THICK METRIC SPACES, RELATIVE HYPERBOLICITY, AND QUASI-ISOMETRIC RIGIDITY

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Abstract. We study the geometry of nonrelatively hyperbolic groups. Generalizing a result of Schwartz, any quasi-isometric image of a non-relatively hyperbolic space in a relatively hyperbolic space is contained in a bounded neighborhood of a single peripheral subgroup. This implies that a group being relatively hyperbolic with nonrelatively hyperbolic peripheral subgroups is a quasi-isometry invariant. As an application, Artin groups are relatively hyperbolic if and only if freely decomposable.

We also introduce a new quasi-isometry invariant of metric spaces called *metrically thick*, which is sufficient for a metric space to be nonhyperbolic relative to any nontrivial collection of subsets. Thick finitely generated groups include: mapping class groups of most surfaces; outer automorphism groups of most free groups; certain Artin groups; and others. Nonuniform lattices in higher rank semisimple Lie groups are thick and hence nonrelatively hyperbolic, in contrast with rank one which provided the motivating examples of relatively hyperbolic groups. Mapping class groups are the first examples of nonrelatively hyperbolic groups having cut points in any asymptotic cone, resolving several questions of Drutu and Sapir about the structure of relatively hyperbolic groups. Outside of group theory, Teichmüller spaces for surfaces of sufficiently large complexity are thick with respect to the Weil-Peterson metric, in contrast with Brock–Farb's hyperbolicity result in low complexity.

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1. Introduction

Three of the most studied families of groups in geometric group theory are the mapping class group of a surface of finite type, $\mathcal{MCG}(S)$; the outer automorphism group of a finite rank free group, $\mathrm{Out}(F_n)$; and the special linear group, $SL_n(\mathbb{Z})$. Despite the active interest in these groups, much of their quasi-isometric structure remains unknown, particularly for the first two families. We introduce the notion of a *thick* group (or more generally, metric space), a property which is enjoyed by all groups in each of the families $\mathcal{MCG}(S)$, $\mathrm{Out}(F_n)$, and $SL_n(\mathbb{Z})$ except in the lowest complexity cases where the groups are actually hyperbolic. The notion of thickness helps unify the study of these groups and casts light on some of their geometric properties.

Before proceeding, we recall some relevant developments. In [Gro2], M. Gromov introduced the notion of a relatively hyperbolic group. The theory of relatively hyperbolic groups was developed by Farb in [Far], then further developed in [Bow2], [Dah], [Osi], [Yam], and [DS1]. Several alternate characterizations of relative hyperbolicity have been formulated, all of them more or less equivalent to each other. We recall the definition due to Farb. In the sequel G denotes a finitely generated group endowed with a word metric, $\mathcal{H} = \{H_1, ..., H_n\}$ is a finite family of subgroups of G and $\mathcal{LH}$ denotes the collection of left cosets of $\{H_1, ..., H_n\}$ in G. The group G is *weakly hyperbolic relative to* $\mathcal{H}$ if collapsing the left cosets in $\mathcal{LH}$ to finite diameter sets, in a Cayley graph of G, yields a δ–hyperbolic space. The subgroups $H_1, ..., H_n$ are called *peripheral subgroups.*

The group G is *(strongly) hyperbolic relative to* $\mathcal{H}$ if it is weakly hyperbolic relative to $\mathcal{H}$ and if it has the *bounded coset property.* This latter property, roughly speaking, requires that in a Cayley graph of G with the sets in $\mathcal{LH}$ collapsed to bounded diameter sets, a pair of quasigeodesics with the same endpoints travels through the collapsed $\mathcal{LH}$ in approximately the same manner.

In [DS1, §8 and Appendix], Druţu, Osin and Sapir provide a geometric condition which characterizes relative hyperbolicity of a group. They show that G is hyperbolic relative to $\mathcal{H}$ if and only if any *asymptotic cone* of G is *tree-graded* with respect to the collection of *pieces* given by ultralimits of elements in $\mathcal{LH}$ (see Section 2 for definitions). In particular any asymptotic cone of G has (global) cut-points.

The asymptotic characterization of relative hyperbolicity mentioned above is in turn equivalent to three metric properties in the Cayley graph of G (formulated without asymptotic cones), which are approximately as follows:

- (α_1) Finite radius neighborhoods of distinct elements in $\mathcal{LH}$ are either disjoint or intersect in sets of uniformly bounded diameter;
- (α_2) geodesics diverging slower than linearly from a set gH_i in $\mathcal{LH}$ must intersect a finite radius neighborhood of gH_i;
- (α_3) fat geodesic polygons must stay close to a set in $\mathcal{LH}$ ("fat" here is the contrary of "thin" in its metric hyperbolic sense; see Definition 2.7).

This definition of relative hyperbolicity also makes sense in a general metric setting: a geodesic metric space X is said to be *asymptotically tree-graded* (ATG in short) with respect to a collection $\mathcal{A}$ of subsets of X (called *peripheral subsets*) if the three conditions above hold with G replaced by X and $\mathcal{LH}$ replaced by $\mathcal{A}$ (see also [BF] for another metric version of the notion of relative hyperbolicity). For instance, the complementary set in $\mathbb{H}^3$ of any family of pairwise disjoint open horoballs is asymptotically tree-graded with respect to the collection of boundary horospheres. It was recently proven by Druţu that if a group is asymptotically tree-graded in a metric sense, that is with respect to a collection $\mathcal{A}$ of subsets, then it is relatively hyperbolic with respect to some family of subgroups [Dru3] (see Theorem 2.11 in this paper). The converse of the above statement was shown in [DS1] (see Theorem 2.10).

Convention 1.1. *Throughout the paper, we exclude the trivial case of a metric space X asymptotically tree-graded with respect to a collection $\mathcal{A}$ where some finite radius neighborhood of some subset $A \in \mathcal{A}$ equals X. In the case of an infinite group, G, hyperbolic relative to a collection of subgroups, the trivial case we are excluding is where one of the subgroups is G.*

When a group contains no collection of proper subgroups with respect to which it is relatively hyperbolic, we say the group is not relatively hyperbolic (NRH)*.*

Thickness is, in many respects, opposite to relative hyperbolicity. The notion of thickness is built up inductively. A geodesic metric space is *thick of order zero* if it is *unconstricted*, in the terminology of [DS1], that is: for at least one sequence of scaling constants $d = (d_n)$ and one ultrafilter, all asymptotic cones constructed by means of d and ω are without (global) cut-points. If the metric space is a group then this is equivalent to the condition that *at least one* asymptotic cone is without cut-points. See Section 3 for details, and for a list of examples of groups thick of order zero (unconstricted). A metric space is *thick of order n* if, roughly speaking, it can be expressed as a coarse union of a network of subspaces thick of order $n-1$, each quasi-isometrically embedded, so that two adjacent subspaces in this network have infinite coarse intersection. The exact definition of thickness can be found in Section 7. Because thickness is a quasi-isometry invariant, thickness of a finitely generated group G is well-defined by requiring that the Cayley graph of a finite generating set of G be a thick metric space. Thick metric spaces behave very rigidly when embedded into asymptotically tree-graded metric spaces in particular we obtain (see Theorem 7.8 for a generalization of this result):

Corollary 7.9 (Thick spaces are not asymptotically tree-graded). *If X is a thick metric space, then X is not asymptotically tree-graded. In particular, if X is a finitely generated thick group, then X is not relatively hyperbolic.*

The following result puts strong restrictions on how NRH groups can be quasi-isometrically embedded in ATG spaces.

Theorem 4.1 (NRH subgroups are peripheral). *Let (X, dist_X) be a metric space asymptotically tree-graded with respect to a collection $\mathcal{A}$ of subsets. For every $L \geq 1$ and $C \geq 0$ there exists $R = R(L, C, X, \mathcal{A})$ such that the following holds. If G is a finitely generated group endowed with a word metric* dist *and G is not relatively hyperbolic, then for any (L, C)-quasi-isometric embedding $\mathfrak{q}\colon (G, \mathrm{dist}) \to (X, \mathrm{dist}_X)$ the image $\mathfrak{q}(G)$ is contained in the radius R neighborhood of some $A \in \mathcal{A}$.*

Note that in the theorem above the constant R *does not* depend on the group G.

This theorem shows that the presence of NRH (in particular thick) peripheral subgroups in a relatively hyperbolic group "rigidifies" the structure. A similar rigidity result, with additional hypotheses on both the domain and the range plays a key role in Schwartz's quasi-isometric classification of rank one non-uniform lattices in semisimple Lie groups [Sch]. Druţu–Sapir proved a similar rigidity result under the assumption that the domain is unconstricted [DS1]; using work of [DS1] allows one to obtain the following theorem. (For special cases of this result see also Theorem 3.6 and Theorem 7.8 in this paper or other results in [DS1].)

Theorem 4.8 (Quasi-isometric rigidity of hyperbolicity relative to NRH subgroups). *If Γ is a finitely generated group hyperbolic relative to a finite collection of finitely generated subgroups $\mathcal{G}$ for which each $G \in \mathcal{G}$ is not relatively hyperbolic, then any finitely generated group Γ' which is quasi-isometric to Γ is hyperbolic relative to a finite collection of finitely generated subgroups $\mathcal{G}'$ where each subgroup in $\mathcal{G}'$ is quasi-isometric to one of the subgroups in $\mathcal{G}$.*

In [Dru3] is proved the quasi-isometry invariance of relative hyperbolicity (see Theorem 2.12 in this paper), but without establishing any relation between the peripheral subgroups (which is impossible to do in full generality, see the discussion following Theorem 2.12). Theorem 4.8 resolves this question. Moreover, it advances towards a classification of relatively hyperbolic groups. By results in [PW], the classification of relatively hyperbolic groups reduces to the classification of one-ended relatively hyperbolic groups. Theorem 4.8 points out a fundamental necessary condition for the quasi-isometry of two one-ended relatively hyperbolic groups with NRH peripheral subgroups: that the peripheral subgroups define the same collection of quasi-isometry classes. Nevertheless the condition is not sufficient, as can be seen in [Sch], where it is proved for instance that two fundamental groups of finite volume hyperbolic three-manifolds are quasi-isometric if and only if they are commensurable (while all their peripheral subgroups are isomorphic to $\mathbb{Z}^2$, when there is no torsion). This raises the question on what finer invariants of quasi-isometry may exist for relatively hyperbolic groups (besides the q.i. classes of peripherals) which would allow advancing further in the classification.

Theorems 4.1 and 4.8 motivate the study of non-relative hyperbolicity and, in particular, thickness. In order to verify thickness of a finitely generated group, we formulate an algebraic form of thickness in the setting of groups endowed with word metrics and their undistorted subgroups (see Definition 7.3). Many important groups turn out to have this property, and therefore are NRH:

Theorem 1.2. *The following finitely generated groups (keyed to section numbers) are algebraically thick with respect to the word metric:*

- §8. $\mathcal{MCG}(S)$*, when S is an orientable finite type surface with* $3 \cdot \mathrm{genus}(S) + \#\,\mathrm{punctures} \geq 5$;
- §9. $\mathrm{Aut}(F_n)$ *and* $\mathrm{Out}(F_n)$*, when $n \geq 3$;*
- §10. *A freely indecomposable Artin group with any of the following properties: the integer labels on the Artin presentation graph are all even; the Artin presentation graph is a tree; the Artin presentation graph has no triangles; the associated Coxeter group is finite or affine of type $\widetilde{A}_n$.*
- §11. *Fundamental groups of 3-dimensional graph manifolds;*
- §13. *Non-uniform lattices in semisimple groups of rank at least two.*

The failure of strong relative hyperbolicity for $SL_n(\mathbb{Z})$ when $n \geq 3$ was first proved in [KN]. For the case of mapping class groups, the failure of strong relative hyperbolicity is also proved in [AAS], [Bow3], and [KN]; see the discussion after Corollary 8.3. If one is solely interested in disproving strong relative hyperbolicity, there are more direct approaches which avoid asymptotic cones, such as the one taken in [AAS]. In Propositions 5.4 and 5.5 we also give such results, generalizing the main theorem of [AAS].

In the particular case of Artin groups, more can be proved concerning relative hyperbolicity. The following is an immediate consequence of Proposition 5.5 and Example 10.1:

Proposition 1.3. *Except for the integers, any Artin group with connected Artin presentation graph is not relatively hyperbolic.*

Note that this gives a complete classification of which Artin groups are relatively hyperbolic, since any group with a disconnected presentation graph is freely decomposable and hence relatively hyperbolic with respect to the factors in the free decomposition.

Remark 1.4. For the Artin groups which are not in the list of Theorem 1.2 we do not know whether they are thick or not. Possibly some of them might turn out to be examples of NRH groups that are not thick. This would provide a nice class of examples, as the groups we know which are NRH, but not thick, are fairly pathological, cf. the end of Section 7.

Theorem 1.2 and Proposition 1.3 are interesting also because some of the listed groups are known to be weakly relatively hyperbolic. Examples include: mapping class groups [MM1], certain Artin groups [KS], and fundamental groups of graph manifolds. Thus the study we begin in this paper, of thick groups from the point of view of quasi-isometric rigidity, may also be perceived as a first attempt to study quasi-isometric rigidity of weakly relatively hyperbolic groups. Note that up to now there is no general result on the quasi-isometric behavior of weakly relatively hyperbolic groups. In [KL1], [Pap], [DS], [MSW1] and [MSW2] strong quasi-isometric rigidity results are proved for some particular cases of weakly relatively hyperbolic groups—in fact all of them are fundamental groups of some graphs of groups (fundamental groups of Haken manifolds, groups with a JSJ decomposition, fundamental groups of finite graphs of groups with Bass-Serre tree of finite depth).

Some of the groups mentioned in Theorem 1.2 present even further similarities with (strongly) relatively hyperbolic groups, in that all their asymptotic cones are tree-graded metric spaces. This is the case for the mapping class groups, where it was proved by Behrstock [Beh]; and for the fundamental groups of 3-dimensional graph manifolds, where it follows from results in [KL2] and [KKL]; the latter class includes right angled Artin groups whose Artin presentation graph is a tree of diameter at least three (see Proposition 10.9).

In particular these examples answer in the negative two questions of Druţu and Sapir (see [DS1, Problem 1.18]) regarding a finitely generated group G for which every asymptotic cone is tree-graded: Is G relatively hyperbolic? And is G asymptotically tree-graded with respect to some collection of subsets of G? The negative answers to these questions indicate that a supplementary condition on the pieces in the asymptotic cones is indeed necessary.

Another question resolved by the example of mapping class groups is whether every relatively hyperbolic group is in fact hyperbolic relative to subgroups that are *unconstricted* (see [DS1, Problem 1.21]). Indeed, consider the finitely presented relatively hyperbolic group $\Gamma = \mathcal{MCG}(S) * \mathcal{MCG}(S)$. Suppose that it is hyperbolic relative to a finite collection of unconstricted peripheral subgroups $\mathcal{H}$. Corollary 4.7 implies that each $H \in \mathcal{H}$ must be contained in a conjugate $\gamma\mathcal{MCG}(S)\gamma^{-1}$ of one of the two free factors isomorphic to $\mathcal{MCG}(S)$ in Γ. Applying Corollary 4.7 again to Γ seen as hyperbolic relative to the subgroups in $\mathcal{H}$ we obtain that $\gamma\mathcal{MCG}(S)\gamma^{-1}$ is contained in a conjugate of a subgroup $H_1 \in \mathcal{H}$. This implies that H is contained in a conjugate of H_1, a situation which can occur only if H coincides with the conjugate of H_1. Thus the two inclusions above are equalities, in particular $H = \gamma\mathcal{MCG}(S)\gamma^{-1}$. On the other hand, all asymptotic cones of $\mathcal{MCG}(S)$ have (global) cut-points, and hence the same holds for $\gamma\mathcal{MCG}(S)\gamma^{-1}$ (see [Beh]); this contradicts the hypothesis that H is unconstricted. Note that in the previous argument $\mathcal{MCG}(S)$ can be replaced by any group which is thick (or more generally not relatively hyperbolic) and with (global) cut-points in any asymptotic cone (i.e., *constricted*, in the terminology of [DS1]).

In Section 6, we answer a related weaker question, namely, does any relatively hyperbolic group admit a family of peripheral subgroups which are not relatively hyperbolic? The answer is no, with Dunwoody's inaccessible group providing a counterexample. Since finitely presented groups are accessible, this raises the following natural question.

Question 1.5. *Is there any example of a finitely presented relatively hyperbolic group such that every list of peripheral subgroups contains a relatively hyperbolic group?*

A similar question can be asked for groups without torsion, as these groups are likewise accessible.

Thickness can be studied for spaces other than groups. As an example of this we prove the following:

Theorem 12.3 *For any surface S with $3 \cdot \mathrm{genus}(S) + \#\,\mathrm{punctures} \geq 9$, the Teichmüller space with the Weil-Petersson metric is thick.*

In particular the Teichmüller space is not asymptotically tree-graded. An interesting aspect of this theorem is that although these higher complexity Teichmüller spaces are not asymptotically tree-graded, they do have tree-graded asymptotic cones as proven in [Beh]. We also note that the lack of relative hyperbolicity contrasts with the cases with $3 \cdot \mathrm{genus}(S) + \#\,\mathrm{punctures} \leq 5$ where it has been shown that Teichmüller space is δ–hyperbolic with the Weil-Petersson metric (see [BF], and also [Ara], [Beh]). It also contrasts with the relative hyperbolicity of Teichmüller space in the cases where $3 \cdot \mathrm{genus}(S) + \#\,\mathrm{punctures} = 6$, as recently shown in [BM].

The paper is organized as follows. Section 2 provides background on asymptotic cones and various tools developed in [DS1] for studying relatively hyperbolic groups. In Section 3 we discuss the property of (not) having cut-points in asymptotic cones.

Section 4 contains some general results regarding quasi-isometric embeddings of NRH groups into relatively hyperbolic groups and our main theorem of rigidity of relatively hyperbolic groups. Motivated by these results we provide examples of NRH groups, and in Section 5 we describe a way to build NRH groups. In Section 6 we discuss an example of a relatively hyperbolic group such that any list of peripheral subgroups contains a relatively hyperbolic group.

In Section 7 we define metric and algebraic thickness, we provide results on the structure and rigidity of thick spaces and groups and we discuss an example of an NRH group which is not thick.

The remaining sections of this work establish thickness for various groups and metric spaces. For the mapping class groups, the automorphism group of a free group, and the outer automorphism group of a free group we prove thickness in all cases except when these groups are virtually free (and hence are not thick), this is done in Sections 8 and 9. Artin groups are studied in Section 10. Graph manifolds and Teichmüller space are shown to be thick in Sections 11 and 12. Finally in Section 13, we establish thickness for non-uniform lattices (thickness in the uniform case follows from [KL]).

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2. Preliminaries

A *non-principal ultrafilter* on the positive integers, denoted by ω, is a nonempty collection of sets of positive integers with the following properties:

(1) If $S_1 \in \omega$ and $S_2 \in \omega$, then $S_1 \cap S_2 \in \omega$.
(2) If $S_1 \subset S_2$ and $S_1 \in \omega$, then $S_2 \in \omega$.
(3) For each $S \subset \mathbb{N}$ exactly one of the following must occur: $S \in \omega$ or $\mathbb{N} \setminus S \in \omega$.
(4) ω does not contain any finite set.

Convention: The adjective "non-principal" refers to item (4). Since we work only with non-principal ultrafilters, we shall tacitly drop this adjective throughout the sequel.

For an ultrafilter ω, a topological space X, and a sequence of points $(x_i)_{i\in\mathbb{N}}$ in X, we define x to be the *ultralimit of* $(x_i)_{i\in\mathbb{N}}$ *with respect to* ω, and we write $x = \lim_\omega x_i$, if and only if for any neighborhood $\mathcal{N}$ of x in X the set $\{i \in \mathbb{N} : x_i \in \mathcal{N}\}$ is in ω. Note that when X is compact any sequence in X has an ultralimit [Bou]. If moreover X is Hausdorff then the ultralimit of any sequence is unique. Fix an ultrafilter ω and a family of based metric spaces $(X_i, x_i, \mathrm{dist}_i)$. Using the ultrafilter, a pseudo-distance on $\prod_{i\in\mathbb{N}} X_i$ is provided by:

$$\mathrm{dist}_\omega((a_i),(b_i)) = \lim_\omega \mathrm{dist}_i(a_i,b_i) \in [0,\infty]\,.$$

One can eliminate the possibility of the previous pseudo-distance taking the value $+\infty$ by restricting to sequences $y = (y_i)$ such that $\mathrm{dist}_\omega(y,x) < \infty$, where $x = (x_i)$.

A metric space can be then defined, called the *ultralimit of* $(X_i, x_i, \mathrm{dist}_i)$, by:

$$\lim_\omega (X_i, x_i, \mathrm{dist}_i) = \left\{ y \in \prod_{i\in\mathbb{N}} X_i : \mathrm{dist}_\omega(y, x) < \infty \right\} / \sim ,$$

where for two points $y, z \in \prod_{i\in\mathbb{N}} X_i$ we define $y \sim z$ if and only if $\mathrm{dist}_\omega(y, z) = 0$. The pseudo-distance on $\prod_{i\in\mathbb{N}} X_i$ induces a complete metric on $\lim_\omega(X_i, x_i, \mathrm{dist}_i)$.

Let now (X, dist) be a metric space. Consider $x = (x_n)$ a sequence of points in X, called *observation points*, and $d = (d_n)$ a sequence of positive numbers such that $\lim_\omega d_n = +\infty$, called *scaling constants*. First defined in [Gro1] and [dDW], the *asymptotic cone of* (X, dist) *relative to the ultrafilter* ω *and the sequences* x *and* d is given by:

$$\mathrm{Cone}_\omega(X, x, d) = \lim_\omega \left(X, x_n, \frac{1}{d_n}\mathrm{dist} \right).$$

When the group of isometries of X acts on X so that all orbits intersect a fixed bounded set, the asymptotic cone is independent of the choice of observation points. An important example of this is when X is a finitely generated group with a word metric; thus, when X is a finitely generated group we always take the observation points to be the constant sequence (1) and we drop the observation point from our notation.

