(SEMI)SIMPLE EXERCISES IN QUANTUM COHOMOLOGY

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Abstract. The paper is dedicated to the study of algebraic manifolds whose quantum cohomology or a part of it is a semisimple Frobenius manifold. Theorem 1.8.1 says, roughly speaking, that the sum of (p, p)–cohomology spaces is a maximal Frobenius submanifold that has chances to be semisimple. Theorem 1.8.3 provides a version of the Reconstruction theorem, assuming semisimplicity but not H^2–generation. Theorem 3.6.1 establishes the semisimplicity for all del Pezzo surfaces, providing an evidence for the conjecture that semisimplicity is related to the existence of a full system of exceptional sheaves of the appropriate length. Finally, in §2 we calculate special coordinates for three families of Fano threefolds with minimal cohomology.

§0. Introduction

0.1. Role of semisimplicity. This paper is a contribution to the study of algebraic manifolds whose quantum cohomology (or at least its appropriate subspace) is a generically semisimple Frobenius manifold. This class of manifolds is important in many respects. For example, one expects that the higher genus correlators and the correlators with gravitational descendants for such manifolds can be calculated entirely in terms of the Frobenius structure, when it is semisimple (see [DuZh], [G2], and the references therein). Unfolding spaces of the hypersurface singularities (Landau–Ginzburg models) are generically semisimple as well, therefore any mirror isomorphism between such an unfolding space and (a part of) quantum cohomology must involve semisimple cohomology. Finally, to establish such an isomorphism, we have to compute for both Frobenius manifolds only finitely many constants (monodromy data, or special coordinates) whereas one generally needs infinitely many numbers in order to describe a non–semisimple manifold (like the quantum cohomology of a quintic).

In this paper, we focus on the values of Dubrovin's canonical coordinates (u_i) at the points of H^2 that is, parameter space of the small quantum cohomology. As soon as their identification at a tame semisimple points is made, the remaining part of the mirror picture requires a choice of the flat metric compatible with multiplication. Unless it is explicitly given, we have to recourse to one of the following equivalent data describing it in an implicit way:

(i) Values of the diagonal coefficients of the flat metric $\sum_i \eta_i (du^i)^2$ at a tame semisimple point completed by the values of its first derivatives η_{ij}, that is, initial data for the second structure connection (cf. [Ma2], II.3).

(ii) Monodromy data for the first structure connection and oscillating integrals for the deformed flat coordinates (cf. [G1], [Du] and the references therein).

(iii) Choice of one of K. Saito's primitive forms.

(iv) Choice of a filtration on the cohomology space of the Milnor fiber (M. Saito, cf. [He2] and the references therein).

(v) Use of the semi–infinite Hodge structure (this is a far–reaching refinement of (iii), cf. [Bar]).

In this paper we discuss only (i).

0.2. General notation for quantum cohomology. Let V be a smooth projective manifold. We choose a homogeneous basis of its cohomology space (Δ_a), $\Delta_a \in H^{|\Delta_a|}(V)$ where a runs over a set of indices such that Δ_0 always denotes the dual class of $[V]$. The dual coordinates on $H^*(V)$ are denoted (x_a). The Poincaré form $g_{ab} = (\Delta_a, \Delta_b)$ allows us to raise indices, e. g. $\Delta^a := \sum_b g^{ab}\Delta_b$. The letter β generally denotes a variable algebraic element of the group $H_2(V, \mathbf{Z})/(tors)$, and q^β is the respective element of the Novikov ring. We write k or $k(V)$ for $c_1(V)$ and $k(\beta)$ for $(c_1(V), \beta)$.

The genus zero Gromov–Witten invariants are written as correlators, that is polylinear functions indexed by β:

$$H^*(V, \mathbf{Q})^{\otimes n} \to \mathbf{Q}: \ \Delta_{a_1} \otimes \dots \otimes \Delta_{a_n} \mapsto \langle \Delta_{a_1} \dots \Delta_{a_n} \rangle_\beta .$$

For $n = 0$ we write the respective correlator as $\langle \emptyset \rangle_\beta$. If $\dim V \geq 3$, we have $\langle \emptyset \rangle_\beta = 0$.

Sometimes it is convenient to consider the total correlators with values in the completed Novikov ring: $\langle \Delta_{a_1}, \dots, \Delta_{a_n} \rangle := \sum_\beta \langle \Delta_{a_1}, \dots, \Delta_{a_n} \rangle_\beta \, q^\beta$ and extend them to the completed tensor algebra of $H^*(V)$. The potential of quantum cohomology is the formal series in (x_a) over the Novikov ring which can be compactly written in this notation as $\Phi := \langle e^{\sum_a x_a \Delta_a} \rangle$. The quantum multiplication table is given by $\Delta_a \circ \Delta_b = \sum_c \Phi_{abc} \Delta^c$ where $\Phi_{abc} = \partial_a \partial_b \partial_c \Phi$, $\partial_a = \partial/\partial x_a$. Using the standard properties of the Gromov–Witten invariants (symmetry, divisor axiom and identity axiom) we can rewrite this in the following form convenient for further calculations:

$$\Delta_a \circ \Delta_b = \Delta_a \cup \Delta_b +$$
$$\sum_{\beta \neq 0} \sum_{c \neq 0} \langle \Delta_a \Delta_b \Delta_c \, e^{\sum_{k:\, |\Delta_k| > 2} x_k \Delta_k} \rangle_\beta \Delta^c \, e^{(\sum_{k:\, |\Delta_k| = 2} x_k \Delta_k, \beta)} q^\beta . \tag{0.1}$$

Since $\beta \mapsto e^{(\sum_{k:\, |\Delta_k| = 2} x_k \Delta_k, \beta)} q^\beta$ is a generic character of $H_2(V, \mathbf{Z})/(tors)$, as well as $\beta \mapsto q^\beta$, we will allow ourselves in the future to use (0.1) omitting $e^{(\sum_{k:\, |\Delta_k| = 2} x_k \Delta_k, \beta)}$ and instead interpreting q^β as a monomial function on the algebraic torus T_V with the character group $H^2(V, \mathbf{Z})/(tors)$.

0.3. Plan of the paper. We start §1 with a review of general properties of semisimple Frobenius manifolds. Actually, a natural context for this discussion is the realm of F–manifolds, that is, "Frobenius manifolds without metric", introduced in [HeMa]. We recall their definition and the main properties related to the semisimplicity in 1.1–1.7.

In 1.8 we then prove the first main result of this paper, Theorem 1.8.1. Roughly, it says that the most natural candidate for semisimple quantum cohomology is not the whole cohomology of a Fano manifold, but only its (p,p)–part. The second result, Theorem 1.8.3, is a new version of the First Reconstruction Theorem in [KoMa]. Finally, we discuss the special coordinates for Fano manifolds with minimal (p,p)–cohomology (rank one in every dimension).

In §2 we provide explicit canonical coordinates and the initial data for the second structure connection for three families of Fano threefolds with minimal (p,p)–cohomology. They can be useful for finding Landau–Ginzburg superpotentials and primitive forms for them. Since these threefolds are not toric, no clear–cut prescription is available for doing this.

The whole §3 is dedicated to the quantum cohomology of the del Pezzo surfaces. This subject was already treated in several papers, see [GP] and the references therein. We systematically use the symmetry properties of the quantum cohomology with respect to the Weyl groups, established in [GP]. This allows us to encode a large amount of numerical data in rather compact tables and, more important, to perform calculations in quantum cohomology in terms of the geometry of root systems and reflections. Since the expected mirror picture involving the Lagrangian vanishing cycles is also naturally formulated in this way, we feel that this is the right language.

The principal result of §3 is the Theorem 3.6, establishing generic semisimplicity of the quantum cohomology of all del Pezzo surfaces V_r. For $r \leq 3$ when V_r are toric, the point $q = 1$ turns out to be tame semisimple. This ceases to be true for $r \geq 5$, and instead we study the behavior of quantum cohomology near the boundary of the partial compactification of the torus T_{V_r}, where the exponentiated coordinate corresponding to an exceptional curve vanishes.

For this proof, the calculations of 3.3–3.5 are not needed, and the reader may prefer to skip them. On the other hand, they might be useful for finding the Landau–Ginzburg superpotentials in the non–toric cases.

Acknowledgement. We are thankful to Lothar Göttsche who provided us with several values of Gromov–Witten invariants missing in [GP], and to A. Bondal, who allowed us to use the tables of the Gromov–Witten invariants of certain Fano threefolds, made by him, A. Kuznetsov, and D. Orlov.

§1. Semisimplicity and F–manifolds

1.1. Definition. *An F–manifold is a triple $(M, \circ, e)$, where M is a manifold and $\circ$ is an associative supercommutative $\mathcal{O}_M$–bilinear multiplication $\mathcal{T}_M \times \mathcal{T}_M \to \mathcal{T}_M$ with identity vector field e, satisfying the following axiom: for any (local) vector fields X, Y, Z, W we have*

$$[X \circ Y, Z \circ W] - [X \circ Y, Z] \circ W - Z \circ [X \circ Y, W]$$

$$-X \circ [Y, Z \circ W] + X \circ [Y, Z] \circ W + X \circ Z \circ [Y, W]$$

$$-Y \circ [X, Z \circ W] + Y \circ [X, Z] \circ W + Y \circ Z \circ [X, W] = 0 \qquad (1.1)$$

This notion makes sense in any of the standard categories of manifolds, and also supermanifolds, if appropriate signs are introduced in the structural identity, cf. [HeMa]. Below we will assume that M is a complex manifold.

A shorter formulation of (1.1) is

$$\mathrm{Lie}_{X \circ Y}(\circ) = X \circ \mathrm{Lie}_Y(\circ) + Y \circ \mathrm{Lie}_X(\circ) \qquad (1.2)$$

where X, Y are any two local vector fields, and $\circ$ is considered as a tensor.

1.1.1. Question. *Does the quantum multiplication in K–theory (whose construction is sketched by Givental in [G3]) satisfy (1.1)–(1.2)?*

Returning to F–manifolds, we will say that $x \in M$ is a *semisimple point* of the F–manifold M, if the following equivalent conditions are satisfied:

(i) *T_xM endowed with the multiplication $\circ$ is a (commutative) semisimple algebra, that is, isomorphic to $\mathbf{C}^n$, $n = \dim M$, with componentwise multiplication.*

(ii) *In a neighborhood of x, there exists a basis of vector fields e_i, $i = 1, \dots, n$, such that $e_i \circ e_j = \delta_{ij} e_i$ and $[e_i, e_j] = 0$ for all i, j.*

$(M, \circ, e)$ is called *generically semisimple*, if its set of semisimple points is open and dense.

¿From (ii) it follows that in a neighborhood of any semisimple point, M admits a system of Dubrovin's canonical coordinates $(u_1, \dots, u_n)$. They have the following property: $e_i := \partial/\partial u_i$ form a complete system of pairwise orthogonal idempotents with respect to $\circ$. Any two local systems of canonical coordinates differ by a constant shift $(u_j + c_j)$ followed by a permutation.

