CHIRAL DE RHAM COMPLEX

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Introduction

0.1. The aim of this note is to define certain sheaves of vertex algebras on smooth manifolds. In this note, "vertex algebra" will have the same meaning as in Kac's book [K]. Recall that these algebras are by definition $\mathbb{Z}/(2)$-graded. "Smooth manifold" will mean a smooth scheme of finite type over $\mathbb{C}$.

For each smooth manifold X, we construct a sheaf Ω_X^{ch}, called the **chiral de Rham complex** of X. It is a sheaf of vertex algebras in the Zarisky topology, i.e. for each open $U \subset X$, $\Gamma(U; \Omega_X^{ch})$ is a vertex algebra, and the restriction maps are morphisms of vertex algebras. It comes equipped with a $\mathbb{Z}$-grading by *fermionic charge*, and the *chiral de Rham differential* d_{DR}^{ch}, which is an endomorphism of degree 1 such that $(d_{DR}^{ch})^2 = 0$. One has a canonical embedding of the usual de Rham complex

$$(\Omega_X, d_{DR}) \hookrightarrow (\Omega_X^{ch}, d_{DR}^{ch}) \tag{0.1}$$

The sheaf Ω_X^{ch} has also another $\mathbb{Z}$-grading, by *conformal weight*, and the subsheaf Ω_X coincides with its conformal weight zero component. The wedge multiplication on Ω_X may be restored from the operator product in Ω_X^{ch}, see 1.3. The map (0.1) is a quasiisomorphism, cf. Theorem 4.4. Each component of Ω_X^{ch} of fixed conformal weight admits a canonical finite filtration whose graded factors are symmetric powers of the tangent bundle $\mathcal{T}_X$, of the bundle of 1-forms Ω_X^1, and tensor products thereof.

The sheaf Ω_X^{ch} has the formal structure of the De Rham complex of a manifold with S^1-action. One may want to regard Ω_X^{ch} as a "semiinfinite de Rham complex of the space of maps from the punctured formal disk to X".

Similar sheaves exist in complex-analytic and C^∞ settings.

If X is Calabi-Yau, then the sheaf Ω_X^{ch} has a structure of a *topological vertex algebra* (i.e. it admits $N = 2$ supersymmetry, cf. 2.1), cf. 4.5. (For an arbitrary X, the obstruction to the existence of this structure is expressible in terms of the first Chern class of $\mathcal{T}_X$, cf. Theorem 4.2.)

One may hope that the vertex algebra $R\Gamma(X; \Omega_X^{ch})$ defines the conformal field theory which is Witten's "A-model" associated with X.

0.2. One can also try to define a purely even sheaf $\mathcal{O}_X^{ch}$ of vertex algebras, which could be called **chiral structure sheaf**. Here the situation is more subtle than in the case of Ω_X^{ch}, where "fermions cancel the anomaly". On can define this sheaf for curves, cf. 5.5. If $\dim(X) > 1$, then there exists a non-trivial obstruction of cohomological nature to the construction of $\mathcal{O}_X^{ch}$. This obstruction can be expressed in terms of a certain homotopy Lie algebra, cf. §5 A.

However, one can define $\mathcal{O}_X^{ch}$ for the flag manifolds $X = G/B$ (G being a simple algebraic group and B a Borel subgroup), cf. §5 B, C. This sheaf admits a structure of a $\widehat{\mathfrak{g}}$-module at the critical level (here $\mathfrak{g} = Lie(G)$ and $\widehat{\mathfrak{g}}$ is the corresponding affine Lie algebra). The space of global sections $\Gamma(X; \mathcal{O}_X^{ch})$ is the irreducible vacuum $\widehat{\mathfrak{g}}$-module for $\mathfrak{g} = sl(2)$; conjecturally, it is also true for any $\mathfrak{g}$, cf. 5.13. The sheaf $\mathcal{O}_X^{ch}$ may be regarded as localization of Feigin-Frenkel bosonization. More generally, one may hope that "chiral $\mathcal{D}$-modules on X" provide localization of $\widehat{\mathfrak{g}}$-modules on the critical level.

0.3. A few words about the plan of the note. In §1 we recall the basic definitions, to fix terminology and notations. In §2, some "free field representation" results are described. No doubt, they are all well known (although for some of them we do not know the precise reference). They are a particular case of the construction of Ω_X^{ch}, given in §3. In §4, we discuss the topological structure, and in §5 the chiral structure sheaf.

0.4. The idea of this note arose from reading the papers [LVW] and [LZ]. We have learned the first example of "localization along the target space" from B. Feigin, to whom goes our deep gratitude.

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§1. Recollections on vertex algebras

For more details on what follows, see [K]. For a coordinate-free exposition, see [BD1].

1.0. For a vector space V, $V[[z, z^{-1}]]$ will denote the space of all formal sums $\sum_{i \in \mathbb{Z}} a_i z^i$, $a_i \in V$. $V((z))$ will denote the subspace of Laurent power series, i.e. the sums as above, with $a_i = 0$ for $i << \infty$. We denote by ∂_z the operator of differentiation by z acting on $V[[z, z^{-1}]]$. We set $\partial_z^{(k)} := (\partial_z)^k / k!$, for an integer $k \geq 0$. For $a(z) \in V[[z, z^{-1}]]$, we will also use the notation $f(z)'$ for $\partial_z f(z)$, and $\int a(z)$ for the coefficient at z^{-1} (i.e. the residue).

V^* will denote the dual space. In the sequel, we will omit the prefix "super" in the words like "superalgebra", etc. For example, "a Lie algebra" will mean "a Lie superalgebra".

1.1. Let us define a **vertex algebra** following [K], 4.1. Thus, a vertex algebra is the data $(V, Y, L_{-1}, \mathbf{1})$. Here V is a $\mathbb{Z}/(2)$-graded vector space $V = V^{ev} \oplus V^{odd}$. The $\mathbb{Z}/(2)$-grading is called *parity*; for an element $a \in V$, its parity will be denoted by $\tilde{a}$. The space of endomorphisms $\mathrm{End}(V)$ inherits the $\mathbb{Z}/(2)$-grading from V.

$\mathbf{1}$ is an even element of V, called *vacuum*. Y is an even linear mapping

$$Y: \; V \longrightarrow \mathrm{End}(V)[[z, z^{-1}]] \tag{1.1}$$

For $a \in V$, the power series $Y(a)$ will be denoted $a(z)$, and called the *field* corresponding to a. The coefficients of the series $a(z)$ are called *Fourier modes*.

L_{-1} is an even endomorphism of V.

The following axioms must be satisfied.

Vacuum Axiom. $L_{-1}(\mathbf{1}) = 0; \mathbf{1}(z) = \mathrm{id}$ (the constant power series). For all $a \in V$, $a(z)(\mathbf{1})\big|_{z=0} = a$ (in particular, the components of $a(z)$ at the negative powers of z act as zero on the vacuum).

Translation Invariance Axiom. For all $a \in V$, $[L_{-1}, a(z)] = \partial_z a(z)$

Locality Axiom. For all $a, b \in V$, $(z-w)^N [a(z), b(w)] = 0$ for $N >> 0$.

The meaning of this equality is explained in [K].

Given two fields $a(z), b(z)$, their *operator product expansion* is defined as in [K], (2.3.7a)

$$a(z)b(w) = \sum_{j=1}^{N} \frac{c^j(w)}{(z-w)^j} + :a(z)b(w):$$

where

$$:a(z)b(w): := a(z)_+ b(w) + (-1)^{\tilde{a}\tilde{b}} b(w) a(z)_- \tag{1.2}$$

Here

$$a(z)_- = \sum_{n \geq 0} a_{(n)} z^{-n-1}; \quad a(z)_+ = \sum_{n<0} a_{(n)} z^{-n-1}$$

for $a(z) = \sum a_{(n)} z^{-n-1}$, cf. [K], (2.3.5), (2.3.3).

A *morphism* of vertex algebras $(V, \mathbf{1}, L_{-1}, Y) \longrightarrow (V', \mathbf{1}', L'_{-1}, Y')$ is an even linear map $f: V \longrightarrow V'$ taking $\mathbf{1}$ to $\mathbf{1}'$, such that $f \circ L_{-1} = L'_{-1} \circ f$, and for each $a \in V$, if a_n is a Fourier mode of the field $a(z)$, we have $f(a)_n \circ f = f \circ a_n$.

1.2. A **conformal vertex algebra** (cf. [K], 4.10) is a vertex algebra $(V, Y, L_{-1}, \mathbf{1})$, together with an even element $L \in V$ such that if we write the field $L(z)$ as

$$L(z) = \sum_n L_n z^{-n-2},$$

the endomorphisms L_n satisfy the Virasoro commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3 - n}{12} \cdot c \cdot \delta_{n,-m} \tag{1.3}$$

Here c is a complex number, to be called the *Virasoro central charge* of V (in [K] it is called the rank).

The component L_{-1} of the series (1.2) must coincide with the endomorphism L_{-1} from the definition of a vertex algebra.

The endomorphism L_0 must be diagonalizable, with integer eigenvalues. Set $V^{(n)} = \{a \in V |\ L_0(a) = na\}$. We must have $V = \oplus_{n \in \mathbb{Z}} V^{(n)}$. For $a \in V^{(n)}$, the number n is called the **conformal weight** of a and denoted $|a|$. This $\mathbb{Z}$-grading induces a $\mathbb{Z}$-grading on the space $\mathrm{End}(V)$, also to be called the conformal weight. By definition, an endomorphism has conformal weight n if it maps $V^{(i)}$ to $V^{(i+n)}$ for all i.

The conformal weight grading on V should be compatible with the parity grading, i.e. both gradings should come from a $\mathbb{Z} \times \mathbb{Z}/(2)$-bigrading.

We require that if a has conformal weight n, then the field $a(z)$ has the form

$$a(z) = \sum_i a_i z^{-i-n}, \tag{1.4}$$

with a_i having the conformal weight $-i$.

We have the important **Borcherds formula**. If a, b are elements in a conformal vertex algebra, with $|a| = n$ then

$$a(z)b(w) = \sum_i \frac{a_i(b)(w)}{(z-w)^{i+n}} + :a(z)b(w): \tag{1.5}$$

cf. [K], Theorem 4.6.