Every sequence $(A_n)_{n\in\mathbb{N}}$ of non-empty subsets of X has a *limit set* in the asymptotic cone $\mathrm{Cone}_\omega(X, x, d)$, denoted by $\lim_\omega A_n$ and defined as the set of images in the asymptotic cone of sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n \in A_n$ for every n. The set $\lim_\omega A_n$ is empty when $\lim_\omega \frac{\mathrm{dist}(x_n, A_n)}{d_n} = \infty$, otherwise it is a closed subset of $\mathrm{Cone}_\omega(X, x, d)$. In the latter case, $\lim_\omega A_n$ is isometric to the ultralimit of $(A_n, y_n, \frac{\mathrm{dist}}{d_n})_{n\in\mathbb{N}}$ with the metric dist on A_n induced from X, and with basepoints $y_n \in A_n$ such that $\lim_\omega \frac{\mathrm{dist}(x_n, y_n)}{d_n} < \infty$.

Given a collection $\mathcal{P}$ of subsets in X and an asymptotic cone $\mathrm{Cone}_\omega(X, x, d)$ of X, we denote by $\lim_\omega(\mathcal{P})$ the collection of non-empty limit sets $\lim_\omega P_n$ where $(P_n)_{n\in\mathbb{N}}$ is a sequence of subsets $P_n \in \mathcal{P}$. We will often consider the case where $X = G$ is a group and $\mathcal{H}$ is a fixed collection of subgroups of G, in this case we take $\mathcal{P}$ to be the collection of left cosets gH, with $g \in G$ and $H \in \mathcal{H}$. We denote the latter collection also by $\mathcal{LH}$. We now recall a notion introduced in [DS1, §2].

Definition 2.1. Let $\mathbb{F}$ be a complete geodesic metric space and let $\mathcal{P}$ be a collection of closed geodesic subsets (called *pieces*). The space $\mathbb{F}$ is said to be *tree-graded with respect to* $\mathcal{P}$ when the following two properties are satisfied:

(T_1) The intersection of each pair of distinct pieces has at most one point.
(T_2) Every simple non-trivial geodesic triangle in $\mathbb{F}$ is contained in one piece.

When the collection of pieces $\mathcal{P}$ is understood then we say simply that $\mathbb{F}$ is *tree-graded.*

Lemma 2.2 (Druţu–Sapir [DS1])**.** *Let $\mathbb{F}$ be a complete geodesic metric space which is tree-graded with respect to a collection of pieces $\mathcal{P}$.*

(1) *For every point $x \in F$, the set T_x of topological arcs originating at x and intersecting any piece in at most one point is a complete real tree (possibly reduced to a point). Moreover if $y \in T_x$ then $T_y = T_x$.*

(2) *Any topological arc joining two points in a piece is contained in the same piece. Any topological arc joining two points in a tree T_x is contained in the same tree T_x.*

A tree as in Lemma 2.2 (1) is called a *transversal tree*, and a geodesic in it is called a *transversal geodesic*. Both of these notions are defined relative to the collection of pieces $\mathcal{P}$, which when understood is suppressed.

The notion of tree-graded metric space is related to the existence of cut-points.

Convention: By cut-points we always mean *global* cut-points. We consider a singleton to have a cut-point.

Lemma 2.3 (Druţu–Sapir [DS1], Lemma 2.31)**.** *Let X be a complete geodesic metric space containing at least two points and let $\mathcal{C}$ be a non-empty set of cut-points in X. There exists a uniquely defined (maximal in an appropriate sense) collection $\mathcal{P}$ of subsets of X such that*

- *X is tree-graded with respect to $\mathcal{P}$;*
- *any piece in $\mathcal{P}$ is either a singleton or a set with no cut-point in $\mathcal{C}$.*

Moreover the intersection of any two distinct pieces from $\mathcal{P}$ is either empty or a point from $\mathcal{C}$.

Definition 2.4. Let X be a metric space and let $\mathcal{A}$ be a collection of subsets in X. We say that X is *asymptotically tree-graded (ATG)* with respect to $\mathcal{A}$ if

(I) every asymptotic cone $\mathrm{Cone}_\omega(X)$ of X is tree-graded with respect to $\lim_\omega(\mathcal{A})$;
(II) X is not contained in a finite radius neighborhood of any of the subsets in $\mathcal{A}$.

The subsets in $\mathcal{A}$ are called *peripheral subsets*.

The second condition does not appear in [DS1]. It is added here to avoid the trivial cases, like that of X asymptotically tree-graded with respect to $\mathcal{A} = \{X\}$. For emphasis, one could refer to an ATG structure satisfying (II) as being a *proper asymptotically tree-graded structure*. Since we always assume that the tubular neighborhoods of peripheral subsets are proper subsets (see Convention 1.1), we suppress the use of the adjective "proper." Similarly, we assume that relative hyperbolicity is always with respect to a collection of proper peripheral subgroups.

As mentioned in the introduction, Druţu–Sapir provide a characterization of ATG metric spaces, further simplified by Druţu in [Dru3], in terms of three metric properties involving elements of $\mathcal{A}$, geodesics, and geodesic polygons. There are several versions of the list of three properties, we recall here those that we shall use most, keeping the notation in [Dru3].

First we recall the notion of fat polygon introduced in [DS1]. This notion is in some sense the opposite of the notion of "thin" polygon (i.e., a polygon behaving metrically like a polygon in a tree, up to bounded perturbation).

Throughout the paper $\mathcal{N}_r(A)$ denotes the set of points x satisfying $\mathrm{dist}(x, A) < r$ and $\overline{\mathcal{N}}_r(A)$ the set of points x with $\mathrm{dist}(x, A) \leq r$.

Notation 2.5. For every quasi-geodesic $\mathfrak{p}$ in a metric space X, we denote the origin of $\mathfrak{p}$ by $\mathfrak{p}_-$ and the endpoint of $\mathfrak{p}$ by $\mathfrak{p}_+$.

Given $r > 0$ we denote by $\breve{\mathfrak{p}}_r$ the set $\mathfrak{p} \setminus \mathcal{N}_r\left(\{\mathfrak{p}_-\,,\, \mathfrak{p}_+\}\right)$.

A *geodesic (quasi-geodesic) k-gonal line* is a set P which is the union of k geodesics (quasi-geodesics) $\mathfrak{q}_1, ..., \mathfrak{q}_k$ such that $(\mathfrak{q}_i)_+ = (\mathfrak{q}_{i+1})_-$ for $i = 1, ..., k-1$. If moreover $(\mathfrak{q}_k)_+ = (\mathfrak{q}_1)_-$ then we say that P is a *geodesic (quasi-geodesic) k-gon.*

Notation 2.6. Given a vertex $x \in \mathcal{V}$ and $\mathfrak{q}, \mathfrak{q}'$ the consecutive edges of P such that $x = \mathfrak{q}_+ = \mathfrak{q}'_-$, we denote the polygonal line $P \setminus (\mathfrak{q} \cup \mathfrak{q}')$ by $\mathcal{O}_x(P)$. When there is no possibility of confusion we simply denote it by $\mathcal{O}_x$.

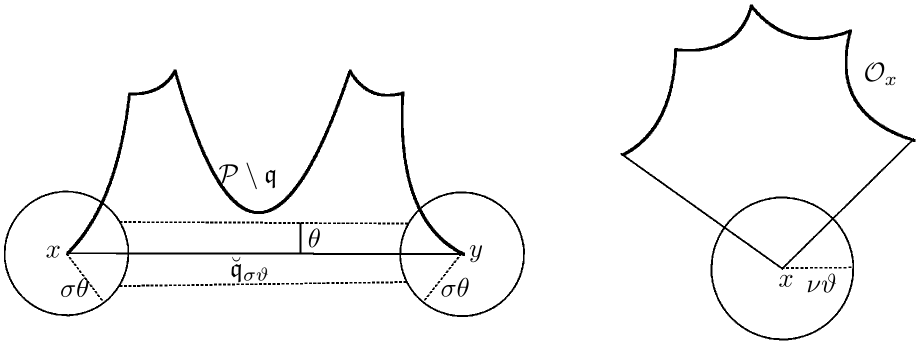

FIGURE 1. Properties (F_1) and (F_2).

Definition 2.7 (fat polygons)**.** Let $\vartheta > 0$, $\sigma \geq 1$ and $\nu \geq 4\sigma$. We call a k-gon P with quasi-geodesic edges (ϑ, σ, ν)*–fat* if the following properties hold:

(F_1) for every edge $\mathfrak{q}$ we have, with the notation 2.5, that

$$\operatorname{dist}(\breve{\mathfrak{q}}_{\sigma\vartheta}\,,\, P \setminus \mathfrak{q}) \geq \vartheta;$$

(F_2) for every vertex x we have

$$\operatorname{dist}(x, \mathcal{O}_x) \geq \nu\vartheta.$$

When $\sigma = 2$ we say that P is (ϑ, ν)*–fat.*

Theorem 2.8 ([DS1], [Dru3])**.** *Let (X, dist) be a geodesic metric space and let $\mathcal{A}$ be a collection of subsets of X. The metric space X is asymptotically tree-graded with respect to $\mathcal{A}$ if and only if the following properties are satisfied:*

(α_1) *For every $\delta > 0$ the diameters of the intersections $\mathcal{N}_\delta(A) \cap \mathcal{N}_\delta(A')$ are uniformly bounded for distinct pairs of $A, A' \in \mathcal{A}$.*

(α_2) *There exists ε in $\left[0, \frac{1}{2}\right)$ and $M > 0$ such that for every geodesic $\mathfrak{g}$ of length ℓ and every $A \in \mathcal{A}$ with $\mathfrak{g}(0), \mathfrak{g}(\ell) \in \mathcal{N}_{\varepsilon\ell}(A)$ we have $\mathfrak{g}([0,\ell]) \cap \mathcal{N}_M(A) \neq \emptyset$.*

(β_3) *There exists $\vartheta > 0$, $\nu \geq 8$ and $\chi > 0$ such that any (ϑ, ν)–fat geodesic hexagon is contained in $\mathcal{N}_\chi(A)$, for some $A \in \mathcal{A}$.*

Remark 2.9. In Theorem 2.8, property (α_2) can be replaced by the following stronger property:

(β_2) There exists $\epsilon > 0$ and $M \geq 0$ such that for any geodesic $\mathfrak{g}$ of length ℓ and any $A \in \mathcal{A}$ satisfying $\mathfrak{g}(0), \mathfrak{g}(\ell) \in \mathcal{N}_{\epsilon\ell}(A)$, the middle third $\mathfrak{g}\left(\left[\frac{\ell}{3}\,,\, \frac{2\ell}{3}\right]\right)$ is contained in $\mathcal{N}_M(A)$.

The notion of asymptotically tree-graded space relates to the standard definition of (strong) relative hyperbolicity by the following.

Theorem 2.10 (Druţu–Osin–Sapir [DS1])**.** *A finitely generated group G is hyperbolic relative to a finite collection of finitely generated subgroups $\mathcal{H}$ if and only if G is asymptotically tree-graded with respect to $\mathcal{LH}$.*

The converse statement of the above theorem can be strengthened as follows.

Theorem 2.11 (Druţu [Dru3])**.** *If G is a finitely generated group which is asymptotically tree-graded with respect to a collection $\mathcal{A}$ of subsets, then G is either hyperbolic or it is relatively hyperbolic with respect to a finite family of finitely generated subgroups $\{H_1, ..., H_m\}$ such that every H_i is contained in $\mathcal{N}_{\varkappa}(A_i)$ for some $A_i \in \mathcal{A}$, where $\varkappa$ is the maximum between the constant M in (β_2) and the constant χ in (β_3).*

A consequence of this is the following result:

Theorem 2.12 (relative hyperbolicity is rigid, Druţu [Dru3])**.** *If a group G' is quasi-isometric to a relatively hyperbolic group G then G' is also relatively hyperbolic.*

Note that formulating a relation between the peripheral subgroups of G and of G' is, in general, nontrivial. This can be seen for instance when $G = G' = A*B*C$, since G is hyperbolic relative to $\{A, B, C\}$, and also hyperbolic relative to $\{A*B, C\}$.

3. Unconstricted and constricted metric spaces

Definition 3.1. A metric space B is *unconstricted* if the following two properties hold:

(1) there exists an ultrafilter ω and a sequence d such that for every sequence of observation points b, $\mathrm{Cone}_\omega(B, b, d)$ does not have cut-points;
(2) for some constant c, every point in B is at distance at most c from a bi-infinite geodesic in B.

When B is an infinite finitely generated group, being unconstricted means simply that *at least one* of its asymptotic cones does not have cut-points. Opposite to it, a *constricted* group is a group with cut-points in every asymptotic cone. See the list following Definition 3.4 for examples of unconstricted groups.

Remark 3.2. Theorem 2.10 implies that relatively hyperbolic groups are constricted. Thus, unconstricted groups are particular cases of NRH groups. They play an essential part in the notion we introduce, of thick group.

Note that the definition above slightly differs from the one in [DS1] in that property (2) has been added. We incorporate this condition into the definition as it is a required hypothesis for all the quasi-isometry rigidity results we obtain. Since, up to bi-Lipschitz homeomorphism, the set of asymptotic cones is a quasi-isometry invariant of a metric space B, it follows that constrictedness and unconstrictedness are quasi-isometry invariants.

The property of being constricted is related to the divergence of geodesics [Ger]. Let X be a geodesic metric space. Given a geodesic segment $\mathfrak{c}\colon [-R, R] \to X$, its *divergence* is a function $div_{\mathfrak{c}}\colon (0, R] \to \mathbb{R}_+$, where for every $r > 0$ we define $div_{\mathfrak{g}}(r)$ as the distance between $\mathfrak{c}(-r)$ and $\mathfrak{c}(r)$ in $X \setminus B(\mathfrak{c}(0), r)$ endowed with the length metric (with the assumption that $\mathfrak{c}(-r)$ and $\mathfrak{c}(r)$ can be joined in $X \setminus B(\mathfrak{c}(0), r)$ by a path of finite length). To a complete minimizing geodesic $\mathfrak{g} : \mathbb{R} \to X$ is associated a function $div_{\mathfrak{g}}$ defined similarly on $\mathbb{R}_+$. By a slight abuse of terminology, it is standard to refer to the growth rate of the function $div_{\mathfrak{g}}$ as *the divergence of* $\mathfrak{g}$.

A geodesic in a metric space X is called *periodic* if its stabilizer in the group of isometries of X is co-bounded. By combining Proposition 4.2 of [KKL] with Lemma 2.3, we obtain:

Lemma 3.3. *Let $\mathfrak{g} : \mathbb{R} \to X$ be a periodic geodesic. If $\mathfrak{g}$ has superlinear divergence, then in any asymptotic cone, $\mathrm{Cone}_\omega(X)$, for which the limit of $\mathfrak{g}$ is nonempty there exists a collection of proper subsets of $\mathrm{Cone}_\omega(X)$ with respect to which it is tree-graded. Furthermore, in this case one has that the limit of $\mathfrak{g}$ is a transversal geodesic.* □

Definition 3.4. A collection of metric spaces, $\mathcal{B}$, is *uniformly unconstricted* if:
(1) for some constant c, every point in every space $B \in \mathcal{B}$ is at distance at most c from a bi-infinite geodesic in B;
(2) for every sequence of spaces (B_i, dist_i) in $\mathcal{B}$, there exists an ultrafilter ω and a sequence of scaling constants d so that for every sequence of basepoints $b = (b_i)$ with $b_i \in B_i$, $\lim_\omega(B_i, b_i, 1/d_i\,\mathrm{dist}_i)$ does not have cut-points.

Recall that a group is *elementary* if it is virtually cyclic.

Examples of uniformly unconstricted collections of spaces:
(1) The collection of all cartesian products of geodesic metric spaces of infinite diameter. This follows from the fact that every ultralimit of a sequence of such spaces appears as cartesian product of two non-trivial geodesic metric spaces. Such a cartesian product cannot have a global cut-point, because Euclidean rectangles do not have cut-points.
(2) The collection of finitely generated non-elementary groups with a central element of infinite order is uniformly unconstricted [DS1, Theorem 6.7].
(3) The collection of finitely generated non-elementary groups satisfying the same identity is uniformly unconstricted [DS1, Theorem 6.12]. Recall that a group G is said to *satisfy an identity (a law)* if there exists a word $w(x_1, ..., x_n)$ in n letters $x_1, ..., x_n$, and their inverses, such that if x_i are replaced by arbitrary elements in G then the word w becomes 1.

    In particular this applies to the collection of all solvable groups of class at most $m \in \mathbb{N}$, and to the collection of Burnside groups with a uniform bound on the order of elements.
(4) The collection of uniform (or cocompact) lattices in semisimple groups of rank at least 2 and at most $m \in \mathbb{N}$ is uniformly unconstricted [KL].
(5) Every finite collection of unconstricted metric spaces is uniformly unconstricted, as is, more generally, every collection of unconstricted metric spaces containing only finitely many isometry classes.

Remark 3.5. Uniform unconstrictedness is a quasi-isometry invariant in the following sense. Consider two collections of metric spaces $\mathcal{B}, \mathcal{B}'$ which are uniformly quasi-isometric, meaning that there are constants $L \geq 1$ and $C \geq 0$ and a bijection between $\mathcal{B}, \mathcal{B}'$ such that spaces that correspond under this bijection are (L, C)-quasi-isometric. It follows that $\mathcal{B}$ is uniformly unconstricted if and only if $\mathcal{B}'$ is uniformly unconstricted.

One of the main interests in (uniformly) unconstricted metric spaces resides in their rigid behavior with respect to quasi-isometric embeddings into ATG metric spaces.

Theorem 3.6 (Druţu–Sapir [DS1])**.** *Let X be ATG with respect to a collection of subsets $\mathcal{A}$. Let $\mathcal{B}$ be a collection of uniformly unconstricted metric spaces. For every (L, C) there exists M depending only on L, C, X, $\mathcal{A}$ and $\mathcal{B}$, such that for every (L, C)–quasi-isometric embedding $\mathfrak{q}$ of a metric space B from $\mathcal{B}$ into X, $\mathfrak{q}(B)$ is contained in an M-neighborhood of a peripheral subset $A \in \mathcal{A}$.*

4. Non-relative hyperbolicity and quasi-isometric rigidity

In the particular case when all the metric spaces in $\mathcal{B}$ are finitely generated groups endowed with word metrics, Theorem 3.6 can be greatly improved: its conclusion holds when $\mathcal{B}$ is the collection of *all* NRH groups.

Theorem 4.1. *Let (X, dist_X) be ATG with respect to a collection $\mathcal{A}$ of subsets. For every $L \geq 1$ and $C \geq 0$ there exists $R = R(L, C, X, \mathcal{A})$ such that the following holds. If (G, dist) is an NRH group endowed with a word metric, and $\mathfrak{q} \colon (G, \mathrm{dist}) \to (X, \mathrm{dist}_X)$ is an (L, C)-quasi-isometric embedding, then $\mathfrak{q}(G)$ is contained in $\mathcal{N}_R(A)$ for some $A \in \mathcal{A}$.*

Remark 4.2. The first result of this kind appeared in Schwartz's proof of the classification of non-uniform lattices in rank one semisimple Lie groups [Sch]. In that case, one of the key technical steps is showing that any quasi-isometry of a neutered space coarsely preserves the collection of boundary horospheres. To do this he proved the "Quasi-flat Lemma" which, reformulated in the language of this paper, states that the quasi-isometric image of an unconstricted metric space into a neutered space must stay in a uniformly bounded neighborhood of a single boundary horosphere.

This theorem was later generalized by Druţu–Sapir [DS1] who kept the unconstricted hypothesis on the domain, but replaced the hypothesis that the image is in a neutered space by only assuming relative hyperbolicity of the target space.

Remark 4.3. Theorem 4.1 also holds in the case that G is replaced by a metric space which is not ATG. In this case though, the constant R will additionally depend on the choice of metric space and the choice of quasi-isometry.

Remark 4.4. By Stallings' Ends Theorem [Sta] a finitely generated group has more than one end if and only if it splits nontrivially as an amalgamated product or HNN-extension with finite amalgamation. A group which splits in this manner is obviously hyperbolic relative to its vertex subgroups. Consequently if a group is NRH then it is one-ended.

Remark 4.5. A result similar to Theorem 4.1 has been obtained in [PW, §3], for G a one-ended group and X the fundamental group of a graph of groups with finite edge groups. Although NRH groups are one-ended, the hypothesis in Theorem 4.1 cannot be weakened to "G a one-ended group," as illustrated by the case when $G = X$ and G is the fundamental group of a finite volume real hyperbolic manifold.

Before proving Theorem 4.1, we state some consequences of it, and give a list of examples of NRH groups.

Corollary 4.6. *Let G be an infinite group which admits an (L, C)-quasi-isometric embedding into a geodesic metric space X which is asymptotically tree-graded with* *respect to a collection of subsets $\mathcal{A}$. Then either $\mathfrak{q}(G)$ is contained in $\mathcal{N}_R(A)$ for some $A \in \mathcal{A}$ and $R = R(L, C, X, \mathcal{A})$ or G is relatively hyperbolic.*

Another consequence is a new proof of the following which was first established in [DS1, Theorem 1.8].

Corollary 4.7 (see also [DS1], Theorem 1.8)**.** *Let G be a finitely generated group hyperbolic relative to $\mathcal{H} = \{H_1, ..., H_m\}$. Let H be an undistorted finitely generated subgroup of G. Then either H is contained in a conjugate of H_i, $i \in \{1, 2, ..., m\}$, or H is relatively hyperbolic.*

Perhaps the most important consequence of Theorem 4.1 is the following quasi-isometric rigidity theorem for groups hyperbolic relative to NRH subgroups.