F–manifolds admit a natural product operation:

$$(M_1, \circ_1, e_1) \times (M_2, \circ_2, e_2) := (M_1 \times M_2, \circ_1 \boxplus \circ_2, , e_1 \boxplus e_2).$$

The main structural identity (1.1) can be used as an integrability condition, in order to show the following result ([He1], Theorem 4.2): *for any $x \in M$, the canonical* *decomposition of $(T_xM, \circ)$ into a product of local algebras uniquely extends to the direct decomposition of the germ of (M, x) into the product of irreducible germs of F–manifolds.*

1.2. Euler fields. A semisimple point x endowed with local canonical coordinates u_i is called *tame semisimple*, if $u_i(x) \neq u_j(x)$ for all $i \neq j$. There are two basic setups in which the eventual constant shifts of canonical coordinates can be eliminated and the notion of tame semisimplicity made coordinate–independent.

First, it might happen that the fundamental group of the submanifold of semisimple points acts transitively on the elements of a local idempotent system (e_i). Then (u_i) can be normalized up to a common shift $(u_i + c)$ and a permutation, and tameness or otherwise with respect to such coordinates is defined unambiguously.

Second, it might happen that the F–manifold $(M, \circ, e)$ is endowed with a structure Euler vector field E (this is the case of quantum cohomology). In the context of F–manifolds, a local vector field E is called an *Euler field of (constant) weight* d_0 if $\mathrm{Lie}_E(\circ) = d_0\circ$ that is, for all local vector fields X, Y we have

$$[E, X \circ Y] - [E, X] \circ Y - X \circ [E, Y] = d_0 X \circ Y. \tag{1.3}$$

Euler fields form a sheaf of Lie algebras, and weight is a linear function on this sheaf. The commutator of two Euler fields has weight zero.

It is easy to describe all Euler fields in a neighborhood of a semisimple point endowed with canonical coordinates (u_i). Namely, writing (1.3) for $X = e_i$, $Y = e_j$, we see that any local Euler field of weight d_0 is of the form

$$E = d_0 \sum_i u_i e_i + \sum_j c_j e_j, \tag{1.4}$$

where c_j are arbitrary constants. Reversing this argument, we see that given an Euler field E of non–zero weight, we can define uniquely up to permutation canonical coordinates (u_i) by the condition $E = d_0 \sum_i u_i e_i$. Moreover, functions $(d_0 u_i)$ can be then invariantly described as the spectrum of the operator $E\circ$ acting upon local vector fields. Notice that in quantum cohomology E is normalized in such a way that $d_0 = 1$, and we will adopt this convention.

1.3. The spectral cover. Let $(M, \circ, e)$ be an F–manifold. By definition, its *spectral cover* is the relative analytic spectrum $L := \mathrm{Specan}\,(\mathcal{T}_M, \circ)$, together with two structure maps $\pi : L \to M$ and $\sigma : \mathcal{T}_M \to \pi_*(\mathcal{O}_L)$. Clearly, σ is an isomorphism of sheaves of $\mathcal{O}_M$–algebras. A point $x \in M$ is semisimple iff π is étale at x.

Moreover, L is endowed with a canonical closed embedding $i : L \to T^*M$ to the complex analytic cotangent bundle of M. This embedding is induced by the map of sheaves of algebras $S(\mathcal{T}_M) \to (\mathcal{T}_M, \circ)$ identical on $\mathcal{T}_M$.

The definition of L, as well as the definition of a semisimple point, does not require $\circ$ to satisfy (1.1). Generic semisimplicity is equivalent to the fact that L is reduced (has no nilpotents in the structure sheaf). If this condition is satisfied, (1.1) becomes equivalent to the condition that L is Lagrangian.

1.4. K. Saito's frameworks and Landau–Ginzburg models. A large class of generically semisimple F–manifolds is furnished by the construction due to K. Saito in the theory of singularities and axiomatized in [Ma1], [Ma2] (for the richer Frobenius structure) under the name of *K. Saito's frameworks.* Briefly, let $p: N \to M$ be a submersion of complex manifolds, generally non–compact. Then N carries the complex of sheaves of relative holomorphic forms $(\Omega^*(N/M), d_{N/M})$. Let Ψ be a closed relative 1–form, and $C \subset N$ the closed analytic subspace $\Psi = 0$ of N. We can define a map $s: \mathcal{T}_M \to p_*(\mathcal{O}_C)$ in the following way. Let X be a local vector field on M, defined in a sufficiently small open subset U. Cover $p^{-1}(U)$ by small open subsets U_j in such a way that on each U_i there exists a lift X_j of X. Then the maps $X \mapsto i_{X_j}(\Psi)$ on U_j glue together and produce a well defined map of sheaves $s: \mathcal{T}_M \to p_*(\mathcal{O}_C)$. (In [Ma1], [Ma2], only the case $\Psi = d_{N/M}F$ was considered, which is not a restriction locally on N but overlooks more global effects).

Assume that s is an isomorphism of $\mathcal{O}_M$–modules, in particular, p restricted to C is flat. Then s induces a multiplication $\circ$ on $\mathcal{T}_M$ coming from $p_*(\mathcal{O}_C)$ and endowed with the identity vector field $e = s^{-1}(1)$. If $p: C \to M$ is generically étale, $(M, \circ, e)$ is an F–manifold. Its spectral cover L is then canonically identified with C, and s with σ. When these conditions are satisfied, we will call $(p: N \to M, \Psi)$ a K. Saito's framework (in the context of F–manifolds).

If an F–manifold $(M, \circ, e)$ is initially defined by another construction, e. g. is the quantum cohomology of a manifold V, then any isomorphism of it with the base of some Saito's framework $(p: N \to M, \Psi)$ is called *a Landau–Ginzburg representation* of it. When $\Psi = d_{N/M}F$ for some local function F on M, F is called the respective *LG–superpotential* of $(M, \circ, e)$. Canonical coordinates on a simply connected $U \subset M$ over which C is étale, are precisely the critical values of F, that is, restrictions of F on various connected components of $p^{-1}(U)$.

If V is a Fano manifold with generically semisimple quantum cohomology, any Landau–Ginzburg representation of it is traditionally considered as a *mirror construction.*

1.5. Example: unfolding of isolated hypersurface singularities. Let $f = f(z_1, \dots, z_m)$ be the germ of an analytic function with an isolated singularity at zero. Let $g_1, \dots, g_n$ be germs of analytic functions at zero whose classes modulo the Jacobian ideal $J := (\partial f/\partial z_i)$ form a basis of the Milnor ring of f. Put $F = F(z,t) = f(z) + \sum_{j=1}^n t_j g_j(z)$. Consider F as a function on the germ N at zero of the analytic space with coordinates (z,t) endowed with the projection $(z,t) \mapsto (t)$ to the germ M of the (t)–space. Then $(N \to M, p, d_{N/M}F)$ is a K. Saito's framework. It produces a generically semisimple F–structure on the base of miniversal deformation M whose spectral cover C is smooth. Moreover, this (germ of) F–manifold is *irreducible*, that is, cannot be represented as a product of F–manifolds, because its tangent space at zero is a local algebra.

Conversely, any irreducible germ of a generically semisimple F–manifold *with smooth spectral cover* L is isomorphic to the miniversal unfolding base of an isolated hypersurface singularity, and this singularity is unique up to stable right equivalence, that is, adding squares of extra z–coordinates and performing invertible coordinate changes. This beautiful theorem was proved in [He1] (see Theorem 16.6), using earlier results by V. Arnold and A. Givental.

It gives a very neat intrinsic description of irreducible germs of F–manifolds allowing the Landau–Ginzburg representations of the well studied type.

If we only know that an F–manifold $(M, \circ, e)$ has a smooth spectral cover L, then Hertling's local decomposition theorem together with the result above produces a rather detailed local information about possible LG–potentials, but neither proves its existence, nor characterizes it uniquely.

1.6. Example: LG–superpotentials of the quantum cohomology of projective spaces. Quantum cohomology of $H^*(\mathbf{P}^r)$ is semisimple at zero. The canonical coordinates of this point defined as the eigenvalues of the multiplication by the standard Euler field E are simply

$$u_k = \zeta^k (r+1),\ \zeta = e^{\frac{2\pi i}{r+1}}. \tag{1.5}$$

The respective K. Saito's framework which is traditionally considered as the mirror of $\mathbf{P}^r$ is constructed as follows (cf. [G1], [Bar]). Denote by M the germ of $\mathbf{C}^{r+1}$ at zero, with local coordinates $t_0, \dots, t_r$. Denote by N the direct product of M and the torus $z_0 \dots z_r = 1$, and by p its projection on M. Put $f(z) := z_0 + \dots + z_r$. Finally, consider the function

$$F(z;t) := f(z) + \sum_{j=0}^{r} t_j\, f(z)^j. \tag{1.6}$$

An easy calculation shows that at $t = 0$, the critical points of $F(z;0) = f(z)$ are $z_0 = \dots = z_r = \zeta^k,\ k = 0, \dots, r+1$, with the respective critical values $\zeta^k(r+1)$. Therefore, there exists a unique isomorphism of the F–manifold $(M, \circ, e)$ produced by this framework with the germ of the quantum cohomology of $H^*(\mathbf{P}^r)$ extending the above identification of the canonical coordinates at zero.

The deformation of $f(z)$ explicitly described by (1.6) is formally similar to the usual construction of the unfolding space of the isolated singularity: in fact, the first $r+1$ powers of $f(z)$ generate a basis of the Milnor ring of f (which is semisimple rather than local this time). However, theoretical reasons for choosing this deformation are less compelling than in the classical case: we are dealing with a global but non–compact deformation problem.

For a thorough discussion of a class of such problems, see [Sa] and the references therein.

1.7. Metric and flat coordinates. Let again $(M, \circ, e)$ be an F–manifold with semisimple base point. Choose also an Euler field E whose eigenvalues constitute a normalized system of canonical coordinates u_i on M. An additional structure which turns M into a Frobenius manifold is the choice of a (complex analytic) flat metric g such that $g(X \circ Y, Z) = g(X, Y \circ Z)$ for all local vector fields X, Y, Z. Dubrovin has shown that in the canonical coordinates such a metric can be expressed as $\sum_i e_i \eta \, (du_i)^2$ where η is a local function defined up to adding a constant. For quantum cohomology, g is the Poincaré form, and one can take for η the flat coordinate dual to the cohomology class of a point. For the latter, cf. [Ma2], Ch. I, Proposition 3.5 c).

If the base point of M is tame semisimple, then the metric can be uniquely reconstructed from the values of $\eta_i = e_i \eta$ and $\eta_{ij} = e_i e_j \eta$ at this point. For a proof, see [Ma2], I.3.

The essential (and difficult) problem in this case consists in studying the global analytic properties of flat coordinates near the boundary of the tame semisimple domain. An appropriate language for such a study is the language of monodromy data for the first and the second structure connections of $(M, \circ, g)$ for which we refer to [Du], [Guz], and references therein (first connection and the Stokes data), and [Ma2], [He2] (second connection, Schlesinger's equations).