The Virasoro commutation relations (1.3) are equivalent to the operator product

$$L(z)L(w) = \frac{c}{2(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{L(w)'}{z-w} \tag{1.6}$$

(we omit from now on the regular part).

1.3. Let V be a conformal vertex algebra, such that $V^{(n)} = 0$ for $n < 0$. For $a \in V^{(n)},\ b \in V^{(m)}$, consider the operator product

$$a(z)b(w) = \sum_i \frac{a_i(b)(w)}{(z-w)^{i+n}} \tag{1.7}$$

We have $|a_i(b)| = m - i$. Therefore, $a_m(b)z^{-n-m}$ is the most singular term in (1.5).

If $n = m = 0$ then $a(z)b(w)$ is non-singular at $z = w$. Define the multiplication on the space $V^{(0)}$ by the rule

$$a \cdot b = \left(a(z)b(w)\big|_{z=w} \mathbf{1} \right)\Big|_{z=0} \tag{1.8}$$

This endows $V^{(0)}$ with a structure of an associative and commutative algebra with unity $\mathbf{1}$.

1.4. Heisenberg algebra. In this Subsection, we will define a conformal vertex algebra, to be called **Heisenberg vertex algebra**.

Fix an integer $N > 0$. Let H_N be the Lie algebra which as a vector space has the base $a^i_n,\ b^i_n,\ i = 1, \ldots, N;\ n \in \mathbb{Z}$, and C, all these elements being even, with the brackets

$$[a^j_m, b^i_n] = \delta_{ij}\delta_{m,-n}C, \tag{1.9}$$

all other brackets being zero.

Our vertex algebra, to be denoted V_N, as a vector space is the **vacuum representation** of H_N. As an H_N-module, it is generated by one vector $\mathbf{1}$, subject to the relations

$$b^i_n\mathbf{1} = 0 \ \text{ if } n > 0;\ a^i_m\mathbf{1} = 0 \ \text{ if } m \geq 0;\ C\mathbf{1} = \mathbf{1} \tag{1.10}$$

The mapping

$$P(b^i_n, a^j_m) \mapsto P(b^i_n, a^j_m) \cdot \mathbf{1}$$

identifies V_N with the ring of commuting polynomials on the variables $b^i_n,\ a^j_m,\ n \leq 0,\ m < 0,\ i,j = 1, \ldots, N$. We will use this identification below.

Let us define the structure of a vertex algebra on V_N. The $\mathbb{Z}/(2)$ grading is trivial: everything is even. The vacuum vector is $\mathbf{1}$.

The fields corresponding to the elements of V_N are defined by induction on the degree of a monomial. First we define the fields for the degree one monomials by setting

$$b^i_0(z) = \sum_{n\in\mathbb{Z}} b^i_n z^{-n};\ a^j_{-1}(z) = \sum_{n\in\mathbb{Z}} a^j_n z^{-n-1} \tag{1.11}$$

Here $b^i_n,\ a^j_n$ in the right hand side are regarded as operating on V_N by multiplication.

We set

$$b^i_{-n}(z) = \partial^{(n)}_z b^i_0(z),\quad a^j_{-n-1}(z) = \partial^{(n)}_z a^i_{-1}(z) \tag{1.12}$$

for $n > 0$.

The fields corresponding to the monomials of degree > 1 are defined by induction, using the *normal ordering*. Let us call the operators $b^i_n,\ n > 0$, and $a^j_n,\ n \geq 0$, acting on V_N, *annihilation operators*.

For $x = b^i_n$ or $a^j_n,\ n \in \mathbb{Z}$, and $b \in \mathrm{End}(V_N)$, the **normal ordered product** $:xb:$ is given by

$$:xb: := \begin{cases} bx & \text{if } x \text{ is an annihilation operator} \\ xb & \text{otherwise} \end{cases} \tag{1.13}$$

Define by induction

$$: x_1 \cdot \ldots \cdot x_k :=: x_1 \cdot (: x_2 \cdot \ldots \cdot x_k :) :, \tag{1.14}$$

for $x_p = b^i_n$ or a^j_n, $p = 1, \ldots, k$.

Given two series $x(z) = \sum_{n\in\mathbb{Z}} x_n z^{-n+p}$ and $y(z) = \sum_{n\in\mathbb{Z}} y_n z^{-n+q}$, with x_n as above, we set

$$: x(z)y(w) := \sum_{n,m\in\mathbb{Z}} : x_n y_m :\ z^{-n+p} w^{-m+q} \tag{1.15}$$

For any finite sequence $x_1, \ldots, x_p$, where each x_j is equal to one of a^i_n or b^i_n, we define the series $: x_1(z)x_2(z) \cdot \ldots \cdot x_p(z) :\in \mathrm{End}(V_N)[[z, z^{-1}]]$ by induction, as in (1.14). This expression does not depend on the order of x_i's.

Given a monomial $x_1 \cdot \ldots \cdot x_p \mathbf{1} \in V_N$, with x_i as above, we define the corresponding field by

$$x_1 \cdot \ldots \cdot x_p(z) =: x_1(z) \cdot \ldots \cdot x_p(z) : \tag{1.16}$$

Since every element of V_N is a finite linear combination of monomials as above, this completes the definition of the mapping (1.1).

We will use the shorthand notations

$$b^i(z) = b^i_0(z);\ a^j(z) = a^j_{-1}(z) \tag{1.17}$$

The operator products of these basic fields are

$$a^j(z)b^i(w) = \frac{\delta_{ij}}{z-w} + (\text{regular}) \tag{1.18a}$$

$$b^i(z)b^j(w) = (\text{regular});\ a^i(z)a^j(w) = (\text{regular}) \tag{1.18b}$$

where "(regular)" means the part regular at $z = w$. These operator products are equivalent to the commutation relations (1.9).

Other operator products are computed by differentiation of (1.18), and using Wick theorem, cf. [K], 3.3.

One can say that the vertex algebra V_N is generated by the even fields $b^i(z),\ a^j(z)$, of conformal weights 0 and 1 respectively, subject to the relations (1.18).

The Virasoro field is given by

$$L(z) = \sum_{i=1}^{N} :b^i(z)' \cdot a^i(z): \tag{1.19}$$

The central charge is equal to $2N$. Let us check this. Assume for simplicity that $N = 1$, and let us omit the index 1 at the fields a, b. Thus, we have

$$L(z) =: b(z)'a(z):$$

Let us compute the operator product $L(z)L(w)$ using the Wick theorem. We have

$$:b(z)'a(z)::b(w)'a(w): = [b(z)'a(w)][a(z)b(w)']+$$

$$+[b(z)'a(w)]:a(z)b(w)': +[a(z)b(w)']:b(z)'a(w): =$$

(we have $b(z)'a(w) = a(z)b(w)' = -1/(z-w)^2$)

$$= \frac{1}{(z-w)^4} - \frac{2:b(w)'a(w):}{(z-w)^2} - \frac{:b(w)''a(w): + :b(w)'a(w)':}{z-w}$$

Hence,

$$L(z)L(w) = \frac{1}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{L(w)'}{z-w}$$

which is (1.6) with $c = 2$.

1.5. Clifford algebra. Let Cl_N be the Lie algebra which as a vector space has the base $\phi^i_n,\ \psi^i_n,\ i = 1, \ldots, N,\ n \in \mathbb{Z}$, and C, all these elements being *odd*, with the brackets

$$[\phi^j_m, \psi^i_n] = \delta_{ij}\delta_{m,-n} \cdot C \tag{1.20}$$

Clifford vertex algebra Λ_N is defined as in the previous subsection, starting with the odd fields

$$\psi^i(z) = \sum_{n\in\mathbb{Z}} \psi^i_n z^{-n-1};\ \phi^j(z) = \sum_{n\in\mathbb{Z}} \phi^j_n z^{-n} \tag{1.21}$$

and repeating the definitions of *loc. cit.*, with a (resp. b) replaced by ψ (resp. ϕ). One must put the obvious signs in the definition of the normal ordering. Thus, Λ_N is generated by the odd fields $\phi^i(z), \psi^i(z)$, subject to the relations

$$\phi^i(z)\psi^j(w) = \frac{\delta_{ij}}{z-w} + \text{regular}; \tag{1.22a}$$

$$\phi^i(z)\phi^j(w) = \text{regular}; \ \ \psi^i(z)\psi^j(w) = \text{regular} \tag{1.22b}$$

The Virasoro field is given by

$$L(z) = \sum_{i=1}^{N} \ :\phi^i(z)' \cdot \psi^i(z): \tag{1.23}$$

The central charge is equal to $-2N$.

1.6. If $\mathcal{A}$ and $\mathcal{B}$ are vertex algebras then their tensor product $\mathcal{A}\otimes\mathcal{B}$ admits a canonical structure of a vertex algebra, cf. [K], 4.3. The Virasoro element is given by

$$L_{\mathcal{A}\otimes\mathcal{B}} = L_{\mathcal{A}}\otimes 1 + 1\otimes L_{\mathcal{B}} \tag{1.24}$$

We will use in the sequel the tensor product of the Heisenberg and Clifford vertex algebras $\Omega_N = V_N \otimes \Lambda_N$. Its Virasoro central charge is equal to 0.

1.7. Let $\mathcal{A}$ be a vertex algebra. A linear map $f:\ \mathcal{A} \longrightarrow \mathcal{A}$ is called *derivation* of $\mathcal{A}$ if for any $a\in\mathcal{A}$,

$$f(a)(z) = [f, a(z)] \tag{1.25}$$

Note that an invertible map $g:\ \mathcal{A}\longrightarrow\mathcal{A}$ is an automorphism of $\mathcal{A}$ iff

$$g(a)(z) = ga(z)g^{-1} \tag{1.26}$$

and (1.25) is the infinitesimal version of (1.26).

It follows from Borcherds formula that for every $a\in\mathcal{A}$, the Fourier mode $\int a(z)$ is a derivation, cf. Lemma 1.3 from [LZ]. Consequently, if the endomorphism $\exp(\int a(z))$ is well-defined, it is an automorphism of $\mathcal{A}$.

§2. De Rham chiral algebra of an affine space

2.1. A **topological vertex algebra of rank d** is a conformal vertex algebra $\mathcal{A}$ of Virasoro central charge 0, equipped with an even element J of conformal weight 1, and two odd elements, Q, of conformal weight 1, and G, of conformal weight 2.