Theorem 4.8. *Let G be a finitely generated group which is hyperbolic relative to a finite family of finitely generated subgroups $\mathcal{H}$ such that each $H \in \mathcal{H}$ is not relatively hyperbolic. If a group G' is quasi-isometric to G then G' is hyperbolic relative to $\mathcal{H}' = \{H_1, ..., H_m\}$, where each H_i is quasi-isometric to some $H \in \mathcal{H}$.*

Proof. The proof is almost identical to the proof of Theorem 5.13 in [DS1]. Indeed let $X = G$ and let $\mathcal{A} = \{gH \ : \ g \in G/H \text{ and } H \in \mathcal{H}\}$. The pair $(X, \mathcal{A})$ satisfies all the hypotheses of Theorem 5.13 in [DS1], except (1). Still, hypothesis (1) is used in that proof only to ensure that for every quasi-isometry constants $L \geq 1$ and $C \geq 0$ there exists a constant $M = M(L, C, X, \mathcal{A})$ such that for every $A \in \mathcal{A}$ and for every (L, C)–quasi-isometric embedding $\mathfrak{q} \colon A \to X$ there exists $B \in \mathcal{A}$ for which $\mathfrak{q}(A) \subset \mathcal{N}_M(B)$. In our case, each $H \in \mathcal{H}$ is known to be undistorted since it is a peripheral subgroup (see, for instance, [DS1] for details). Thus, the hypothesis that all $H \in \mathcal{H}$ are NRH implies via Theorem 4.1 that for every $L \geq 1$ and $C \geq 0$ there exists a constant M as above depending only on L, C, and the undistorsion constants of each H in G. □

In view of Theorems 4.1 and 4.8, it becomes interesting to consider examples of NRH groups. We do this below. In Section 5 we give a procedure allowing one to build NRH groups from smaller NRH groups (see Proposition 5.4).

Examples of NRH groups:

(I) Non-elementary groups without free non-abelian subgroups. This follows from the fact that non-elementary relatively hyperbolic groups contain a free non-abelian subgroup.

The class of groups without free non-abelian subgroups contains the non-elementary amenable groups, but it is strictly larger than that class; indeed, a well known question attributed to J. von Neumann [Neu] is whether these two classes coincide (this is known as *the von Neumann problem*). The first examples of non-amenable groups without free non-abelian subgroups were given in [Ol']. Other examples were later given in [Ady] and in [OS].

(II) Non-elementary groups with infinite center. Indeed, if G is hyperbolic then its center is finite. Assume that G is relatively hyperbolic with respect to $H_1, ..., H_m$ and at least one H_i is infinite (otherwise G would be hyperbolic). Since $G \neq H_i$ there exists a left coset $gH_i \neq H_i$. For every $z \in Z(G)$, H_i and $zH_i = H_i z$ are at Hausdorff distance at most $\mathrm{dist}(1, z)$. This and Theorem 2.8, (α_1), imply that $zH_i = H_i$, thus $Z(G) \subset H_i$. Similarly it can be proved that $Z(G) \subset gH_ig^{-1}$. If follows that $Z(G) \subset gH_ig^{-1} \cap H_i$, hence that it is finite (see for instance [DS2, Lemma 4.20]).

(III) Unconstricted groups.

(IV) Inductive limits of small cancellation groups (see Section 7.1).

The remainder of this section provides the proof of Theorem 4.1, thus we let $(X, \mathrm{dist}_X), \mathcal{A}, L$, $C, \mathfrak{q}$ and G be as in the statement of the theorem. We will proceed by using the quasi-isometric embedding $\mathfrak{q}$ to construct an asymptotically tree-graded structure on (G, dist).

In order to produce an asymptotically tree-graded structure on G we first search for a constant τ such that the following set is non-empty:

$$\mathcal{A}_\tau = \{A \in \mathcal{A} \; ; \; \mathcal{N}_\tau(A) \cap \mathfrak{q}(G) \neq \emptyset\} \,. \tag{1}$$

Then, for every $A \in \mathcal{A}_\tau$ we consider the pre-image $B_A = \mathfrak{q}^{-1}\left(\mathcal{N}_\tau(A)\right)$ and the set

$$\mathcal{B}_\tau = \{B_A \; ; \; A \in \mathcal{A}_\tau\} \,. \tag{2}$$

For an appropriate choice of τ, we will show that the collection $\mathcal{B}_\tau$ defines an asymptotically tree-graded structure on (G, dist). We begin with the following lemmas which will allow us to choose the constant τ.

Lemma 4.9 ([DS1], Theorem 4.1 and Remark 4.2, (2))**.**

(a) *There exists $M' > 0$ such that for every (L, C)–quasi-geodesic $\mathfrak{p} \colon [0, \ell] \to X$ and every $A \in \mathcal{A}$ satisfying $\mathfrak{p}(0)\,,\ \mathfrak{p}(\ell) \in \mathcal{N}_{\ell/3L}(A)$, the tubular neighborhood $\mathcal{N}_{M'}(A)$ intersects $\mathfrak{p}\left([0, \ell]\right)$.*

(b) *For every $\sigma \geq 1$ and $\nu \geq 4\sigma$ there exists ϑ_0 satisfying the following: for every $\vartheta \geq \vartheta_0$ there exists χ with the property that every hexagon with (L, C)-quasi-geodesic edges which is (ϑ, σ, ν)–fat is contained in $\mathcal{N}_\chi(A)$ for some $A \in \mathcal{A}$.*

Lemma 4.10. *Let $\mathfrak{p} \colon Y \to X$ be an (L, C)–quasi-isometric embedding. Let $\sigma = 4L^2 + L \geq 1$, $\nu = 4\sigma$ and $\vartheta \geq C$. If P is a $(2L\vartheta, \nu + 1)$–fat geodesic hexagon, then $\mathfrak{p}(P)$ is a hexagon with (L, C)–quasi-geodesic edges which is (ϑ, σ, ν)–fat.*

Proof. $(\mathbf{F_1})$ Let $\mathfrak{g}$ be an edge of P, of endpoints x, y. Let $x \in \mathfrak{p}(\mathfrak{g}) \backslash \mathcal{N}_{\sigma\vartheta}\left(\{\mathfrak{p}(x), \mathfrak{p}(y)\}\right)$. Then $x = \mathfrak{p}(t)$ with $t \in \mathfrak{g}$ at distance at most $\frac{1}{L}\sigma\vartheta - C$ from x and y. Since $\frac{1}{L}\sigma\vartheta - C = (4L + 1)\vartheta - C \geq 4L\vartheta$, property (F_1) for P implies that t is at distance at least $2L\vartheta$ from any edge $\mathfrak{p} \neq \mathfrak{g}$ of P. Then $\mathfrak{p}(x)$ is at distance at least $\frac{1}{L}2L\vartheta - C = 2\vartheta - C \geq \vartheta$ from $\mathfrak{q}(\mathfrak{p})$.

$(\mathbf{F_2})$ Let v be an arbitrary vertex of P. Property (F_2) for P grants that $\mathrm{dist}\left(v, \mathcal{O}_v(\,P)\right) \geq (\nu + 1)(2L\vartheta)$, hence $\mathrm{dist}\left(\mathfrak{p}(v), \mathcal{O}_{\mathfrak{p}(v)}\left(\,\mathfrak{p}(P)\right)\right) \geq \frac{1}{L}(\nu + 1)(2L\vartheta) - C = 2(\nu + 1)\vartheta - C \geq \nu\vartheta$. □

For the remainder of the proof, we fix the following constants:

- σ and ν as in Lemma 4.10;
- if ϑ_0 is the constant provided by Lemma 4.9 for σ and ν above, it is no loss of generality to assume further that $\vartheta_0 \geq C$;
- let $\vartheta = 2L\vartheta_0$;
- let χ the constant given by Lemma 4.9 for ϑ_0;
- $\tau = \max\left(\chi, M'\right)$, where M' is the constant from Lemma 4.9.

If G does not contain a $(\vartheta, \nu+1)$–fat geodesic hexagon or if all such hexagons have uniformly bounded diameter, then G is hyperbolic by Corollary 4.20 in [Dru3]. This contradicts our hypothesis on G. We may thus henceforth assume that for every $\eta > 0$, the space G contains a $(\vartheta, \nu+1)$–fat geodesic hexagon of diameter at least η. For every such hexagon P, Lemma 4.10 and the above choice of constants imply that $\mathfrak{q}(P) \subset \mathcal{N}_\chi(A) \subset \mathcal{N}_\tau(A)$ for some $A \in \mathcal{A}$. In particular the set $\mathcal{A}_\tau$ is non-empty.

Lemma 4.11. *The metric space* (G, dist) *is asymptotically tree-graded with respect to the set* $\mathcal{B}_\tau$ *defined in (2).*

Proof. We start with the simple remark that if $x \in \mathcal{N}_t(B_A)$ then $\mathfrak{q}(x) \in \mathcal{N}_{Lt+C+\tau}(A)$.

According to Theorem 2.8 it suffices to verify conditions $(\boldsymbol{\alpha_1})$, $(\boldsymbol{\alpha_2})$, $(\boldsymbol{\beta_3})$.

We first establish $(\boldsymbol{\alpha_1})$. Let $A, A' \in \mathcal{A}_\tau$, $A \neq A'$, and let $x, y \in \mathcal{N}_\delta(B_A) \cap \mathcal{N}_\delta(B_{A'})$. Then $\mathfrak{q}(x)$ and $\mathfrak{q}(y)$ are in $\mathcal{N}_{L\delta+C+\tau}(A) \cap \mathcal{N}_{L\delta+C+\tau}(A')$. Since X is asymptotically tree-graded, $\mathrm{diam}(\mathcal{N}_{L\delta+C+\tau}(A) \cap \mathcal{N}_{L\delta+C+\tau}(A')) = D$ is uniformly bounded. Thus

$$\mathrm{dist}(x,y) \leq L\,[\mathrm{dist}_X(\mathfrak{q}(x), \mathfrak{q}(y)) + C] \leq L\,(D + C)\ .$$

We prove $(\boldsymbol{\alpha_2})$ for $\varepsilon = \frac{1}{6L^2}$ and $M = \left(3L + \frac{1}{L}\right)(C+\tau)$. Let $\mathfrak{g}\colon [0,\ell] \to G$ be a geodesic with endpoints in $\mathcal{N}_{\varepsilon\ell}(B_A)$ for some $A \in \mathcal{A}_\tau$. Then $\mathfrak{q}\circ\mathfrak{g}$ is an (L,C)-quasi-geodesic with endpoints in $\mathcal{N}_{L\varepsilon\ell+C+\tau}(A)$. If $C+\tau \geq \frac{\ell}{6L}$, that is $\ell \leq 6L(C+\tau)$ then $\mathfrak{g} \subset \overline{\mathcal{N}}_{3L(C+\tau)}(\{\mathfrak{g}(0), \mathfrak{g}(\ell)\}) \subset \mathcal{N}_{(3L+1/L)(C+\tau)}(B_A)$. If $C+\tau < \frac{\ell}{6L}$ then Lemma 4.9 implies that $\mathfrak{q}\circ\mathfrak{g}([0,\ell])$ intersects $\mathcal{N}_{M'}(A)$. It follows that $\mathfrak{g}([0,\ell])$ intersects B_A.

We prove $(\boldsymbol{\beta_3})$ for $(\vartheta, \nu+1)$ as above and for $\chi = 0$. Let P be a $(\vartheta, \nu+1)$–fat geodesic hexagon in G. Then by Lemma 4.10, $\mathfrak{q}(P)$ is a $(\vartheta_0, \sigma, \nu)$–fat hexagon with (L,C)–quasi-geodesic edges. Lemma 4.9 implies that $\mathfrak{q}(P)$ is contained in $\mathcal{N}_\chi(A)$. It follows that $A \in \mathcal{A}_\tau$ and that $P \subset B_A$. □

Let M be the maximum between the constant from (β_2) and the constant χ from (β_3), for $(G, \mathcal{B}_\tau)$. Note that the constants in (β_2) and (β_3) for $(G, \mathcal{B}_\tau)$ can be obtained from the constants in the same properties for $(X, \mathcal{A})$, as well as from τ, L and C. Consequently $M = M(X, \mathcal{A}, L, C)$.

Lemma 4.11, Theorem 2.11 and the hypothesis that G is NRH imply that $G \subset \mathcal{N}_M(B_A)$ for some $A \in \mathcal{A}_\tau$. Hence $\mathfrak{q}(G) \subset \mathcal{N}_{LM+C+\tau}(A)$, completing the proof of Theorem 4.1.

5. Networks of subspaces

We begin by defining the notions of networks of subspaces and of subgroups.

Definition 5.1. (network of subspaces).

Let X be a metric space and $\mathcal{L}$ a collection of subsets of X. Given $\tau \geq 0$ we say that X is a *τ–network with respect to the collection* $\mathcal{L}$ if the following conditions are satisfied:

$(\mathbf{N}_1)$ $X = \bigcup_{L \in \mathcal{L}} \mathcal{N}_\tau(L)$;

($\mathbf{N}_2$) Any two elements L, L' in $\mathcal{L}$ can be *thickly connected in* $\mathcal{L}$: there exists a sequence, $L_1 = L, L_2, \ldots, L_{n-1}, L_n = L'$, with $L_i \in \mathcal{L}$ and with $\operatorname{diam}(\mathcal{N}_\tau(L_i) \cap \mathcal{N}_\tau(L_{i+1})) = \infty$ for all $1 \leq i < n$.

We now define a version of the above notion in the context of finitely generated groups with word metrics. Recall that a finitely generated subgroup H of a finitely generated group G is *undistorted* if any word metric of H is bi-Lipschitz equivalent to a word metric of G restricted to H.

Definition 5.2 (**algebraic network of subgroups**). Let G be a finitely generated group, let $\mathcal{H}$ be a finite collection of subgroups of G and let $M > 0$. The group G is an *M–algebraic network with respect to $\mathcal{H}$* if:

($\mathbf{AN}_0$) All subgroups in $\mathcal{H}$ are finitely generated and undistorted in G.

($\mathbf{AN}_1$) There is a finite index subgroup G_1 of G such that $G \subset \mathcal{N}_M(G_1)$ and such that a finite generating set of G_1 is contained in $\bigcup_{H \in \mathcal{H}} H$.

($\mathbf{AN}_2$) Any two subgroups H, H' in $\mathcal{H}$ can be *thickly connected in* $\mathcal{H}$: there exists a finite sequence $H = H_1, \ldots, H_n = H'$ of subgroups in $\mathcal{H}$ such that for all $1 \leq i < n$, $H_i \cap H_{i+1}$ is infinite.

Proposition 5.3. *If a finitely generated group G is an M–algebraic network with respect to $\mathcal{H}$ then it is an M–network with respect to the collection of left cosets*

$$\mathcal{L} = \{\, gH : g \in G_1 \,,\, H \in \mathcal{H} \,\} \,.$$

Proof: Property ($\mathbf{N}_1$) is trivial. We prove property ($\mathbf{N}_2$). Since it is equivariant with respect to the action of G it suffices to prove it for $L = H$ and $L' = gH'$, $H, H' \in \mathcal{H}$ and $g \in G_1$. Fix a finite generating set S of the finite index subgroup G_1 of G so that $S \subset \bigcup_{H \in \mathcal{H}} H$; all lengths in G_1 will be measured with respect to this generating set. We argue by induction on $|g| = |g|_S$. If $|g| = 1$, then $g \in S$. By hypothesis, g is contained in a subgroup $\widetilde{H}$ in $\mathcal{H}$. We take a sequence $H = H_1, H_2, ..., H_k = \widetilde{H}$ as in ($\mathbf{AN}_2$), and a similar sequence $\widetilde{H} = \overline{H}_1, \overline{H}_2, ..., \overline{H}_m = H'$. Then the sequence

$$H = H_1, H_2, ..., H_k = \widetilde{H} = g\widetilde{H} = g\overline{H}_1, g\overline{H}_2, ..., g\overline{H}_m = gH'$$

satisfies the properties in ($\mathbf{N}_2$). We now assume the inductive hypothesis that for all $g \in G_1$ with $|g| \leq n$ and all $H, H' \in \mathcal{H}$ the cosets H and gH' can be connected by a sequence satisfying ($\mathbf{N}_2$) with $\tau = M$. Take $g \in G_1$ such that $|g| = n+1$; thus $g = \hat{g}s$, where $s \in S$ and $\hat{g} \in G_1$, $|\hat{g}| = n$. By hypothesis there exists some $\widetilde{H} \in \mathcal{H}$ containing s. Take arbitrary $H, H' \in \mathcal{H}$. In order to show that H and gH' can be connected by a good sequence it suffices to show, by the inductive hypothesis, that $\hat{g}\widetilde{H} = g\widetilde{H}$ and gH' can be connected by a good sequence. This holds because $\widetilde{H}$ and H' can be so connected, according to ($\mathbf{AN}_2$). $\square$

One of the reasons for which one can be interested in the notion of network of groups is that it represents a way of building up NRH groups. More precisely the following holds:

Proposition 5.4. *Let G be a finitely generated group which is an M–algebraic network with respect to $\mathcal{H}$, such that each of the subgroups in $\mathcal{H}$ is not relatively hyperbolic.*

If G is an undistorted subgroup of a group Γ hyperbolic relative to $\widetilde{H}_1, .., \widetilde{H}_m$, then G is contained in a conjugate of some subgroup $\widetilde{H}_i, i \in \{1, ..., m\}$.

In particular G is not relatively hyperbolic.

Proof: According to Corollary 4.7, any subgroup $H \in \mathcal{H}$ is contained in the conjugate of some $\widetilde{H}_i, i \in \{1, ..., m\}$. Since distinct conjugates of subgroups $\widetilde{H}_i$ have finite intersections, it follows from $(\mathbf{AN}_2)$ that all subgroups in $\mathcal{H}$ are in the same conjugate $\gamma\widetilde{H}_i\gamma^{-1}$. Hence, condition $(\mathbf{AN}_1)$ implies that G has a finite index subgroup G_1 which is completely contained in the same conjugate $\gamma\widetilde{H}_i\gamma^{-1}$. Given M the constant in $(\mathbf{AN}_1)$, for any $g \in G$, $gG_1g^{-1} \subset \mathcal{N}_M(G_1) \subset \mathcal{N}_M(\gamma\widetilde{H}_i\gamma^{-1})$. It follows that $g(\gamma\widetilde{H}_i\ \gamma^{-1})g^{-1} \cap \mathcal{N}_M(\gamma\widetilde{H}_i\gamma^{-1})$ has infinite diameter. From this it can be deduced, by [MSW2, Lemma 2.2], that $g(\gamma\widetilde{H}_i\gamma^{-1})g^{-1} \cap \gamma\widetilde{H}_i\gamma^{-1}$ is also infinite. This implies that the two conjugates coincide and thus $g \in \gamma\widetilde{H}_i\gamma^{-1}$. We have thereby shown that $G < \gamma\widetilde{H}_i\gamma^{-1}$. □

In Proposition 5.4 the hypotheses of undistortedness (of G in Γ and of every subgroup $H \in \mathcal{H}$ in G) can be removed, if the hypothesis "all subgroups in $\mathcal{H}$ are NRH" is strengthened to "all subgroups in $\mathcal{H}$ are non-elementary and without free non-Abelian subgroups". The latter condition implies the former but they are not equivalent: for instance uniform lattices in semisimple groups of rank at least two are unconstricted hence NRH and they have many non-Abelian free subgroups.

Thus, the following statement, generalizing the main result of [AAS], holds:

Proposition 5.5. *Let G be a finitely generated group with a finite collection $\mathcal{H}$ of finitely generated subgroups satisfying* $(\mathbf{AN}_1)$ *and* $(\mathbf{AN}_2)$*. Assume moreover that all $H \in \mathcal{H}$ are non-elementary and do not contain free non-Abelian subgroups.*

If G is a subgroup of a group Γ hyperbolic relative to $\widetilde{H}_1, .., \widetilde{H}_m$, then G is contained in a conjugate of some subgroup $\widetilde{H}_i, i \in \{1, ..., m\}$.

In particular G is not relatively hyperbolic.

Proof: We use the Tits alternative in relatively hyperbolic groups: a subgroup in Γ is either virtually cyclic, parabolic (i.e. contained in a conjugate of some subgroup $\widetilde{H}_i$), or it contains a free non-Abelian subgroup; the proof follows from [Tuk] and [Bow2]. Hence, with our hypotheses, any subgroup $H \in \mathcal{H}$ is parabolic. The rest of the proof is identical to the proof of Proposition 5.4. □

6. Relative hyperbolicity and Dunwoody's inaccessible group

Having a quasi-isometric rigidity theorem for relatively hyperbolic groups whose peripheral subgroups are not relatively hyperbolic, one might think to ask:

Question 6.1. *Given a finitely generated relatively hyperbolic group G, is G hyperbolic relative to some finite collection of subgroups* none of which are relatively hyperbolic*?*

Remark 6.2. Note that if G is hyperbolic relative to $\{H_i\ ;\ i=1,2,..,m\}$ and if each H_i is hyperbolic relative to $\{H_i^j\ ;\ j=1,2,..,n_i\}$ then G is hyperbolic relative to $\{H_i^j\ ;\ j=1,2,..,n_i\,,\ i=1,2,..,m\}$. Examples where such process never terminates are easily found (for instance when G is a free non-Abelian group with H_i finitely generated non-Abelian subgroups and H_i^j finitely generated non-Abelian subgroups of H_i.). Still, one might ask if in every relatively hyperbolic group there exists a terminal point for the process above (like $H=\{1\}$ in the case of a free group). This is the meaning of Question 6.1.

We answer this question in the negative, using Dunwoody's example J of an inaccessible group [Dun]:

Proposition 6.3. *Dunwoody's group J is relatively hyperbolic. If J is hyperbolic relative to a finite collection of subgroups $A_1,\dots,A_I$, at least one of the subgroups $A_1,\dots,A_I$ is relatively hyperbolic.*

This proposition shows that J satisfies a kind of "relatively hyperbolic inaccessibility": whenever J is written as a relatively hyperbolic group, one of the peripheral subgroups A is also relatively hyperbolic and so A can be replaced by its list of peripheral subgroups, giving a new relatively hyperbolic description of J; this operation can be repeated forever, giving an infinite sequence of finer and finer relatively hyperbolic descriptions of J.