1.8. Semisimple quantum cohomology. We now turn to the main subject of this section, F–manifolds and Frobenius manifolds of quantum cohomology.

First of all, semisimple Frobenius manifolds cannot have odd coordinates. Hence we should restrict ourselves from the start by the even–dimensional quantum cohomology. It is well known that it is endowed with the induced F–manifold structure, Euler field, flat invariant metric and flat identity.

1.8.1. Theorem. *a) Let V be a projective manifold whose even–dimensional quantum cohomology is generically semisimple as a formal Frobenius manifold over the Novikov ring. Then*

$$h^{p,q}(V) = 0 \text{ for all } p \neq q, \ p + q \equiv 0 \bmod 2. \tag{1.7}$$

b) Generally, for an arbitrary projective manifold V, the formal completion at of $\oplus_{p \in \mathbf{Z}} H^{p,p}(V)$ at the point of the classical limit is endowed with an induced structure of Frobenius submanifold with flat identity and Euler field.

Recall that at the point of the classical limit vanish all the flat coordinates outside H^2 and exponentiated coordinates on H^2. More precisely, the H^2 component of this point lies in the partial compactification of the respective torus corresponding to the Kähler cone.

1.8.2. Dubrovin's conjecture and a generalization. Classical calculations show that there exist Fano complete intersections in projective spaces for which (1.7) fails. For example, $2m$–dimensional intersections of three quadrics have

$$h^{m-1,m+1} = h^{m+1,m-1} = m(m+1)/2.$$

Hence their even–dimensional quantum cohomology cannot be semisimple.

Thus, the best result that one can expect is that $\oplus_{p\in\mathbf{Z}} H^{p,p}(V)$ is semisimple for an interesting class of Fano manifolds.

B. Dubrovin ([Du], 4.2.2) conjectured that the even quantum cohomology of V is semisimple *if and only if* V is Fano and $Der^b(Coh\,V)$ admits a full system of exceptional objects whose length equals $\dim H^*(V)$. This agrees with Theorem 1.8.1, because on such a manifold all cohomology classes are algebraic, and in particular, $H^{odd}(V) = 0$ and $h^{p,q} = 0$ for $p \neq q$. Theorem 1.8.1 suggests the following strengthening of this conjecture: $\oplus_{p\in\mathbf{Z}} H^{p,p}(V)$ *(with quantum multiplication) is semisimple if and only if V is Fano and $Der^b(Coh\,V)$ admits a full system of exceptional objects whose length equals* $\sum_p h^{p,p}(V)$.

For some positive results on the semisimplicity in a more narrow sense, see [Beau], [TX], and the references in these papers. In particular, in [TX] it is shown that for certain Fano complete intersections the operator of the quantum multiplication by the canonical class is generically semisimple on the classical subring of $\oplus_{p\in\mathbf{Z}} H^{p,p}(V)$ generated by $H^{1,1}(V)$. However, we do not know, whether this subring is stable with respect to the quantum multiplication.

Proof of the Theorem 1.8.1. a) It is based on the consideration of Euler fields, discussed in 1.2. Namely, from (1.4) one easily deduces that if on a generically semisimple F–manifold two Euler fields of non–zero weights commute, they must be proportional.

Now let us turn to the quantum cohomology. As Sheldon Katz remarked, it admits generally at least two commuting Euler fields. In order to write them down explicitly, choose a bigraded homogeneous basis (Δ_a) of $H = H^{*,*}(V, \mathbf{C})$ considered as the space of flat vector fields, and let (x_a) be the dual flat coordinates vanishing at zero. Let $\Delta_a \in H^{p_a,q_a}(V)$. Put $-K_V = \sum_{p_b+q_b=2} r_b\Delta_b$. Then

$$E_1 := \sum_a (1-p_a)x_a\Delta_a + \sum_b r_b\Delta_b, \qquad (1.8)$$

$$E_2 := \sum_a (1-q_a)x_a\Delta_a + \sum_b r_b\Delta_b \qquad (1.9)$$

are Euler of weight 1. To check this, one can repeat the calculations made in [Ma2], III.5.4 with obvious changes. In the final count, the reason for this is that Gromov–Witten correspondences are algebraic, and therefore are represented by the cohomology classes of types (p,p).

Assuming now that $h^{p,q}(V)\ne 0$ for some $p\ne q$, $p+q\equiv 0 \bmod 2$, we see that the restrictions of (1.8) and (1.9) to the even–dimensional cohomology cannot be proportional. This contradiction concludes the proof of a).

b) With the same notation, a correlator $\langle\Delta_{a_1}\dots\Delta_{a_n}\rangle_\beta$ vanishes unless $\sum_i p_{a_i}=\sum_i q_{a_i}$ because the Gromov–Witten correspondences are represented by algebraic cycles. Let Φ be the potential of the quantum cohomology. Assume that $p_a=q_a$ and $p_b=q_b$, but $p_c\ne q_c$ for some a,b,c. From the preceding remark it follows that Φ_{ab}^c vanishes after restriction to the subspace $\oplus_p H^{p,p}$ given by the equations $x_e=0$ for all e such that $p_e\ne q_e$. Hence the $\circ$ product of two vector fields tangent to this submanifold is again tangent to it. It is also clear that the Poincaré form restricts to a non–degenerate flat metric, and the identity field and the Euler field are tangent to it as well.

We will now prove a theorem which establishes a result of the same type as the First Reconstruction Theorem of [KoMa], but under the different assumptions.

1.8.3. Reconstruction Theorem. *Assume that the (p,p)–part of the quantum cohomology of V is generically semisimple and, moreover, admits a tame semisimple point lying in the subspace $H^2(V)$ (parameter space of the small quantum deformation).*

In this case all genus zero Gromov–Witten invariants of $\oplus_p H^{p,p}(V)$ can be reconstructed from the correlators $\langle\gamma_1\otimes\dots\otimes\gamma_n\rangle_\beta$ with $n\le 4$, $\gamma_i\in H^{a_i,a_i}(V)$, and $\beta\in H_2(V,Z)$ which can be non–vanishing only if

$$k(\beta):=(c_1(V),\beta)=\sum_{i=1}^n(a_i-1)+3-\dim V. \tag{1.10}$$

In particular, if $\pm c_1(V)$ is numerically effective, finitely many correlators suffice for the complete reconstruction.

Proof. As was explained in 1.7, the whole germ of the Frobenius manifold at such tame semisimple point of H^2 is determined by its canonical coordinates (u_i^0), the values η_i^0 at this point of the coefficients η_i of the metric (Poincaré form) $\sum\eta_i(du_i)^2$, and the values η_{ij}^0 of their first derivatives $\eta_{ij}=e_i\eta_j$.

One easily sees that in order to calculate these data, it suffices to know the expressions of (u_i) through some flat coordinates modulo J^2 where J is the ideal of equations of H^2 inside $\oplus H^{p,p}$. Now, (u_i) are the eigenvalues of the operator $E\circ$ acting on vector fields. To calculate them modulo J^2, it suffices to know the potential Φ of quantum cohomology modulo J^5 because the structure constants of the $\circ$–multiplication are the third derivatives of the potential. With this precision, Φ is determined by all four–point correlators. Of course, (1.10) is a special case of the Dimension Axiom.

1.9. Fano manifolds with minimal (p,p)**–cohomology.** The finite family of numbers $(u_i^0, \eta_i^0, \eta_{ij}^0)$ essentially coincides with what was called *special coordinates* of the tame semisimple germ of Frobenius manifold, cf. [Ma2], III.7.1.1. In this subsection, we will show how to calculate them for the $\oplus H^{p,p}(V)$–part of the quantum cohomology of those Fano manifolds, for which $\dim H^{p,p}(V) = 1$ for all $1 \le p \le \dim V := r$. This generalizes our old computation for projective spaces.

We will work with a homogeneous basis $\Delta_p \in H^{p,p}(V)$ consisting of rational cohomology classes satisfying the following conditions: $\Delta_0 =$ the dual class of $[V]$, $\Delta_1 = c_1(V)/\rho$ is the ample generator of $\operatorname{Pic} V$, and ρ is called *the index* of V. Furthermore, Δ_{r-p} is dual to Δ_p, that is $(\Delta_p, \Delta_{r-p}) = 1$, $\Delta_r =$ the dual class of a point. The dual coordinates are denoted $x_0, \dots, x_r$. The nonvanishing $\beta = 0$ correlators are coefficients of the cubic self–intersection form

$$(x_0\Delta_0 + \dots + x_r\Delta_r)^3.$$

We put

$$[d;\, a_1, \dots, a_k] := \langle \Delta_{a_1} \dots \Delta_{a_k} \rangle_{d\Delta_{r-1}}\,.$$

These symbols satisfy the following relations:

(i) If $r \ge 3$, $[d;\, a_1, \dots, a_k] \ne 0$ and $d > 0$, then necessarily

$$k > 0,\ a_i > 0 \text{ for all } i, \text{ and } d\rho = \sum_{i=1}^k (a_i - 1) + 3 - r \tag{1.11}$$

(see (1.10)).

(ii) $[d;\, a_1, \dots, a_k]$ is symmetric with respect to the permutations of $a_1, \dots, a_k$.

(iii) $[d;\, 1, a_2, \dots, a_k] = d\,[d;\, a_2, \dots, a_k]$.

(iv) Associativity relations, expressing the associativity of the multiplication (0.1).

The multiplication table in the first infinitesimal neighborhood of $H^2(V)$ involves only up to four–point correlators and looks as follows:

$$\Delta_a \circ \Delta_b = \Delta_a \cup \Delta_b +$$

$$\sum_{d\ge1}\sum_{c\ge1} \left([d;\, a,b,c] + \sum_{f\ge2} [d;\, a,b,c,f]\, x_f \right) \Delta_{r-c} q^d\,. \tag{1.12}$$

Finally, the (restricted) Euler field of weight 1 is

$$E = \sum_{p=0}^{r} (1-p)\, x_p \Delta_p + \rho\, \Delta_1 \tag{1.13}$$

As we already remarked, under the assumptions of the Theorem 1.8.3, the eigenvalues of $E\circ$ at the generic point of H^2 are pairwise distinct and determine the canonical coordinates of this point. We will have to calculate in the first infinitesimal neighborhood of H^2 and therefore we will consider all the relevant quantities as consisting of two summands: restriction to H^2 and the linear (in x_a) correction term, in particular

$$u_i := u_i^{(0)} + u_i^{(1)}.$$

The remaining special coordinates are given by the following formulas.

1.9.1. Theorem. *Put*

$$e_i := \frac{\prod_{j\neq i}(E - u_j)}{\prod_{j\neq i}(u_i - u_j)} = e_i^{(0)} + e_i^{(1)}\,. \tag{1.14}$$

where the multiplication is understood in the sense of quantum cohomology with the coefficient ring extended by (u_i) and $(u_i - u_j)^{-1}$.

Then we have on H^2:

$$\eta_i = e_i^{(0)}(x_r),\ \ \eta_{ij} = e_i^{(0)} e_j\,(x_r) \tag{1.15}$$

where in (1.15) e_i are considered as vector fields acting upon coordinates via $\Delta_a = \partial/\partial x_a$.