The following operator products must hold

$$L(z)L(w) = \frac{2L(w)}{(z-w)^2} + \frac{L(w)'}{z-w} \tag{2.1a}$$

$$J(z)J(w) = \frac{d}{(z-w)^2}; \ L(z)J(w) = -\frac{1}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{J(w)'}{z-w} \tag{2.1b}$$

$$G(z)G(w)=0;\ L(z)G(w)=\frac{2G(w)}{(z-w)^2}+\frac{G(w)'}{z-w};\ J(z)G(w)=-\frac{G(w)}{z-w} \tag{2.1c}$$

$$Q(z)Q(w)=0;\ L(z)Q(w)=\frac{Q(w)}{(z-w)^2}+\frac{Q(w)'}{z-w};\ J(z)Q(w)=\frac{Q(w)}{z-w} \tag{2.1d}$$

$$Q(z)G(w)=-\frac{d}{(z-w)^3}+\frac{J(w)}{(z-w)^2}-\frac{L(w)}{z-w} \tag{2.1e}$$

Note the following consequence of (2.1e),

$$[Q_0, G(z)]=-L(z) \tag{2.2}$$

2.2. In this subsection, we will intorduce a structure of a topological vertex algebra of rank N on the vertex algebra Ω_N from 1.6. This topological vertex algebra will be called **de Rham chiral algebra of the affine space** $\mathbb{A}^N$.

Recall that the Virasoro element is given by

$$L=\sum_{i=1}^{N}\left(b^i_{-1}a^i_{-1}+\phi^i_{-1}\psi^i_{-1}\right) \tag{2.3a}$$

Define the elements J, Q, G by

$$J=\sum_{i=1}^{N}\ \phi^i_0\psi^i_{-1};\ Q=\sum_{i=1}^{N}\ a^i_{-1}\phi^i_0;\ G=\sum_{i=1}^{N}\ \psi^i_{-1}b^i_{-1} \tag{2.3b}$$

The corresponding fields are

$$L(z)=\sum\left(:b^i(z)'a^i(z):+:\phi^i(z)'\psi^i(z):\right) \tag{2.4a}$$

and

$$J(z)=\sum\ :\phi^i(z)\psi^i(z):,\ Q(z)=\sum\ :a^i(z)\phi^i(z):,\ G(z)=\sum\ :\psi^i(z)b^i(z)': \tag{2.4b}$$

The relations (2.1) are readily checked using the Wick theorem.

2.3. Let us define the **fermionic charge** operator acting on Ω_N, by

$$F:=J_0=\sum_i\sum_n\ :\phi^i_n\psi^i_{-n}: \tag{2.5}$$

We have

$$F\mathbf{1}=0 \tag{2.6}$$

and

$$[F,\phi^i_n]=\phi^i_n;\ [F,\psi^i_n]=-\psi^i_n;\ [F,a^i_n]=[F,b^i_n]=0 \tag{2.7}$$

We set

$$\Omega^p_N=\{\omega\in\Omega_N|\ F\omega=p\omega\} \tag{2.8}$$

Obviously,

$$\Omega_N = \oplus_{p\in\mathbb{Z}}\ \Omega^p_N \tag{2.9}$$

We define an endomorphism d of the space Ω_N by

$$d := -Q_0 = -\sum_{i,n}\ : a^i_n \phi^i_{-n} : \tag{2.10}$$

(we could omit the normal ordering in the last formula since the letters a and ϕ commute anyway). We have $d^2 = 0$. Indeed, by Wick theorem $Q(z)Q(w) =$ regular, hence all Fourier modes of $Q(z)$ (anti)commute. The map d is called **chiral de Rham differential**.

The map d increases the fermionic charge by 1, by (2.7). Thus, the space Ω_N equipped with the fermionic charge grading and the differential d, becomes a complex (infinite in both directions), called **chiral de Rham complex** of $\mathbb{A}^N$.

Consider the usual algebraic de Rham complex $\Omega(\mathbb{A}^N) = \oplus^N_{p=0}\ \Omega^p(\mathbb{A}^N)$ of the affine space $\mathbb{A}^N$. We identify the coordinate functions with the letters $b^1_0, \ldots, b^N_0$, and their differentials with the fermionic variables $\phi^1_0, \ldots, \phi^N_0$.

Thus, we identify the commutative dg algebras

$$\Omega(\mathbb{A}^N) = \mathbb{C}[b^1_0, \ldots, b^N_0] \otimes \Lambda(\phi^1_0, \ldots, \phi^N_0), \tag{2.12}$$

the second factor being the exterior algebra. The grading is defined by assigning to the letters b^i_0 (resp. ϕ^j_0) the degree 0 (resp. 1).

The usual de Rham differential is given by

$$d_{DR} = \sum_i\ a^i_0 \phi^i_0, \tag{2.13}$$

as follows from the relations (1.9).

2.4. Theorem. *The obvious embedding of complexes*

$$i:\ (\Omega(\mathbb{A}^N), d_{DR}) \longrightarrow (\Omega_N, d) \tag{2.14}$$

is compatible with the differentials, and is a quasiisomorphism.

We identify the space Ω_N with the space of polynomials in the letters b^i_n, ϕ^i_n ($n \le 0$) and a^i_n, ψ^i_n ($n < 0$). One sees that on the subspace $\mathbb{C}[b^i_0, \phi^i_0]$, all the summands $a^i_n \phi^i_{-n}$ with $n \neq 0$ act trivially. It follows that the map i is compatible with the differentials.

To prove that it is a quasiisomorphism, let us split d in two commuting summands $d = d_+ + d_-$ where

$$d_+ = \sum_i \sum_{n\ge 0}\ a^i_n \phi^i_{-n},\ d_- = \sum_i \sum_{n<0}\ a^i_n \phi^i_{-n} \tag{2.15}$$

We think of the space Ω_N as of the tensor product $\mathbb{C}[a^i_n, \psi^i_n] \otimes \mathbb{C}[b^j_m, \phi^j_m]$. The differential d_- acts trivially on the second factor, and on the first one it is the Koszul differential. So, the cohomology of d_- is $\mathbb{C}[b^j_m, \phi^j_m]_{m\le 0}$.

Now, we have to compute the cohomology of this space with respect to d_+. For this purpose, split d_+ once again as $d_+ = d_{DR} + d_+^{\geq 0}$, and our space as $\mathbb{C}[b_0^j, \phi_0^j] \otimes \mathbb{C}[b_m^j, \phi_m^j]_{m<0}$. The differential $d_+^{\geq 0}$ acts trivially on the first factor, and on the second one, it is the de Rham differential. Hence (by Poincaré lemma), taking the cohomology of $d_+^{\geq 0}$ kills all non-zero modes, and we are left precisely with the usual de Rham complex.

Alternatively, it follows from (2.2) that

$$[G_0, d] = L_0 \tag{2.16}$$

The operator G_0 commutes with L_0, and it follows from (2.16) that it gives a homotopy to 0 for the operator d on all the subcomplexes of non-zero conformal weight. Therefore, all cohomology lives in the conformal weight zero subspace. $\triangle$

2.5. The vertex algebra Ω_N satisfies the assumptions of 1.3. The subspace $\Omega(\mathbb{A}^N)$ coincides with the conformal weight zero component of it. If we apply the definition of 1.3, we get the structure of a commutative algebra on $\Omega(\mathbb{A}^N)$ which is given by the usual wedge product of differential forms.

§3. Localization

3.1. Consider the Heisenberg vertex algebra V_N defined in 1.4. As in *loc. cit.*, we will identify the space V_N with the space of polynomials $\mathbb{C}[b^i_{-n}, a^j_{-m}]_{n\geq 0,\ m>0}$. To simplify the notations below, let us denote the zero mode variables b^i_0 by b^i.

Let A_N denote the algebra of polynomials $\mathbb{C}[b^1, \dots, b^N]$. The space A_N is identified with the subspace $V_N^{(0)} \subset V_N$ of conformal weight zero. The space V_N has an obvious structure of an A_N-module. Let $\widehat{A}_N$ denote the algebra of formal power series $\mathbb{C}[[b^1, \dots, b_N]]$. Set

$$\widehat{V}_N = \widehat{A}_N \otimes_{A_N} V_N \tag{3.1}$$

We are going to introduce a structure of a conformal vertex algebra on the space $\widehat{V}_N$. Let us define the map (1.1).

Let $f(b^1, \dots, b^N)$ be a power series from $\widehat{A}_N$. We claim that the expression $f(b^1(z), \dots, b^N(z))$ makes sense as an element of $\mathrm{End}(\widehat{V}_N)[[z, z^{-1}]]$. (We are grateful to Boris Feigin who has shown us a particular case of the following construction.) Let us express the power series $b^i(z)$ as

$$b^i(z) = b^i + \Delta b^i(z) \tag{3.2}$$

Thus,

$$\Delta b^i(z) = \sum_{n>0} (b^i_n z^{-n} + b^i_{-n} z^n) \tag{3.3}$$

Let us define $f(b^1(z), \dots, b^N(z))$ by the Taylor formula

$$f(b^1(z), \dots, b^N(z)) = \sum \Delta b^1(z)^{i_1} \cdot \ldots \cdot \Delta b^N(z)^{i_N} \partial^{(i_1, \dots, i_N)} f(b^1, \dots, b^N) \tag{3.4}$$

where

$$\partial^{(i_1,\dots,i_N)} = \frac{\partial_{b^1}^{i_1}}{i_1!} \cdot \ldots \cdot \frac{\partial_{b^N}^{i_N}}{i_N!} \tag{3.5}$$

We will show that the series (3.4) gives a well-defined element of $\mathrm{End}(\widehat{V}_N)[[z,z^{-1}]]$. Let us write

$$\Delta b^1(z)^{i_1} \cdot \ldots \cdot \Delta b^N(z)^{i_N} = \sum_k c_k^{i_1,\dots,i_N} z^{-k}$$

The coefficient $c_k^{i_1,\dots,i_N}$ is an infinite sum of the monomials

$$b_{k_1}^{j_1} \cdot \ldots \cdot b_{k_I}^{j_I}, \tag{3.6}$$

with $I = i_1 + \ldots + i_N$, $k_1 + \ldots + k_I = k$. Pick an element $v \in \widehat{V}^N$. There exists M such that $b_{l_1}^{j_1} \cdot \ldots \cdot b_{l_N}^{j_N} v = 0$ if $\sum l_i > M$.