First we review Dunwoody's construction of J. Let H be the group of permutations of $\mathbb{Z}$ generated by the transposition $t=(0,1)$ and the shift map $s(i)=i+1$. Each element $\sigma\in H$ agrees outside a finite set with a unique power s^p, and the map $\sigma\mapsto p$ defines a homomorphism $\pi\colon H\mapsto\mathbb{Z}$ whose kernel denoted H_ω is the group of finitely supported permutations of $\mathbb{Z}$. Let $H_i\subset H_\omega$ be the group of permutations supported on $[-i,i]=\{-i,-i+1,\dots,0,1,\dots,i-1,i\}$, so $H_\omega=\cup_{i=0}^\infty H_i$. Let V be the group of all functions from $\mathbb{Z}$ to $\mathbb{Z}_2=\{\pm1\}$ with finite support and the usual group law. Let V_i be the subgroup of all such maps with support $[-i,i]$. Let $z_i\in V_i$ be the map defined by $z_i(n)=-1$ if and only if $n\in[-i,i]$. The group H_i acts on the left of V_i by ${}^h v(j)=v(h^{-1}(j))$, and so we can form the semidirect product $G_i'=V_i\rtimes H_i$, each of whose elements can be written uniquely as vh with $v\in V_i$ and $h\in H_i$, and the group law is $(v_0h_0)\cdot(v_1h_1)=(v_0\,{}^{h_0}v_1)(h_0h_1)$. The element z_i is central in G_i' and so we have a direct product subgroup $K_i=\langle z_i\rangle\times H_i\approx\mathbb{Z}/2\times H_i<G_i'$. For $i=1,2,\dots$ choose G_i to be an isomorphic copy of G_i', with the G_i pairwise disjoint. The group K_i being a subgroup of G_i' and of G_{i+1}', we may identify K_i with its images in G_i and G_{i+1}, which defines the following graph of groups whose fundamental group is denoted P:

$$G_1 \xrightarrow{K_1} G_2 \xrightarrow{K_2} G_3 \xrightarrow{K_3} G_4 \xrightarrow{K_4} \cdots \tag{3}$$

We shall need below the following equation which can be regarded as taking place within G_{i+1}:

$$K_i\cap K_{i+1}=H_i \tag{4}$$

Collapsing all edges in (3) to the right of the one labelled K_n produces another decomposition of P as the fundamental group of the graph of groups

$$G_1 \xrightarrow{K_1} G_2 \xrightarrow{K_2} G_3 \cdots\cdots\cdots G_{n-1} \xrightarrow{K_{n-1}} G_n \xrightarrow{K_n} Q_n \tag{5}$$

and then collapsing all edges except the one labelled K_n we get a decomposition $P = P_n *_{K_n} Q_n$. Noting that P contains $H_1 < H_2 < H_3 < \cdots < H_\omega$, we can form the amalgamated product

$$J = P *_{H_\omega} H \tag{6}$$

Since $H_\omega \subset Q_n$, the group J also has the decomposition

$$J = P_n *_{K_n} \underbrace{(Q_n *_{H_\omega} H)}_{J_n} \tag{7}$$

Applying (5) and the definition of P_n we obtain a decomposition of J as the fundamental group of the graph of groups

$$G_1 \xrightarrow{K_1} G_2 \xrightarrow{K_2} G_3 \cdots\cdots\cdots\cdots G_{n-1} \xrightarrow{K_{n-1}} G_n \xrightarrow{K_n} J_n \tag{8}$$

From both (7) and (8) we see that J is relatively hyperbolic: in either of these graph of groups presentations, each edge group is finite and includes properly into both adjacent vertex groups, and so J is hyperbolic relative to the vertex groups. This proves the first clause of Proposition 6.3.

To prepare for the rest of the proof we need some additional facts about the group H.

- H is the intersection of the nested family of subgroups $J_1 > J_2 > J_3 > \cdots$

To prove this, since $J_n = Q_n *_{H_\omega} H$, it suffices to show that the intersection of the nested family $Q_1 > Q_2 > Q_3 > \cdots$ equals H_ω. Consider an element $x \in P = Q_0$ that is contained in this intersection. Since Q_0 is generated by its subgroups $G_1, G_2, \ldots$, the element x can be written as a word w_1 whose letters are elements of $G_1, G_2, \ldots, G_n$ for some n. Since $x \in Q_1$, x can also be written as a product of elements of $G_2, G_3, \ldots$. By uniqueness of normal forms in a graph of groups, any letter of w_1 that is in the subgroup G_1 is also in the subgroup K_1; each such letter can be pulled across the K_1 edge into the subgroup G_2, and so x can be written as a word w_2 whose letters are elements of $G_2, \ldots, G_n$. Continuing inductively in this fashion, we see that $x \in G_n$. Going one more step, since $x \in Q_n$, it can be written as a product of elements of $G_{n+1}, G_{n+2}, G_{n+3}, \ldots$, and so by uniqueness of normal forms we have $x \in K_n < G_{n+1}$. And going one more step again, x can be written as a product of elements of $G_{n+2}, G_{n+3}, \ldots$, and so $x \in K_{n+1}$. Applying (4) we have $x \in K_n \cap K_{n+1} = H_n < H_\omega$.

Next we need:

- H satisfies the hypotheses of Proposition 5.5.

To see this, let H_{even} be the abelian subgroup of H generated by the transpositions $(2n, 2n+1)$, $n \in \mathbb{Z}$, and let H_{odd} be the abelian subgroup generated by the transpositions $(2n+1, 2n+2)$, $n \in \mathbb{Z}$. The squared shift map s^2 preserves each of these subgroups, and so we have subgroups $H_{\text{even}} \rtimes \langle s^2 \rangle$ and $H_{\text{odd}} \rtimes \langle s^2 \rangle$ of H, each solvable and therefore non-elementary, without free non-Abelian subgroups. These two subgroups generate the index 2 subgroup $\pi^{-1}(2\mathbb{Z})$, thus $(\mathbf{AN}_1)$ is satisfied. Also $\langle s^2 \rangle$ is contained in both subgroups, whence $(\mathbf{AN}_2)$.

Now we prove the second clause of Proposition 6.3. Arguing by contradiction, suppose that J is relatively hyperbolic with respect to peripheral subgroups $L_1, \ldots, L_m$ none of which is relatively hyperbolic. By Proposition 5.5, the group H must be contained in some conjugate of some L_i, so we have $H < L_i' = g L_i g^{-1}$

for some $g \in J$. Since H is infinite, L'_i is infinite. By combining Corollary 4.7 with the relatively hyperbolic description (8), the NRH subgroup L'_i must be contained in a conjugate of one of $G_1, \ldots, G_n, J_n$, but only J_n is infinite and so $L'_i < hJ_nh^{-1}$ for some $h \in J$. We therefore have $H < J_n \cap hJ_nh^{-1}$, and so by malnormality $J_n = hJ_nh^{-1}$. Thus $L'_i < J_n$ for all n, and so $L'_i < H$. We have therefore proved that $L'_i = H$, and so J is hyperbolic relative to a collection of subgroups that includes H.

Now note that $H \cap z_nHz_n^{-1}$ contains H_n whose cardinality goes to $+\infty$ as $n \to +\infty$. Since H is a peripheral subgroup of J, the intersection of H with its distinct conjugates has uniformly bounded cardinal. Thus for n large enough we have $H = z_nHz_n^{-1}$. In particular H and z_nH are at finite Hausdorff distance $|z_n|$, which together with the fact that H is infinite and with property (α_1) imply that $H = z_nH$, hence that $z_n \in H$, a contradiction.

7. Thick spaces and groups

A particular case of NRH groups are those obtained by using the construction in Proposition 5.4 inductively, with unconstricted groups as a starting point. This particular case of groups are the thick groups. We begin by introducing the notion of thickness in the general metric setting.

Definition 7.1. (metric thickness and uniform thickness).

(M$_1$) A metric space is called *thick of order zero* if it is unconstricted. A family of metric spaces is *uniformly thick of order zero* if it is uniformly unconstricted.

(M$_2$) Let X be a metric space and $\mathcal{L}$ a collection of subsets of X. Given $\tau \geq 0$ and $n \in \mathbb{N}$ we say that X is *τ–thick of order at most $n+1$ with respect to the collection $\mathcal{L}$* if X is a τ-network with respect to $\mathcal{L}$, and moreover:

(θ) when the subsets in $\mathcal{L}$ are endowed with the restricted metric on X, then the collection $\mathcal{L}$ is uniformly thick of order at most n.

We say X is *thick of order at most n* if it is *τ–thick of order at most n* with respect to some collection $\mathcal{L}$ for some τ. Further, X is said to be *τ–thick of order n (with respect to the collection $\mathcal{L}$)* if it is *τ–thick of order at most n (with respect to the collection $\mathcal{L}$)* and for no choices of τ and $\mathcal{L}$ is it thick of order at most $n-1$. When the choices of $\mathcal{L}$, τ, and n are irrelevant, we simply say that X is *thick*.

(M$_3$) A family $\{X_i \mid i \in I\}$ of metric spaces is *uniformly thick of order at most $n+1$* if the following hold.

($\upsilon\theta_1$) There exists $\tau > 0$ such that every X_i is τ–thick of order at most $n+1$ with respect to a collection $\mathcal{L}_i$ of subsets of it;

($\upsilon\theta_2$) $\bigcup_{i\in I} \mathcal{L}_i$ is uniformly thick of order at most n, where each $L \in \mathcal{L}_i$ is endowed with the induced metric.

Remark 7.2. Thickness is a quasi-isometry invariant in the following sense. Let X, X' be metric spaces, let $\mathfrak{q} \colon X \to X'$ be a (L, C)–quasi-isometry, let $\mathcal{L}$ be a collection of subsets of X, let $\mathcal{L}'$ be a collection of subsets of X', and suppose that there is a bijection $\mathfrak{q}_\# \colon \mathcal{L} \to \mathcal{L}'$ such that the subsets $\mathfrak{q}(L)$ and $\mathfrak{q}_\#(L)$ have Hausdorff distance $\leq C$ in X', for each $L \in \mathcal{L}$. For example, one could simply take $\mathcal{L}' = \{\mathfrak{q}(L) \bigm| L \in \mathcal{L}\}$. If we metrize each space in $\mathcal{L}$ or in $\mathcal{L}'$ by restricting the ambient metric, it follows that $\mathcal{L}$ and $\mathcal{L}'$ are uniformly quasi-isometric, and so

$\mathcal{L}$ is uniformly unconstricted if and only if $\mathcal{L}'$ is uniformly unconstricted. This is the basis of an easy inductive argument which shows that X is τ–thick of order n with respect to $\mathcal{L}$ if and only if X' is τ'–thick of order n with respect to $\mathcal{L}'$, where $\tau' = \tau'(L, C, \tau)$.

We now define a stronger version of thickness in the context of finitely generated groups with word metrics.

Definition 7.3 (**algebraic thickness**). Consider a finitely generated group G.

($\mathbf{A}_1$) G is called *algebraically thick of order zero* if it is unconstricted.

($\mathbf{A}_2$) G is called *M–algebraically thick of order at most $n+1$ with respect to $\mathcal{H}$*, where $\mathcal{H}$ is a finite collection of subgroups of G and $M > 0$, if:

- G is an M-algebraic network with respect to $\mathcal{H}$;
- all subgroups in $\mathcal{H}$ are algebraically thick of order at most n.

G is said to be *algebraically thick of order $n+1$ with respect to $\mathcal{H}$*, when n is the smallest value for which this statement holds.

Remark 7.4. The algebraic thickness property does not depend on the word metric on G, moreover it holds for any metric quasi-isometric to a word metric. Hence in what follows, when mentioning this property for a group we shall mean that the group is considered endowed with some metric quasi-isometric to a word metric. See Section 8 for an example where we use a proper finite index subgroup G_1 to verify thickness.

Examples: Examples of groups that are algebraically thick of order one are provided by mapping class groups (see Section 8 and [Beh]), right angled Artin groups whose presentation graph is a tree of diameter greater than 2 (Corollary 10.8 and Proposition 10.9), and fundamental groups of graph manifolds (see Section 11 and [KKL]). An example of a metric space thick of order one is the Teichmüller space with the Weil-Petersson metric (see Section 12 and [Beh]). An example of a group thick of order two is described in [BDM], see also Remark 11.3.

Question 7.5. *Since the order of metric thickness is a quasi-isometry invariant (see Remark 7.2), we ask whether the order of algebraic thickness is also a quasi-isometry invariant.*

Proposition 7.6.

(a) *If a finitely generated group G is M–algebraically thick of order at most n then it is M–metrically thick of order at most n. Moreover if $n \geq 1$ and G is M–algebraically thick of order at most n with respect to $\mathcal{H}$ then it is M–metrically thick of order at most n with respect to the collection of left cosets*
$$\mathcal{L} = \{ gH : g \in G_1 , H \in \mathcal{H} \} .$$

(b) *Let $G_1, G_2, ..., G_n$ be finitely generated groups algebraically thick of order at most n. Then any family $\{X_i \mid i \in I\}$ of metric spaces such that each X_i is isometric to G_k for some $k \in \{1, 2, ..., n\}$ is uniformly metrically thick of order at most n.*

Proof: We prove the proposition inductively on n. The statements (a) and (b) are true for $n = 0$. Suppose that they are true for all $k \leq n$. We prove them for $n + 1$.

(a) Since all groups in $\mathcal{H}$ are undistorted and algebraically thick of order at most n with respect to their own word metrics, by Remark 7.4 it follows that they are algebraically thick of order at most n also when endowed with the restriction of the metric on G. This and (b) for n imply that $\mathcal{L}$ is uniformly metrically thick of order at most n, verifying condition (θ). This and Proposition 5.3 allow to finish the argument.

(b) Each group G_i is M_i–algebraically thick of order at most n with respect to some collection $\mathcal{H}_i$ of subgroups, where $M_i > 0$. Each $H \in \mathcal{H}_i$ is thick of order at most $n - 1$. Property $(\upsilon\theta_1)$ holds for $\{X_i \mid i \in I\}$, with the constant $\tau = \max\{M_i \mid i \in \{1, 2, ..., n\}\}$. Each metric space X_i, $i \in I$, is isometric to some G_k, hence by (a) it is metrically thick with respect to the family of isometric images of $\{gH \mid g \in G_k^1,\, H \in \mathcal{H}_k\}$, where G_k^1 is a finite index subgroup in G_k. Property (b) applied to the finite family of groups $\bigcup_{k=1}^{n} \mathcal{H}_k$ yields property $(\upsilon\theta_2)$ for the family of metric spaces $\{X_i \mid i \in I\}$. □

A consequence of Proposition 7.6 is that the order of algebraic thickness is at least the order of metric thickness. Thus, we ask the following strengthening of Question 7.5.

Question 7.7. *For a finitely generated group is the order of algebraic thickness equal to the order of metric thickness?*

A motivation for the study of thickness is that it provides a metric obstruction to relative hyperbolicity. In particular, it gives us examples to which one can apply Theorem 4.1.

In the sequel, we shall not mention the collection of subsets/subgroups with respect to which thickness is satisfied, when irrelevant to the problem.

Theorem 7.8. *Let $\mathcal{X}$ be a collection of uniformly thick metric spaces, and let Y be a metric space asymptotically tree-graded with respect to a collection $\mathcal{P}$ of subsets. Then there is a constant $M = M(L, C, \mathcal{X}, Y, \mathcal{P})$ such that for any $X \in \mathcal{X}$ and any (L, C)–quasi-isometric embedding $\mathfrak{q}\colon X \hookrightarrow Y$, the image $\mathfrak{q}(X)$ is contained in $\mathcal{N}_M(P)$ for some $P \in \mathcal{P}$.*

Proof: We prove the statement by induction on the order of thickness. If $n = 0$, then the family $\mathcal{X}$ is uniformly unconstricted and the statement follows from Theorem 3.6. Assume that the statement is true for n. We prove it for $n + 1$. Let $\mathcal{X}$ be a collection of metric spaces uniformly thick of order at most $n + 1$. For each $X \in \mathcal{X}$ let $\mathcal{L}_X$ be the collection of subsets with respect to which X is thick. The family $\mathcal{L} = \bigcup_{X \in \mathcal{X}} \mathcal{L}_X$ is uniformly thick of order at most n. By the inductive hypothesis, there exists $M = M(L, C, \mathcal{L}, Y, \mathcal{P})$ such that for any $L \in \mathcal{L}$, any (L, C)–quasi-isometric embedding of L into Y is contained into the radius M neighborhood of a set $P \in \mathcal{P}$. Let X be any metric space in $\mathcal{X}$ and let $\mathfrak{q} : X \to Y$ be an (L, C)–quasi-isometric embedding. For every $L \in \mathcal{L}_X$, the subset $\mathfrak{q}(L)$ is contained in $\mathcal{N}_M(P_L)$ for some $P_L \in \mathcal{P}$. Further, hypothesis $(\mathbf{N}_2)$ is satisfied also by the collection of subsets $\{\mathfrak{q}(L) \mid L \in \mathcal{L}\}$. Theorem 2.8, (α_1), implies that P_L is the same for all $L \in \mathcal{L}$. It follows that $\mathfrak{q}\left(\bigcup_{L \in \mathcal{L}} L\right)$ is contained in the M–neighborhood of P. Properties $(\upsilon\theta_1)$ and $(\mathbf{N}_1)$ together imply that $\mathfrak{q}(X)$ is contained in the $(M + L\tau + C)$–neighborhood of the same P. □

Taking $Y = X$ this immediately implies:

Corollary 7.9. *If X is a thick metric space, then X is not asymptotically tree-graded. In particular, if X is a finitely generated group, then X is not relatively hyperbolic.*

7.1. **NRH groups which are not thick.** Thick groups provide an important class of NRH groups. It is therefore natural to ask whether there exist examples of NRH groups which are not thick. A construction in [TV] (of which a more elaborated version can be found in [DS1, §7]) provides an example of a two-generated group, recursively (but not finitely) presented, which is NRH and not metrically thick.

Notation 7.10. Given an alphabet A and a word w in this alphabet, $|w|$ denotes the length of the word.

Definition 7.11 (property $C^*(\lambda)$)**.** Let F_A denote the set of reduced words in an alphabet A. A set $\mathcal{W} \subset F_A$ which is assumed to be closed under cyclic permutations and taking inverses, is said to satisfy *property* $C^*(\lambda)$ if the following hold:

(1) if u is a subword in a word $w \in \mathcal{W}$ so that $|u| \geq \lambda|w|$ then u occurs only once in w;
(2) if u is a subword in two distinct words $w_1, w_2 \in \mathcal{W}$ then $|u| \leq \lambda \min(|w_1|, |w_2|)$.

Let $A = \{a\, ,\, b\}$ and let $k_n = 2^{2^n}$. In the alphabet A consider the sequence of words $w_n = \left(a^{k_n} b^{k_n} a^{-k_n} b^{-1}\right)^{k_n}$. Note that $|w_n| = k_n(3k_n + 1)$. In what follows we denote this length by d_n and the sequence (d_n) by d.

A standard argument gives the following result (see [TV] and [Bow1] for versions of it).

Lemma 7.12. *If $\mathcal{W}$ is the minimal collection of reduced words in F_A containing $\{w_n\ ;\ n \in \mathbb{N}\, ,\, n \geq 4\}$, closed with respect to cyclic permutations and taking inverses, then the following hold:*
(1) *$\mathcal{W}$ can be generated recursively;*
(2) *$\mathcal{W}$ satisfies $C^*(1/500)$;*
(3) *for every $n \in \mathbb{N}$, the set $\{w \in \mathcal{W}\ ;\ |w| \geq d_n\}$ satisfies $C^*(1/k_n)$.*

Proposition 7.13. *(*[TV], [DS1]*) The two-generated and recursively presented group $G = \langle a, b \mid w_n\, ,\, n \geq 4\rangle$, has the following properties.*
(1) *Any asymptotic cone of G is either a real tree or a tree-graded space with pieces isometric to the same circle with the arc distance.*
(2) *The group G is not relatively hyperbolic.*

Proof: **(1)** Let $\Re_n$ be the loop through 1 in the Cayley graph of G, labeled by the word w_n starting from 1.

In [DS1, §7] it is proved that the asymptotic cone $\mathrm{Cone}_\omega(G; 1, d)$ is tree-graded, with the set of pieces composed of ultralimits of sequences of the form $(g_n \Re_n)$ where $g_n \in G$. In our case these ultralimits are all isometric to the unit circle. The same proof works in fact not only for (d_n) but for any scaling sequence, thus giving the statement in (1), since for other scaling sequences the ultralimits can be either circles, points or lines. A version of the last part of the argument can also be found in [TV].

(2) Assume that the group G is hyperbolic relative to a finite family of finitely generated subgroups $\mathcal{H}$. Then $\mathrm{Cone}_\omega(G;1,d)$ is tree-graded with set of pieces ultralimits of left cosets of subgroups in $\mathcal{H}$. According to Lemma 2.15 in [DS1], the subset without cut-point $\lim_\omega(\Re_n)$ is contained in some $\lim_\omega g_n H$ where $H \in \mathcal{H}$.

Let $\mathfrak{p}_n$ be an arbitrary sub-path in $\Re_n$, of length $\frac{1}{6}d_n$. This sub-path is a geodesic in the Cayley graph of G [DS1, §7.2]. Let $\mathfrak{p}_n'$ and $\mathfrak{p}_n''$ be the first and the last third of $\mathfrak{p}_n$. Since both have length $\frac{1}{18}d_n$ and are contained ω-almost surely in $\mathcal{N}_{o(d_n)}(g_nH)$, property (α_2) implies that both intersect a tubular neighborhood of radius $O(1)$ of g_nH. The quasi-convexity of g_nH ([DS1, §4], [Dru3, §4.3]) implies that ω-almost surely the middle third of $\mathfrak{p}_n$ is contained in $\mathcal{N}_M(g_nH)$, for some uniform constant M. Now the loop $\Re_n$ can be divided into 18 sub-paths of length $\frac{1}{18}d_n$, each of which appears as the middle third of a larger sub-path. We may conclude that $\Re_n$ is ω-almost surely contained in $\mathcal{N}_M(g_nH)$. In particular $1 \in \mathcal{N}_M(g_nH)$, hence it may be assumed that $g_n \in B(1,M)$. Since $B(1,M)$ is finite, the ultrafilter allows us to assume that g_n is a constant sequence.

