\geq 0}(1-q^jz)\prod_{j>0}(1-q^jz^{-1})$$

Then $\dfrac{\theta(\xi-\alpha)}{\theta(\xi)\theta(\alpha)}$ is a section of L_α with the required divisor $(a)-(0)$.

Our recipe for the calculation of the element

$$<(a)-(0),(a)-(0)> \in (T_0^*E)^{\otimes 2}\otimes_{k^*}[a\cdot a]$$

leads to the expression

$$\frac{d\xi\cdot\theta'(0)}{\theta^2(\alpha)}\left(\frac{\theta(-\alpha)}{d\xi\cdot\theta'(0)\theta(\alpha)}\right)^{-1} = -\left(\frac{d\xi\cdot\theta'(0)}{\theta(\alpha)}\right)^2.$$

Notice that $\theta'(0) = 2\pi i\eta^2(\tau)$ and the chosen analytic trivialization of $(T_0^*E)^2$ is $(2\pi i)^2\eta(\tau)^4(d\xi)^2$. So the final answer is $-\theta(\alpha)^{-2}$.

2. The Tate curves. Let K be a field complete with respect to a discrete valuation v. Let $\mathcal{O} = \{a \in K | v(a) \leq 0\}$ be the ring of integers, $I = \{a \in K | v(a) < 0\}$ the maximal ideal, $k = \mathcal{O}/I$ the residue field.

Let $q \in I$. According to Tate, the group $K^*/q^{\mathbb{Z}}$ is isomorphic to the group of points of the elliptic curve E_q over K given by equation $y^2 + xy = x^3 + a_4 x + a_6$, where

$$a_4 = -5 \sum_{j \geq 1} \frac{j^3 q^j}{1 - q^j}; \quad a_6 = -\frac{1}{12} \sum_{j \geq 1} \frac{(7j^5 + 5j^3) q^j}{1 - q^j}$$

The discriminant and j-invariant of this curve are: $\Delta = q \prod_{j \geq 1} (1 - q^j)^{24}, \quad j = \frac{1}{q} + 744 + 196884q + \cdots$. The map of $K^*/q^{\mathbb{Z}}$ to the group of points of E_q is defined by the following expressions:

$$x(u) = \sum_{j \in \mathbb{Z}} \frac{q^j u}{(1 - q^j u)^2} - 2 \sum_{j \geq 1} \frac{j q^j}{1 - q^j}; \quad y(u) = \sum_{j \in \mathbb{Z}} \frac{q^{2j} u^2}{(1 - q^j u)^3} + \sum_{j \geq 1} \frac{j q^j}{1 - q^j}.$$

The unity 1 of K^* maps to neutral element 0 of the curve.

Define a function $T(u)$ on K^* by the formula:

$$T(u) = \prod_{j \geq 0} (1 - q^j u) \prod_{j > 0} (1 - q^j u^{-1})$$

This function vanishes on $\{q^{\mathbb{Z}}\}$. It is quasiperiodic: $T(uq) = -u^{-1} T(u)$.

Let $a \in \mathcal{O} \setminus q\mathcal{O}$. Denote by the same symbol its image in E_q. Then a section s of the bundle $\mathcal{O}_{E_q}((a) - (0))$ can be represented by a function f on K^* such that $f(uq) = af(u)$. Indeed, the periodic function $f(u)T(u)(T(ua^{-1}))^{-1}$ has the required divisor.

Like in the analytic case, the total space of the Poincaré line bundle $T_{(a,b)}$ is isomorphic to the quotient $K^* \times K^* \times K^*$ modulo the action of the group $\mathbb{Z} \oplus \mathbb{Z}$ generated by the following transformations:

$$(a, b, \lambda) \to (qa, b, b\lambda); \quad (a, b, \lambda) \to (a, qb, a\lambda).$$

The corresponding group structure on the collection $T_{(a,\cdot)}$ is defined by the law: $(b_1, \lambda_1) \times (b_2, \lambda_2) = (b_1 b_2, \lambda_1 \lambda_2)$.