We have

$$k = \sum k_i = {}^+\sum |k_i| - {}^-\sum |k_i|$$

where ${}^+\sum$ (resp. ${}^-\sum$) denotes the sum of all positive (resp. negative) summands. If $b_{k_1}^{j_1} \cdot \ldots \cdot b_{k_I}^{j_I} v \neq 0$, then

$${}^+\sum |k_i| \le M \tag{3.7a}$$

On the other hand,

$${}^-\sum |k_i| = {}^+\sum |k_i| - k \le M - k \tag{3.7b}$$

There exists only a finite number of tuples $(k_1, \dots, k_I)$ satisfying (3.7a) and (3.7b). Therefore, $c_k^{i_1,\dots,i_N}$ are well-defined endomorphisms of $\widehat{V}_N$.

We have

$$f(b^1(z), \dots, b^N(z)) = \sum_k \Bigg(\sum_{i_1,\dots,i_N} c_k^{i_1,\dots,i_N} \partial^{(i_1,\dots,i_N)} f(b^1, \dots, b^N) \Bigg) z^{-k} \tag{3.8}$$

All numbers k_i are non-zero. Let I_+ (resp., I_-) be the number of positive (resp. negative k_i's). We have $I_- \le {}^-\sum k_i \le M - k$, hence $M \ge {}^+\sum k_i \ge I_+ = I - I_- \ge I - M + k$. Therefore,

$$I \le 2M - k \tag{3.9}$$

Therefore, when we apply the series (3.8) to the element v, only a finite number of terms in the sum over $(i_1, \dots, i_N)$ survives. Therefore, the series (3.8) is a well-defined element of $\mathrm{End}(\widehat{V}_N)[[z,z^{-1}]]$.

Every element of $\widehat{V}_N$ is a finite sum of products $g(a)f(b)$ where $g(a)$ is a polynomial in the letters a and $f(b)$ is a power series as above. We have already defined $f(b)(z)$. The definition of $g(a)(z)$ is the same as in the case of V_N. We define $\big(g(a)f(b)\big)(z)$ by

$$\big(g(a)f(b)\big)(z) =: g(a)(z)f(b)(z) : \tag{3.10}$$

where the normal ordering is defined in (1.2). This completes the definition of the mapping

$$Y:\ \widehat{V}_N \longrightarrow \mathrm{End}(\widehat{V}_N)[[z,z^{-1}]] \tag{3.11}$$

The following version of the definition of the map (3.11) is helpful in practice. Every element $c \in \widehat{V}_N$ is a limit of the elements of $c_i \in V_N$ (in the obvious topology). We can regard the fields $c_i(z)$ as the elements of $\mathrm{End}(\widehat{V}_N)[[z,z^{-1}]]$. The field $c(z)$ is the limit of the fields $c_i(z)$.

We define the vacuum and Virasoro element $\mathbf{1}, L \in \widehat{V}_N$ as the image of the corresponding element of V_N under the natural map $V_N \longrightarrow \widehat{V}_N$.

3.2. Theorem. *The construction of* 3.1 *defines a structure of a conformal vertex algebra on the space* $\widehat{V}_N$.

This follows from [K], Theorem 4.5.

3.3. Let $A_N^{an} \subset \widehat{A}_N$ denote the subalgebra of power series, convergent in a neighbourhood of the origin. Set

$$V_N^{an} = A_N^{an} \otimes_{A_N} V_N \subset \widehat{V}_N \tag{3.12}$$

It is clear from the inspection of the Taylor formula (3.4) that for $f(a,b) \in V_N^{an}$, the Fourier modes of $f(a,b)(z)$ which belong to $\mathrm{End}(\widehat{V}_N)$, respect the subspace V_N^{an}. Therefore, the conformal vertex algebra structure on $\widehat{V}_N$ defined in 3.1 induces the structure of a conformal vertex algebra on V_N^{an}.

Let A_N^{sm} denote the algebra of germs of smooth (C^∞) functions. Set

$$V_N^{sm} = A_N^{sm} \otimes_{A_N} V_N \tag{3.13}$$

For every $f \in A_N^{sm}$, there exists a sequence $\{f_i\}$ in A_N, converging to f together with all derivatives, up to any given order (this follows from the Stone-Weierstrass theorem). We can regard $f_i(z)$ as elements of $\mathrm{End}(V_N^{sm})[[z,z^{-1}]]$. Note that these fields are defined using the Taylor formula (3.4). It follows that the limit of $\{f_i(z)\}$ exists in $\mathrm{End}(V_N^{sm})[[z,z^{-1}]]$. We take this limit as the definition of the field $f(z)$. This gives a conformal vertex algebra structure on V_N^{sm}.

3.4. Let $f \in A_N$ be a nonzero polynomial. Let $A_{N;f}$ denote the localization $A_N[f^{-1}]$. Set

$$V_{N;f} = A_{N;f} \otimes_{A_N} V_N \tag{3.14}$$

Consider the Taylor formula (3.4) applied to the function f^{-1}. We have evidently

$$\left[\partial^{(i_1,\dots,i_N)} f^{-1}\right](b^1,\dots,b^N) \in A_{N;f}$$

Therefore, the construction 3.1 provides a conformal vertex algebra structure on the space $V_{N;f}$.

Let X denote the affine space $\mathrm{Spec}(A_N)$. Let $\mathcal{O}_X^{ch}$ be the $\mathcal{O}_X$-quasicoherent sheaf corresponding to the A_N-module V_N. We have just defined the structure of a conformal vertex algebra on the spaces $V_{N;f} = \Gamma(U_f;\mathcal{O}_X^{ch})$ where $U_f = \mathrm{Spec}(A_{N;f})$.

If $U_f \subset U_g$ then the restriction map $V_{N;g} \longrightarrow V_{N;f}$ is a morphism of conformal vertex algebras.

If $U \subset X$ is an arbitrary open, we have $U = \bigcup U_f$, and

$$\Gamma(U; \mathcal{O}_X^{ch}) = \mathrm{Ker}\big(\prod V_{N;f} \rightrightarrows \prod V_{N;fg}\big)$$

Using this formula, we get a structure of a conformal vertex algebra on the space $\Gamma(U; \mathcal{O}_X^{ch})$. Therefore, $\mathcal{O}_X^{ch}$ gets a structure of a sheaf of conformal vertex algebras.

3.5. We can add fermions to the picture. Consider the spaces $\widehat{\Omega}_N = \widehat{A}_N \otimes_{A_N} \Omega_N$, $\Omega_N^{an} = A_N^{an} \otimes_{A_N} \Omega_N \subset \widehat{\Omega}_N$, $\Omega_N^{sm} = A_N^{sm} \otimes_{A_N} \Omega_N$. The construction 3.1 provides a structure of a topological vertex algebras on these spaces.

Let X be as in 3.4; let Ω_X^{ch} denote the $\mathcal{O}_X$-quasicoherent sheaf associated with the A_N-module Ω_N. The construction 3.4 provides a structure of a sheaf of topological vertex algebras of rank N on Ω_X^{ch}.

3.6. Now we want to study coordinate changes in our vertex algebras. Let X be the formal scheme $\mathrm{Spf}(\mathbb{C}[[b^1, \ldots, b^N]])$. Consider the formal $N|N$-dimensional superscheme $\tilde{X} = \Pi TX$ (here TX is the total space of the tangent bundle, Π is the parity change functor). Thus, $\tilde{X}$ has the same underlying space as X, and the structure sheaf of $\tilde{X}$ coincides with the de Rham complex of X. On $\tilde{X}$, we have N even coordinates $b^1, \ldots, b^N$ and odd ones $\phi^1 = db^1, \ldots, \phi^N = db^N$.

To this superscheme $\tilde{X}$, with the above coordinates, we have assigned a (super)vertex algebra $\widehat{\Omega}_N$, generated by the fields $b^i(z), a^i(z)$ (even ones) and $\phi^i(z), \psi^i(z)$ (odd ones). The fields $a^i(z)$ (resp. $\psi^i(z)$) correspond to the vector fields ∂_{b^i} (resp. ∂_{ϕ^i}) on $\tilde{X}$.

These fields satisfy the relations (cf. (1.18), (1.22))

$$a^i(w)b^j(w) = \frac{\delta_{ij}}{z-w} \tag{3.15a}$$

$$b^i(z)b^j(w) = (\text{regular});\ \ a^i(z)a^j(w) = (\text{regular}) \tag{3.15b}$$

$$\phi^i(z)\psi^j(w) = \frac{\delta_{ij}}{z-w} \tag{3.15c}$$

$$\phi^i(z)\phi^j(w) = (\text{regular});\ \ \psi^i(z)\psi^j(w) = (\text{regular}) \tag{3.15d}$$

$$b^i(z)\phi^j(w) = (\text{regular});\ \ b^i(z)\psi^j(w) = (\text{regular}) \tag{3.15e}$$

$$a^i(z)\phi^j(w) = (\text{regular});\ \ a^i(z)\psi^j(w) = (\text{regular}) \tag{3.15f}$$

Consider an invertible coordinate transformation on X,

$$\tilde{b}^i = g^i(b^1, \ldots, b^N);\ \ b^i = f^i(\tilde{b}^1, \ldots, \tilde{b}^N) \tag{3.16a}$$

where $g^i \in \mathbb{C}[[b^j]];\ f^i \in \mathbb{C}[[\tilde{b}^j]]$. It induces the transformation of the odd coordinates $\phi^i = db^i$,

$$\tilde{\phi}^i = \frac{\partial g^i}{\partial b^j}\phi^j;\ \ \phi^i = \frac{\partial f^i}{\partial \tilde{b}^j}\tilde{\phi}^j \tag{3.16b}$$

(the summation over the repeating indices is tacitly assumed). The vector fields transform as follows,

$$\partial_{\tilde{b}^i} = \frac{\partial f^j}{\partial \tilde{b}^i}(g(b))\partial_{b^j} + \frac{\partial^2 f^k}{\partial \tilde{b}^i \partial \tilde{b}^l}(g(b)) \cdot \frac{\partial g^l}{\partial b^r} \cdot \phi^r \partial_{\phi^k} \tag{3.16c}$$

and

$$\partial_{\tilde{\phi}^i} = \frac{\partial f^j}{\partial \tilde{b}^i}(g(b))\partial_{\phi^j} \tag{3.16d}$$

We want to lift the transformation (3.16a) to the algebra Ω_N. Define the tilded fields by

$$\tilde{b}^i(z) = g^i(b)(z) \tag{3.17a}$$

$$\tilde{\phi}^i(z) = \left(\frac{\partial g^i}{\partial b^j}\phi^j\right)(z) \tag{3.17b}$$

$$\tilde{a}^i(z) = \left(a^j \frac{\partial f^j}{\partial \tilde{b}^i}(g(b))\right)(z) + \left(\frac{\partial^2 f^k}{\partial \tilde{b}^i \partial \tilde{b}^l}(g(b))\frac{\partial g^l}{\partial b^r}\psi^k \phi^r\right)(z) \tag{3.17c}$$

$$\tilde{\psi}^i(z) = \left(\frac{\partial f^j}{\partial \tilde{b}^i}(g(b))\psi^j\right)(z) \tag{3.17d}$$

3.7. Theorem. *The fields $\tilde{b}^i(z), \tilde{a}^i(z), \tilde{\phi}^i(z)$ and $\tilde{\psi}^i(z)$ satisfy the relations* (3.15).