Thus we obtained that for some $g \in B(1,M)$ and some $H \in \mathcal{H}$ the left coset gH contains in its M-tubular neighborhood ω-a.s. the loop $\Re_n$. It follows that $a\Re_n \subset \mathcal{N}_M(agH)$ and $b\Re_n \subset \mathcal{N}_M(bgH)$ ω-a.s. The loop $a\Re_n$ has in common with $\Re_n$ the path $a\mathfrak{p}_a$, where $\mathfrak{p}_a$ is the path of origin 1 and label a^{k_n-1}. It follows that ω-a.s $\mathcal{N}_M(gH)$ and $\mathcal{N}_M(agH)$ intersect in a set of diameter at least k_n-1. Property (α_1) implies that $gH = agH$, thus $a \in gHg^{-1}$.

Likewise, the remark that $b\Re_n$ and $\Re_n$ have in common the path $b\mathfrak{p}_a$, together with (α_1), implies that $b \in gHg^{-1}$. It follows that G coincides with gHg^{-1}, hence with H, therefore the relative hyperbolic structure defined by $\mathcal{H}$ is not proper. □

Remark 7.14. The arguments in the proof of statement (2), Proposition 7.13, can be carried out for a much more general construction of the group G than the one considered here. Thus, the techniques described in [DS1, §7] (following an idea from [Ol'] further developed in [EO]) allow the construction of a large class of new examples of NRH groups.

Corollary 7.15. *The group G does not contain any subspace B which endowed with the restriction of a word metric on G becomes unconstricted.*

In particular G is not metrically thick.

Proof: Assume that G contains an unconstricted subspace B. Then there exists an ultrafilter ω and a sequence δ of positive numbers such that for every sequence of observation points b in B the asymptotic cone $\mathrm{Cone}_\omega(B;b,\delta)$ does not have cut-points. Since B is endowed with the restriction of a word metric on G, $\mathrm{Cone}_\omega(B;b,\delta)$ can be seen as a subset of $\mathrm{Cone}_\omega(G;b,\delta)$.

If $\mathrm{Cone}_\omega(G;b,\delta)$ is a real tree then all arc-connected subsets in it have cut-points, thus it cannot contain a subset $\mathrm{Cone}_\omega(B;b,\delta)$ as above.

Assume that $\mathrm{Cone}_\omega(G;b,\delta)$ is a tree-graded space with pieces isometric to a circle. Lemma 2.15 in [DS1] implies that $\mathrm{Cone}_\omega(B;b,\delta)$ is contained in some piece. This is impossible since $\mathrm{Cone}_\omega(B;b,\delta)$ is infinite diameter, by Definition 3.1, (2). □

Remark 7.16. Note that the group G displays a sort of generalized version of metric thickness with respect to the collection of subspaces $\{g\Re_n \; ; \; g \in G\,, n \geq 4\}$. Indeed this collection satisfies one of the two necessary conditions for uniform unconstrictedness (condition (1) in Definition 3.4), property ($\mathbf{N}_1$) of a metric network obviously holds, and a weaker version of property ($\mathbf{N}_2$) is satisfied: the diameters of the intersections between neighborhoods of consecutive subspaces L_i, L_{i+1} in a sequence connecting thickly are no longer infinite, but increase with the minimum between the diameters of the starting and the target subspaces L and L'.

Question 7.17. *Can the construction above be adapted to give an example of a group which is metrically thick (and thus NRH) but not algebraically thick?*

8. Mapping class groups

Let $S = S_{g,p}$ denote an orientable surface of genus g with p punctures. We parameterize the complexity of S by $\xi(S) = 3g+p-3$ which is the cardinality of any set of closed curves subdividing S into pairs of pants, that is, any maximal, pairwise disjoint, pairwise nonhomotopic set of essential, nonperipheral closed curves on S. Note that every surface with $\xi(S) \leq 1$ either has $\mathcal{MCG}(S)$ finite or virtually free; in particular, these groups are all δ–hyperbolic. This section provides our first example of an algebraically thick group:

Theorem 8.1. *$\mathcal{MCG}(S)$ is algebraically thick of order one when $\xi(S) \geq 2$.*

It is known that the mapping class group is not thick of order 0 (i.e., unconstricted) by the following:

Theorem 8.2 (Behrstock [Beh])**.** *For every surface S, every asymptotic cone of $\mathcal{MCG}(S)$ has cut-points.*

$\mathcal{MCG}(S)$ is not hyperbolic when $\xi(S) \geq 2$ since for any set of curves subdividing S into pairs of pants, the subgroup generated by Dehn twisting along these curves is a free abelian subgroup of $\mathcal{MCG}(S)$ of rank $\xi(S)$. Indeed, according to [BLM], $\xi(S)$ is the maximal rank of a free abelian subgroup of $\mathcal{MCG}(S)$. Moreover, it has been shown that these abelian subgroups are quasi-isometrically embedded in $\mathcal{MCG}(S)$ (see [FLM] and [Mos]). Masur and Minsky showed that $\mathcal{MCG}(S)$ is weakly relatively hyperbolic with respect to a finite collection of stabilizers of curves [MM1]. (The subgroup stabilizing a curve γ will be denoted $\mathrm{stab}(\gamma)$.) Further, it is easily verified that $\mathcal{MCG}(S)$ is not relatively hyperbolic with respect to such a collection of subgroups. This motivates the question of whether there exists a collection of subgroups of $\mathcal{MCG}(S)$ for which this group is relatively hyperbolic (see [Beh]). That no such collection exists is an immediate consequence of Theorem 8.1:

Corollary 8.3. *If S is any surface with $\xi(S) \geq 2$, then there is no finite collection of finitely generated proper subgroups with respect to which $\mathcal{MCG}(S)$ is relatively hyperbolic.*

Anderson, Aramayona, and Shackleton have an alternative proof of Corollary 8.3 using an algebraic characterization of relative hyperbolicity due to Osin [AAS]. This result also appears in both [Bow3] and [KN] although it is not stated as such as it appears under the guise of a fixed point theorem for actions of the mapping class group. We note that the techniques of each of [AAS], [Bow3], and [KN] rely in an essential way on the group structure.

Before giving the proof of Theorem 8.1 we recall some well known results concerning mapping class groups. For closed surfaces the mapping class group was first shown to be finitely generated by Dehn [Deh] in a result which was later independently rediscovered by Lickorish [Lic]; both gave generating sets consisting of finite collections of Dehn twists. For the mapping class group $\mathcal{MCG}(S)$ of a punctured surface S, the finite index subgroup which fixes the punctures pointwise is generated by a finite set of Dehn twists [Bir]; this latter group is also called the *pure mapping class group*, and denoted by $\mathcal{PMCG}(S)$. The extended mapping class group, $\mathcal{MCG}^{\pm}(S)$, is the group of orientation preserving and reversing mapping classes. This is a finite extension of the mapping class group. (See [Bir], [Iva], [Hum]). Since these groups are all quasi-isometric, Remark 7.2 implies that if we can show that the pure mapping class group is algebraically thick of order one, it implies that the same holds for the mapping class group and the extended mapping class group.

Introduced by Harvey, a useful tool in the study of $\mathcal{MCG}(S)$ is the *complex of curves* $\mathcal{C}(S)$ [Har]. When $\xi(S) \geq 2$ the complex $\mathcal{C}(S)$ is a simplicial complex with one vertex corresponding to each homotopy class of nontrivial, nonperipheral simple closed curves in S, and with an n-simplex spanning each collection of $n+1$ vertices whose corresponding curves can be realized on S disjointly.

For later purposes we also need to define $\mathcal{C}(S)$ when $\xi(S) = 1$, in which case the surface S is either a once-punctured torus or a four-punctured sphere: the vertex set of $\mathcal{C}(S)$ is defined as above, with an edge attached to each pair of vertices whose corresponding curves can be realized on S with minimal intersection number, that number being 1 on a once-punctured torus and 2 on a four-punctured sphere.

In either case the complex $\mathcal{C}(S)$ is connected (see for example [MM1]). The distance $d_{\mathcal{C}(S)}(\alpha, \beta)$ between two vertices α, β in $\mathcal{C}(S)$ is the usual simplicial metric, defined to be the length of the shortest edge path between α and β.

Proof of Theorem 8.1. We start by remarking that for any essential simple closed curve γ, the subgroup $\mathrm{stab}(\gamma)$ is a central extension of $\mathcal{MCG}(S \setminus \gamma)$ by the infinite cyclic subgroup generated by a Dehn twist about γ. Thus, if $\xi(S) \geq 2$ then $\mathrm{stab}(\gamma)$ is non-elementary and has a central infinite cyclic subgroup. Consequently $\mathrm{stab}(\gamma)$ is unconstricted. It is an easy consequence of the distance estimates in [MM2, Theorem 6.12], that $\mathrm{stab}(\gamma)$ is undistorted for any essential simple closed curve γ. Select a finite collection of curves Γ_0 such that the Dehn twists along these curves generate $\mathcal{PMCG}(S)$. Connectivity of the curve complex implies that there is a finite connected subgraph of $\mathcal{C}(S)$ containing the vertices in Γ_0; let Γ denote the set of vertices in this new graph. Since $\xi(S) \geq 2$, if α, β are curves representing vertices at distance 1 in $\mathcal{C}(S)$ then α and β are disjoint, and so the subgroup $\mathrm{stab}(\alpha) \cap \mathrm{stab}(\beta) = \mathrm{stab}(\alpha \cup \beta)$ is infinite. It follows that $\mathcal{MCG}(S)$ is algebraically thick of order at most 1 with respect to $\mathcal{H} = \{\mathrm{stab}(\gamma) \mid \gamma \in \Gamma\}$. By Theorem 8.2, $\mathcal{MCG}(S)$ is not unconstricted and thus it is thick of order 1. $\square$

9. $\mathrm{Aut}(F_n)$ AND $\mathrm{Out}(F_n)$

We start by fixing a set of generators $\{x_1, ..., x_n\}$ for the free group F_n. We denote the automorphism and outer automorphism groups of F_n by $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n) = \mathrm{Aut}(F_n)/\mathrm{Inn}(F_n)$, respectively, where $\mathrm{Inn}(F_n)$ is the group of inner automorphisms. Recall that an element of $\mathrm{Aut}(F_n)$ is a *special* automorphism if the induced automorphism of $\mathbb{Z}^n$ has determinant 1. The subgroup $\mathrm{SAut}(F_n)$ of special automorphisms has index two in $\mathrm{Aut}(F_n)$.

Notation: All indices in this section are taken modulo n, where n is the rank of the free group we are considering.

We denote the following Dehn twists in $\mathrm{Aut}(F_n)$:

- $r_i = \begin{cases} x_{i+1} \mapsto x_{i+1}x_i & \\ x_j \mapsto x_j & \text{for } j \neq i+1\,, \end{cases}$
- $l_i = \begin{cases} x_{i+1} \mapsto x_i x_{i+1} & \\ x_j \mapsto x_j & \text{for } j \neq i+1\,, \end{cases}$
- $n_i = \begin{cases} x_{i+2} \mapsto x_{i+2}\, x_i & \\ x_j \mapsto x_j & \text{for } j \neq i+2\,. \end{cases}$

Culler and Vogtmann proved that the set S composed of all r_i and l_i is a set of generators of $\mathrm{SAut}(F_n)$, see [CV]. Note that all elements in S have infinite order. The elementary argument in Example 2.4 of [Ali] yields the following.

Lemma 9.1. *Let $n \geq 3$. The $\mathbb{Z}^2$ subgroup of $\mathrm{Aut}(F_n)$ generated by the pair $\langle \phi_i, \phi_j \rangle$ is undistorted when $\phi_i \in \{r_i, l_i\}$, $\phi_j \in \{r_j, l_j\}$, and $\mathrm{dist}(i,j) \geq 2$, where $\mathrm{dist}(i,j)$ is measured in $\mathbb{Z}/n\mathbb{Z}$. The $\mathbb{Z}^2$ subgroups $\langle r_i, l_i \rangle$, $\langle n_i, r_i \rangle$, and $\langle n_i, l_{i+1} \rangle$ are also undistorted for all i. These subgroups also inject to undistorted subgroups of $\mathrm{Out}(F_n)$.*

Theorem 9.2. *If $n \geq 3$, then both $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ are algebraically thick of order at most one.*

Proof. We consider $\mathcal{H}$ the set of all subgroups $\langle\, \phi_i \,,\, \phi_j \,\rangle$, where $\phi_i \in \{r_i, l_i\}$, $\phi_j \in \{r_j, l_j\}$, and $\mathrm{dist}(i,j) \geq 2$, and we also include in $\mathcal{H}$ the subgroups $\langle n_i, r_i \rangle$ and $\langle n_i, l_{i+1} \rangle$. We may regard these as subgroups of $\mathrm{Aut}(F_n)$ or, since they each intersect $\mathrm{Inn}(F_n)$ trivially, as subgroups of $\mathrm{Out}(F_n)$.

We shall now prove that both $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ are algebraically thick of order one with respect to the subgroups in $\mathcal{H}$.

Each subgroup $H = \langle\, \phi \,,\, \psi \,\rangle$ in $\mathcal{H}$ is isomorphic to $\mathbb{Z}^2$, hence unconstricted. Lemma 9.1 shows that each such subgroup is undistorted.

By [CV], the r_i and l_i provide a complete set of generators for $\mathrm{SAut}(F_n)$, and $\mathrm{SAut}(F_n)$ is a subgroup of $\mathrm{Aut}(F_n)$ of index two, thus we have shown that property $(\mathbf{AN}_1)$ is satisfied for $\mathrm{SAut}(F_n)$ and thus for $\mathrm{Aut}(F_n)$.

We verify property $(\mathbf{AN}_2)$ in the definition of algebraic thickness. Note that since $\langle \phi, l_j \rangle \cap \langle \phi, r_j \rangle \supset \langle \phi \rangle$, it suffices to show that the subgroups generated by r_i and n_i can be thickly connected. For every $\langle r_i, r_j \rangle$ with $\mathrm{dist}(i,j) \geq 2$, Lemma 9.1 shows that the subgroup $\langle r_i, r_j \rangle$ thickly connects any pair of subgroups of $\mathcal{H}$ where one contains r_i and the other r_j. Thus, to finish the verification of property $(\mathbf{AN}_2)$ it remains to find sequences joining a pair of subgroups, where one contains r_i and the other r_{i+1}. Observe that the sequence of subgroups $\langle r_i, n_i \rangle$, $\langle n_i, l_{i+1} \rangle$, $\langle l_{i+1}, r_{i+1} \rangle$ each intersects the next in an infinite diameter subset. This shows that any subgroup containing r_i can be thickly connected to one containing r_{i+1} through a sequence of subgroups in $\mathcal{H}$, thereby completing our verification of property $(\mathbf{AN}_2)$.

All the subgroups of $\mathrm{Aut}(F_n)$ that are used above to prove thickness are mapped, via the canonical epimorphism, injectively and without distortion to $\mathrm{Out}(F_n)$. Thus the hypotheses of Definition 7.3 hold as well in $\mathrm{Out}(F_n)$, whence $\mathrm{Out}(F_n)$ is algebraically thick of order one for $n \geq 3$. □

10. Artin groups

An *Artin group* is a group given by a presentation of the following form:

$$A = \langle x_1, ..., x_n \mid (x_i, x_j)_{m_{ij}} = (x_j, x_i)_{m_{ji}} \rangle \tag{9}$$

where, for all $i \neq j$ in $\{1, \ldots, n\}$,

$$m_{ij} = m_{ji} \in \{2, 3, \ldots, \infty\} \qquad \text{and} \qquad (x_i, x_j)_{m_{ij}} = \begin{cases} \mathrm{Id} & \text{if} \quad m_{ij} = \infty\,, \\ \underbrace{x_i x_j x_i ...}_{m_{ij} \text{ terms}} & \text{if} \quad m_{ij} < \infty\,. \end{cases}$$

Such a group can be described by a finite (possibly disconnected) graph $\mathcal{G}_A$, the *Artin presentation graph*, where the vertices of $\mathcal{G}_A$ are labeled $1, \ldots, n$ in correspondence with the generators $x_1, \ldots, x_n$, and the vertices i and j are joined by an edge labeled by the integer m_{ij} whenever $m_{ij} < \infty$. When $m_{ij} = \infty$ there is no associated relator in the presentation (9), and $\mathcal{G}_A$ has no edge between vertices i and j.

A subgroup generated by a subset S of $\{x_1, ..., x_n\}$ is called a *special subgroup* of A and it is denoted by A_S. Any special subgroup A_S is itself an Artin group with presentation given by the relations in (9) containing only generators in S, and such that $\mathcal{G}_{A_S}$ is the subgraph of $\mathcal{G}_A$ spanned by the vertices corresponding to S. This has been proved by Van der Lek in [dL, Chapter II, Theorem 4.13]. See also [Par] for an elementary proof as well as for a history of the result.

In particular the two generator special subgroup A_{ij} generated by x_i, x_j is an Artin group: if $m_{ij} = \infty$ then A_{ij} is free of rank 2; whereas if $m_{ij} < \infty$ then A_{ij} is defined by the single relator $(x_i, x_j)_{m_{ij}} = (x_j, x_i)_{m_{ji}}$.

The *Coxeter group* W associated to an Artin group A has a presentation obtained from (9) by adding relations saying that each x_i^2 is the identity.

Example 10.1. A two generator Artin group $\langle x, y \mid (x, y)_m = (y, x)_m \rangle$ with $m < \infty$ is unconstricted. This holds since the element $(x, y)_{2m}$ is central, and it is of infinite order since it projects to a nonzero element of $\mathbb{Z}$ under the exponent sum homomorphism $A \to \mathbb{Z}$.

In [KS] the following has been proven.

Theorem 10.2 (I. Kapovich–P. Schupp)**.** *An Artin group A defined as in (9) with $m_{ij} \geq 7$ for all $i \neq j$ is weakly hyperbolic relatively to the collection of two generator special subgroups*

$$\mathcal{H} = \{A_{ij} \mid m_{ij} < \infty\}\,.$$

As noted in the same paper, the above result cannot be improved to say that A is strongly hyperbolic relative to $\mathcal{H}$. Nevertheless the question remained whether A was strongly hyperbolic relative to other groups, or at least metrically hyperbolic relative to some collection of subsets. Our methods give a partial answer to this question, with the interesting outcome that when our methods work, A turns out to be algebraically thick of order at most 1 with respect to the exact same collection $\mathcal{H}$. We do not go as far as to check thickness for all of the Artin groups in Theorem 10.2, but in Corollary 10.8 below we show thickness as long as the graph $\mathcal{G}_A$ has no triangles. Here are some other special classes of Artin groups.

Free decompositions. The graph $\mathcal{G}_A$ with n points and no edges describes the group with n generators and no relators, i.e., the free group on n generators. More generally, if $\mathcal{G}_A$ is disconnected then A decomposes into a free product, one factor for each connected component in the defining graph. The converse is true as well: if $\mathcal{G}_A$ is connected then A is freely indecomposable, in fact A is a one-ended group. This follows for example from Proposition 1.3 and Remark 4.4. Since any nontrivial free product is relatively hyperbolic with the free factors as peripheral subgroups, we henceforth restrict our attention to one-ended Artin groups, those whose defining graphs have only one connected component.

Right angled Artin groups and even Artin groups. The complete graph on n vertices with each $m_{ij} = 2$ describes the group with n commuting generators, i.e., $\mathbb{Z}^n$. More generally, a *right angled Artin group* is one for which $m_{ij} \in \{2, \infty\}$ for all i, j. Recently there has been interest in the quasi-isometric classification of right angled Artin groups (see [BN] and [BKS]). Generalizing a right angled Artin group, an Artin group is *even* if each m_{ij} is an even integer or infinity.

Finite type Artin groups. An Artin group is of *finite type* if the associated Coxeter group W is finite. For example, the *braid group* on n strands is the Artin group with n generators, with $m_{i,i+1} = 3$, and $m_{ij} = 2$ if $|i - j| > 1$ — in this case the associated Coxeter group W is just the symmetric group on n symbols. An Artin group of finite type is unconstricted, since it has an infinite cyclic central subgroup of infinite index, as proven in [BS] and [Del].

Affine type Artin groups. An Artin group A is of *affine type* if the associated Coxeter group W is a Euclidean crystallographic group. For example, when $\mathcal{G}_A$ is a cycle of $n+1$ edges with a 3 on each edge then W is the full group of symmetries of a tiling of $\mathbf{R}^n$ by cubes, in which case we denote $A = \widetilde{A}_n$.

The reason for so many different special classes of Artin groups seems to be a proliferation of techniques for studying various aspects of Artin groups, and a concomitant lack of any single technique that works on all Artin groups — most theorems about Artin groups carry extra hypotheses on the Artin presentation. For example, there are various constructions in the literature of biautomatic and/or CAT(0) structures on Artin groups (we refer the reader to [ECH$^+$] for the definition of a biautomatic structure):

- Every right angled Artin group is CAT(0), in fact it is the fundamental group of a nonpositively curved cube complex [BB], and so it is biautomatic [NR].
- Braid groups are biautomatic [ECH$^+$]. More generally, Artin groups of finite type are biautomatic [Cha].
- If $\mathcal{G}_A$ has no triangles then A is CAT(0) [BM] and biautomatic (combining [Pri] and [GS1]; see comments in [BM]).
- A is CAT(0) and biautomatic if the edges of $\mathcal{G}_A$ can be oriented so that each triangle has an orientation agreeing with the orientations of all three edges, and in each square the orientations of the four edges do not alternate when going around the square [BM].

- Artin groups for which each $m_{ij} \geq 4$ are biautomatic [Pei].
- Artin groups of affine type $\widetilde{A}_n$, also known as the *affine braid groups*, are biautomatic [CP].

We shall prove thickness for some of these groups. The method we use is:

Lemma 10.3. *If the graph $\mathcal{G}_A$ is connected, and if each two generator special subgroup A_{ij} with $m_{ij} < \infty$ is undistorted in A, then A is algebraically thick of order ≤ 1.*

Proof. For i, j, k all distinct, the subgroup $A_{ij} \cap A_{ik}$ contains the infinite order element x_i. Since $\mathcal{G}_A$ is connected, and since the two generator special subgroups A_{ij} with $m_{ij} < \infty$ are undistorted and unconstricted (see Example 10.1), the lemma follows. □

One can verify undistortedness of two generator special subgroups in different cases by using a variety of methods: retractions; nonpositive curvature methods; the Masur–Minsky distance estimates for mapping class groups; or automatic group methods.