Let us calculate the expression $< (a) - (0), (a) - (0) >$. The divisor of the section $T(u)^{-1}(T(ua^{-1}))$ equals $(a) - (0)$. The regularized value of this expression at the divisor $(a) - (0)$ is equal to:

$$\frac{-\frac{du}{u}(\prod_{j>0}(1-q^j))^2}{\prod_{j\geq 0}(1-q^j a)\prod_{j>0}(1-q^j a^{-1})} \times \left(\frac{\prod_{j\geq 0}(1-q^j a^{-1})\prod_{j>0}(1-q^j a)}{-\frac{du}{u}(\prod_{j>0}(1-q^j))^2} \right)^{-1} =$$

$$= -\left(\frac{\frac{du}{u}(\prod_{j>0}(1-q^j))^2}{a^{-\frac{1}{2}}\prod_{j\geq 0}(1-q^ja)\prod_{j>0}(1-q^ja^{-1})}\right)^2.$$

In this calculation notice that $(1-u)\cdot u = -(u-1)$ modulo $(u-1)^2$, so the factor $1-u$ leads to $-\frac{du}{u}$.

The trivialization of $(T_0^*E)^{\otimes 12}$ is defined by the section

$$\left(\frac{du}{u}\right)^{12}\Delta = \left(\left(\frac{du}{u}\right)^2 q^{\frac{1}{6}}(\prod_{j>0}(1-q^j))^4\right)^6$$

Hence the needed expression is:

$$-\left(\frac{\frac{du}{u}(\prod_{j>0}(1-q^j))^2}{a^{-\frac{1}{2}}\prod_{j\geq 0}(1-q^ja)\prod_{j>0}(1-q^ja^{-1})}\right)^2\left(\left(\frac{du}{u}\right)^2 q^{\frac{1}{6}}(\prod_{j>0}(1-q^j))^4\right)^{-1}$$

$$= -\left(q^{\frac{1}{12}}a^{-\frac{1}{2}}T(a)\right)^{-2}.$$

3. Calculation of the canonical height. a) *Archimedean case.* Let us calculate the archimedean height. The torsor $T_{(a,0)}$ is trivialized. So the group $T_{(a,\cdot)}$ is isomorphic to the quotient of $\mathbb{C}\times\mathbb{C}^*$ with coordinates (β,λ) by the action of the group $\mathbb{Z}+\mathbb{Z}\tau$:

$$1\colon(\beta,\lambda)\to(\beta+1,\lambda),\quad \tau\colon(\beta,\lambda)\to(\beta+\tau,\lambda\times\exp(2\pi i\alpha))$$

A homomorphism $|(\cdot,\cdot)|_\alpha : \mathbb{C}\times\mathbb{C}^*\to\mathbb{R}^+$ which is invariant under the action of $\mathbb{Z}+\mathbb{Z}\tau$ and coincides with the norm $|\cdot|$ on $\mathbb{C}^*$ if $\beta=0$ is given by

$$|(\beta,\lambda)|_\alpha = \exp(-\pi i\frac{(\alpha-\bar\alpha)(\beta-\bar\beta)}{\tau-\bar\tau})|\lambda|$$

It defines a homomorphism $T_{(\alpha,\cdot)}\to\mathbb{R}^+$. Therefore the value of our height at α equals to

$$\log|(\alpha,-(\frac{1}{\theta(\alpha)})^2)|_\alpha = 2[-(\log|\theta(\alpha)| + \frac{\pi i}{2}\frac{(\alpha-\bar\alpha)^2}{\tau-\bar\tau})]$$

It coincides with 2 times the canonical Néron height from [Sil].

b) *The nonarchimedean case.* Let $q \in I^n \setminus I^{n+1}$. The Tate curve over $Spec(\mathcal{O})$ has a singular fiber over $Spec(k)$. The singular fiber $C = \cup C_j$ of the minimal Neron model is an n-gon. The set $I^j \backslash I^{j+1}$ represents the points of the curve E over $Spec(K)$ whose restrictions to the singular fiber belong to the j-th component C_j of the n-gon.