We will use the relations

$$h(b)(z)a^i(w) = \frac{\partial h/\partial b^i(w)}{z-w};\ a^i(z)h(b)(w) = \frac{\partial h/\partial b^i(w)}{z-w} \tag{3.18}$$

for each $h \in \widehat{A}_N$, which follow from (3.15a) and the Wick theorem. Let us check (3.15a) for the tilded fields. We have

$$\tilde{b}^i(z)\tilde{a}^j(w) = g^i(b)(z)a^k\frac{\partial f^k}{\partial \tilde{b}^j}(g(b))(w) +$$

$$+ g^i(b)(z)\frac{\partial^2 f^k}{\partial \tilde{b}^j \partial \tilde{b}^l}(g(b))\frac{\partial g^l}{\partial b^r}\psi^k\phi^r(w)$$

By the Wick theorem, the first summand is equal to

$$\frac{\partial g^i}{\partial b^k}\frac{\partial f^k}{\partial \tilde{b}^j}(g(b))(w) \cdot \frac{1}{z-w} = \frac{\delta_{ij}}{z-w},$$

by (3.18). The second summand is zero. Let us check (3.15b). The first identity is clear. We have

$$\tilde{a}^i(z)\tilde{a}^j(w) = a^k\frac{\partial f^k}{\partial \tilde{b}^i}(g(b))(z)a^n\frac{\partial f^n}{\partial \tilde{b}^j}(g(b))(w) + a^k\frac{\partial f^k}{\partial \tilde{b}^i}(g(b))(z)\frac{\partial^2 f^n}{\partial \tilde{b}^j \partial \tilde{b}^l}(g(b))\frac{\partial g^l}{\partial b^r}\psi^n\phi^r(w) +$$

$$+ \frac{\partial^2 f^k}{\partial \tilde{b}^i \partial \tilde{b}^l}(g(b))\frac{\partial g^l}{\partial b^r}\psi^k\phi^r(z)a^n\frac{\partial f^n}{\partial \tilde{b}^j}(g(b))(w) +$$

$$+\frac{\partial^2 f^k}{\partial\tilde{b}^i\partial\tilde{b}^l}(g(b))\frac{\partial g^l}{\partial b^r}\psi^k\phi^r(z)\frac{\partial^2 f^n}{\partial\tilde{b}^j\partial\tilde{b}^l}(g(b))\frac{\partial g^l}{\partial b^r}\psi^n\phi^r(w)$$

When we compute each term using the Wick theorem, there appear single and double pairings. The part corresponding to the single pairings coincides with the expression of the bracket $[\partial_{\tilde{b}^i}, \partial_{\tilde{b}^j}]$ in old coordinates b^p, so it vanishes. The "anomalous" part comes from the double pairings.

One double pairing appears in the first term and is equal to

$$-\frac{\partial}{\partial b^n}\left[\frac{\partial f^k}{\partial\tilde{b}^i}(g(b))\right](z)\frac{\partial}{\partial b^k}\left[\frac{\partial f^n}{\partial\tilde{b}^j}(g(b))\right](w)\cdot\frac{1}{(z-w)^2}$$

another one appears in the fourth term and equals

$$\delta_{km}\delta_{rn}\frac{\partial^2 f^k}{\partial\tilde{b}^i\partial\tilde{b}^l}(g(b))\frac{\partial g^l}{\partial b^r}(z)\frac{\partial^2 f^n}{\partial\tilde{b}^j\partial\tilde{b}^p}(g(b))\frac{\partial g^p}{\partial b^m}(w)\cdot\frac{1}{(z-w)^2}=$$

$$=\frac{\partial^2 f^k}{\partial\tilde{b}^i\partial\tilde{b}^l}(g(b))\frac{\partial g^l}{\partial b^n}(z)\frac{\partial^2 f^n}{\partial\tilde{b}^j\partial\tilde{b}^p}(g(b))\frac{\partial g^p}{\partial b^k}(w)\cdot\frac{1}{(z-w)^2}.$$

We see that these terms cancel out.

The remaining relations, (3.15c - f), contain only single pairings, and are easily checked. △

Thus, for each automorphism $g=(g^1,\dots,g^N)$ of $\mathbb{C}[[b^1,\dots,b^N]]$, (3.16a), the formulas (3.17) determine a morphism of vertex algebras

$$\tilde{g}:\ \widehat{\Omega}_N\longrightarrow\widehat{\Omega}_N \tag{3.19}$$

More precisely, each element c of $\widehat{V}_N$ is an (infinite) sum of finite products of $a^i_{-n}, b^j_{-m}, \psi^i_{-n}, \psi^j_{-m}$. We have $c=\big(c(z)\mathbf{1}\big)(0)$. By definition,

$$\tilde{g}(c)=\Big(\tilde{g}(c(z))\mathbf{1}\Big)(0) \tag{3.20}$$

Thus, we have to define the field $\tilde{g}(c(z))$ for each $c\in\widehat{V}_N$. If c is one of the generators $a^i_{-1}, b^i_0, \phi^i_0, \psi^i_{-1}$, we define $\tilde{g}(c(z))$ by formulas (3.17). We set

$$\tilde{g}(a^i_{-1-n}(z))=\partial_z^{(n)}\tilde{g}(a^i_{-1}(z)), \tag{3.21}$$

and the same with b,ϕ,ψ, cf. (1.12). Finally, if $c=c^1c^2\cdot\ldots\cdot c^p$ where each c^i is one of the letters a,b,ϕ or ψ, we set

$$\tilde{g}(c(z))=:\tilde{g}(c^1(z))\tilde{g}(c^2(z))\cdot\ldots\cdot\tilde{g}(c^p(z)): \tag{3.22}$$

where the normal ordered product of two factors is defined by (1.2), and if $p>2$, we use the inductive formula (1.14).

Equivalently, if $c^j=x^j_{k_j}$ where $x^j=a^i,b^i,\phi^i$ or ψ^i, we have

$$\tilde{g}(c^1\cdot\ldots\cdot c^p\mathbf{1})=\big[\tilde{g}(x^1(z))\big]_{k_1}\cdot\ldots\cdot\big[\tilde{g}(x^p(z))\big]_{k_p}\mathbf{1} \tag{3.23}$$

(Here we use the following notation. If $a(z)=\sum_i a_i z^{-i-n}$ is a field corresponding to an element of conformal weight n, we denoted the Fourier mode a_i by $a(z)_i$.)

Let G_N denote the group of automorphisms (3.16a).

3.8. Theorem. *The assignment $g \mapsto \tilde{g}$ defines the group homomorphism $G_N \longrightarrow Aut(\Omega_N)$.*

Let us consider two coordinate transformations, $'b^i = g_1^i(b)$, and $''b^i = g_2^i('b)$. Let f_j denote the transformation inverse to g_j. We have to check that

$$\widetilde{g_2 g_1} = \tilde{g}_2 \tilde{g}_1 \tag{3.24}$$

By Theorem 4.5 from [K], it suffices to check this equality on the generators. Let us begin with a^i. The element $''a^i_{-1}\mathbf{1}$ is expressed in the coordinates $'a$, etc., as follows

$$''a^i_{-1}\mathbf{1} = \left('a^j_{-1}\frac{\partial f_2^j}{\partial'' b^i}(g_2('b_0)) + \frac{\partial^2 f_2^k}{\partial'' b^i \partial'' b^l}(g_2('b_0))\frac{\partial g_2^l}{\partial' b^r}('b_0)\ '\psi^k_{-1}\ '\phi^r_0 \right)\mathbf{1}$$

Expressing it in the coordinates a, etc., we get the element

$$\tilde{g}_2\tilde{g}_1(a^i) =$$

$$= \left[a^p \frac{\partial f_1^p}{\partial' b^j}(g_1(b))(z) + \frac{\partial^2 f_1^p}{\partial' b^j \partial' b^q}(g_1(b))\frac{\partial g_1^q}{\partial b^s}\psi^p \phi^s(z) \right]_{-1} \frac{\partial f_2^j}{\partial'' b^i}(g_2 g_1(b))(z)_0 \mathbf{1} +$$

$$+ \frac{\partial^2 f_2^k}{\partial'' b^i \partial'' b^l}(g_2 g_1(b))\frac{\partial g_2^l}{\partial' b^r}(g_1(b))(z)_0 \left[\frac{\partial_1^p}{\partial' b^k}(g_1(b))\psi^p(z) \right]_{-1} \left[\frac{\partial g_1^r}{\partial b^q}\phi^q(z) \right]_0 \mathbf{1} \tag{3.25}$$

(we have used (3.23)). Now, the action of our group G_N on the classical de Rham complex is associative. It follows that the expression (3.25) is equal to $\widetilde{g_2 g_1}(a^i)$ plus two anomalous terms:

$$a_0^p \left[\frac{\partial f_1^p}{\partial' b^j}(g_1(b))(z) \right]_{-1} \frac{\partial f_2^j}{\partial'' b^i}(g_2 g_1(b_0))\mathbf{1} = - \left[\frac{\partial f_1^p}{\partial' b^j}(g_1(b))(z) \right]_{-1} \frac{\partial}{\partial b_0^k}\frac{\partial f_2^j}{\partial'' b^i}(g_2 g_1(b_0))\mathbf{1}$$

coming from the first summand, and

$$\frac{\partial^2 f_2^k}{\partial'' b^i \partial'' b^l}(g_2 g_1(b_0))\frac{\partial g_2^l}{\partial' b^r}(g_1(b_0)) \left[\frac{\partial f_1^p}{\partial' b^k}(g_1(b))(z) \right]_{-1} \frac{\partial g_1^r}{\partial b^q}(b_0)\psi_0^p \phi_0^q \mathbf{1} =$$

$$= \frac{\partial^2 f_2^k}{\partial'' b^i \partial'' b^l}(g_2 g_1(b_0))\frac{\partial g_2^l}{\partial' b^r}(g_1(b_0)) \left[\frac{\partial f_1^p}{\partial' b^k}(g_1(b))(z) \right]_{-1} \frac{\partial g_1^r}{\partial b^p}(b_0)\mathbf{1}$$

coming from the second one. One sees that these two terms cancel out, which proves (3.24) for a^i. For the generators b, ψ, and ϕ, the anomaly does not appear at all. $\triangle$

3.9. Theorems 3.7 and 3.8 allow one to define the sheaf of conformal vertex algebras Ω_X^{ch} for each smooth manifold X in an invariant way, by gluing the sheaves defined in 3.5. This can be done in each of the three settings: in algebraic, complex analytic or smooth one.