Proof. The elements e_i are the basic pairwise orthogonal idempotents in the quantum cohomology ring at the considered point satisfying $E \circ e_i = u_i e_i$. The metric potential η is x_r.

Here is an efficient way of computing $e_i^{(1)}$. First, compute ω_i defined by the identity in the first neighborhood:

$$e_i^{(0)} \circ e_i^{(0)} = e_i^{(0)} + \omega_i. \tag{1.16}$$

Then we have

$$e_i^{(1)} = -\frac{\omega_i}{2e_i^{(0)} - 1} = \omega_i \circ (1 - 2e_i^{(0)}) \tag{1.17}$$

where the division resp. multiplication is again made in the first neighborhood.

In fact, this follows from

$$(e_i^{(0)} + e_i^{(1)})^{\circ 2} = e_i^{(0)} + e_i^{(1)}$$

and (1.16).

§2. Fano threefolds with minimal cohomology

2.1. Notation. Let V be a Fano threefold. We keep the general notation of 1.3, but now consider only the case $r = 3$. Besides the index ρ, we consider the degree $\delta := (c_1(V)^3)/\rho^3$ of V.

There exist four families of Fano threefolds $V = V_\delta$ with cohomology $H^{p,p}(V, \mathbf{Z}) \cong \mathbf{Z}$ for $p = 0, \dots, 3$ and $H^{p,q}(V, \mathbf{Z}) = 0$ for $p \neq q$. Besides $V_1 = \mathbf{P}^3$ and the quadric $V_2 = Q$, they are V_5 and V_{22}, with degree as subscript; their indices are, respectively, $4, 3, 2, 1$. One can get a V_5 by considering a generic codimension three linear section of the Grassmannian of lines in $\mathbf{P}^5$ embedded in $\mathbf{P}^9$.

The nonvanishing $\beta = 0$ correlators are coefficients of the cubic self–intersection form

$$(x_0\Delta_0 + \cdots + x_3\Delta_3)^3 = \delta\, x_1^3 + 3x_0^2x_3 + x_0x_1x_2\,.$$

In this section, we will deal only with Q, V_5 and V_{22}, since projective spaces of any dimension were treated by various methods earlier: see [Ma3] for special coordinates, [Du], 4.2.1 and [Guz] for monodromy data, [Bar] for semiinfinite Hodge structures.

2.2. Tables of correlators. The following tables provide the coefficients of the multiplication table (1.12).

It suffices to tabulate the primitive correlators, where primitivity means that $a_i > 1$ and $a_i \leq a_{i+1}$. The symmetry and the divisor identities furnish the remaining correlators.

Manifold Q:

[1; 2,3]	[1; 2,2,2]	[2; 3,3,3]	[2; 2,2,3,3]
1	1	1	1

Manifold V_5:

[1; 3]	[1; 2,2]	[2; 3,3]	[2; 2,2,3]	[3; 3,3,3]	[2; 2,2,2,2]	[3; 2,2,3,3]	[4; 3,3,3,3]
3	1	1	1	1	1	2	3

Manifold V_{22}:

[1; 2]	[2; 3]	[2; 2,2]	[3; 2,3]	[4; 3,3]	[3; 2,2,2]	[4; 2,2,3]	[5; 2,3,3]
2	6	1	3	10	1	4	16

[6; 3,3,3]	[4; 2,2,2,2]	[5; 2,2,2,3]	[6; 2,2,3,3]	[7; 2,3,3,3]	[8; 3,3,3,3]
65	2	9	41	186	840

The tables were compiled in the following way. First, (1.10) furnishes the list of all primitive correlators that might be (and actually are) non–vanishing. Second, several correlators corresponding to the smallest values of n in (1.10) must be computed geometrically: $n = 2$ for Q, $n = 1,2$ for V_5, and V_{22}. These values were computed by A. Bondal, D. Kuznetsov and D. Orlov. Third, the associativity equations uniquely determine all the remaining correlators, in the spirit of the First Reconstruction Theorem of [KoMa].

2.3. Canonical coordinates. The canonical coordinates on $H^2 \cap \{x_0 = 0\}$ expressed in terms of the flat coordinates are the roots $u_0, \dots, u_3$ of the following characteristic equations of the operator $E\circ$:

$$Q: \qquad u^4 - 108\, q\, u = 0, \tag{2.1}$$

$$V_5: \qquad u^4 - 44\, q\, u^2 - 16\, q^2 = 0, \tag{2.2}$$

$$V_{22}: \qquad (u + 4\, q)\, (u^3 - 8\, qu^2 - 56\, q^2 u - 76\, q^3) = 0. \tag{2.3}$$

If $x_0 \neq 0$, one must simply add x_0 to the values above.

2.3.1. Question. Find "natural" functions $f(z)$ whose critical values at 0 are roots of (2.1)–(2.3) and whose unfolding space carries an appropriate flat metric.

2.4. Multiplication tables, idempotents, and metric coefficients. The remaining special coordinates η_i, η_{jk} were calculated using the multiplication tables in the first neighborhood of H^2 obtained by specializing (1.12). We calculated $e_i^{(0)}$ by determining the eigenvectors of ad E; then we used equation (1.17) to get $e_i^{(1)}$.

Manifold Q:

$$\begin{aligned}
\Delta_1^2 &= 2\,\Delta_2 + q\,\Delta_1 x_3 + q\,\Delta_0 x_2,\\
\Delta_1\Delta_2 &= \Delta_3 + q\,\Delta_0 + q\,\Delta_2 x_3 + q\,\Delta_1 x_2,\\
\Delta_1\Delta_3 &= q\,\Delta_1 + q\,\Delta_2 x_2 + 2q^2\Delta_0 x_3,\\
\Delta_2^2 &= q\,\Delta_1 + q\,\Delta_2 x_2 + q^2\Delta_0 x_3,\\
\Delta_2\Delta_3 &= q\,\Delta_2 + q^2\Delta_0 x_2 + q^2\Delta_1 x_3,\\
\Delta_3^2 &= q^2\Delta_0 + q^2\Delta_1 x_2 + 2q^2\Delta_2 x_3.
\end{aligned}$$

Let ξ_i, $i = 1, \dots, 3$ be the three roots of $\xi^3 = 4q$. Then the respective idempotents have the following form:

$$e_0 = \frac{1}{2}\Delta_0 - \frac{1}{2q}\Delta_3 + \frac{x_2}{4}\Delta_1 + \frac{x_3}{2}\Delta_2$$

and, for $i = 1, \dots, 3$,

$$e_i = \frac{1}{6}\Delta_0 + \frac{\xi_i^2}{12q}\Delta_1 + \frac{\xi_i}{6q}\Delta_2 + \frac{1}{6q}\Delta_3 - \frac{\xi_i x_2}{36}\Delta_0$$

$$-\left(\frac{x_2}{12}+\frac{\xi_i x_3}{12}\right)\Delta_1-\left(\frac{\xi_i^2 x_2}{18q}+\frac{x_3}{6}\right)\Delta_2-\left(\frac{\xi_i x_2}{27q}+\frac{\xi_i^2 x_3}{12q}\right)\Delta_3.$$

Probably the most direct and efficient test of our computations is to simply compute the pairwise products of these idempotents using the multiplication table above. This verifies the formulas of the e_j while checking at the same time that our multiplication table yields an associative product.

Note that $\Delta_i = \frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_1} q = q$ since we identify q with e^{x_1} to get a proper (non–formal) Frobenius manifold. So we get as special coordinates (where $i,j \in \{1,2,3\}$):

$$\eta_0 = -\frac{1}{2q}, \quad \eta_i = \frac{1}{6q}, \quad \eta_{00} = 0,$$

$$\eta_{i0} = \frac{\xi_i^2}{12q}\Delta_1\left(\frac{-1}{2q}\right) = \frac{\xi_i^2}{24q^2}, \quad \eta_{0i} = \frac{-1}{2q}\Delta_3\left(-\frac{\xi_i^2 x_3}{12q}\right) = \frac{\xi_i^2}{24q^2},$$

$$\eta_{ij} = \frac{\xi_i^2}{12q}\Delta_1\left(\frac{1}{6q}\right) - \frac{\xi_i}{6q}\Delta_2\left(\frac{\xi_j x_2}{27q}\right) - \frac{1}{6q}\Delta_3\left(\frac{\xi_j^2 x_3}{12q}\right) = -\frac{\xi_i^2}{72q^2} - \frac{\xi_i\xi_j}{162q^2} - \frac{\xi_j^2}{72q^2}.$$

The symmetry of η_{ij} is just an additional check of our computations, as we know in general that e_i are commuting vector fields.

Manifold V_5:

$$\Delta_1^2 = 5\Delta_2 + 3q\,\Delta_0 + 3q\,\Delta_2 x_3 + q\,\Delta_1 x_2 + 4q^2\Delta_0 x_3,$$
$$\Delta_1\Delta_2 = \Delta_3 + q\,\Delta_1 + q\,\Delta_2 x_2 + 2q^2\Delta_1 x_3 + 2q^2\Delta_0 x_2,$$
$$\Delta_1\Delta_3 = 3q\,\Delta_2 + 2q^2\Delta_0 + 4q^2\Delta_2 x_3 + 2q^2\Delta_1 x_2 + 3q^3\Delta_0 x_3,$$
$$\Delta_2^2 = q\,\Delta_2 + q^2\Delta_0 + 2q^2\Delta_2 x_3 + q^2\Delta_1 x_2 + 2q^3\Delta_0 x_3,$$
$$\Delta_2\Delta_3 = q^2\Delta_1 + 2q^2\Delta_2 x_2 + 2q^3\Delta_1 x_3 + 2q^3\Delta_0 x_2,$$
$$\Delta_3^2 = 2q^2\Delta_2 + q^3\Delta_0 + 3q^3\Delta_2 x_3 + 2q^3\Delta_1 x_2 + 3q^4\Delta_0 x_3.$$

Let u_i be the roots of $u^4 - 44\,q\,u^2 - 16\,q^2$. The idempotents are given by

$$4000q^3 e_i = 1440q^3\Delta_0 - 20q^2{u_i}^2\Delta_0 + 70qu_i^3\Delta_1 - 3040q^2u_i\Delta_1$$
$$-880q^2\Delta_2 + 40q{u_i}^2\Delta_2 + 4920qu_i\Delta_3 - 110{u_i}^3\Delta_3$$
$$-1968u_iq^3x_2\Delta_0 + 44{u_i}^3q^2x_2\Delta_0 + 352q^4x_3\Delta_0 - 16{u_i}^2q^3x_3\Delta_0$$
$$+176q^3x_2\Delta_1 - 8q^2{u_i}^2\Delta_1x_2 - 5412u_iq^3\Delta_1x_3 + 121{u_i}^3q^2\Delta_1x_3$$
$$-2864u_iq^2x_2\Delta_2 + 62q{u_i}^3x_2\Delta_2 + 1056q^3\Delta_2x_3 - 48x_3{u_i}^2q^2\Delta_2$$

$$-16qu_i^2x_2\Delta_3 + 6036q^2u_ix_3\Delta_3 + 352q^2x_2\Delta_3 - 138q{u_i}^3x_3\Delta_3.$$

The special coordinates η_{ii} are

$$\eta_{ii} = \frac{-964\,q + 21\,{u_i}^2}{800\,q^3} = \frac{-251 \pm 105\sqrt{5}}{400\,q^2}.$$

Now since the Galois group of $u^4 - 44\,q\,u^2 - 16\,q^2$ obviously does not act transitively on the pairs of roots, we have to distinguish two cases in determining the η_{ij}. So we fix a root u_1; the other roots are given by $u_2 = -u_1$ and $u_{3,4}^2 - 11\,q + u_1^2 = 0$. We calculated

$$\eta_{12} = \eta_{21} = \frac{932\,q - 21\,{u_1}^2}{800\,q^3} = \frac{47 \mp 21\,\sqrt{5}}{80\,q^2} \quad \text{and}$$
$$\eta_{13} = \frac{-3\,u_3u_1 + 4\,q}{200q^3}.$$

All other coordinates are obtained from these via Galois permutations.