Consider the map

$$\tilde{h}_v : K^* \times K^* \times K^* \to \mathbb{Z}; \quad \tilde{h}_v(a, b, \lambda) = v(q)v(\lambda) - v(a)v(b).$$

This map is invariant with respect to the action of $\mathbb{Z} \oplus \mathbb{Z}$; therefore it detemines a map h_v from $T_{(\cdot,\cdot)}$ to $\mathbb{Z}$. The map h_v is a group homomorphism on $T_{(a,\cdot)}$, its image is discrete, hence its kernel is a maximal compact subgroup.

Let $v(a) = j$. The value of this map on $< (a) - (0), (a) - (0) >$ equals to:

$$v(q)v(\theta(a)^{-2}) - v(a)v(a) = n \cdot (-2) \cdot (\frac{1}{12}n - \frac{1}{2}j + v(1-a)) - j^2$$

$$= -\frac{1}{6}n^2 + jn - j^2 - 2nv(1-a) = -n^2 B_2(\frac{j}{n}) - 2nv(1-a)$$

where $B_2(x) = x^2 - x + 1/6$ is the second Bernoulli polynomial. (Compare with the integrality condition given by the third Bernoulli polynomial).

4. Functoriality of the groups $B_2(E)_{\mathbb{Q}}$ under the isogenies. Let n be an integer prime to the characteristic of k. Let $\lambda : E_1 \to E_2$ be an isogeny of order n between the elliptic curves E_1 and E_2.

i) *Pull back* $\lambda_2^* : B_2(E_2) \to B_2(E_1)[\frac{1}{n}]$. Let $\hat{\lambda}$ be the dual isogeny $J_{E_2} \to J_{E_1}$. The pull back of the basic extension $0 \to k^* \to B_2(E_1) \to S^2 J_{E_1} \to 0$ (considered modulo n-torsion) under the homomorphism $\frac{1}{n}\hat{\lambda} \cdot \hat{\lambda} : S^2 J_{E_2} \to S^2 J_{E_1}$ provides the map λ_2^*:

$$\begin{array}{ccccccccc} 0 & \to & k^* & \to & B_2(E_2) & \to & S^2 J_{E_2} & \to & 0 \\ & & || & & \downarrow \lambda_2^* & & \downarrow \frac{1}{n}\hat{\lambda} \cdot \hat{\lambda} & & \\ 0 & \to & k^* & \to & B_2(E_1)[\frac{1}{n}] & \to & S^2 J_{E_1}[\frac{1}{n}] & \to & 0 \end{array}$$

A more direct description of λ_2^* can be spelled as follows. We have a natural morphism of k^*- torsors:

$$\lambda^* : [L, M]^{\otimes n} \to [\lambda^* L, \lambda^* M]; \quad < s_1, s_2 >^{\otimes n} \quad \to \quad < \lambda^{-1} s_1, \lambda^{-1} s_2 >$$

To check that it is a map of torsors notice that

$$(< f \cdot s_1, s_2 >)^{\otimes n} = f(s_2)^n (< s_1, s_2 >)^{\otimes n}$$

and $f(s_2)^n = (\lambda^{-1} f)(\lambda^{-1} s_2)$. It is easy to see that these maps provide the pull-back map:

$$\lambda_2^* : B_2(E_2) \to B_2(E_1)[\frac{1}{n}]; \quad < s_1, s_2 > \to < \lambda^{-1} s_1, \lambda^{-1} s_2 >^{\otimes \frac{1}{n}} .$$

Theorem 4.1 $n \cdot \left(\lambda_2^* \{\lambda(a)\}_2 - \sum_{\gamma \in Ker\lambda} \{a + \gamma\}_2 \right) = 0$

Proof. Any isogeny can be presented as the composition of cyclic isogenies. So we can assume that λ is cyclic. The expression

$$f_n(a) := n \cdot \left(\lambda_2^* \{\lambda(a)\}_2 - \sum_{\gamma \in Ker\lambda} \{a + \gamma\}_2 \right)$$

is a nonvanishing function on the noncompact curve $E_1 \setminus Ker\lambda$.