Retractions. Our first results on Artin groups use a simple algebraic method to prove undistortedness:

Proposition 10.4. *Let A be an Artin group. Suppose that for each 2 generator special subgroup A_{ij} with $m_{ij} < \infty$, there exists a retraction $p\colon A \to A_{ij}$. Then A is algebraically thick of order ≤ 1.*

Proof. This is a consequence of Lemma 10.3 and the observation that for any finitely generated group G and any finitely generated subgroup $H < G$, if there exists a retraction $G \to H$ then H is undistorted. □

In each application of this proposition, the retraction from an Artin group A generated by S to a special subgroup A' generated by $S' \subset S$ will be induced by a retraction from $S \cup \{\mathrm{Id}\}$ to $S' \cup \{\mathrm{Id}\}$.

Even Artin groups. Consider first the case of an Artin group A presented by (9) so that each m_{ij} is an even integer or $+\infty$. For each word w in S define $p(w)$ by deleting all letters of w not in S'. We need only prove that $p(w)$ is well defined in A'. To do this, consider a defining relator in the presentation (9), of the form

$$[x_i, x_j]_{m_{ij}} = [x_j, x_i]_{m_{ij}}$$

Bring one side of this relator over to the other, to obtain

$$R_{ij} = [x_i, x_j]_{m_{ij}}([x_j, x_i]_{m_{ij}})^{-1}$$

Then conjugate by an arbitrary word v in S to get a relator $vR_{ij}v^{-1}$. It suffices to check that $p(vR_{ij}v^{-1})$ defines the identity in A', and this is done in two cases. If the indices i, j are in the index set of S' then

$$p(vR_{ij}v^{-1}) = wR_{ij}w^{-1}$$

where $w = p(v)$, and this is a relator in A'. On the other hand, if one or both of the indices i, j are not in the index set of S' then, since m_{ij} is even, the word $p(R_{ij})$ is freely equal to the identity, and so $wp(R_{ij})w^{-1}$ is freely equal to the identity. By Proposition 10.4 it follows that:

Theorem 10.5. *Even Artin groups are algebraically thick of order at most 1.*

Trees. Consider next the case that $\mathcal{G}_A$ is a tree. There is a unique retraction $p\colon \mathcal{G}_A \mapsto \mathcal{G}_{A'}$ so that each component of $\mathcal{G}_A - \mathcal{G}_{A'}$ maps to the unique vertex of $\mathcal{G}_{A'}$ incident to that component. This induces a map $p\colon S \mapsto S'$. Extend p to a map from words in S to words in S'. Again we need only prove that given a relator $vR_{ij}v^{-1}$ for A as above, $p(vR_{ij}v^{-1}) = wp(R_{ij})w^{-1}$ defines the identity in A'. Consider the edge e of $\mathcal{G}_{A'}$ connecting s_i to s_j. If $e \subset A'$ then $p(R_{ij}) = R_{ij}$ and we are done. If e is contained in a component of $\mathcal{G}_A - \mathcal{G}_{A'}$ incident to a vertex s_k of $\mathcal{G}_{A'}$ then $p(R_{ij})$ is a word in the single generator s_k with exponent sum equal to zero and so is freely equal to the identity. By Proposition 10.4, A is algebraically thick of order ≤ 1.

Other examples. There seem still to be numerous other examples to which Proposition 10.4 applies. For example, consider the case that the group $\mathcal{G}_A$ has rank 1, meaning that it deformation retracts onto a circular subgroup $\mathcal{G}_{A'}$. Suppose furthermore that each integer that occurs as a label m_{ij} on some edge of $\mathcal{G}_{A'}$ occurs for at least two different edges.

For any edge of $\mathcal{G}_A$ not in $\mathcal{G}_{A'}$ there is a retraction defined as in the example above where the graph is a tree. For any edge $A_{ij} = e \subset \mathcal{G}_{A'}$, let $f \subset \mathcal{G}_{A'}$ be another edge with the same integer label. Removing the interiors of e and f from $\mathcal{G}_A$ results in two connected subgraphs $\mathcal{G}_i$, $\mathcal{G}_j$, with notation chosen so that $x_i \in \mathcal{G}_i$. Let y_i, y_j denote the endpoints of f, with notation chosen so that $y_i \in \mathcal{G}_i$. There is a retract $A \to e$ defined by taking $\mathcal{G}_i$ to x_i, and taking f to e so that y_i goes to x_i. This retract restricts to a retraction of the generating set S onto $\{x_i, x_j\}$. This map extends to a well defined retraction $A \to A_{ij}$ for the following reasons: for edges not equal to f the corresponding Artin relation maps to a word freely equal to the identity; and the Artin relation for the edge f maps to the Artin relation for the edge e because those two edges are labeled by the same integer.

We have not investigated the full extent to which Proposition 10.4 applies, but on the other hand we can easily construct somewhat random examples to which it seems not to apply, for example an Artin group whose presentation graph is the complete graph on four vertices and whose six edges are labeled by six pairwise relatively prime integers.

Nonpositive curvature. A good reference for nonpositively curved groups is [BH]. A geodesic metric on a cell complex C is a *polyhedral Euclidean metric* if for each cell c there is a compact, convex Euclidean polyhedron P and a characteristic map $P \mapsto c$ so that the metric on P pushes forward to the given metric on c. A polyhedral spherical metric is similarly defined, using spheres of constant curvature $+1$ instead of Euclidean space. The link of each vertex in a polyhedral Euclidean metric inherits a polyhedral spherical metric.

If C comes equipped with a polyhedral Euclidean metric then we say that C is a *piecewise Euclidean cell complex.* Furthermore, if the link of each vertex $v \in C$ has no closed geodesic of length $< 2\pi$ then we say that C is *nonpositively curved.* A subcomplex $D \subset C$ is *locally convex* if for each vertex $v \in D$, the link of v in D is a geodesically convex subset of the link of v in C.

Proposition 10.6 ([BH]). *If C is a finite piecewise Euclidean non-positively curved cell complex, and if D is a locally convex subcomplex, then the inclusion of universal covers $\widetilde{D} \to \widetilde{C}$ is globally isometric. It follows that the inclusion $D \hookrightarrow C$ induces an injection $\pi_1(D) \to \pi_1(C)$ with undistorted image.* □

Although right angled Artin groups are already considered in Theorem 10.5, the following gives a different approach.

Theorem 10.7. *If the Artin group A is right angled, or if it satisfies Pride's condition that $\mathcal{G}_A$ has no triangles, then A is the fundamental group of a piecewise Euclidean non-positively curved cell complex C_A so that each 2 generator special subgroup A_{ij} is the inclusion induced image of a locally convex subcomplex of C_A.*

The proof is given below. Combining Theorem 10.7 with Lemma 10.3 and Proposition 10.6 we obtain:

Corollary 10.8. *Artin groups A which are right angled or for which $\mathcal{G}_A$ has no triangles are algebraically thick of order ≤ 1.* □

In one case we can compute the order to be exactly 1:

Corollary 10.9. *Any right angled Artin group A for which $\mathcal{G}_A$ is a tree of diameter at least 3 has cut-points in every asymptotic cone, and so A is thick of order 1.*

Proof. Once we construct a compact, non-Seifert fibered, 3-dimensional graph manifold M whose fundamental group is isomorphic to A, the result follows by work of [KL2] and [KKL] (see Lemma 3.3 and Section 11). The manifold M will be a "flip manifold" in the terminology of Section 11.

Consider first a right angled Artin group A' for which $\mathcal{G}_{A'}$ is a *star graph*, meaning a tree of diameter 2, with valence 1 vertices $v_1, \ldots, v_k$ for $k \geq 2$, and a valence k vertex v_0 called the *star vertex*. The group A' is the product of a rank k free group with $\mathbb{Z}$. We can realize A' as the fundamental group of a 3-manifold M' which is the product of a "horizontal" $k+1$-holed sphere crossed with a "vertical" circle, so that the generators $v_1, \ldots, v_k$ correspond to the horizontal circles in k of the boundary tori, and the generator v_0 corresponds to the vertical circle.

Suppose now that A is a right angled Artin group and $\mathcal{G}_A$ is a tree of diameter ≥ 3. Let $v_1, \ldots, v_m$ be the vertices of $\mathcal{G}_A$ of valence ≥ 2, and note that $m \geq 2$. Let $\mathcal{G}_{A_i}$ denote the maximal star subgraph of $\mathcal{G}_A$ with star vertex v_i. The graph $\mathcal{G}_{A_i}$ presents a special subgroup A_i which is isomorphic to the fundamental group of a 3-manifold M_i as above, homeomorphic to the product of a sphere with holes crossed with the circle. We have $\mathcal{G}_A = \mathcal{G}_{A_1} \cup \cdots \cup \mathcal{G}_{A_m}$. When $i \neq j$ and $\mathcal{G}_{A_i}$, $\mathcal{G}_{A_j}$ are not disjoint then $\mathcal{G}_{A_i} \cap \mathcal{G}_{A_j}$ is a single edge of $\mathcal{G}_A$, in which case M_i and M_j each have a torus boundary whose fundamental group corresponds to the $\mathbb{Z}^2$ special subgroup generated by v_i and v_j; we now glue these two tori so that the horizontal circle on one torus glues to the vertical circle on the other. The result of gluing $M_1, \ldots, M_m$ in this manner is the desired 3-manifold M, and M is not Seifert fibered because $m \geq 2$. □

Proof of Theorem 10.7. Suppose first that A is right angled. For each subset I of the set of generator indices $\{1, \ldots, n\}$ for which the generators $\{x_i \mid i \in I\}$ all commute with each other, let T_I be the Cartesian product of $|I|$ copies of the unit circle. Glue these tori together using the obvious injection $T_{I'} \hookrightarrow T_I$ whenever

$I' \subset I$, with base point $T_\emptyset$. The result is a nonpositively curved piecewise Euclidean cell complex C_A with fundamental group A.

Consider a special subgroup $A' \subset A$ with the property that if e is an edge of $\mathcal{G}_A$ whose endpoints are in $\mathcal{G}_{A'}$ then e is in $\mathcal{G}_{A'}$. For example, $\mathcal{G}'_A$ could be a single edge of $\mathcal{G}_A$. Then by construction $C_{A'}$ may be regarded as a subcomplex of C_A, and clearly $C_{A'}$ is locally convex.

Suppose next that A is an Artin group for which $\mathcal{G}_A$ has no triangles. We use the construction of Brady–McCammond [BM] to produce the desired piecewise Euclidean cell complex C_A, and to verify local convexity of the appropriate subcomplexes. This verification is considerably more delicate than for right angled Artin groups. The standard presentation of a 2 generator Artin group

$$\langle y_1, y_2 \mid (y_1, y_2)_m = (y_2, y_1)_m \rangle$$

can be transformed into the presentation

$$\langle d, y_1, y_2, \ldots, y_m \mid d = y_1 y_2, d = y_2 y_3, \ldots, d = y_{m-1} y_m, d = y_m y_1 \rangle \tag{10}$$

by triangulating the relator $(y_1, y_2)_m = (y_2, y_1)_m$ and in the process introducing new generators $d, y_3, \ldots, y_m$ [BM]. Note than when each $m \geq 3$, the ordering y_i, y_j is essential to the description of the presentation (10): the word $y_i y_j$ is a subword of some relator, but the reversed word $y_j y_i$ is not.

The presentation complex of (10) has one vertex, $1 + m$ edges, and m triangular faces. The link of the unique vertex is given in Figure 2. Note that the vertices come in four layers: the *first layer* d, the *second layer* $\{y_1, \ldots, y_n\}$, the *third layer* $\{\bar{y}_1, \ldots, \bar{y}_n\}$, and the *bottom layer* $\bar{d}$. Also, the edges come in three horizontal layers: the *top edges* connecting first to second layer vertices; the *middle layer* connecting second to third layer vertices; and the *bottom layer* connecting third to fourth layer vertices. Consider now an Artin group A presented as in (9). Choose an orientation

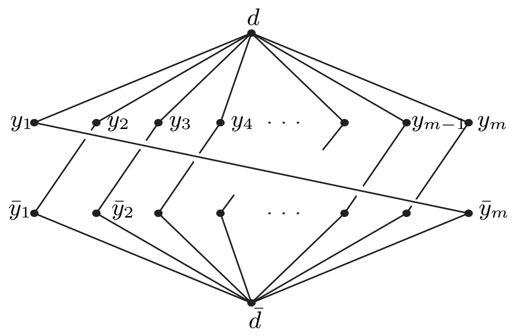

FIGURE 2

on each edge of $\mathcal{G}_A$, which determines an ordering of the endpoints of each edge of $\mathcal{G}_A$; henceforth, when we consider the 2 generator subgroup $A_{ij} = \langle x_i, x_j \mid (x_i, x_j)_{m_{ij}} = (x_j, x_i)_{m_{ij}}$ we will assume that the ij edge points from x_i to x_j. Now rewrite the presentation (9) to produce the *Brady–McCammond presentation* of A, by triangulating each Artin relator $(x_i, x_j)_{m_{ij}} = (x_j, x_i)_{m_{ij}}$ and introducing new generators following the pattern of (10), where we carefully choose notation so that new generators associated to distinct A_{ij} are distinct, as follows:

$$A_{i,j} = \langle d_{i,j}, x_i, x_j, x_{i,j,3}, x_{i,j,4} \ldots, x_{i,j,m} \mid d_{ij} = x_i x_j, d_{i,j} = x_j x_{i,j,3}, d_{i,j} = x_{i,j,3} x_{i,j,4} \ldots, d_{i,j} = x_{i,j,m} x_i \rangle \tag{11}$$

Let C_{ij} be the presentation complex for this presentation of A_{ij}, and let L_{ij} be the link of the unique vertex of C_{ij}. The two vertex pairs $\{x_i, \bar{x}_i\}$ and $\{x_j, \bar{x}_j\}$ in L_{ij} will be called the *peripheral vertex pairs* in L_{ij}.

Let C_A be the presentation complex for the Brady–McCammond presentation of A, and note that C_A is the union of its subcomplexes C_{ij} for $m_{ij} < \infty$. Also, let L_A be the link of the unique vertex of C_A, and note that L_A is the union of its subcomplexes L_{ij}. When $m_{ij}, m_{kl} < \infty$ and $\{i,j\} \neq \{k,l\}$, then either $\{i,j\} \cap \{k,l\} = \emptyset$ in which case $C_{ij} \cap C_{kl}$ is the unique vertex of C_A and $L_{ij} \cap L_{kl} = \emptyset$, or $\{i,j\} \cap \{k,l\}$ is a singleton, say $i = k$, in which case $C_{ij} \cap C_{il}$ is a single edge of C_A, labeled say by x_i, and $L_{ij} \cap L_{il}$ is a peripheral vertex pair, say $\{x_i, \bar{x}_i\}$. It follows that the layering of vertices and edges of the sublinks L_{ij} extends to a layering of all vertices and edges of L_A.

To organize L_A, note that there is a map L_A to $\mathcal{G}_A$ so that the inverse image of the ij edge of $\mathcal{G}_A$ is L_{ij}, and the inverse image of the vertex of $\mathcal{G}_A$ labeled x_i is the peripheral vertex pair of L_{ij} labeled $x_i, \bar{x}_i$.

Now we use the condition that $\mathcal{G}_A$ has no triangles. In this case Brady and McCammond choose a metric on C_A so that each edge labeled d_{ij} has length $\sqrt{2}$, each edge labeled x_i or x_{ijk} has length 1, and each triangle is a $\pi/2, \pi/4, \pi/4$ Euclidean triangle; they prove that C_A is nonpositively curved. Note that each top and bottom layered edge in L_A has spherical length $\pi/4$, and each middle layer edge has spherical length $\pi/2$.

To verify that C_{ij} is a locally convex subcomplex of C_A we must verify that for any locally injective edge path γ in L_A with endpoints in L_{ij} but with no edge in L_{ij}, the spherical length of γ is at least π.

If γ has at least four edges then we are done. If γ has three edges then it must connect some 2nd layer vertex to some 3rd layer vertex, and so at least one of the edges of γ is a middle layer vertex of length $\pi/2$, and we are done. The path γ cannot have one edge because L_A does not have an edge outside of L_{ij} connecting a 2nd and 3rd layer vertex of L_{ij}.

Suppose γ has two edges. Since $\mathcal{G}_A$ has no triangles, γ must project to a single edge of $\mathcal{G}_A$ and so γ is entirely contained in some L_{kl} distinct from but intersecting L_{ij}, and hence $\{i,j\} \cap \{k,l\}$ is a singleton. We assume that $k = i$, the other cases being handled identically. Then γ must connect one of the vertices labeled $x_i, \bar{x}_i$ to itself. However, L_{il} contains no locally injective edge path of length two with both endpoints at x_i or both at $\bar{x}_i$. □

Artin groups of affine type $\widetilde{A}_n$. Our next verification of undistortedness uses a different method, relying ultimately on distance estimates in mapping class groups.

Theorem 10.10. *If $n \geq 3$ then the Artin group $\widetilde{A}_n$ is algebraically thick of order at most* 1.

One possible approach to proving undistortedness of special subgroups of $\widetilde{A}_n$ is using the automatic group methods, which will be reviewed briefly below. Charney and Peifer prove in [CP] that $\widetilde{A}_n$ is biautomatic, and it would suffice then to prove that the two generator special subgroups of $\widetilde{A}_n$ are rational with respect to the Charney—Peifer biautomatic structure. Instead we shall consider an embedding of $\widetilde{A}_n$ into a braid group B, and we shall prove that all special subgroups of $\widetilde{A}_n$ are undistorted in B. This trick was suggested to us by our conversations with Ruth Charney. Our thanks to Ruth Charney for very helpful suggestions and comments on this proof.

Proof. We abbreviate $\widetilde{A}_n$ to $\widetilde{A}$, and we write its presentation in the form

$$\begin{aligned}\widetilde{A} = \langle x_0, x_1, \dots, x_n \mid\ & x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} \quad \text{for all} \quad i \in \mathbb{Z}/(n+1)\mathbb{Z}, \\ & x_i x_j = x_j x_i \quad \text{for all} \quad i, j \in \mathbb{Z}/(n+1)\mathbb{Z} \quad \text{such that} \quad j - i \not\equiv \pm 1 \rangle\end{aligned}$$

where index arithmetic takes place in $\mathbb{Z}/(n+1)\mathbb{Z}$. The cyclic permutation of the generators $x_0, x_1, \dots, x_n$ induces an automorphism of $\widetilde{A}$, and this automorphism cyclically permutes the two generator special subgroups $\langle x_i, x_{i+1} \rangle$. It therefore suffices to show that one of these two generator special subgroups is undistorted.

Consider the braid group B on $n+2$ strands, an Artin group with $n+1$ generators $y_0, y_1, \dots, y_n$ and with presentation

$$\begin{aligned}B = \langle y_0, y_1, \dots, y_n \quad \mid\ & y_i y_j = y_j y_i \quad \text{if} \quad 0 \le i \le j-2 \le j \le n, \\ & y_i y_{i+1} y_i = y_{i+1} y_i y_{i+1} \quad \text{if} \quad 0 \le i \le n-1 \rangle\end{aligned}$$

Let $h \colon \widetilde{A} \to B$ denote the homomorphism defined on the generators by $h(x_0) = \delta y_n \delta^{-1}$ and $h(x_i) = y_i$ for $i = 1, \dots, n$, where $\delta = y_0^{-2} y_1^{-1} \cdots y_n^{-1}$. By combining [KP] with the discussion at the beginning of [CP], it follows that h is injective. To obtain this expression for δ, we refer to [CP, Figure 4], which shows δ as an element of the annular braid group on $n+1$ strands. As explained in [CP], this latter group is isomorphic to the index $n+2$ subgroup of B in which the 0^{th} strand does not move, and from this viewpoint [CP, Figure 4] can be redrawn as in Figure 3 below, which gives the desired expression for δ.

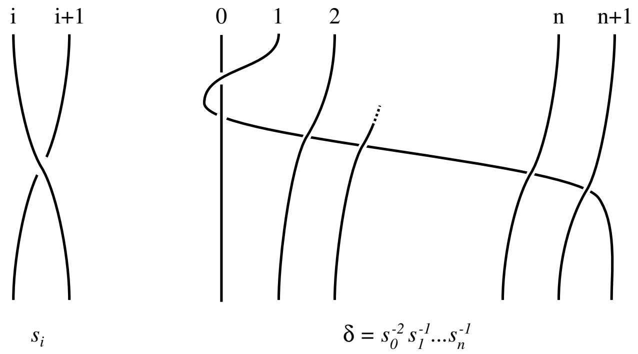

FIGURE 3

Clearly h maps the special subgroup of $\widetilde{A}$ generated by x_1, x_2 isomorphically to the special subgroup of B generated by y_1, y_2. It therefore suffices to show that special subgroups in B are undistorted, because of the following trick: given any finitely generated groups $K < H < G$, if K is undistorted in G then K is undistorted in H.

The group B is the mapping class group of a punctured disc D, and any special subgroup of B is the subgroup of mapping classes supported on a subsurface $F \subset D$ whose boundary is a collection of essential simple closed curves in D. But the fact that the inclusion of the mapping class group of F into the mapping class group of D is a quasi-isometric embedding is an immediate consequence of [MM2, Theorem 6.12]. □

Biautomatic groups methods. We close this section with a discussion of the following:

Question 10.11. *Are all Artin groups algebraically thick?*

One of the most important problems about Artin groups is the following:

Question 10.12. *Are all Artin groups biautomatic?*

In an automatic or biautomatic group, a "rational subgroup" is a subgroup with a particularly simple relation to the (bi)automatic structure. See [ECH$^+$] for the definition; see also [GS2] for a detailed study of rational subgroups of biautomatic groups. The key fact that we propose using is:

Theorem 10.13. [ECH$^+$, Theorems 3.3.4 and 8.3.1] *If G is an automatic group and H is a rational subgroup then H is undistorted in G.* □

By using Theorem 10.13, an affirmative answer to Question 10.11 reduces to an affirmative answer to the following refinement of Question 10.12:

Question 10.14. *Does every Artin group have a biautomatic structure so that every special subgroup is rational?*

What makes this a reasonable question to pursue are the many special classes of Artin groups known to be biautomatic (see the earlier list). It would be interesting to check some of these classes for rationality of special subgroups. For example, we have checked that all special subgroups of the braid group B are rational with respect to the symmetric biautomatic structure defined in [ECH$^+$, Theorem 9.3.6] — incidentally, this would provide another proof of Theorem 10.10, but to save space we opted for quickly quoting the Masur–Minsky distance estimates for mapping class groups.