Manifold V_{22}:

$$\Delta_1^2 = 22\Delta_2 + 2q\,\Delta_1 + 24q^2\Delta_0$$
$$+2q\,\Delta_2x_2 + 48q^2\Delta_2x_3 + 4q^2\Delta_1x_2 + 27q^3\Delta_1x_3 + 27q^3\Delta_0x_2 + 160q^4\Delta_0x_3,$$
$$\Delta_1\Delta_2 = \Delta_3 + 2q\,\Delta_2 + 2q^2\Delta_1 + 9q^3\Delta_0$$
$$+4q^2\Delta_2x_2 + 27q^3\Delta_2x_3 + 3q^3\Delta_1x_2 + 16q^4\Delta_1x_3 + 16q^4\Delta_0x_2 + 80q^5\Delta_0x_3,$$
$$\Delta_1\Delta_3 = 24q^2\Delta_2 + 9q^3\Delta_1 + 40q^4\Delta_0$$
$$+27q^3\Delta_2x_2 + 160q^4\Delta_2x_3 + 16q^4\Delta_1x_2 + 80q^5\Delta_1x_3 + 80q^5\Delta_0x_2 + 390q^6\Delta_0x_3,$$
$$\Delta_2^2 = 2q^2\Delta_2 + q^3\Delta_1 + 4q^4\Delta_0$$
$$+3q^3\Delta_2x_2 + 16q^4\Delta_2x_3 + 2q^4\Delta_1x_2 + 9q^5\Delta_1x_3 + 9q^5\Delta_0x_2 + 41q^6\Delta_0x_3,$$
$$\Delta_2\Delta_3 = 9q^3\Delta_2 + 4q^4\Delta_1 + 16q^5\Delta_0$$
$$+16q^4\Delta_2x_2 + 80q^5\Delta_2x_3 + 9q^5\Delta_1x_2 + 41q^6\Delta_1x_3 + 41q^6\Delta_0x_2 + 186q^7\Delta_0x_3,$$
$$\Delta_3^2 = 40q^4\Delta_2 + 16q^5\Delta_1 + 65q^6\Delta_0$$
$$+80q^5\Delta_2x_2 + 390q^6\Delta_2x_3 + 41q^6\Delta_1x_2 + 186q^7\Delta_1x_3 + 186q^7\Delta_0x_2 + 840q^8\Delta_0x_3.$$

The first idempotent is given by

$$e_0 = \frac{\Delta_0}{2} + \frac{2\Delta_2}{q^2} - \frac{\Delta_3}{2q^3} + qx_2\Delta_0 + 2\,q^2x_3\Delta_0$$

$$+\frac{x_2}{4}\,\Delta_1+\frac{qx_3}{2}\Delta_1+\frac{2\,x_2}{q}\Delta_2+\frac{3x_3}{2}\Delta_2-\frac{x_2}{2q^2}\Delta_3.$$

Now let $u_i, i=1,\ldots,3$ be the roots of $u^3-8qu^2-56q^2u-76q^3$. Then the respective idempotents are given by

$$5324\,q^4\,e_i=$$

$$\left(-71742q^4-24552q^3u_i+2354q^2{u_i}^2\right)\Delta_0+\left(-30272q^3-10186q^2u_i+979q{u_i}^2\right)\Delta_1$$

$$+\left(-118712q^2-38126qu_i+3696{u_i}^2\right)\Delta_2+\left(49346q+16192u_i-1562\frac{{u_i}^2}{q}\right)\Delta_3$$

$$+\left(-130876q^5-43283u_iq^4+4168{u_i}^2q^3\right)x_2\Delta_0$$

$$+\left(-483464q^6-161648u_iq^5+15528{u_i}^2q^4\right)x_3\Delta_0$$

$$+\left(-38977q^4-12940u_iq^3+1245{u_i}^2q^2\right)x_2\Delta_1$$

$$+\left(-145898q^5-48889u_iq^4+4694{u_i}^2q^3\right)x_3\Delta_1$$

$$+\left(-143992q^3-46818u_iq^2+4522{u_i}^2q\right)x_2\Delta_2$$

$$+\left(-491334q^4-164832u_iq^3+15822{u_i}^2q^2\right)x_3\Delta_2$$

$$+\left(35042q^2+11348u_iq-1098{u_i}^2\right)x_2\Delta_3$$

$$+\left(112272q^3+37824u_iq^2-3633{u_i}^2q\right)x_3\Delta_3.$$

¿From this we compute the special coordinates

$$\eta_0=-\frac{1}{2q^3},\qquad \eta_i=49346q+16192u_i-1562\frac{{u_i}^2}{q},$$

$$\eta_{00}=-\frac{1}{q^4},\qquad \eta_{i0}=\eta_{0i}=-\frac{-2536q^2-688\,u_iq+69\,{u_i}^2}{968\,q^6},$$

$$\eta_{ii}=\frac{-3412q^2-260\,u_iq+41\,{u_i}^2}{484\,q^6}$$

$$\eta_{ij}=\frac{404}{121\,q^4}-\frac{34}{121}\,\frac{u_i+u_j}{q^5}+\frac{13}{968}\,\frac{u_ju_i}{q^6}.$$

§3. Del Pezzo surfaces

3.1. Notation. Let V be a surface. For surfaces, we will denote the dual class of a point by Δ_2, and the dual coordinate by z. Assuming $h^{2,0}(V)=0$, we may and will identify the group $H_2(V,\mathbf{Z})/(tors)$ with $H^{1,1}(V,\mathbf{Z})$.

We start again with calculating the structure constants of the quantum multiplication of classes in $H^{p,p}$ restricted to the first infinitesimal neighborhood of $H^{1,1}$. Essentially, this means that we must calculate the third derivatives of the potential modulo z^2. The canonical coordinates of this "very small quantum cohomology" are the eigenvalues of the operator $k\circ$, $k=c_1(V)$, calculated at the points of $H^{1,1}$ so that only triple correlators are involved in their calculation.

3.2. Quantum multiplication table. Since Δ_0 is the identity with respect to $\circ$, we need only the following formulas, which follow from (0.1). Let D_1, $D_2\in H^2(V)$, then

$$D_1\circ D_2=(D_1,D_2)\,\Delta_2+$$

$$\sum_{\beta\neq 0}\langle e^{z\Delta_2}\rangle_\beta(D_1,\beta)\,(D_2,\beta)\,q^\beta\beta+\left(\sum_{\beta\neq 0}\langle\Delta_2e^{z\Delta_2}\rangle_\beta(D_1,\beta)\,(D_2,\beta)q^\beta\right)\Delta_0. \tag{3.1}$$

Similarly, for $D\in H^2(V)$,

$$D\circ\Delta_2=\sum_{\beta\neq 0}\langle\Delta_2e^{z\Delta_2}\rangle_\beta(D,\beta)\,q^\beta\beta+\left(\sum_{\beta\neq 0}\langle\Delta_2^2e^{z\Delta_2}\rangle_\beta(D,\beta)\,q^\beta\right)\Delta_0. \tag{3.2}$$

Finally,

$$\Delta_2\circ\Delta_2=\sum_{\beta\neq 0}\langle\Delta_2^2e^{z\Delta_2}\rangle_\beta\,q^\beta\beta+\left(\sum_{\beta\neq 0}\langle\Delta_2^3e^{z\Delta_2}\rangle_\beta\,q^\beta\right)\Delta_0. \tag{3.3}$$

If we want only terms modulo z^2, (1.5) can be used in order to restrict the summation over β:

$$D_1\circ D_2=(D_1,D_2)\,\Delta_2+$$

$$\sum_{\beta:\,k(\beta)=1}\langle\emptyset\rangle_\beta(D_1,\beta)\,(D_2,\beta)\,q^\beta\beta+\left(\sum_{\beta:\,k(\beta)=2}\langle\Delta_2\rangle_\beta(D_1,\beta)\,(D_2,\beta)q^\beta\right)\Delta_0+$$

$$z\sum_{\beta:\,k(\beta)=2}\langle\Delta_2\rangle_\beta(D_1,\beta)\,(D_2,\beta)\,q^\beta\beta+z\left(\sum_{\beta:\,k(\beta)=3}\langle\Delta_2^2\rangle_\beta(D_1,\beta)\,(D_2,\beta)q^\beta\right)\Delta_0+O(z^2), \tag{3.4}$$

$$D \circ \Delta_2 = \sum_{\beta:\, k(\beta)=2} \langle \Delta_2 \rangle_\beta (D,\beta)\, q^\beta \beta + \left(\sum_{\beta:\, k(\beta)=3} \langle \Delta_2^2 \rangle_\beta (D,\beta)\, q^\beta \right) \Delta_0 +$$

$$z \sum_{\beta:\, k(\beta)=3} \langle \Delta_2^2 \rangle_\beta (D,\beta)\, q^\beta \beta + z \left(\sum_{\beta:\, k(\beta)=4} \langle \Delta_2^3 \rangle_\beta (D,\beta)\, q^\beta \right) \Delta_0 + O(z^2). \tag{3.5}$$

In the case when one of the divisor classes is k, (3.4) and (3.5) further simplify:

$$k \circ D = k(D)\, \Delta_2 +$$

$$\sum_{\beta:\, k(\beta)=1} \langle \emptyset \rangle_\beta (D,\beta)\, q^\beta \beta + 2 \left(\sum_{\beta:\, k(\beta)=2} \langle \Delta_2 \rangle_\beta (D,\beta)\, q^\beta \right) \Delta_0 +$$

$$2z \sum_{\beta:\, k(\beta)=2} \langle \Delta_2 \rangle_\beta (D,\beta)\, q^\beta \beta + 3z \left(\sum_{\beta:\, k(\beta)=3} \langle \Delta_2^2 \rangle_\beta (D,\beta)\, q^\beta \right) \Delta_0 + O(z^2). \tag{3.6}$$