Let $p_n : \mathcal{E} \to X_0(n)$ be the universal family of elliptic curves over the modular curve $X_0(n)$, and $\Lambda : \mathcal{E} \to \mathcal{E}$ the universal cyclic n-isogeny. The curves E_1 and E_2 are the fibers of $\mathcal{E}$ over the points $(E_1, Ker\lambda)$ and $(E_2, E_2[n]/Ker\lambda)$ of $X_0(n)$, and λ is the restriction of Λ. The construction above defines an algebraic function F on the universal curve $\mathcal{E}_1$.

Consider the punctured formal neighborhood of the cusp point in which the universal isogeny is totally ramified. The j-invariant of the restriction of the universal curves to this neighborhood has a pole at the cusp; hence, this restriction can be described by the *Tate* curves $E_q \to E_{q^n}$ ([Sil]).

Let us prove that F_n equals 1 for the Tate curves. We are dealing with the isogeny $K^*/(q^n)^{\mathbb{Z}} \to K^*/q^{\mathbb{Z}}$. For the Tate curve we expressed in s. 4.2 the pairing $< (a) - (0), (a) - (0) >$ in terms of the θ-function

$$\theta_q(a) := q^{\frac{1}{12}} a^{-\frac{1}{2}} T(a) = q^{1/12} a^{-\frac{1}{2}} \prod_{j \geq 0} (1 - q^j a) \prod_{j > 0} (1 - q^j a^{-1}).$$

Remark. $\theta_q(a)$ is defined only up to a choice of sign, so only $(\theta_q(a))^2$ makes sense.

So we need to prove the following proposition.

Proposition 4.2

$$\left(\frac{\prod_{0 \leq k < n} \theta_q(t \cdot q^k)}{\left(\prod_{0 \leq k < n} \theta_{q^n}(t \cdot q^k) \right)^n} \right)^2 = 1 \qquad (37)$$

It shows that the restriction of the function F_n to the preimage of neighborhood of the cusp point equals 1. Hence this function is equal to 1 on all universal curve $X_0(N)$; the function f_n is the restriction of F_n to the fiber E; therefore $f_n = 1$.

Proof of the proposition. The θ-function has the following property:

$$\theta_q(t \cdot q^l) = (-1)^k a^{-k} q^{-k^2/2} \theta_q(t)$$

Therefore $\prod_{0\leq k<n} \theta_q(t \cdot q^k)$ equals to

$$(-1)^{s_1(n)} t^{-s_1(n)} q^{-s_2(n)/2} q^{n/12} t^{-n/2} \prod_{j\geq 0}(1-q^j t)^n \prod_{j>0}(1-q^j t^{-1})^n \tag{38}$$

Using the definition and notations

$$s_1(n) := 1+...+(n-1) = \frac{n(n-1)}{2}; \quad s_2(n) := 1^2+...+(n-1)^2 = \frac{(n-1)n(2n-1)}{6}$$

we have

$$\theta_{q^n}(t \cdot q^k) = q^{n/12} t^{-1/2} q^{-k/2} \prod_{j\geq 0}\left(1 - q^{nj} \cdot tq^k\right) \prod_{j>0}\left(1 - q^{nj} \cdot t^{-1} q^{-k}\right)$$

On the other hand $\prod_{0\leq k<n} \theta_{q^n}(t \cdot q^k)$ is equal to

$$q^{\frac{n^2}{12} - \frac{s_1(n)}{2}} t^{\frac{-n}{2}} \cdot \prod_{0\leq k<n} \prod_{j\geq 0}\left(1 - q^{nj+k} t\right) \prod_{0\leq k<n} \prod_{j>0}\left(1 - q^{nj-k} t^{-1}\right) =$$

$$q^{\frac{n^2}{12} - \frac{s_1(n)}{2}} t^{\frac{-n}{2}} \prod_{j'\geq 0}(1 - q^{j'} t) \prod_{j'>0}(1 - q^{j'} t^{-1}) \tag{39}$$