11. Fundamental groups of graph manifolds

Recall that graph manifolds are compact Haken manifolds of zero Euler characteristic, such that all their geometric components are Seifert manifolds. In this section when referring to graph manifolds we always assume that they are connected, and we rule out the case of Nil and Sol manifolds. Hence all graph manifolds we consider are obtained by gluing finitely many Seifert components with hyperbolic base orbifolds along boundary tori or Klein-bottles, where the gluing does not identify the fibers.

In the universal cover $\widetilde{M}$ of such a manifold M, a flat projecting onto a torus or Klein bottle along which different Seifert components are glued is called a *separating flat*. A copy of a universal cover of a Seifert component is called a *geometric component*. Note that separating flats bound and separate geometric components.

A particular case of graph manifolds are the *flip manifolds*, in the terminology of [KL2]. Each Seifert component of a flip manifold is the product of a compact, oriented surface-with-boundary (the base) and S^1 (the fiber). Wherever two Seifert components are glued along a boundary torus the gluing interchanges the basis and the fiber directions.

Every flip manifold admits a non-positively curved metric, as follows. For each Seifert component, put a hyperbolic metric with geodesic boundary on the base so that each boundary component has length 1, pick a metric on the fiber to have length 1, and use the Cartesian product metric; each gluing map of boundary tori is then an isometry. Note that each Seifert component is locally convex.

Not every graph manifold admits a nonpositively curved metric [Lee]. On the other hand, according to [KL2], the fundamental group of any graph manifold is quasi-isometric to the fundamental group of some flip manifold. Moreover, the induced quasi-isometry between the universal covers of the two manifolds preserves the geometric decomposition, namely, the image of any geometric component is a uniformly bounded distance from a geometric component. We prove the following.

Theorem 11.1. *The fundamental group $G = \pi_1(M)$ of a non-Seifert fibered graph manifold is algebraically thick of order* 1 *with respect to the family $\mathcal{H}$ of fundamental groups of its geometric (Seifert) components.*[1]

Remark 11.2. Note that G is weakly hyperbolic relative to the family $\mathcal{H}$, because for any finite graph of finitely generated groups, the fundamental group is weakly hyperbolic relative to the vertex groups.

Proof. By [KL2, Theorem 1.1], G is quasi-isometric to the fundamental group of a compact non-positively curved flip manifold with totally geodesic flat boundary, and the images under a quasi-isometry of subgroups in $\mathcal{H}$ are a bounded distance from fundamental groups of geometric components. Since Seifert components of flip manifolds are locally convex, an application of Proposition 10.6 shows that the subgroups in $\mathcal{H}$ are undistorted in G.

Any subgroup H in $\mathcal{H}$ has a finite index subgroup H_1 which is the fundamental group of a trivial circle bundle over an orientable surface of genus at least two. Thus H is quasi-isometric to the direct product of $\mathbb{R}$ with a convex subset in $\mathbb{H}^2$. Consequently any asymptotic cone of H is bi-Lipschitz equivalent to a direct product of an $\mathbb{R}$-tree with $\mathbb{R}$, therefore H is unconstricted.

The group G decomposes as a fundamental group of a graph of groups, with vertex groups in $\mathcal{H}$ and edge groups commensurable to $\mathbb{Z}^2$. Any two subgroups in $\mathcal{H}$ can be thickly connected using a path in this graph of groups. We conclude that G is algebraically thick of order ≤ 1 with respect to $\mathcal{H}$. In [KL2] it is proven that the fundamental group of a non-Seifert fibered graph manifold manifolds has superlinear divergence. Hence, G is constricted by Lemma 3.3. Thus we have that G is algebraically thick of order 1 with respect to $\mathcal{H}$. □

Remark 11.3. Using graph manifolds one can construct an example of a group that is thick of order two but not of order zero or one. Indeed, consider a manifold N obtained by doubling a compact flip manifold M along a periodic geodesic $g \subset M$ that is not contained in a Seifert component of M. The fundamental group of N is

[1] This result was suggested to us by Kleiner, who had observed that these groups are not relatively hyperbolic with respect to any collection of finitely generated subgroups [Kle].

algebraically thick of order 2, but not of order 0 or 1. (Details of this construction and further results will be provided in [BDM].)

12. Teichmüller space and the pants graph

Let $S = S_{g,p}$ be an orientable surface of genus g with p punctures, with complexity $\xi = \xi(S) = 3g+p-3$. We let $\mathcal{C}_P(S)$ denote the pants graph of S, defined as follows. A vertex of $\mathcal{C}_P(S)$ is an isotopy class of pants decompositions of S. Given a pants decomposition $T = \{\gamma_1, \ldots, \gamma_\xi\}$, associated to each γ_i is the unique component R_i of $S - \cup\{\gamma_j \bigm| 1 \leq j \leq \xi, j \neq i\}$ which is not a pair of pants. This subsurface R_i has complexity 1, it is either a once punctured torus or a 4 punctured sphere, and we refer to R_i as a *complexity* 1 *piece* of T. Two pairs of pants decompositions $T, T' \in \mathcal{C}_P(S)$ with $T = \{\gamma_1, \ldots, \gamma_\xi\}$ and $T' = \{\gamma'_1, \ldots, \gamma'_\xi\}$ are connected by an edge of $\mathcal{C}_P(S)$ if they differ by an *elementary move*, which means that T and T' can be reindexed so that, up to isotopy, the following conditions are satisfied:

1. $\gamma_i = \gamma'_i$ for all $2 \leq i \leq \xi$
2. Letting X be the common complexity 1 piece of T and T' associated to both γ_1 and γ'_1, we have $d_{\mathcal{C}(X)}(\gamma_1, \gamma'_1) = 1$.

We now recall two results concerning the pants complex. The first relates the geometry of the pants complex to that of Teichmüller space, see [Bro].

Theorem 12.1 (Brock)**.** *$\mathcal{C}_P(S)$ is quasi-isometric to Teichmüller space with the Weil-Petersson metric.*

Bowditch [Bes] asked whether Teichmüller space with the Weil-Petersson metric was a δ–hyperbolic metric space. This was first answered by Brock and Farb in [BF] where they showed:

Theorem 12.2 (Brock–Farb)**.** *$\mathbb{R}^n$ can be quasi-isometrically embedded into $\mathcal{C}_P(S)$ for all $n \leq \lfloor \frac{\xi(S)+1}{2} \rfloor$.*

Combined with Theorem 12.1, this result showed the answer to Bowditch's question is "no" when the surface is sufficiently complex, i.e., satisfies $\xi(S) > 2$. By contrast, Brock–Farb gave an affirmative answer to Bowditch's question when $\xi(S) \leq 2$ (see also [Ara] and [Beh] for alternative proofs of δ–hyperbolicity in these low complexity cases). Brock–Farb proved hyperbolicity by showing that these Teichmüller spaces are strongly relatively hyperbolic with respect to a collection of subsets which themselves are hyperbolic; they then showed that this implies hyperbolicity of Teichmüller space. This raises the question of whether the presence of relative hyperbolicity generalizes to the higher complexity cases: we show that it does not, for sufficiently high complexity.

Theorem 12.3. *For any surface S of finite type with $\xi(S) \geq 6$, Teichmüller space with the Weil-Petersson metric is not asymptotically tree-graded as a metric space with respect to any collection of subsets.*

This is particularly interesting in light of the following:

Theorem 12.4 (Behrstock [Beh])**.** *For every surface S, every asymptotic cone of Teichmüller space with the Weil-Petersson metric is tree-graded.*

Together Theorems 12.3 and 12.4 say that the pieces in the tree-graded structure of an asymptotic cone do not merely arise by taking ultralimits of a collection of subsets.

Theorem 12.3 will follow from Theorem 7.9, once we establish:

Theorem 12.5. *For any surface S of finite type with $\xi(S) \geq 6$, Teichmüller space with the Weil-Petersson metric is metrically thick of order one.*

Proof: Let S denote a surface with $\xi(S) \geq 6$. Brock–Farb proved Theorem 12.2 by explicitly constructing quasiflats of the desired dimension. We shall use these same quasiflats to prove thickness, so we now recall their construction. Cut S along a pairwise disjoint family of simple closed curves into a collection of subsurfaces each of which is either a thrice punctured sphere or of complexity 1. Let $\mathcal{R} = \{R_1, \ldots, R_k\}$ be the subcollection of complexity 1 subsurfaces, and we assume that $k \geq 2$. Note that this is possible for a given k if and only if $2 \leq k \leq \lfloor \frac{\xi(S)+1}{2} \rfloor$. For each i, let g_i denote a geodesic in the curve complex $\mathcal{C}(R_i)$. One obtains a pants decomposition of S by taking the union of the curves ∂R_i and one curve from each g_i.

Theorem 12.2 is proven by showing that, for a fixed collection of subsurfaces and geodesics as above, the collection of all such pants decompositions is a quasiflat of rank k. If $\mathcal{G}$ denotes the family of geodesics $\{g_1, g_2, ..., g_k\}$, the above quasiflat is denoted by $Q_{\mathcal{R},\mathcal{G}}$. For a fixed surface S, all such quasi-isometric embeddings of $\mathbb{R}^k$ have uniformly bounded quasi-isometry constants. Let $\mathcal{L}$ be the collection of all quasiflats $Q_{\mathcal{R},\mathcal{G}}$. Note that when $\xi(S) \geq 6$ every element of $\mathcal{C}_P(S)$ is contained in some $Q_{\mathcal{R},\mathcal{G}}$, thus this collection satisfies condition $(\mathbf{N}_1)$ of metric thickness. Further, since each $Q_{\mathcal{R},\mathcal{G}}$ is a quasiflat of dimension at least two with uniform quasi-isometry constants, this collection is uniformly unconstricted.

It remains to verify condition $(\mathbf{N}_2)$. We proceed in two steps:

(1) Any pair $T, T' \in \mathcal{C}_P(S)$ which differ by an elementary move lie in some quasiflat in $\mathcal{L}$.
(2) Any pair of quasiflats in $\mathcal{L}$ which intersect can be thickly connected in $\mathcal{L}$.

Since the pants complex is connected by elementary moves, the first step implies that given any two pants decompositions $T, T' \in \mathcal{C}_P(S)$, one can find a sequence of quasiflats in $\mathcal{L}$ each intersecting the next in at least one point, such that the first quasiflat contains T and the last contains T'. The second step then implies that this sequence is a subsequence of a sequence of quasiflats in $\mathcal{L}$ where each intersects its successor in an infinite diameter set. This establishes condition $(\mathbf{N}_2)$.

(Step 1). Fix two pair of pants decompositions $T, T' \in \mathcal{C}_P(S)$ which differ by an elementary move. This elementary move is supported in a subsurface R_1 with $\xi(R_1) = 1$. Since $\xi(S) \geq 6$ there exists a curve α of T and T' disjoint from R_1. Let R_2 be the union of the pants of T and T' on either side of α, so R_1, R_2 have disjoint interior and $\xi(R_2) = 1$. Let $g_1 \subset \mathcal{C}(R_1)$ be an infinite geodesic extending the elementary move in R_1. The product of g_1 with a geodesic g_2 supported on R_2 is a two-dimensional quasiflat $Q_{\mathcal{R},\mathcal{G}}$, $\mathcal{R} = \{R_1, R_2\}$, $\mathcal{G} = \{g_1, g_2\}$, an element of $\mathcal{L}$, containing both T and T'.

(Step 2). Consider $Q = Q_{\mathcal{R},\mathcal{G}}$ and $Q' = Q'_{\mathcal{R}',\mathcal{G}'}$ an arbitrary pair of quasiflats in $\mathcal{L}$, with non-empty intersection containing $T \in \mathcal{C}_P(S)$.

(a). Assume first that there exists $R \in \mathcal{R} \cap \mathcal{R}'$. Let $g \in \mathcal{G}$ and $g' \in \mathcal{G}'$ denote the corresponding geodesics in Q and Q', respectively. Consider the quasiflat

$Q'' = Q''_{\mathcal{R},\mathcal{G}''}$, where $\mathcal{G}''$ is obtained from $\mathcal{G}$ by replacing g by g'. Then Q'' has infinite diameter intersection with both Q and Q' thus providing the desired thickly connecting sequence.

(b). Fixing $T \in \mathcal{C}_P(S)$, define a finite 1-complex as follows. A vertex is a collection of pairwise disjoint subsurfaces $\mathcal{R} = \{R_1, \ldots, R_k\}$ with cardinality $k \geq 2$, such that each R_i is a complexity 1 piece of T. Two such collections $\mathcal{R}, \mathcal{R}'$ are connected by an edge if they are not disjoint. By part *(a)* it suffices to show that this 1-complex is connected, and we prove this using that $\xi(S) \geq 6$. Consider two vertices $\mathcal{R}, \mathcal{R}'$. By removing elements of each we may assume that each has cardinality 2. Let $\mathcal{R} = \{R_1, R_2\}$, $\mathcal{R}' = \{R'_1, R'_2\}$. We may also assume that $\mathcal{R}, \mathcal{R}'$ are disjoint. If some element of $\mathcal{R}$ is disjoint from some element of $\mathcal{R}'$, say $R_1 \cap R'_1 = \emptyset$, then both are connected by an edge to $\{R_1, R'_1\}$; we may therefore assume that $R_i \cap R'_j \neq \emptyset$ for $i, j = 1, 2$. If some element of $\mathcal{R}$ or $\mathcal{R}'$ is a once-punctured torus, say R_1 with boundary curve α, then the only possible element of $\mathcal{R}'$ that can intersect R_1 is the one obtained by removing α, contradicting that there are two elements of $\mathcal{R}'$ that intersect R_1; we may therefore assume that each R_i and each R'_j is a four punctured sphere. It follows now that T has four distinct pairs of pants P_1, P_2, P_3, P_4 such that $R_1 = P_1 \cup P_2$, $R'_1 = P_2 \cup P_3$, $R_2 = P_3 \cup P_4$, $R'_2 = P_4 \cup P_1$. Since $\xi(S) \geq 6$, there is a curve α of T not incident to any of $P_1, \ldots, P_4$, and letting R'' be the complexity 1 piece of T obtained by removal of α, it follows that $\mathcal{R} = \{R_1, R_2\}$ is connected by an edge to $\{R_1, R_2, R''\}$ which is connected to $\{R'_1, R'_2, R''\}$ which is connected to $\{R'_1, R'_2\}$.

We have now shown that Teichmüller space with the Weil-Petersson metric is thick of order at most 1 when $\xi(S) \geq 6$. That it is thick of order exactly 1 follows from Theorem 12.4. ☐

Remark 12.6. With a little more work one can prove Step *(2)(b)* under the weaker assumption that $\xi(S) \geq 5$. Condition $(\mathbf{N}_1)$ can also be proved when $\xi(S) \geq 5$: the proof of $(\mathbf{N}_1)$ given here has an unnecessarily strong conclusion, namely that each point of $\mathcal{C}_P(S)$ lies in the union of $\mathcal{L}$. However, we do not know how to weaken the proof of Step *(1)* for any case when $\xi(S) < 6$.

Remark 12.7. We have recently been informed that in the cases with $\xi(S) = 3$, the Weil-Petersson metric on Teichmüller space is relatively hyperbolic [BM]. It remains unknown whether Teichmüller space with the Weil-Petersson metric is relatively hyperbolic when $\xi(S) = 4$ or 5. These cases do not fall under the cases where Teichmüller space is hyperbolic (see [BF], or for alternate arguments, [Ara] or [Beh]) or under the cases of Theorem 12.5 where Teichmüller space is metrically thick and hence not relatively hyperbolic. This motivates the following question:

Question 12.8. *In the cases where* $\xi(S) = 4$ *or* 5 *is the Weil-Petersson metric on Teichmüller space relatively hyperbolic?*

13. Subsets in symmetric spaces and lattices

Subsets in symmetric spaces. Let X be a product of symmetric spaces and Euclidean buildings of rank at least two, and let dist_X be a product metric on it (uniquely defined up to rescaling in the factors). Given a geodesic ray r in X, *the*

Busemann function associated to r is defined by

$$f_r : X \to \mathbb{R}\,,\ f_r(x) = \lim_{t\to\infty}[\mathrm{dist}_X(x, r(t)) - t]\,.$$

Remark 13.1. The Busemann functions of two asymptotic rays in X differ by a constant [BH].

The level hypersurface $H(r) = \{x \in X\ ;\ f_r(x) = 0\}$ is called the *horosphere determined by* r, the sublevel set $Hbo(r) = \{x \in X\ ;\ f_r(x) < 0\}$ is the *open horoball determined by* r.

Proposition 13.2. *Let $\mathcal{R}$ be a family of geodesic rays in X, of cardinality at least three, such that no ray is contained in a rank one factor of X and such that the open horoballs in the family $\{Hbo(r) \mid r \in \mathcal{R}\}$ are pairwise disjoint. Then for any $M > 0$, any connected component C of $\bigcup_{r\in\mathcal{R}} \mathcal{N}_M(H(r))$ endowed with dist_X is M-thick of order one with respect to*

$$\mathcal{L} = \{H(r) \mid r \in \mathcal{R}, H(r) \subset C\}\,.$$

Proof: The fact that $\{Hbo(r) \mid r \in \mathcal{R}\}$ are pairwise disjoint implies that their basepoints $\{r(\infty) \mid r \in \mathcal{R}\}$ are pairwise opposite. Since the previous set has cardinality at least three, this implies by [Dru1, proof of Proposition 5.5, b] that all the rays in $\mathcal{R}$ are in the same orbit of the group $\mathrm{Isom}(X)$. Hence all horospheres $H(r)$ with $r \in \mathcal{R}$ are isometric. Thus in order to have $(\mathbf{N}_2)$ it suffices to prove that one such horosphere endowed with the restriction of dist_X is unconstricted.

Let H be such a horosphere. According to [Dru1] and [Dru2], any asymptotic cone $H_\infty = \mathrm{Cone}_\omega(H, h, d)$ is a horosphere in the asymptotic cone $X_\infty = \mathrm{Cone}_\omega(X, h, d)$. The cone X_∞ is a Euclidean building having the same rank as X [KL]. Let r_∞ be the ray in X_∞ such that $H_\infty = H(r_\infty)$. The hypothesis that rays in $\mathcal{R}$ are not contained in a rank one factor of X implies that r_∞ is not contained in a rank one factor of X_∞. The arguments in [Dru1] and [Dru2] imply that under this hypothesis any two points x, y in H_∞ can be joined by a pair of topological arcs in H_∞ intersecting only in their endpoints. In particular H_∞ can not have cut-points. We conclude that H is unconstricted.

The fact that C is connected implies that for every $H(r), H(r') \in \mathcal{L}$ there exists a finite sequence $r_1 = r, r_2, ..., r_n = r'$ such that $\mathcal{N}_M(H(r_i)) \cap \mathcal{N}_M(H(r_{i+1}))$ is non-empty. Then $\mathrm{dist}(H(r_i), H(r_{i+1})) \le 2M$. There exists a maximal flat F in X containing rays asymptotic to both r_i and r_{i+1}. Remark 13.1 implies that one may suppose that both r_i and r_{i+1} are contained in F. Since $r_i(\infty)$ and $r_{i+1}(\infty)$ are opposite, $F \cap H(r_i)$ and $F \cap H(r_{i+1})$ are two parallel hyperplanes, at distance at most $2M$. It follows that $\mathcal{N}_M(H(r_i)) \cap \mathcal{N}_M(H(r_{i+1}))$ has infinite diameter. Thus $H(r)$ and $H(r')$ are thickly connected by the sequence $H(r_1), H(r_2), ..., H(r_n)$. $\square$

Higher rank lattices. Particularly interesting examples of spaces C satisfying the hypotheses of Proposition 13.2 are those on which some $\mathbb{Q}$–rank one lattice acts cocompactly. In this case, the space C is quasi-isometric to the lattice, and one can prove more than just metric thickness.

We recall first some known facts about lattices. In rank one semisimple groups, uniform lattices are hyperbolic, while non-uniform lattices are relatively hyperbolic with respect to their maximal unipotent subgroups (this in particular implies that maximal unipotent subgroups are undistorted in the lattice). Thus in both cases lattices cannot be thick.

In higher rank semisimple groups, uniform lattices have as asymptotic cones Euclidean buildings of higher rank [KL] so they are unconstricted, thus in particular they are thick of order zero.

In what follows we prove that non-uniform lattices in higher rank semisimple groups are algebraically thick of order at most one. In our arguments we also use unipotent subgroups. Unlike in the rank one case, these subgroups are distorted in the ambient lattice, therefore we have to embed them into solvable undistorted subgroups of the lattice, in order to prove thickness. For details on the notions and the results mentioned in this section we refer to [BT], [Bor], [Mar] and [Mor].

Let G be a connected semisimple group. Then G has a unique decomposition, up to permutation of factors, as an almost direct product $G = \prod_{i=1}^{m} \mathbf{G}_i(k_i)$, where k_i is a locally compact non-discrete field and $\mathbf{G}_i(k_i)$ is a connected group almost simple over k_i. Recall that:

- An algebraic group defined over a field k is called *almost simple over k* if all the proper k-algebraic normal subgroups of it are finite.
- An algebraic group is called *absolutely almost simple* if any proper algebraic normal subgroup of it is finite.
- An algebraic group G is an *almost direct product* of its subgroups $H_1, .., H_m$ if the multiplication map $H_1 \times ... \times H_m \to G$ is surjective and of finite kernel (an *isogeny*).