In the complex analytic situation, we have our sheaves of vertex algebras over the coordinate charts, and the formulas (3.17) allow to glue these sheaves in a sheaf over X.

In the algebraic situation, it is convenient to use the language of localization of $(\mathfrak{g}, K)$-modules, cf. [BS], 4.2, 4.4; [BFM], §3. Let K be a (pro)algebraic group, and $(\mathfrak{g}, K)$ be a Harish-Chandra pair. Thus, $\mathfrak{g}$ is a Lie algebra, and $\mathfrak{k} = Lie(K)$ is embedded in $\mathfrak{g}$. Let $\hat{X}$ be a $(\mathfrak{g}, K)$-scheme, and $\hat{X} \longrightarrow X$ be a K-torsor over a smooth scheme X. If M is a $(\mathfrak{g}, K)$-module, i.e. a $\mathfrak{g}$-module, with the $\mathfrak{k}$-action integrated, then one assigns to it the K-equivariant $\mathcal{D}_{\hat{X}}$-module $\Delta_{\hat{X}}(M) = \mathcal{D}_{\hat{X}} \otimes_{U\mathfrak{g}} M$, and the $\mathcal{D}_X$-module $\Delta(M) = \Delta_{\hat{X}}(M)^K$.

If, in addition, M is a commutative algebra on which $\mathfrak{g}$ acts by derivations, then $\Delta_{\hat{X}}(M)$ is a $\mathcal{D}_{\hat{X}}$-algebra, and $\Delta(M)$ is a $\mathcal{D}_X$-algebra. In other words, $\mathrm{Spec}(\Delta(M))$ is a $\mathcal{D}_X$-scheme, that is, a scheme over X equipped with an integrable connection. There arises a sheaf of sets on X such that to any open $U \subset X$ there corresponds the set of horizontal sections of $\mathrm{Spec}(\Delta(M))$ over U. The latter set is in fact the set of $\mathbb{C}$-points of an ind-scheme: it equals $\mathrm{Spec}(H_\nabla(U; \Delta(M)))$ where $H_\nabla(U; \Delta(M))$ is a topological commutative algebra; for details, see [BD1], 2.3.15.

For example, let $\hat{X}$ be the scheme of formal coordinate systems on a smooth N-dimensional manifold X. A point of $\hat{X}$ is a pair (a point $x \in X$, an isomorphism $\widehat{\mathcal{O}}_{X;x} \xrightarrow{\sim} \mathbb{C}[[b^1, \dots, b^N]]$). Let $K = \mathrm{Aut}(\mathbb{C}[[b^1, \dots, b^N]])$ be the group of formal coordinate changes (3.16a) preserving the origin, and $\mathfrak{g} = \mathrm{Der}(\mathbb{C}[[b^1, \dots, b^N]])$ the Lie algebra of infinitesimal coordinate changes not necessarily preserving the origin. The obvious projection $\hat{X} \longrightarrow X$ is a K-torsor, and $\hat{X}$ is a $(\mathfrak{g}, K)$-scheme.

Let E be a finite dimensional $GL(N)$-module. Consider it as a K-module using the canonical projection $K \longrightarrow GL(N)$. The space $E \otimes \mathbb{C}[[b^1, \dots, b^N]]$ is naturally a K-module, with the diagonal action of K; we equip it with a structure of a $(\mathfrak{g}, K)$-module, by letting the constant vector fields act through the second factor. Let M_E be the algebra of functions on the last linear space; it is also a $(\mathfrak{g}, K)$-module. Hence we get the commutative $\mathcal{D}_X$-algebra $\Delta(M_E)$, cf. [BD1], Chapter 2, §3; [BD2], 1.2.5, and the sheaf of ind-schemes $U \mapsto \mathrm{Spec}(H_\nabla(U; \Delta(M_E)))$ on X. A little thought shows that this all is a fancy way to define the sheaf of sections of the bundle $V_E = \hat{X} \times_K E \longrightarrow X$ of tensor fields associated with E: namely, the space of sections $\Gamma(U; V_E)$ is equal to the set of $\mathbb{C}$-points of the scheme $\mathrm{Spec}(H_\nabla(U; \Delta(M_E)))$. The $\mathcal{D}_X$-module $\Delta(M_E)$ itself is equal to the sheaf of functions on the corresponding jet bundle.

Returning to our vertex algebras, we play the same game, starting with the algebra M_N of functions on $\widehat{\Omega}_N$ instead of M_E, equipped with the $(\mathfrak{g}, K)$-module structure given by the formulas (3.17). We define Ω_X^{ch} to be the sheaf of $\mathbb{C}$-points of ind-schemes

$$U \mapsto \Gamma(U; \Omega_X^{ch}) = \mathrm{Spec}(H_\nabla(U; \Delta(M_N)))(\mathbb{C}) \tag{3.26}$$

Note that the last schemes are in fact linear spaces by 3.10 below. Further, it is a sheaf of vertex algebras since the automorphisms (3.17) respect the vertex algebra structure. Let us show that this sheaf admits a canonical filtration whose graded factors are the sheaves of tensor fields discussed in the previous paragraph.

3.10. Placing the formulae (3.16) and (3.17), (3.23) on the desk next to each other, one realizes that the "symbols of fields" transform in the same way as the corresponding geometric quantities: functions, 1-forms, and vector fields. To be more precise, introduce a filtration on $\widehat{\Omega}_N$ as follows.

The space $\widehat{\Omega}_N$ is a free $\widehat{A}_N$-module with a base consisting of monomials in letters a^i_n, ψ^i_n $(n<0)$; b^i_m, ϕ^i_m $(m \leq 0)$. Define a partial ordering on this base by

(a) $a>\phi,\ a>\psi,\ a>b,\ \psi>\phi,\ \psi>b,\ \phi>b$; $x^i_n > x^j_m$ if $n<m$, x being a, b, ϕ or ψ;

(b) extending this order to the whole set of monomials lexicographically.

This partial order on the base naturally determines an increasing exhausting filtration on the spaces of fixed conformal weight,

$$F_0\widehat{\Omega}^{(i)}_N \subset F_1\widehat{\Omega}^{(i)}_N \subset \ldots \tag{3.27}$$

For example, $F_0\widehat{\Omega}^{(0)}_N = \widehat{A}_N,\ F_0\widehat{\Omega}^{(-1)}_N = \oplus_{i=1}^N \widehat{A}_N b^i_{-1}$, etc. A glance at (3.17), (3.23) shows that the corresponding graded object $\mathrm{Gr}^F_\bullet\Omega^{(i)}_N$ is a direct sum of symmetric powers of the tangent bundle, symmetric powers of the bundle of 1-forms, and tensor products thereof. For example, the image of b^i_0 in $\mathrm{Gr}^F_\bullet\widehat{\Omega}^{(0)}_N$ is a function, that of b^i_{-n} $(n>0)$ is a 1-form, that of a^i_{-n} is a vector field, etc.

It remains to notice that this filtration is stable under coordinate changes and therefore induces the desired filtration on Ω^{ch}_X.

§4. Conformal and topological structure

4.1. Let us return to the formal setting 3.6. Recall that we have in our vertex algebra $\widehat{\Omega}_N$ the fields $L(z), J(z), Q(z)$ and $G(z)$, defined by the formulas (2.4), which make it a topological vertex algebra. Let us study the effect of the coordinate changes (3.16a) on these fields.

Let us denote by $\tilde{L}(z)$, etc., the field $L(z)$, etc., written down using formulas (2.4), in terms of the tilded fields $\tilde{a}^i(z)$, etc., and then expressed in terms of the old fields $a^i(z)$, etc.