Comparing (38) and (39) we see that it remains to check that

$$\left(q^{\frac{n^2}{12} - \frac{s_1(n)}{2}} t^{-n/2}\right)^n = q^{-s_2(n)/2 + \frac{n}{12}} \cdot t^{-\frac{n}{2} - s_1(n)}$$

The statement of the proposition follows.

In particular when $\lambda[m]$ is the isogeny of multiplication by m the theorem gives

Corollary 4.3 *Suppose $\bar{k} = k$. Then for any $a \in E(k)$ one has the "distribution relation"*

$$m(\{a\}_2 - \sum_{mb=a} \{b\}_2) = 0 \tag{40}$$

Remark. In this case one can define $[m]^* : B_2(E_2) \to B_2(E_1)[\frac{1}{m}]$ using the map $\frac{1}{m}[m] \circ \frac{1}{m}[m]$ in the diagram defining λ_2. Thus we have the factor m instead of m^2 in the formula (40).

ii) *The transfer map* $\lambda_{2*} : B_2(E_1)[\frac{1}{n}] \to B_2(E_2)[\frac{1}{n}]$. Notice that the group $B_2(E)_{\mathbb{Q}}$ does not satisfy the descent property. Namely, if $k \subset K$ is a finite Galois extension then

$$B_2(E/k)_{\mathbb{Q}} \hookrightarrow B_2(E/K)_{\mathbb{Q}}^{Gal(K/k)}$$

but this inclusion is not an isomorphism because the group $S^2J(k)_{\mathbb{Q}}$ does not have the descent property.

Suppose $k = \bar{k}$. The transfer map λ_{2*} should satisfy the projection formula

$$\lambda_2^* \circ \lambda_{2*} = n \cdot Id \tag{41}$$

and should fit into the following diagram, considered modulo n-torsion:

$$\begin{array}{ccccccccc}
0 & \to & k^* & \to & B_2(E_1) & \to & S^2J_{E_1} & \to & 0 \\
 & & \downarrow m_n & & \downarrow \lambda_{2*} & & \downarrow \lambda \cdot \lambda & & \\
0 & \to & k^* & \to & B_2(E_2) & \to & S^2J_{E_2} & \to & 0
\end{array}$$

Here $m_n : x \to x^n$. This is necessary in order to have (41) on the subgroup $k^* \subset B_2(E)$.

Let us define the transfer map as follows:

$$\lambda_{2*}\{a\}_2 := \{\lambda(a)\}_2 - \Big(\lambda_2^*\{\lambda a\}_2 - n\{a\}_2\Big) \tag{42}$$

Remark. Projection of $\lambda_2^*\{\lambda a\}_2 - n\{a\}_2$ to S^2J equals $\frac{1}{n}\hat{\lambda} \circ \lambda(a \cdot a) - na \cdot a = 0$, so $\lambda_2^*\{\lambda a\}_2 - n\{a\}_2 \in k^*$.

Let us show that formula (42) provides a transfer homomorphism. Suppose that $\sum\{a_i\}_2 + c = 0$ in the group $B_2(E_1)$, where $c \in k^*$ (we write the group $B_2(E)$ additively). We have to prove that

$$\sum_i \{\lambda(a_i)\}_2 - \Big(\sum_i \lambda_2^*\{\lambda(a_i)\}_2 - \sum_i n\{a_i\}_2\Big) + nc = 0$$

By the assumption $\sum_i\{a_i\}_2 = c^{-1} \in k^*$. As we have shown before, the expression in brackets always belongs to the subgroup $k^* \subset B_2(E_2)$. Therefore $\sum_i \lambda_2^*\{\lambda(a_i)\}_2 \in k^*$. Notice that λ_2^* is injective modulo n-torsion and is the identity on the subgroup $k^* \subset B_2(E)$. Therefore modulo n-torsion $\sum_i \{\lambda(a_i)\}_2 \in k^*$ and $\sum_i \{\lambda(a_i)\}_2 = \sum_i \lambda_2^* \{\lambda(a_i)\}_2$.