In particular,

$$k \circ k = (k,k)\, \Delta_2 + \sum_{\beta:\, k(\beta)=1} \langle \emptyset \rangle_\beta\, q^\beta \beta + 4 \left(\sum_{\beta:\, k(\beta)=2} \langle \Delta_2 \rangle_\beta\, q^\beta \right) \Delta_0 +$$

$$4z \sum_{\beta:\, k(\beta)=2} \langle \Delta_2 \rangle_\beta\, q^\beta \beta + 9z \left(\sum_{\beta:\, k(\beta)=3} \langle \Delta_2^2 \rangle_\beta\, q^\beta \right) \Delta_0 + O(z^2). \tag{3.7}$$

Furthermore,

$$k \circ \Delta_2 = 2 \sum_{\beta:\, k(\beta)=2} \langle \Delta_2 \rangle_\beta\, q^\beta \beta + 3 \left(\sum_{\beta:\, k(\beta)=3} \langle \Delta_2^2 \rangle_\beta\, q^\beta \right) \Delta_0 +$$

$$3z \sum_{\beta:\, k(\beta)=3} \langle \Delta_2^2 \rangle_\beta\, q^\beta \beta + 4z \left(\sum_{\beta:\, k(\beta)=4} \langle \Delta_2^3 \rangle_\beta\, q^\beta \right) \Delta_0 + O(z^2). \tag{3.8}$$

3.3. Del Pezzo surfaces. Let $V = V_r$ be obtained from $\mathbf{P}^2$ by blowing up r points in general position. We get del Pezzo surfaces (characterized by the ampleness of k) for $r \le 8$. Sums in the right hand sides of (3.4)–(3.8) for larger ranks $r+1$ of $H^{1,1}(V)$ are long and for $r \ge 9$ are infinite. However, for del Pezzo surfaces (and to a certain degree, for more general rational surfaces) the situation becomes simpler because a Weyl group acts upon $\beta \in H^{1,1}(V)$ leaving $k(\beta)$, (β,β) and the correlators $\langle \Delta_2^{k(\beta)-1}\rangle_\beta$ invariant. For $k(\beta) \le 4$ the number of orbits is quite small, and the resulting expressions become manageable. We start with some classical results on the structure of $H^2(V,\mathbf{Z})$ (see e. g. [Ma3]).

Fix a representation of V_r as blow up of $\mathbf{P}^2$. Then k and the classes of the resulting exceptional curves $l_1,\dots,l_r$ generate a sublattice of index 3 in the Néron–Severi group coinciding with $H^2(V,\mathbf{Z})$. The whole lattice N_r is generated by the orthonormal basis $(l_0;\, l_1,\dots,l_r)$ where l_0 is the class of a line lifted to V. We identify $\beta = al_0 - \sum_i b_i l_i$ with the vector $(a;\, b_1,\dots,b_r)$ so that $a = a(\beta) = (\beta, l_0)$ and $b_i = b_i(\beta) = (\beta, l_i)$. We have $k = k_r = (3;1,\dots,1)$ in this notation.

Denote by R_r the set $\{\beta \,|\, k(\beta) = 0,\, (\beta,\beta) = -2\}$. It is a system of roots, lying entirely in the orthogonal complement O_r to k_r. Reflections with respect to the roots generate the Weyl group W_r fixing k_r. In our representation, this group for $r \ge 3$ is generated by all permutations of l_r's and by the Cremona transformation

$$(a;\, b_1,\dots,b_r) \mapsto (a+\delta;\, b_1+\delta, b_2+\delta, b_3+\delta, b_4,\dots,b_r), \quad \delta := a - b_1 - b_2 - b_3. \tag{3.9}$$

The function $\beta \mapsto \langle \Delta_2^{k(\beta)-1}\rangle_\beta$ is constant on W_r–orbits. This is well known. One argument is given in [GP], 5.1. It involves geometric Cremona transformations (blowing up and down), which generate W_r, together with permutations of $(l_1,\dots,l_r)$. Alternatively, one can invoke Hirschowitz's theorem ([Hirsch]) which shows that the action of W_r on N_r is induced by the action of W_r on a generic del Pezzo surface and its definition field.

In fact, a related reasoning readily extends to a more general situation. Consider a family of algebraic manifolds over a base whose fundamental group acts on the fiber cohomology in a nontrivial way. A typical situation is the existence of several Lefschetz type transforms associated with various cusps of the moduli space. If they are reflections, a version of the following proposition will hold.

3.3.1. Proposition. *Let ρ be a root in R_r, $s_\rho : \gamma \mapsto \gamma + (\gamma,\rho)\,\rho$ the respective reflection. It induces the involution of the algebraic torus T_{V_r} with the character group $H^2(V_r,\mathbf{Z})$. Let T_ρ be the codimension one subtorus consisting of fixed points of this involution: $q^{s_\rho(\beta)} = q^\beta$ on T_ρ for all β. Then we have, after restriction on T_ρ:*

a) For any $D \in H^2(V)$ orthogonal to ρ

$$D \circ \rho = \mu_{D,\rho}\, \rho,\ \ \mu_{D,\rho} = -\frac{1}{2} \sum_\beta \langle e^{z\Delta_2}\rangle_\beta\, (\rho,\beta)^2 (D,\beta) q^\beta. \tag{3.11}$$

Similarly,

$$\Delta_2 \circ \rho = \nu_\rho\, \rho,\ \ \nu_\rho = -\frac{1}{2} \sum_\beta \langle e^{z\Delta_2}\rangle_\beta\, (\rho,\beta)^2\, q^\beta. \tag{3.12}$$

b) If D_1, D_2 are orthogonal to ρ, then $D_1 \circ D_2$ is orthogonal to ρ on T_ρ; the same is true for $\rho \circ \rho$.

Proof. We have the following general identity. Let S be a W_r–orbit, $D_1, \dots, D_n$ orthogonal to ρ, and m an integer. Then we have on T_ρ

$$\sum_{\beta \in S} (D_1, \beta) \dots (D_n, \beta)\, (\rho, \beta)^{2m+1} q^\beta = 0\,. \tag{3.13}$$

To see this, replace s by $s_\rho(\beta)$ in the summands of (3.13). The whole sum will remain the same, whereas each summand will change sign, because of the factor $(\rho, \beta)^{2m+1}$.

Consider now (3.1) for $D_1 = D$, $D_2 = \rho$. In view of (3.13), the coefficient of Δ_0 vanishes on T_ρ. Take the scalar product with any D' orthogonal to ρ. It will vanish, again in view of (3.13). Hence our vector is proportional to ρ. To calculate the coefficient μ_ρ, it suffices to take the scalar product with ρ. In order to prove (3.12), we treat similarly (3.2) with $D = \rho$.

To establish b), look at (3.1) with both D_i orthogonal to ρ, resp. with $D_1 = D_2 = \rho$. (Notice that in these cases the coefficient of Δ_0 does not vanish, but of course, Δ_0 is still orthogonal to ρ in the Poincaré metric). This ends the proof.

The intersection of all subtori T_ρ is the one–dimensional subgroup $q^\beta = e^{tk_r(\beta)}$, t arbitrary. On this subgroup, and in particular at $t = 0$ (which we write as $q = 1$), the statements of the Proposition 3.3.1 are valid simultaneously for all roots ρ. Hence we get the following corollary.

3.3.2. Corollary. *On the subgroup defined above, $k_r\circ$, $\Delta_2\circ$ act on the orthogonal complement O_r to k_r as multiplication by the scalars (independent on the choice of $\rho \in R_r$) denoted respectively*

$$\mu = -\frac{1}{2} \sum_\beta \langle e^{z\Delta_2} \rangle_\beta\, (\rho, \beta)^2\, k_r(\beta)\, e^{tk_r(\beta)}, \tag{3.14}$$

$$\nu = -\frac{1}{2} \sum_\beta \langle \Delta_2 e^{z\Delta_2} \rangle_\beta\, (\rho, \beta)^2\, e^{tk_r(\beta)}, \tag{3.15}$$

In fact, O_r is spanned by all roots in R_r.

3.4. Structure of the sets $k(\beta) = a$. For each $3 \le r \le 8$ we denote by I_r, F_r, G_r, H_r the following subsets of N_r, each constituting one orbit of W_r. The statements accompanying their definitions are proved, for example, in [Ma3], Ch. 4.

$$I_r := \{\beta \,|\, k(\beta) = (\beta, \beta) = -1\} = W_r(0; -1; 0, \dots, 0). \tag{3.16}$$

Elements of I_r are exactly classes of exceptional curves on a general del Pezzo surface of degree $9-r$. (This is also true for $r=9$ but false for larger values of r: cf. [Hirsch], 3.4).

$$F_r := W_r(l_0 - l_1) = W_r(1;1,0,\dots,0). \tag{3.17}$$

$$G_r := W_r l_0 = W_r(1;0,0,\dots,0). \tag{3.18}$$

$$H_r := W_r(l_1 + l_2) = W_r(0;-1,-1,0,\dots,0). \tag{3.19}$$

Elements of H_r for $r \geq 4$ are exactly classes of cycles $\lambda+\mu$ where λ and μ are two disjoint exceptional curves. For $r=3$, however, there are two W_3–orbits of such cycles: blowing down two disjoint exceptional curves can produce either $\mathbf{P}^2$ blown up at a single point or $\mathbf{P}^1 \times \mathbf{P}^1$. The orbit H_3 corresponds to the first case.

3.4.1. Proposition. *The total support and nonzero values of the function* $\beta \mapsto \langle \Delta_2^{k(\beta)-1} \rangle_\beta$ *for* $k(\beta) \leq 3$ *are given in the following list:*

a) $\langle \emptyset \rangle_\beta = 1$ *on* I_r *for all* $3 \leq r \leq 8$.
In N_8 *in addition* $\langle \emptyset \rangle_{k_8} = 12$ *(recall that* k_r *is* W_r*–invariant for any* r*).*

b) $\langle \Delta_2 \rangle_\beta = 1$ *on* F_r *for all* $3 \leq r \leq 8$.
In N_7 *in addition* $\langle \Delta_2 \rangle_{k_7} = 12$.
In N_8 *in addition* $\langle \Delta_2 \rangle_\beta = 12$ *on* $k_8 + I_8$ *and* $\langle \Delta_2 \rangle_{2k_8} = 90$.

c) $\langle \Delta_2^2 \rangle_\beta = 1$ *on* G_r *for all* $3 \leq r \leq 8$.
In N_6 *in addition* $\langle \Delta_2^2 \rangle_{k_6} = 12$.
In N_7 *in addition* $\langle \Delta_2^2 \rangle_\beta = 12$ *on* $k_7 + I_7$.
In N_8 *there are four additional orbits with the following values:* 12 *on* $k_8 + H_8$*;* 96 *on* $k_8 + F_8$*;* 576 *on* $2k_8 + I_8$*;* 2880 *on* $3k_8$.

Proof. The argument depends on compiling and studying a table, fortunately rather short one.