The *rank* of G is defined as $\operatorname{rank} G = \sum_{i=1}^{m} \operatorname{rank}_{k_i} \mathbf{G}_i$, where $\operatorname{rank}_{k_i} \mathbf{G}_i$ is the dimension of the maximal k_i–split torus of $\mathbf{G}_i$. Recall that a *k_i–split torus* is a subgroup of G which is abelian, closed, connected, with every element semisimple, and which is diagonalizable over the field k_i. We make the following two assumptions on G:

($\mathbf{Hyp}_1$) For every $i = 1, 2, ..., m$, $\operatorname{char} k_i = 0$.

($\mathbf{Hyp}_2$) For every $i = 1, 2, ..., m$, $\operatorname{rank}_{k_i} \mathbf{G}_i \geq 1$, and $\operatorname{rank} G \geq 2$.

Notation: Given two functions f, g defined on a set X and taking real values, we write $f \ll g$ if $f(x) \leq Cg(x)$ for every $x \in X$, where C is a constant uniform in x. We write $f \asymp g$ if $f \ll g$ and $g \ll f$.

The group G can be endowed with a left invariant metric dist_G with the property that for fixed embeddings of each $\mathbf{G}_i(k_i)$ into $SL(n_i, k_i)$,

$$\operatorname{dist}_G(1, g) \asymp \sum_{i=1}^{m} \ln\left(1 + \|g_i - \mathrm{Id}_i\|_{i,max}\right) . \tag{12}$$

See for instance [LMR] for details.

Let Γ be an *lattice* in G, that is, a discrete subgroup of G for which G/Γ carries a G–left invariant finite measure. If the projection of Γ to any direct factor of G is dense then the lattice is called *irreducible*. Otherwise it is commensurable to a product $\Gamma_1 \times \Gamma_2$, where Γ_i are lattices in direct factors of G. Note that in this latter case, Γ is unconstricted, according to the first example following Definition 3.4. Therefore, in what follows we always assume that Γ is irreducible.

The lattice Γ is called *uniform* if G/Γ is compact. Throughout the rest of the section we assume that Γ is a *non-uniform lattice*, that is G/Γ is not compact.

Theorem 13.3 (Lubotzky–Mozes–Raghunathan [LMR], Theorem A)**.** *The word metric on Γ is bi-Lipschitz equivalent to* dist_G *restricted to* Γ.

By Margulis' Arithmeticity Theorem [Mar, Chapter IX], the hypotheses that $\operatorname{rank} G \geq 2$ and that Γ is irreducible imply that Γ is an S–arithmetic group: there exists a global field F, a simply connected absolutely almost simple algebraic group $\mathbf{G}$ defined over F, a finite set $\mathbf{S}$ of valuations of F containing the archimedean ones and a homomorphism $\Phi : \prod_{v\in\mathbf{S}} \mathbf{G}(F_v) \to G$ such that $\ker\Phi$ is compact, $\operatorname{Im}\Phi$ is a closed normal subgroup of G with $G/\operatorname{Im}\Phi$ also compact, and Γ is commensurable with $\Phi(\mathbf{G}(\mathcal{O}_{\mathbf{S}}))$, where $\mathcal{O}_{\mathbf{S}}$ is the ring of $\mathbf{S}$–integers in F, defined by $|\cdot|_v \leq 1$ for every $v \notin \mathbf{S}$. Here $\mathbf{G}(\mathcal{O}_{\mathbf{S}})$ is $\mathbf{G}(F) \cap \mathrm{GL}(n, \mathcal{O}_{\mathbf{S}})$ if we assume that $\mathbf{G}$ is an F-algebraic subgroup of $\mathrm{GL}(n)$, and we identify $\mathbf{G}(\mathcal{O}_{\mathbf{S}})$ to its image under the diagonal embedding in $\prod_{v\in\mathbf{S}} \mathbf{G}(F_v)$. The hypothesis that Γ is non-uniform is equivalent to the property that $\operatorname{rank}_F \mathbf{G} \geq 1$.

Theorem 13.4. *The lattice Γ is algebraically thick of order at most one.*

Proof: It suffices to prove the statement for $G = \prod_{v\in S} \mathbf{G}(F_v)$ and $\Gamma = \mathbf{G}(\mathcal{O}_S)$, where $\mathbf{G}(\mathcal{O}_S)$ is identified to its image under the diagonal embedding in G. We first recall some useful notions and results. A reductive group is F*–anisotropic* if it is defined over F and if it does not contain any non-trivial F–split torus.

Lemma 13.5. *A reductive subgroup R of G which is defined over F intersects Γ in a lattice, that is $R/R\cap\Gamma$ has finite measure. Moreover if R is F–anisotropic then the lattice is uniform, that is $R/R\cap\Gamma$ is compact.*

Let P be a parabolic subgroup of G defined over F. The following can be said:

(1) The *unipotent radical* U of P (i.e. the unique maximal unipotent normal subgroup of P) is algebraic defined over F.
(2) There exists an F–split torus T such that $P = Z(T)U$, where $Z(T)$ is the centralizer of T in G.
(3) In its turn, $Z(T)$ is an almost direct product of a torus T' with $M = [Z(T), Z(T)]$, and the group M is semisimple.
(4) The torus T' is an almost direct product of T with an F–anisotropic torus C.
(5) The groups $Z(T), M, T'$ are algebraic defined over F.
(6) The group M contains countably many F–anisotropic tori D each of which is maximal in M and contains for each $v \in S$ a maximal torus of M defined over F_v [LMR, Lemma 3.10]. For each such torus D, the product TCD is a maximal torus in G.
(7) When the F-parabolic group P is minimal, T is a maximal F-split torus, U is a maximal F-unipotent subgroup of G, and $U\cap\Gamma$ is a maximal unipotent subgroup in Γ.
(8) Conversely, given an F–split torus T there exist finitely many F–parabolic subgroups P that can be written as $P = Z(T)U$, and such that all the above decompositions and properties hold. These parabolic subgroups correspond to finitely many walls (faces of Weyl chambers) composing T.

Let T be a maximal F-split torus in G, and let $T' = TC$, $D < M$ and $\widetilde{T} = TCD$ be tori associated to T as above. Let Λ be the system of F-roots of G with respect to T, and let $\widetilde{\Lambda}$ be the system of roots of G with respect to $\widetilde{T}$. For every $\widetilde{\lambda} \in \widetilde{\Lambda}$, if its restriction to T, $\widetilde{\lambda}|_T$, is not constant equal to 1 then it is in Λ.

In what follows all bases of roots and lexicographic orders on roots will be considered as chosen simultaneously on both Λ and $\widetilde{\Lambda}$ so that they are compatible with respect to the restriction from $\widetilde{T}$ to T.

Let $\mathfrak{g}$ be the Lie algebra of G. For every $\widetilde{\lambda}$ in $\widetilde{\Lambda}$ denote by $\mathfrak{g}_{\widetilde{\lambda}}$ the one dimensional eigenspace $\left\{v \in \mathfrak{g}\ ;\ Ad(t)(v) = \widetilde{\lambda}(t)v\ ,\ \forall t \in \widetilde{T}\right\}$. Here $Ad(t)$ is the differential at the identity of the conjugacy by t. There is a unique one-parameter unipotent subgroup $U_{\widetilde{\lambda}}$ in G tangent to the Lie algebra $\mathfrak{g}_{\widetilde{\lambda}}$. Let $\lambda \in \Lambda$. A multiple of it $n\lambda$ with $n \in \mathbb{N}$ can be in Λ for $n \in \{1, 2\}$. Consider the Lie subalgebra $\mathfrak{u}_\lambda = \bigoplus_{\widetilde{\lambda}|_T=\lambda,2\lambda} \mathfrak{g}_{\widetilde{\lambda}}$ and let U_λ denote the unique T-stable F-unipotent subgroup of G tangent to the Lie algebra $\mathfrak{u}_\lambda$. Let Δ be a basis for Λ (or, in another terminology, a fundamental system of roots). Every root λ in Λ can be written as $\sum_{\alpha\in\Delta} m_\alpha(\lambda)\,\alpha$, where $(m_\alpha(\lambda))_{\alpha\in\Delta}$ are integers either all non-negative or all non-positive.

Let P be an F-maximal parabolic subgroup. There exists a maximal F-split torus T, a basis Δ for the system Λ of F-roots of G with respect to T, and a root $\alpha \in \Delta$ such that the following holds. Let $\Lambda_\alpha^+ = \{\lambda \in \Lambda\ ;\ m_\alpha(\lambda) > 0\}$. The parabolic P decomposes as $P = Z(T_\alpha)\widetilde{U}_\alpha$, where

- $T_\alpha = \{t \in T\ ;\ \beta(t) = 1\ ,\ \forall \beta \in \Delta\ ,\ \beta \neq \alpha\}$;
- $\widetilde{U}_\alpha$ is the unipotent subgroup tangent to the Lie algebra $\widetilde{\mathfrak{u}}_\alpha = \bigoplus_{\widetilde{\lambda}|_T\in\Lambda_\alpha^+} \mathfrak{g}_{\widetilde{\lambda}}$. Note that $\widetilde{\mathfrak{u}}_\alpha = \bigoplus_{\lambda\in\Lambda_\alpha} \mathfrak{u}_\lambda$, where Λ_α is such that any root in Λ_α^+ is either contained in Λ_α or is the double of a root in Λ_α. In particular, the above implies that U_α is a subgroup of $\widetilde{U}_\alpha$.

The following result is proven in [Rag] for $\mathrm{rank}_F\mathbf{G} \geq 2$ and in [Ven] for $\mathrm{rank}_F\mathbf{G} = 1$.

Theorem 13.6 (Raghunathan; Venkataramana)**.** *Let T be a maximal F-split torus in G and let Λ be the system of F-roots of G with respect to T. Then the group generated by the subgroups $U_\lambda \cap \Gamma$, $\lambda \in \Lambda$, has finite index in Γ.*

Note that when $\mathrm{rank}_F\mathbf{G} = 1$, the family of unipotent subgroups $\{U_\lambda\ ;\ \lambda \in \Lambda\}$ contains only two maximal F–unipotent subgroups, which are *opposite* (i.e. with trivial intersection).

Each of the subgroups $U_\lambda \cap \Gamma$ is finitely generated. Thus in order to prove thickness it suffices to construct a family $\mathcal{H}$ of unconstricted subgroups of Γ satisfying properties $(\mathbf{AN}_0)$ and $(\mathbf{AN}_2)$, and such that each subgroup $U_\lambda \cap \Gamma$ is contained in a subgroup in $\mathcal{H}$.

The parabolic groups defined over F compose a spherical building Σ of rank $r = \mathrm{rank}_F\mathbf{G}$. Minimal parabolic groups correspond to chambers in this building, while larger parabolic groups correspond to panels and faces in the building. Maximal parabolic groups correspond to vertices.

In what follows we fix a maximal F-split torus T in G and the system of F–roots Λ of G with respect to T. Let $\mathcal{P}$ be the finite collection of all the maximal F–parabolic subgroups in G containing T. They correspond to the vertices of an apartment in Σ. Each $P \in \mathcal{P}$ decomposes as $P = Z(T_\alpha)\widetilde{U}_\alpha$ for some $\alpha \in \Lambda$. Let $M_P = [Z(T_\alpha), Z(T_\alpha)]$, and let C_P be the F–anisotropic torus such that $Z(T_\alpha)$ is an almost direct product of T_α with C_P and with M_P. Also let D_P be a maximal F-anisotropic torus in M_P. We make the choice of D_P so that if $P, P' \in \mathcal{P}$ correspond to opposite vertices in the building Σ (in which case the corresponding unipotent radicals have trivial intersection, while the corresponding tori T_α and $T_{\alpha'}$ coincide, therefore also $M_P = M_{P'}$) then $D_P = D_{P'}$.

Consider the solvable group $S_P = C_P D_P \widetilde{U}_\alpha$. Since C_P is an F-anisotropic torus, by Lemma 13.5 the intersection $C_P \cap \Gamma$ is a uniform lattice in C_P, likewise for $D_P \cap \Gamma$ in D_P. Also $\widetilde{U}_\alpha \cap \Gamma$ is a (uniform) lattice in $\widetilde{U}_\alpha$. Consequently the semidirect product $\Gamma_P = (C_P \cap \Gamma)(D_P \cap \Gamma)(\widetilde{U}_\alpha \cap \Gamma)$ is a uniform lattice in S_P. Note that $(C_P \cap \Gamma)(D_P \cap \Gamma)$ is never trivial. This is due on one hand to the fact that $T_\alpha C_P D_P$ is a maximal torus in G, so it has dimension at least two, and since T_α has dimension one it follows that $C_P D_P$ is of dimension at least one. On the other hand $(C_P \cap \Gamma)(D_P \cap \Gamma)$ is a uniform lattice in $C_P D_P$.

We show that Γ is algebraically thick of order at most 1 with respect to $\mathcal{H} = \{\Gamma_P \mid P \in \mathcal{P}\}$. Note that for any $\lambda \in \Lambda$ the subgroup U_λ is contained in the unipotent radical $\widetilde{U}_\lambda$ of some $P \in \mathcal{P}$. In particular each $U_\lambda \cap \Gamma$ is contained in at least one Γ_P.

Each group Γ_P, $P \in \mathcal{P}$, is finitely generated and solvable, hence it is unconstricted ([DS1, §6.2] , see also Section 3, Example 3).

Therefore it only remains to prove properties $(\mathbf{AN}_0)$ and $(\mathbf{AN}_2)$.

$(\mathbf{AN}_0)$ We prove that Γ_P is undistorted in Γ.

Notation: In what follows, given a finitely generated group H we write dist_H to denote a word metric on H. Given a Lie group L we denote by dist_L a metric on L defined by a left-invariant riemannian structure.

It suffices to prove that $\mathrm{dist}_{\Gamma_P}(1, g) \ll \mathrm{dist}_\Gamma(1, g)$ for every $g \in \Gamma_P$. An element g in Γ_P decomposes as $g = tu$, where $t \in (C_P \cap \Gamma)(D_P \cap \Gamma)$ and $u \in U_P \cap \Gamma$. We have that

$$\mathrm{dist}_{\Gamma_P}(1, tu) \leq \mathrm{dist}_{\Gamma_P}(1, u) + \mathrm{dist}_{\Gamma_P}(1, t)\,.$$

Note that $\mathrm{dist}_{\Gamma_P}(1, t) \leq \mathrm{dist}_{(C_P\cap\Gamma)(D_P\cap\Gamma)}(1, t) \ll \mathrm{dist}_G(1, t)$. The last inequality follows from the fact that a word metric on $(C_P \cap \Gamma)(D_P \cap \Gamma)$ is bi-Lipschitz equivalent to the restriction of a metric from $C_P D_P$, and from the fact that $C_P D_P$ is undistorted in G.

Lemma 13.7. *For every $u \in U_P \cap \Gamma$, $\mathrm{dist}_{\Gamma_P}(1, u) \ll \mathrm{dist}_G(1, u)$.*

Proof. The group U_P is a group of type $\widetilde{U}_\alpha$ for some $\alpha \in \Lambda$. That is, if $\Lambda_\alpha^+ = \{\lambda \in \Lambda\ ;\ m_\alpha(\lambda) > 0\}$, then the group U_P has Lie algebra $\widetilde{\mathfrak{u}}_\alpha = \bigoplus_{\widetilde{\lambda}|_T \in \Lambda_\alpha^+} \mathfrak{g}_{\widetilde{\lambda}}$. In particular $\widetilde{\mathfrak{u}}_\alpha = \bigoplus_{\lambda \in \Lambda_\alpha} \mathfrak{u}_\lambda$, where Λ_α is such that any root in Λ_α^+ is either contained in or is the double of a root in Λ_α.

Lemma 13.8 (Lubotzky–Mozes–Raghunathan [LMR], §3)**.** *Let $\lambda_1, \lambda_2, ..., \lambda_N$ be the enumeration of the roots in Λ_α in increasing order. The order here is the lexicographic order with respect to some basis Δ of Λ having α as first root.*

(1) *There exist morphisms $f_i : U_P \to U_{\lambda_i}$, $1 \leq i \leq N$, such that for every $u \in U_P$, $u = f_1(u) \cdot f_2(u) \cdot \ldots \cdot f_N(u)$ and*

$$\mathrm{dist}_G(1, u) \leq \sum_{i=1}^{N} \mathrm{dist}_G(1, f_i(u)) \ll \mathrm{dist}_G(1, u)\,.$$

(2) *If Γ_1 is a suitable congruence subgroup of $\Gamma = \mathbf{G}(\mathcal{O}_S)$, then for every $u \in \Gamma_1 \cap U_P$ the components $f_i(u)$ are in $U_{\lambda_i} \cap \Gamma$ for all $i = 1, 2, ..., N$.*

Let Γ_1 be as in Lemma 13.8. It has finite index in Γ, therefore it suffices to prove Lemma 13.7 for $u \in U_P \cap \Gamma_1$. In this case $f_i(u) \in U_{\lambda_i} \cap \Gamma$ for all $i = 1, 2, ..., N$, and $\mathrm{dist}_{\Gamma_P}(1, u) \leq \sum_{i=1}^{N} \mathrm{dist}_{\Gamma_P}(1, f_i(u))$. By Lemma 13.8, (1), it will then suffice to prove Lemma 13.7 for each $f_i(u)$. Hence we may assume in what follows that $u \in U_\lambda \cap \Gamma$, for some $\lambda \in \Lambda_\alpha$.

Consider the solvable subgroup $S_\lambda = C_P D_P U_\lambda$ of S_P, and its uniform lattice $\Gamma_\lambda = (C_P \cap \Gamma)(D_P \cap \Gamma)(U_\lambda \cap \Gamma)$, which is a subgroup of Γ_P. It will suffice to prove that $\mathrm{dist}_{\Gamma_\lambda}(1,u) \ll \mathrm{dist}_G(1,u)$, which is equivalent to proving that $\mathrm{dist}_{S_\lambda}(1,u) \ll \mathrm{dist}_G(1,u)$. With a decomposition similar to the one in Lemma 13.8, (1), we can reduce the problem to the case when u is in the uniparametric unipotent subgroup $U_{\widetilde{\lambda}}$ for some root $\widetilde{\lambda}$ such that its restriction to T is λ. The torus $C_P D_P$ is orthogonal to the one-dimensional F-split torus T_α associated to P, in a maximal torus containing both. If the root $\widetilde{\lambda}$ would be constant equal to 1 on $C_P D_P$, then λ would be constant equal to 1 on $C_P D_P \cap T$, hence the same would be true for α. Consequently α would be equal to 1 on the orthogonal of T_α in T. This implies that the F-structure on G is reducible (see for instance [KL] for a geometric argument), which implies that Γ is reducible, contradicting the hypothesis. Thus there exists at least one uniparametric semisimple subgroup T_1 in $C_P D_P$ on which $\widetilde{\lambda}$ takes all positive values. An argument as in [Gro3, §3.D] then implies that $\mathrm{dist}_{S_\lambda}(1,u) \leq \mathrm{dist}_{T_1 U_{\widetilde{\lambda}}}(1,u) \ll \ln(1 + \|u - I\|) \ll \mathrm{dist}_G(1,u)$. □

Lemma 13.7 together with the considerations preceding it imply that

$$\mathrm{dist}_{\Gamma_P}(1, tu) \ll \mathrm{dist}_G(1,t) + \mathrm{dist}_G(1,u)\,.$$

On the other hand $\mathrm{dist}_G(1,t) + \mathrm{dist}_G(1,u) \ll \mathrm{dist}_G(1,tu)$. This follows from the well known fact that $\mathrm{dist}_G(1,t) \leq \mathrm{dist}_G(1,tu)$ and from the triangular inequality $\mathrm{dist}_G(1,u) \leq \mathrm{dist}_G(1,t) + \mathrm{dist}_G(1,tu)$. Then $\mathrm{dist}_{\Gamma_P}(1,tu) \ll \mathrm{dist}_G(1,tu) \ll \mathrm{dist}_\Gamma(1,tu)$, where the latter estimate follows from Theorem 13.3. This completes the proof of $(\mathbf{AN}_0)$.

$(\mathbf{AN}_2)$ First we suppose that $\mathrm{rank}_F \mathbf{G} = 1$. Then $\mathcal{P}$ has only two elements, P and P', which are opposite. Consequently $\Gamma_P \cap \Gamma_{P'}$ contains $(C_P \cap \Gamma)(D_P \cap \Gamma)$, which is a lattice in a torus of dimension at least one, hence it is infinite.

Suppose now that $\mathrm{rank}_F \mathbf{G} \geq 2$. This implies that the building Σ composed of F-parabolics has rank at least two, therefore it is connected. Let $P, P' \in \mathcal{P}$. The groups P and P' seen as vertices in the same apartment in Σ can be connected by a finite gallery of chambers in the same apartment. This gallery is represented by a sequence of minimal F-parabolic subgroups $B_1, B_2, ..., B_k$, with $B_1 < P$ and $B_k < P'$. For each $i = 1, 2, ..., k-1$ there exists $P_i \in \mathcal{P}$ such that both B_i and B_{i+1} are contained in P_i. In the spherical building Σ the group P_i represents a vertex of the panel that the chamber B_i and the chamber B_{i+1} have in common. Thus one obtains a sequence of maximal parabolics in $\mathcal{P}$, $P_0 = P, P_1, P_2, ..., P_{k-1}, P_k = P'$. For each $i = 0, 1, 2, ..., k-1$, the intersection of the respective unipotent radicals U_{P_i} and $U_{P_{i+1}}$ of P_i and P_{i+1} contains the center $U_{i,i+1}$ of the unipotent radical of B_i. The group $U_{i,i+1}$ can be written as U_α with α the maximal positive root in the basis corresponding to the chamber B_i, in particular it is defined over F. Hence $\Gamma_{P_i} \cap \Gamma_{P_{i+1}}$ contains $U_{i,i+1} \cap \Gamma$, which is a lattice in $U_{i,i+1}$. We conclude that Γ_P and $\Gamma_{P'}$ are thickly connected by the sequence $\Gamma_{P_0} = \Gamma_P$, Γ_{P_1} , Γ_{P_2} , ..., $\Gamma_{P_k} = \Gamma_{P'}$. □

Question 13.9. *Are non-uniform higher rank lattices unconstricted?*

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