Using proposition (4.1) one can easyly see that

$$\lambda_{2*}\{a\}_2 := \{\lambda(a)\}_2 - \frac{1}{n} \sum_{\gamma \in Ker\lambda} \Big(n\{a+\gamma\}_2 - n\{a\}_2 \Big) \tag{43}$$

5. A presentation of the group $B_2(E)$ by generators and relations. Let $\mathcal{P}(a)$ (resp $\mathcal{P}^{'}(a)$) be the x-coordinate (resp. y-coordinate) of the point $a \in E$ in Tate's normal form of an elliptic curve over an arbitrary field k:

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

Proposition 4.4 . *a) If $a \neq b$ then*

$$< (a+b)-(0), (a+b)-(0) > \otimes < (a-b)-(0), (a-b)-(0) >$$

$$\otimes < (a)-(0), (a)-(0) >^{-2} \otimes < (b)-(0), (b)-(0) >^{-2} = (\Delta^{-1/6}(\mathcal{P}(a) - \mathcal{P}(b)))^{-2}$$

b). If $a = b$ but $2a \neq 0$ then the left hand side is equal to $(\Delta^{-1/4}\mathcal{P}'(a))^{-2}$. If $2a = 0$ then we get $(\Delta^{-1/3}\mathcal{P}''(a))^{-2}$

Proof. We will prove part a). Part b) is similar. Let L_a be the line bundle corresponding to the divisor $(a) - (0)$. Evidently

$$[L_{a+b}, L_{a+b}] \otimes [L_{a-b}, L_{a-b}] \otimes [L_a, L_a]^{-2} \otimes [L_b, L_b]^{-2} = k^*$$

so

$$< (a+b)-(0), (a+b)-(0) > \otimes < (a-b)-(0), (a-b)-(0) >$$

$$\otimes < (a)-(0), (a)-(0) >^{-2} \otimes < (b)-(0), (b)-(0) >^{-2} \in k^* \otimes (T_0^* E)^{\otimes -4}$$

We want to calculate this element. One has

$$< (a+b)-(0), (a+b)-(0) > \otimes < (a-b)-(0), (a-b)-(0) >$$

$$\otimes < (a)-(0), (a)-(0) >^{-2} \otimes < (b)-(0), (b)-(0) >^{-2} =$$

$$< (a+b)-(a), (a+b)-(a) > \otimes < (a+b)-(a), (a)-(0) >^2$$

$$\otimes < (a-b)-(a), (a-b)-(a) > \otimes < (a-b)-(a), (a)-(0) >^2$$

$$\otimes < (b)-(0), (b)-(0) >^{-2}$$

We have

$$< (a+b) - (b), (a+b) - (b) > \otimes < (a) - (0), (a) - (0) >^{-1} = 1 \in k^* \quad (44)$$

Indeed, the left hand side is a regular function in b on the elliptic curve and so it is a constant; its value at $b = 0$ is 1.

Therefore the first, third and last terms of the expression above that we need to compute cancel thanks to (44) and we get:

$$< (a+b) + (a-b) - 2(a), (a) - (0) >^2$$

Notice that $(a+b)+(a-b)-2(a)$ is the divisor of the function $\Delta^{-1/6}(\mathcal{P}(\xi - a) - \mathcal{P}(b))$. Its value at the point 0 is $\Delta^{-1/6}(\mathcal{P}(a) - \mathcal{P}(b))$ and its generalized value at the point a is the trivialization we have chosen.