It lists all non–zero 9–uples $\beta = (a; b_1, \dots, b_8)$ in $\mathbf{Z}^9$ with the following properties:

$$1 \leq k(\beta) = 3a - b_1 - \dots - b_9 \leq 3,\ b_1 \geq b_2 \geq \dots \geq b_8 \geq 0,\ a \geq b_1 + b_2 + b_3. \tag{3.20}$$

In addition, it includes $(0;0,-1,0,\dots,0)$ and contains 10 entries altogether. In the notation omitting final zeros and rendering multiplicities of b_i's by superscripts (cf. [GP], p. 87), they are

$$(0;-1),\ (1;),\ (1;1),\ (3;1^6),\ (3;1^7),\ (3;1^8),\ (4;2,1^7),\ (6,2^7,1),\ (6;2^8),\ (9;3^8)\,. \tag{3.21}$$

According to [GP], sections 3 and 5, any $\beta \in N_8$ with $\langle \Delta_2^{k(\beta)-1} \rangle_\beta \neq 0$ and $k(\beta) \leq 3$ is W_8–equivalent to one of the entries of this list. (Actually, to a single entry, because they all are pairwise distinguished by the values of pairs $k(\beta), (\beta,\beta)$). Moreover, if an entry ends with $\geq s$ zeroes, then for $s \geq 3$ its W_{8-s}–orbit contains all $\beta \in N_{8-s}$ with non–zero correlators and respective value of $k(\beta)$.

In fact, for any such β we can first permute b_i's to make them decreasing. If we get then $\delta := a - b_1 - b_2 - b_3 < 0$, we can decrease a by applying the respective Cremona transformation (3.9) or, which is the same, by reflecting β with respect of one of the simple roots in N_{8-s}. If after several steps of this kind we get a vector with $a > 0$ and some negative b_i, or with $a = 0$ but not with exactly one $b_i = -1$ and zeros on the remaining places, then $\langle \Delta_2^{k(\beta)-1} \rangle_\beta = 0$ according to [GP]. Hence we have to consider only the 10 classes listed above.

The values of $\langle \Delta_2^{k(\beta)-1} \rangle_\beta$ for these classes can be read off from the tables on p. 88 of [GP]. One exception is $(9; 3, \dots, 3)$ which must be calculated using recursion; its value was communicated to us by L. Göttsche.

Finally, we can directly identify the relevant W_r–orbits with the sets listed in the Proposition. Notice however that adding zeros changes the invariant description of the vector. For example, $(3; 1^6)$ produces k_6 in N_6, but $k_7 + l_7$ in N_7, and for the respective orbits we have $W_6 k_6 = \{k_6\}$, $W_7(k_7 + l_7) = k_7 + I_7$.

The complete list of F_8 (calculated for other purposes) in this format can be found in [MaTschi] on p. 329. Similarly, the complete list of G_8 can be extracted from the table on p. 330 in the following way: replace the misprinted entry $(10; 5^2, 4^3, 3^2, 2)$ by the correct one $(10; 5^2, 3^5, 2)$ and delete the following six entries belonging to another orbit (with vanishing Gromov–Witten invariant): $(5; 3^2, 1^6)$, $(7; 3^5, 1^3)$, $(9; 5, 3^6, 1)$, $(11; 5^3, 3^5)$, $(13; 5^6, 3^2)$, $(15; 7, 5^7)$.

3.5. The point $q = 1$, $z = 0$. We will state some simple lemmas which use the W_r-invariance and will simplify the calculation of the quantum cup product; at the end of this section we will combine the obtained information to describe the quantum cohomology algebra at $q = 1$, $z = 0$ in the case $r \geq 5$.

First, note the following

3.5.1 Fact. *If $r \geq 5$ and ρ is a root, then the orthogonal subspace $\rho^\perp$ inside the root space O_r is generated by the roots orthogonal to ρ.*

Indeed, if $r = 5$ and $\rho = (0; -1, 1, 0, 0, 0)$, then $(1; 1, 1, 1, 0, 0)$, $(1; 0, 0, 1, 1, 1)$, $(0; 0, 0, -1, 1, 0)$ and $(0; 0, 0, 0, -1, 1)$ are linearly independent roots and all orthogonal to ρ. This argument can easily be extended to higher r.

This fact will actually make a difference in several computations, which is why the Proposition 3.5.5 would be false for $r \leq 4$.

3.5.2. Lemma. *Let S be a W_r orbit in N_r, $k_r(S) := k_r(\beta)$ for an arbitrary $\beta \in S$, and let $\rho \perp \rho'$ be two orthogonal roots. Then we have:*

a) $\sum_{\beta \in S} \beta = k_r(S)\, \dfrac{|S|}{9 - r}\, k_r.$

b) $\sum_{\beta \in S} (\rho, \beta)\beta = -\frac{1}{2} \sum_{\beta \in S} (\rho, \beta)^2 \rho.$

c) $\sum_{\beta \in S} (\rho, \beta)^m (\rho', \beta) = \sum_{\beta \in S} (\rho, \beta)^{2m+1} = 0.$

d) If $r \geq 5$, then $\sum_{\beta\in S}(\rho,\beta)^2\beta = \dfrac{k_r(S)}{9-r}\sum_{\beta\in S}(\rho,\beta)^2\, k_r$.

e) $\sum_{\beta\in S}(\rho,\beta)(\rho',\beta)\beta = 0$.

Proof. a) Since $\sum_{\beta\in S}\beta$ is a W_r–invariant element of N_r, it must be proportional to k_r. To calculate the proportionality coefficient, it remains to take the intersection index with k_r.

b) This follows from symmetrizing over β and $s_\rho(\beta) = \beta + (\rho,\beta)\rho$:

$$\sum_{\beta\in S}(\rho,\beta)\beta = \frac{1}{2}\sum_{\beta\in S}\Big((\rho,\beta)\beta + (\rho, s_\rho(\beta))s_\rho(\beta)\Big) = \frac{1}{2}\sum_{\beta\in S}\Big((\rho,\beta)\beta - (\rho,\beta)(\beta + (\rho,\beta)\rho)\Big).$$

c) These are special cases of (3.13).

d) Since we assume $r \geq 5$, this formula can be checked by taking the intersection index of the right- and left-hand side with k_r, with ρ and with an arbitrary root $\rho'\perp\rho$, respectively. For ρ, ρ' this follows from c), and for k_r it is obvious.

e) As in b), we symmetrize over β and $s_\rho(\beta)$, from which we see that the sum is equal to $-\frac{1}{2}\sum_{\beta\in S}(\rho,\beta)^2(\rho',\beta)\rho$; this is zero according to c).

3.5.3. Cardinalities. The following table lists the cardinalities of the sets introduced above:

r	3	4	5	6	7	8
$\lvert W_r\rvert$	$2^2 3$	$2^3 3\cdot 5$	$2^7 3\cdot 5$	$2^7 3^4 5$	$2^{10} 3^4 5\cdot 7$	$2^{14} 3^5 5^2 7$
$\lvert R_r\rvert$	2^3	$2^2\cdot 5$	$2^3\cdot 5$	$2^3 3^2$	$2\cdot 3^2 7$	$2^4 3\cdot 5$
$\lvert I_r\rvert$	$2\cdot 3$	$2\cdot 5$	2^4	3^3	$2^3 7$	$2^4 3\cdot 5$
$\lvert F_r\rvert$	3	5	$2\cdot 5$	3^3	$2\cdot 3^2 7$	$2^4 3^3 5$
$\lvert G_r\rvert$	2	5	2^4	$2^3 3^2$	$2^6 3^2$	$2^7 3^3 5$
$\lvert H_r\rvert$	$2\cdot 3$	$2\cdot 3\cdot 5$	$2^4\cdot 5$	$2^3 3^3$	$2^2 3^3 7$	$2^6 3\cdot 5\cdot 7$

The orders of W_r, R_r and I_r are well known. The remaining entries can be calculated directly for $r = 3$ and then inductively in the following way. There are exactly r exceptional classes in N_r having zero intersection index with $(1; 0^r)$, and they are pairwise orthogonal. Hence $|G_r|$ contains as many elements as there are maximal contractible r–tuples of exceptional classes in N_r, that is $|I_r|\,|G_{r-1}|/r$. Similarly, $|F_r|$ counts non–maximal contractible $(r-1)$–tuples so that $|F_r| = |I_r|\,|F_{r-1}|/2(r-1)$. Finally, $|H_r|$ counts unordered contractible pairs, contained in maximal contractible r–tuples so that $|H_r| = |I_r|\,|I_{r-1}|/2$.

3.5.4. Proposition. *At $q = 1$, $z = 0$, the operator $k_r\circ$ admits two complementary orthogonal invariant subspaces: one spanned by Δ_0, k_r, Δ_2, the other being O_r.*

On the first subspace, the characteristic polynomial $\det(\lambda\,\mathrm{id} - k_r\circ)$ *is*

$$R(\lambda) = \lambda^3 - B_r\lambda^2 - D_r\lambda - 36\,(9-r)\,C_r \tag{3.22}$$

where

$$B_r = \frac{|I_r|}{9-r}\,,\ C_r = \frac{|G_r|}{12} + \delta_{r,6} + 56\,\delta_{r,7} + 35760\,\delta_{r,8},$$

$$D_r = 8\,(|F_r| + 12\,\delta_{r,7} + 12\,\delta_{r,8}|I_8| + 90\,\delta_{r,8}) \tag{3.23}$$

($\delta_{r,a}$ being the Kronecker symbol).

On the second subspace, $k_r\circ$ is multiplication by a constant μ_r if $r \geq 4$. This constant is given by the following table:

r	4	5	6	7	8
μ_r	-3	-4	-6	-12	-60

Proof. Formulas (3.7) and (3.8), combined with Proposition 3.4.1 and Lemma 3.5.1, lead to the following action of $k_r\circ$ at $q = 1$, $z = 0$ on the first subspace: $k_r \circ \Delta_0 = k_r$ and

$$k_r \circ k_r = 4\left(|F_r| + 12\,\delta_{r,7} + 12\,\delta_{r,8}|I_8| + 90\,\delta_{r,8}\right)\Delta_0 + \left(\frac{|I_r|}{9-r} + 12\,\delta_{r,8}\right) k_r + (9-r)\,\Delta_2, \tag{3.24}$$

$$k_r \circ \Delta_2 = 2\left(\frac{2\,|F_r|}{9-r} + 12\,\delta_{r,7} + 24\,\delta_{r,8}|I_8| + 180\,\delta_{r,8}\right) k_r + \tag{3.25}$$

$$3\left(|G_r| + 12\,\delta_{r,6} + 12\,\delta_{r,7}|I_7| + \delta_{r,8}(12\,|H_8| + 96\,|F_8| + 576\,|I_8| + 2880)\right)\Delta_0.$$

The direct calculation of the characteristic polynomial then gives (3.22) and (3.23).

To treat the orthogonal complement O_r of k_r in $H^2(V)$, notice first of all that it is spanned by roots $\rho \in R_r$. If $r \geq 4$, they form a single W_r–orbit. We calculate $k_r \circ \rho$ at $q = 1, z = 0$ using (3.6). The coefficient of Δ_0 vanishes according to 3.5.2 c), while the first sum in (3.6) is equal to $-\frac{1}{2}\sum_{\beta:k_r(\beta)=1}\langle\emptyset\rangle_\beta(\rho,\beta)^2\rho =: \mu_r$ by lemma 3.5.2 b). In view of W_r–invariance, the coefficient cannot depend on the root ρ if $r \geq 4$.