4.2. Theorem. *We have*

$$\tilde{L}(z) = L(z) \tag{4.1a}$$

$$\tilde{J}(z) = J(z) + \left(Tr\log(\partial g^i/\partial b^j(z))\right)' \tag{4.1b}$$

$$\tilde{Q}(z) = Q(z) + \left(\frac{\partial}{\partial\tilde{b}^r}\left[Tr\log(\partial f^i/\partial\tilde{b}^j)\right]\tilde{\phi}^r(z)\right)' \tag{4.1c}$$

$$\tilde{G}(z) = G(z) \tag{4.1d}$$

We have $J = \phi^i_0\psi^i_{-1}\mathbf{1}$. Therefore, (cf. (3.23)),

$$\tilde{J} = \left[\frac{\partial g^i}{\partial b^j}(z)\right]_0\left[\frac{\partial f^k}{\partial\tilde{b}^i}\psi^k(z)\right]_{-1}\mathbf{1} = J + \delta_{jk}\left[\frac{\partial g^i}{\partial b^j}(z)\right]_{-1}\left[\frac{\partial f^k}{\partial\tilde{b}^i}(z)\right]_0\mathbf{1} =$$

$$= J + \left(\frac{\partial g^i}{\partial b^j}(z)\right)'_0 \left(\frac{\partial f^j}{\partial \tilde{b}^i}(z)\right)_0 \mathbf{1},$$

which implies (4.1b). We have $G = \psi^i_{-1} b^i_{-1}\mathbf{1}$. Therefore,

$$\tilde{G} = \left[\frac{\partial f^j}{\partial \tilde{b}^i}\psi^j(z)\right]_{-1} \left[g^i(b)(z)\right]_{-1}\mathbf{1} = \left[\frac{\partial f^j}{\partial \tilde{b}^i}\psi^j(z)\right]_{-1} \left[\frac{\partial g^i}{\partial b^k} b^k(z)'\right]_0 \mathbf{1} =$$

$$= \delta_{jk}\psi^j_{-1} b^k_{-1}\mathbf{1} = G$$

This proves (4.1d). We have $Q = a^i_{-1}\phi^i_0\mathbf{1}$. Therefore,

$$\tilde{Q} = \left[a^j \frac{\partial f^j}{\partial \tilde{b}^i}(z) + \frac{\partial^2 f^k}{\partial \tilde{b}^i \partial \tilde{b}^l}\frac{\partial g^l}{\partial b^r}\psi^k \phi^r(z)\right]_{-1} \left[\frac{\partial g^i}{\partial b^q}\phi^q(z)\right]_0 \mathbf{1}$$

The classical terms:

$$\left[a^j_{-1}\left(\frac{\partial f^j}{\partial \tilde{b}^i}\right)_0 + \left(\frac{\partial^2 f^k}{\partial \tilde{b}^i \partial \tilde{b}^l}\frac{\partial g^l}{\partial b^r}\right)_0 \psi^k_{-1}\phi^r_0\right]\left(\frac{\partial g^i}{\partial b^q}\phi^q\right)_0 \mathbf{1} = Q,$$

since the second summand is zero, due to the anticommutation of ψ^k_{-1} and ϕ^q_0. Quantum corrections (anomalous terms):

$$\left[\left(\frac{\partial f^j}{\partial \tilde{b}^i}\right)_{-1}\left(\frac{\partial^2 g^i}{\partial b^j \partial b^q}\right)_0 \phi^r_0 + \left(\frac{\partial^2 f^k}{\partial \tilde{b}^i \tilde{b}^l}\frac{\partial g^l}{\partial b^r}\phi^r\right)_{-1}\left(\frac{\partial g^i}{\partial b^k}\right)_0\right]\mathbf{1} =$$

$$= \left(\left[\left(\frac{\partial f^j}{\partial \tilde{b}^i}\right)_{-1}\left(\frac{\partial^2 g^i}{\partial b^j \partial b^r}\right)_0 + \left(\frac{\partial^2 f^k}{\partial \tilde{b}^i \partial \tilde{b}^l}\frac{\partial g^l}{\partial b^r}\right)_{-1}\left(\frac{\partial g^i}{\partial b^k}\right)_0\right]\phi^r_0 + \right.$$

$$\left. + \left(\frac{\partial^2 f^k}{\partial \tilde{b}^i \partial \tilde{b}^l}\frac{\partial g^l}{\partial b^r}\frac{\partial g^i}{\partial b^k}\right)_0 \phi^r_{-1}\right)\mathbf{1} = \left(\frac{\partial f^k}{\partial \tilde{b}^i \partial \tilde{b}^l}\frac{\partial g^l}{\partial b^r}\frac{\partial g^i}{\partial b^k}\phi^r\right)_{-1}\mathbf{1}, \tag{4.2}$$

since

$$\left(\frac{\partial^2 f^k}{\partial \tilde{b}^i \partial \tilde{b}^l}\frac{\partial g^l}{\partial b^r}\right)_0 \left(\frac{\partial g^i}{\partial b^k}\right)_{-1}\mathbf{1} = -\left(\frac{\partial f^k}{\partial \tilde{b}^l}\frac{\partial^2 g^l}{\partial b^t \partial b^r}\frac{\partial f^t}{\partial \tilde{b}^i}\right)_0 \left(\frac{\partial g^i}{\partial \tilde{b}^k}\right)_{-1}\mathbf{1} =$$

$$= \left(\frac{\partial f^k}{\partial \tilde{b}^l}\frac{\partial^2 g^l}{\partial b^t \partial b^r}\right)_0 \left(\frac{\partial f^t}{\partial \tilde{b}^i}\right)_{-1}\left(\frac{\partial g^i}{\partial b^k}\right)_0 \mathbf{1} = \left(\frac{\partial^2 g^i}{\partial b^t \partial b^r}\right)_0 \left(\frac{\partial f^t}{\partial \tilde{b}^i}\right)_{-1}\mathbf{1}$$

Returning to (4.2), we have

$$\frac{\partial f^k}{\partial \tilde{b}^i \partial \tilde{b}^l}\frac{\partial g^l}{\partial b^r}\frac{\partial g^i}{\partial b^k}\phi^r = \frac{\partial^2 f^k}{\partial \tilde{b}^l \partial \tilde{b}^i}\frac{\partial g^i}{\partial g^k}\tilde{\phi}^l = \frac{\partial}{\partial \tilde{b}^l}\left(\operatorname{Tr}\log(\partial f^i/\partial \tilde{b}^j)\right)\tilde{\phi}^l,$$

which proves (4.1c). It follows from (4.1c) that the operator Q_0 is invariant. Hence, (4.1a) follows from (4.1d) and (2.2). $\triangle$

4.3. It follows from (4.1a) that for an arbitrary smooth manifold X, the field $L(z)$ is a well-defined global section of the sheaf $\operatorname{End}(\Omega^{ch}_X)[[z, z^{-1}]]$, i.e. Ω^{ch}_X is canonically a sheaf of conformal vertex algebras.

It follows from (4.1b) and (4.1c) that the Fourier modes $F = J_0$ ("fermionic charge") and $d_{DR}^{ch} = Q_0$ ("BRST charge") are well-defined endomorphisms of the sheaf Ω_X^{ch}. Thus, $(\Omega_X^{ch}, d_{DR}^{ch})$ becomes a complex of sheaves, graded by F. This is a localization of definition 2.3.

4.4. Theorem. *For any smooth manifold X, the obvious embedding of complexes of sheaves*

$$i: \ (\Omega_X, d_{DR}) \longrightarrow (\Omega_X^{ch}, d_{DR}^{ch}) \tag{4.3}$$

is a quasiisomorphism. This is true in algebraic, analytic and C^∞ settings.

Indeed, the problem is local along X, and we are done by Theorem 2.4.

4.5. If X is Calabi-Yau, i.e. $c_1(\mathcal{T}_X) = 0$, then the fields $J(z)$ and $Q(z)$ are globally well defined, by (4.1b) and (4.1c). Here $\mathcal{T}_X$ denotes the tangent bundle. Therefore, in this case the sheaf Ω_X^{ch} is canonically a sheaf of topological vertex algebras.

§5. Chiral structure sheaf

A. OBSTRUCTION

5.1. Consider the formal setting 3.1, 3.2, 3.6. We have the vertex algebra $\widehat{V}_N$ of "chiral functions" over the formal disk $D_N = \mathrm{Spf}(\widehat{A}_N)$ where $\widehat{A}_N = \mathbb{C}[[b^1, \ldots, b^N]]$. Let W_N denote the Lie algebra of vector fields $f^i(b)\partial_{b^i}$ on D_N. Let $\Omega^1(\widehat{A}_N)$ denote the module of one-forms $f^i(b)db^i$. The spaces $\widehat{A}_N$ and $\Omega^1(\widehat{A}_N)$ are naturally W_N-modules. Recall that the action of W_N on $\Omega^1(\widehat{A}_N)$ is given by

$$f^i\partial_{b^i} \cdot g^j db^j = f^i\partial_{b^i} g^j db^j + g^j df^j \tag{5.1}$$

The de Rham differential $d: \ \widehat{A}_N \longrightarrow \Omega^1(\widehat{A}_N)$ is compatible with the W_N-action.

Let us define a map

$$\pi: \ W_N \longrightarrow \mathrm{End}(\widehat{V}_N) \tag{5.2}$$

For a vector field $\tau = f^i(b)\partial_{b^i}$, let $\tau(z)$ denote the field $f^i(b)a^i(z)$ (of conformal weight 1) of our vertex algebra $\widehat{V}_N$. Let $\pi(\tau) \in \mathrm{End}(\widehat{V}_N)$ denote the Fourier mode

$$\pi(\tau) := \int \tau(z) = \tau(z)_0 \tag{5.3}$$

Note that by 1.7, the maps $\pi(\tau)$ are derivations of $\widehat{V}_N$.

The mapping π does *not* respect the Lie bracket. Let us compute the discrepancy. Let $\tau_1 = f^i(b)\partial_{b^i}, \tau_2 = g^i(b)\partial_{b^i}$ be two vector fields. We have the operator product

$$\tau_1(z)\tau_2(w) = -\frac{\partial_{b^j} f^i(b(z))\partial_{b^i} g^j(b(w))}{(z-w)^2} +$$

$$+\frac{f^i(b(w))\partial_{b^i} g^j(b(w))a^j(w) - g^j(b(w))\partial_{b^j} f^i(b(w))a^i(w)}{z-w} =$$

$$= -\frac{\partial_{b^j} f^i(b(z)) \partial_{b^i} g^j(b(w))}{(z-w)^2} + \frac{[\tau_1, \tau_2](w)}{z-w} - \frac{\left[\partial_{b^j} f^i(b(w))\right]' \partial_{b^i} g^j(b(w))}{z-w} \tag{5.4}$$

It follows that

$$[\pi(\tau_1), \tau_2(w)] = [\tau_1, \tau_2](w) - \left[\partial_{b^j} f^i(b(w))\right]' \partial_{b^i} g^j(b(w)) \tag{5.5}$$

In particular,

$$[\pi(\tau_1), \pi(\tau_2)] = \pi([\tau_1, \tau_2]) - \int \left(\partial_{b^j} f^i(b(w))\right)' \partial_{b^i} g^j(b(w)) \tag{5.6}$$

5.2. For $\omega = f^i(b)db^i \in \Omega^1(\widehat{A}_N)$, let us denote by $\omega(z)$ the field $f^i(b)b^i(z)'$ of our vertex algebra. Denote by $\pi(\omega)$ the Fourier mode $\omega(z)_0 = \int \omega(z)$.