Corollary 4.5 *Assume $k = \bar{k}$. Then the homomorphism (7) is surjective.*

Let us denote by $R_2(E)$ the kernel of the homomorphism (7). Then

$$B_2(E) := \frac{\mathbb{Z}[E(k)\backslash 0]}{R_2(E)}$$

Let $\tilde{R}$ be the subgroup of $[E(k)\backslash 0]$ generated by the elements

$$\{a, b\} := \{a+b\} + \{a-b\} - 2\{a\} - 2\{b\}$$

Notice that $\{a, a\} = \{2a\} - 4\{a\}$ and $\{a, a\} - \{a, -a\} = 2(\{a\} - \{-a\})$.

Lemma 4.6 *For any abelian group A the elements $\{a, b\}$ and $\{a\} - \{-a\}$ generate modulo 2-torsion the kernel of the surjective homomorphism $\mathbb{Z}[A] \longrightarrow S^2A \quad \{a\} \longmapsto a \cdot a$.*

We will not use this fact later, so a (simple) proof is omitted.

Consider the homomorphism $\tilde{R} \longrightarrow k^*$ defined by the formulas

$$\{a, b\} \longmapsto \Delta^{-1/3}(\mathcal{P}(a) - \mathcal{P}(b))^2, \quad a \neq b; \qquad \{a, a\} \longmapsto \Delta^{-1/2}(\mathcal{P}'(a))^2, \quad 2a \neq 0$$

and $\{a, a\} \longmapsto (\Delta^{-1/3}\mathcal{P}''(a))^2$ if $2a = 0$. Thanks to corollary (4.5) this homomorphism is well defined. By definition the subgroup $R_2(E)$ is its kernel.

Remark. This is a homomorphism to the *multiplicative* group k^* of the field k defined via the additive structure of k.

6. A remark on the differential in the complex $B^*(E,3)$. The restriction of δ_3 to the subgroup $B_3^*(E)$ can be defined directly, without referring to the group $B_2(E)$ and the homomorphism h. A more complicated formula is the price we pay.

Set for general $a_i \in E(k)$

$$\delta_3\Big((\{a_1\}-\{0\}) * (\{a_2\}-\{0\}) * (\{a_3\}-\{0\}) * (\{a_4\}-\{0\})\Big) = \qquad (45)$$

$$\frac{[\mathcal{P}(a_1+a_2)-\mathcal{P}(a_3-a_4)][\mathcal{P}(a_1+a_3)-\mathcal{P}(a_4)][\mathcal{P}(a_1+a_4)-\mathcal{P}(a_3)]}{[\mathcal{P}(a_1)-\mathcal{P}(a_3-a_4)][\mathcal{P}(a_1+a_2+a_3)-\mathcal{P}(a_4)][\mathcal{P}(a_1+a_2+a_4)-\mathcal{P}(a_3)]} \otimes -1/2 \cdot a_1 + ...$$

where ... means three other terms obtained by cyclic permutation of indices. Here $\mathcal{P}(a) := x(a)$ is the x-coordinate of a point a.

The expression for (45) is symmetric in $a_1, ..., a_4$, which is not obvious from the formula. Over $\mathbb{C}$ one can rewrite the right hand side of (45) in a more symmetric way using the θ-function:

$$\frac{\theta(a_1+a_2+a_3+a_4)\theta(a_1+a_2)\theta(a_1+a_3)\theta(a_1+a_4)}{\theta(a_1+a_2+a_3)\theta(a_1+a_2+a_4)\theta(a_1+a_3+a_4)\theta(a_1)} \otimes a_1 + ...$$

Morally the differential δ_3 is given by the "formula" $\{a\} \longmapsto -\frac{1}{2}\theta(a) \otimes a$ which, unfortunately, makes no sence if we don't use the group $B_2(E)$. The relation with (45) is given by the classical formula

$$\mathcal{P}(a) - \mathcal{P}(b) = \frac{\theta(a+b)\theta(a-b)}{\theta^2(a)\theta^2(b)}, \quad a \neq \pm b$$

REFERENCES

[BT] Bass H., Tate J., *The Milnor ring of a global field* Springer Lect. Notes in Math., vol. 342 p. 389-447.