The value of μ_r can be computed explicitly by using a complete list of the W_r–orbit I_r as given in [Ma3]; see the relevant remark in the proof of the Proposition 3.5.5 for a more elegant way to compute it.

3.5.5. Proposition. *If $r \geq 5$, then the quantum cohomology algebra at the point $q = 1, z = 0$ is isomorphic to $\mathbf{C} \oplus \mathbf{C}[X_1, \ldots, X_r]/(X_i^2 = X_j^2, X_iX_j = 0)$. To establish this isomorphism, we can send $X_1, \ldots, X_r$ to an arbitrary orthonormal basis of O_r, in which case $X_1^2 = \cdots = X_r^2$ gets sent to*

$$\Gamma := \Delta_2 - \frac{1}{2(9-r)}\sum_{\beta:k_r(\beta)=1}\langle\emptyset\rangle_\beta(\rho,\beta)^2 k_r - \frac{1}{2}\sum_{\beta:k_r(\beta)=2}\langle\Delta_2\rangle_\beta(\rho,\beta)^2\Delta_0. \tag{3.26}$$

Proof. Let ρ be a root. Using lemma 3.5.2 d) we compute

$$\rho \circ \rho = -2\Delta_2 + \sum_{\beta : k_r(\beta)=1} \langle\emptyset\rangle_\beta (\rho,\beta)^2 \beta + \sum_{\beta : k_r(\beta)=2} \langle\Delta_2\rangle_\beta (\rho,\beta)^2 \Delta_0 =$$

$$-2\Delta_2 + \frac{1}{9-r} \sum_{\beta : k_r(\beta)=1} \langle\emptyset\rangle_\beta (\rho,\beta)^2 k_r + \sum_{\beta : k_r(\beta)=2} \langle\Delta_2\rangle_\beta (\rho,\beta)^2 \Delta_0 = -2\Gamma.$$

Now let ρ' be a root orthogonal to ρ. By 3.5.2 c) and e) we see that

$$\rho \circ \rho' = \sum_{\beta : k_r(\beta)=1} \langle\emptyset\rangle_\beta (\rho,\beta)(\rho',\beta) \beta + \sum_{\beta : k_r(\beta)=2} \langle\Delta_2\rangle_\beta (\rho,\beta)(\rho',\beta) \Delta_0 = 0.$$

Since ρ and the roots ρ' orthogonal to ρ span O_r, we can conclude that for any D' which lies in this space, we have $\rho \circ D' = (\rho, D')\,\Gamma$. Since the roots span O_r, the formula $D \circ D' = (D, D')\,\Gamma$ holds for all $D, D' \in O_r$.

Note that $\Gamma \circ \rho = \rho' \circ \rho' \circ \rho = 0$ and hence $\Gamma \circ \Gamma = 0 = \rho^{\circ 3}$.

By associativity, Γ must be an eigenvector of $k_r \circ$ to the eigenvalue μ_r. So μ_r must be a root of the characteristic polynomial $R(\lambda)$ defined in (3.22). One can check that it is actually a double root. In fact, if it was a simple root, then the eigenspace of $k_r \circ$ to the eigenvalue μ_r would consist only of nilpotent elements, which is impossible since it is a direct summand of an algebra with identity. (This fact gives the most direct way to determine μ_r: as the double root of $R(\lambda)$.) Since $R(0) < 0$, it can't be a triple root.

Thus we have an $(r+2)$–dimensional eigenspace of $k_r \circ$ and a splitting of $H^*(V)$ into a one–dimensional and an $(r+2)$–dimensional subalgebra. The latter one has a basis consisting of its identity, Γ and an arbitrary orhonormal basis of $k_r^\perp$; in this basis the structure is evidently as described in the proposition.

3.6. Semisimplicity of the quantum cohomology of del Pezzo surfaces. We have seen that if $r \geq 5$, the quantum cohomology is far from being semisimple at the symmetric point $q = 1$; also proposition 3.5.4 shows that this cannot be a tame semisimple point if $r \geq 3$. However, we will prove:

3.6.1. Theorem. *The Frobenius manifold associated to the quantum cohomology of a del Pezzo surface is generically semisimple on H^2.*

Our proof is based on extending the Frobenius manifold associated to V_r to a boundary; on the boundary, it ceases to be a Frobenius manifold, yet the boundary itself will be the Frobenius manifold associated to V_{r-1}.

First, we introduce appropriate coordinates: We write the generic character q^β on H^2 as

$$q^\beta = e^{\sum_{i=0}^r x_i (l_i, \beta)},$$

where the l_i are as defined in the beginning of section 3.3. We define $q_i := e^{x_i}$ and thus can write

$$q^\beta = \prod_{i=0}^{r} q_i^{(l_i,\beta)}.$$

As mentioned in the beginning of section 3.3, all the sums appearing in equations (3.4) to (3.6) are finite, and each summand is a monomial in the variables $q_0, \dots, q_r$ and $q_1^{-1}, \dots, q_r^{-1}$. Hence, the spectral cover map restricted to T_V is the morphism of affine schemes

$$\operatorname{Spec} H^*(V_r, \mathbf{Q})[q_0, q_1, \dots, q_r, q_1^{-1}, \dots q_r^{-1}] \to \operatorname{Spec} \mathbf{Q}[q_0, q_1, \dots, q_r, q_1^{-1}, \dots q_r^{-1}] \tag{3.27}$$

where the left hand side is the small quantum cohomology algebra.

3.6.2. Lemma. *The map (3.27) can be extended in a flat way to the boundary $q_r = 0$. Moreover, the fiber over $q_r = 0$ is isomorphic to a disjoint sum of the identity map and the respective map (3.27) for the case $r - 1$.*

Proof. As we already mentioned, from the properties (P1) and (P2) of Theorem 4.1 in [GP] it follows that the only $\beta \in H_2(V, \mathbf{Z})$ with non–vanishing Gromov–Witten invariant $\langle \Delta_2^{k_r(\beta)-1} \rangle$ and $(\beta, l_r) < 0$ is l_r itself, for which we have $\langle \emptyset \rangle_{l_r} = 1$. This will be the key fact used in our proof.

Let F be the free module over $\mathbf{Q}[q_0, \dots, q_r, q_1, \dots, q_{r-1}^{-1}]$ with basis $\Delta_0, \Delta_2, l_0, \dots, l_{r-1}, q_r l_r$. The module $F[q_r^{-1}]$ is simply the small quantum cohomology algebra; we will show that F is a subalgebra by simply examining all relevant products.

Let D be an arbitrary element in $H^2(V_r)$ orthogonal to l_r, i. e. $D \in H^2(V_{r-1})$, and let T_1 and T_2 be two arbitrary elements in $H^*(V_r)$ orthogonal to l_r, so $T_1, T_2 \in H^*(V_{r-1})$. We will denote their product by $\circ_{V_r}$ or $\circ_{V_{r-1}}$ indicating in which quantum cohomology this product is to be taken.

We claim that:

$$q_r l_r \circ q_r l_r = q_r l_r + O(q_r^2)$$

$$q_r l_r \circ D = O(q_r^2)$$

$$q_r l_r \circ \Delta_2 = O(q_r^2)$$

$$T_1 \circ_{V_r} T_2 = T_1 \circ_{V_{r-1}} T_2 + O(q_r) \tag{3.27}$$

All these formulas follow from (3.4)–(3.6) and the fact mentioned in the beginning of our proof. To give an example of the computations, we treat the case of $q_r l_r \circ D$:

$$l_r \circ D = \sum_{\beta:\, k_r(\beta)=1} \langle \emptyset \rangle_\beta q_0^{(l_0,\beta)} \cdot \ldots \cdot q_r^{(l_r,\beta)} (D, \beta)(l_r, \beta)\beta$$

$$+\sum_{\beta:\,k_r(\beta)=2}\langle\Delta_2\rangle_\beta q_0^{(l_0,\beta)}\cdot\ldots\cdot q_r^{(l_r,\beta)}(D,\beta)(l_r,\beta)\Delta_0$$

The only term that could contribute a negative power of q_r is $\beta = l_r$; however, this term is cancelled by the factor $(D,\beta) = 0$. The terms with q_r^0 as factor do not contribute to the sum either since $(l_r,\beta) = 0$. Hence we get the above formula.

The other cases are dealt with by very similar arguments. In the case of $l_r \circ l_r$, the summand with $\beta = l_r$ gives $\frac{1}{q_r}l_r$ as the only term containing a negative power of q_r, which yields our formula. In the product of T_1, T_2, the summand $\beta = l_r$ either does not appear at all if T_1 or T_2 is Δ_2 because of $k_r(l_r) = 1$; otherwise, if both $T_1, T_2 \in H^2(V_r)$, this summand is cancelled by the factor (T_1,β). Now if we set $q_r = 0$, in the relevant sums computing $T_1 \circ_{V_r} T_2$ only summands with $(\beta, l_r) = 0$ are left, i. e. with $\beta \in H_2(V_{r-1})$. Since the Gromov–Witten invariants of such an β are the same, whether they are computed in V_r or in V_{r-1}, we get the last equation of (3.27).

We have thus seen that the product extends to F. Since it obviously remains associative, F is an algebra over $\mathbf{Q}[q_0,\ldots,q_r,q_1^{-1},\ldots,q_{r-1}^{-1}]$. So we have proved the first assertion of our lemma.

We want to examine the fiber $F_0 := F/q_rF$. Clearly, q_rl_r is an idempotent, so we get a splitting of F_0 into the image and the kernel K of multiplication with q_rl_r. The kernel K is spanned by $H^2(V_{r-1}), \Delta_2$ and $\Delta_0 - q_rl_r$, the image is the span of q_rl_r.

The kernel K is isomorphic to F_0/q_rl_r. From the last equation in the list (3.27) we see that F_0/q_rl_r is isomorphic to the small quantum cohomology algebra of V_{r-1}, which proves the second statement of the lemma.

Proof of Theorem 3.6.1. As mentioned already in 1.6, it is well known that the quantum cohomology of $\mathbf{P}^r$ is generically semisimple. We proceed by induction.

So assume we know that the quantum cohomology of V_{r-1} is generically semisimple on H^2. According to the lemma, this implies that

$$\operatorname{Spec} F \to \operatorname{Spec}\mathbf{Q}[q_0,\ldots,q_r,q_1^{-1},\ldots,q_{r-1}^{-1}]$$

is generically semisimple on the fiber $q_r = 0$. Since semisimplicity is a Zariski–open condition, this map is generically semisimple; restricting to $q_r \neq 0$, we see that the quantum cohomology of V_r is generically semisimple on $H^2(V_r)$.

3.6.3. Remark. It is well known that there exists a full collection of exceptional sheaves for del Pezzo surfaces, see [KuOr]. Hence the Theorem 3.6.1 is a special case of Dubrovin's conjecture mentioned in 1.8.2.

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