[B1] Beilinson A.A.: *Higher regulators and values of L-functions*, VINITI, 24 (1984), 181–238 (in Russian); English translation: J. Soviet Math. 30 (1985), 2036–2070.

[B2] Beilinson A.A.: *Higher regulators of modular curves* Contemp. Math. 55, 1-34 (1986)

[BD2] Beilinson A.A., Deligne P, *Interprétation motivique de la conjecture de Zagier* in Symp. in Pure Math., v. 55, part 2, 1994,

[BL] Beilinson A.A., Levin A.M.: *Elliptic polylogarithms.* Symposium in pure mathematics, 1994, vol 55, part 2, 101-156.

[Bl1] Bloch S.: *Higher regulators, algebraic K- theory and zeta functions of elliptic curves*, Lect. Notes U.C. Irvine, 1977.

[Bl2] Bloch S.: *A note on height pairings, Tamagawa numbers and the Birch and Swinnerton- Dyer conjecture* Invent. Math. 58 (1980) 65-76.

[BG] Bloch S., Grayson D.: *K_2 and L-functions of elliptic curves* Cont. Math. 55, 79-88 (1986)

[De] Deligne P. *La déterminant de la cohomologie* Contemp. Math. 67 93-178, (1987).

[DS] Dupont J., Sah S.H.: *Scissors congruences II*, J. Pure Appl. Algebra, v. 25, (1982), 159–195.

[G1] Goncharov A.B.: *Polylogarithms and motivic Galois groups* Symposium in pure mathematics, 1994, vol 55, part 2, 43 - 96.

[G2] Goncharov A.B.: *Notes on elliptic motivic complexes.* Preprint 1997.

[La] Lang. S.: *Foundations of diophantian geometry* Springer Verlag

[Lev] Levine M. *Relative Milnor K-theory* K-theory, 6 (1992).

[Li] Lichtenbaum S.: *Groups related to scissor congruence groups* Contemporary math., 1989, vol. 83, 151-157.

[M] Milnor J.: *Introduction to algebraic K-theory* Ann. Math. Studies, Princeton, (1971).

[MT] Mazur B., Tate J.: *Biextensions and height pairings* Arithmetic and Geometry, Birkhauser 28, v.1, (1983) 195-237.

[RSS] Rappoport M., Schappacher N., Schneider P.: *Beilinson's conjectures on special values of L-functions*, Perspectives in Mathematics, vol 4, New York: Academic Press, 1988.

[SS] Schappacher N., Scholl, A.: *The boundary of the Eisenstein symbol* Math. Ann. 1991, 290, p. 303-321.

[SS2] Schappacher N., Scholl, A.: *Beilinson's theorem on modular curves* in [RSS]

[Sil] Silverman J.: *Advanced topics in the arithmetic of elliptic curves* Springer Verlag, 1994.

[S] Suslin A.A.: K_3 *of a field and Bloch's group*, Proceedings of the Steklov Institute of Mathematics 1991, Issue 4.

[We] Weil A. : *Elliptic functions according to Eisenstein and Kronecker* Ergebnisse der Mathematik, 88, Springer 77.

[W] Wildeshaus J. : *On an elliptic analog of Zagier's conjecture* Preprint (version of June 1996). To appear in Duke Math. Journal 1997.

[Za] Zarkhin Yu.G. *Néron pairing and quasicharacters* Izvestia Acad. Nauk USSR 36, N2 (1972) 497-509.

[Z] Zagier D.: *The Bloch-Wigner-Ramakrishnan polylogarithm function* Math. Ann. 286, 613-624 (1990)

[Z2] Zagier D.: *Polylogarithms, Dedekind zeta functions and the algebraic K-theory of fields*, Arithmetic Algebraic Geometry (G.v.d.Geer, F.Oort, J.Steenbrink, eds.), Prog. Math., Vol 89, Birkhauser, Boston, 1991, pp. 391–430.

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