For $f = f(b) \in \widehat{A}^N$, let $f(z)$ denote the corresponding field of $\widehat{V}_N$. Its conformal weight is 0, and $f(z)_{-1} = f$. We have

$$df(z) = f(z)' \tag{5.7}$$

Given $\tau = f^i \partial_{b^i} \in W_N,\ \omega = g^j db^j \in \Omega^1(\widehat{A}_N)$, we have the operator product

$$\tau(z)\omega(w) = \frac{f^i(b(z))g^i(b(w))}{(z-w)^2} + \frac{f^i(b(w))\partial_{b^i} g^j b^j(w)'}{z-w} =$$

$$= \frac{f^i(b(w))g^i(b(w))}{(z-w)^2} + \frac{f^i \partial_{b^i} g^j(w) b^j(w)' + f^i(b(w))' g^i(b(w))}{z-w} \tag{5.8}$$

It follows that

$$[\pi(\tau), \omega(z)] = (\tau\omega)(z) \tag{5.9}$$

and

$$[\pi(\tau), \pi(\omega)] = \pi(\tau\omega) \tag{5.10}$$

Let $\tilde{W}_N$ denote the linear subspace of $\mathrm{End}(\widehat{V}_N)$ generated by the Fourier modes $\pi(\tau)$ ($\tau \in W_N$) and $\pi(\omega)$ ($\omega \in \Omega^1(\widehat{A}_N)$). Let $I_N \in \tilde{W}_N$ be the linear subspace generated by the Fourier modes $\pi(\omega)$.

It follows from (5.7) that if ω is exact then $\pi(\omega) = 0$. Thus, π induces an epimorphic map

$$\Omega^1(\widehat{A}_N)/d\widehat{A}_N \longrightarrow I_N \tag{5.11}$$

5.3. Lemma. *The map* (5.11) *is an isomorphism.*

This can be proved by writing down the Fourier mode as an infinite sum of monomials in b^i_n and comparing the coefficients of like terms. In fact, a more general statement, namely, that $\int Q(z) = 0$ if and only if $Q = \mathrm{const} \cdot \mathbf{1}$ or $Q(z) = P(z)'$ for some P, seems to be valid for a broad class of vertex algebras, cf. a similar statement in [FF3].

From our point of view, this phenomenon has topological nature. It is amusing to exhibit an example of a vertex algebra, for which the lemma above is false. Namely, take $b^{-1}b_{-1} \in A_1[b^{-1}]$, see 3.4; then $(b^{-1}b_{-1})(z) = b(z)^{-1}b(z)'$ and $\int b(z)^{-1}b(z)' = 0$, but $b(z)^{-1}b(z)'$ is not a total derivative. △ Content-Length: 20254 X-Lines: 512

5.4. Obviously, $\omega_1(z)\omega_2(w) = 0$ for all $\omega_1, \omega_2 \in \Omega^1(\widehat{A}_N)$. It follows from (5.6) and (5.10) that $\tilde{W}_N$ is a Lie algebra, I_N is its abelian ideal, and we have the canonical extension

$$0 \longrightarrow I_N \longrightarrow \tilde{W}_N \longrightarrow W_N \longrightarrow 0 \tag{5.12}$$

The action of W_N on I_N arising from this extension coincides with the canonical action of W_N on $\Omega^1(\widehat{A}_N)/d\widehat{A}_N$, by (5.10). Note that we have defined this extension together with its splitting (5.2). It is given by the two-cocycle $c \in Z^2(W_N; \Omega^1(\widehat{A}_N)/d\widehat{A}_N)$ of W_N with values in $\Omega^1(\widehat{A}_N)/d\widehat{A}_N$, read from (5.5),

$$c(f^i\partial_{b^i}, g^j\partial_{b^j}) = -\partial_{b^i}g^j d(\partial_{b^j}f^i)(\mathrm{mod}\ d\widehat{A}_N) \tag{5.13}$$

5.5. Let us consider the truncated and shifted de Rham complex

$$\Omega^\bullet:\ 0 \longrightarrow \widehat{A}_N \longrightarrow \Omega^1(\widehat{A}_N) \longrightarrow 0 \tag{5.14}$$

where we place $\Omega^1(\widehat{A}_N)$ in degree zero. It is a complex of W_N-modules. We have an obvious map of complexes of W_N-modules

$$\Omega^\bullet \longrightarrow \Omega^1(\widehat{A}_N)/d\widehat{A}_N \tag{5.15}$$

where the target is regarded as a complex sitting in degree zero.

Let us write down a two-cocycle $\tilde{c} \in Z^2(W_N; \Omega^\bullet)$ which is mapped to c, (5.13), under the map (5.14). Such a cocycle is by definition a pair (c^2, c^3) where $c^2 \in C^2(W_N; \Omega^1(\widehat{A}_N))$ is a two-cochain, and $c^3 \in C^3(W_N; \widehat{A}_N)$ is a three-cochain, such that

$$d_{Lie}(c^2) = d_{DR}(c^3) \tag{5.16a}$$

$$d_{Lie}(c^3) = 0 \tag{5.16b}$$

Let us define

$$c^2(f^i\partial_i, g^j\partial_j) = \partial_i g^j d(\partial_j f^i) - \partial_j f^i d(\partial_i g^j) \tag{5.17}$$

and

$$c^3(f^i\partial_i, g^j\partial_j, h^k\partial_k) = \partial_j f^i \partial_k g^j \partial_i h^k - \partial_k f^i \partial_i g^j \partial_j h^k \tag{5.18}$$

We write for brevity ∂_i instead of ∂_{b^i}. One checks the compatibilities (5.16) directly. Thus, we have defined $\tilde{c}$. One sees that $\tilde{c}$ is mapped to $-2c$ under (5.15).

For $N = 1$ the space $\Omega^1(\widehat{A}_N)/d\widehat{A}_N$ is trivial. This allows one to define the sheaf $\mathcal{O}_X^{ch}$ for curves, acting as in 3.9, and starting from $\widehat{V}_N$ instead of $\widehat{\Omega}_N$.

Assume that $N > 1$. Using the computations of Gelfand-Fuchs, cf. [F], Theorems 2.2.7 and 2.2.4, one can show that the map in cohomology

$$H^2(W_N; \Omega^\bullet) \longrightarrow H^2(W_N; \Omega^1(\widehat{A}_N)/d\widehat{A}_N) \tag{5.19}$$

induced by (5.15) is an isomorphism. We have the canonical short exact sequence

$$0 \longrightarrow H^2(W_N; \Omega^1(\widehat{A}_N)) \longrightarrow H^2(W_N; \Omega^\bullet) \longrightarrow H^3(W_N; \widehat{A}_N) \longrightarrow 0 \tag{5.20}$$

the left- and right-most terms being one-dimensional. Under the second map of this sequence, our cocycle $\tilde{c}$ is mapped to its second component c^3 which is a canonical representative of a generator of the space $H^3(W_N; \widehat{A}_N)$, cf. [F], Theorem 2.2.7′ and Chapter 2, §1, no. 4. In particular, our cocycle $\tilde{c}$ is non-trivial. It follows that the cocycle c defining the extension (5.12) is also non-trivial.

What kind of an object does the cocycle $\tilde{c}$ define? Recall that a *homotopy Lie algebra* $L^\bullet$ is a complex of vector spaces equipped with a collection of brackets

$$[\ ,\dots,\ ]_i:\ \Lambda^i L^\bullet \longrightarrow L^\bullet[-i+2],\ i \geq 2, \tag{5.21}$$

satisfying certain compatibility conditions, cf. for example [HS], §4. In particular $[\ ,]_2$ is a skew symmetric map, satisfying the Jacobi identity up to the homotopy (given by the third bracket).

Let us define a complex $L^\bullet$ as follows. Set $L^0 = W_N \oplus \Omega^1(\widehat{A}_N)$, $L^{-1} = \widehat{A}_N$, the other components being trivial. The differential $L^{-1} \longrightarrow L^0$ is the composition of the de Rham differential and the obvious embedding $\Omega^1(\widehat{A}_N) \hookrightarrow L^0$.

Let the second bracket $[\ ,\ ]_2$ be given by the usual bracket of vector fields, and the action of vector fields on $\widehat{A}_N$ and $\Omega^1(\widehat{A}_N)$. Define the third bracket with the only nontrivial component being the three-cocycle c^3, (5.18). We set the other brackets equal to zero. This way we get a structure of a homotopy Lie algebra on $L^\bullet$.

We have a canonical extension of homotopy Lie algebras

$$0 \longrightarrow \Omega^\bullet \longrightarrow L^\bullet \longrightarrow W_N \longrightarrow 0 \tag{5.22}$$

Here $\Omega^\bullet$ is an abelian ideal in $L^\bullet$ (all brackets are zero). This is a refinement of extension (5.12).

B. PROJECTIVE LINE

5.6. Let X be the projective line $\mathbb{P}^1$. Let us fix a coordinate b on $\mathbb{P}^1$, and consider the open covering $X = U_0 \cup U_1$ where $U_0 = \mathrm{Spec}(\mathbb{C}[b])$, $U_1 = \mathrm{Spec}(\mathbb{C}[b^{-1}])$.

Consider the sheaves $\mathcal{O}^{ch}_{U_0}$ on U_0 with coordinate b, and $\mathcal{O}^{ch}_{U_1}$ on U_1 with coordinate b^{-1}, which were defined in 3.4. Let us glue them on the intersection $U_{01} = U_0 \cap U_1$ using the transition function

$$\tilde{b}(z) = b(z)^{-1} \tag{5.23a}$$

$$\tilde{a}(z) = b^2 a(z) + 2b(z)' \tag{5.23b}$$

In this way, we get the sheaf on the X, to be denoted $\mathcal{O}^{ch}_X$.

5.7. Theorem. *The space of global sections $\Gamma(X; \mathcal{O}^{ch}_X)$ admits a natural structure of an irreducible vacuum $\widehat{sl}_2$-module on the critical level.*

We have to compute $\mathcal{O}^{ch}_{U_0} \cap \mathcal{O}^{ch}_{U_1}$ where both $\mathcal{O}^{ch}_{U_i}$ are regarded as subspaces of $\mathcal{O}^{ch}_{U_{01}}$. It is the essence of the Wakimoto construction, [W], that the fields $a(z)b(z)^2+$

$2b(z)', a(z)$ (resp., $\tilde{a}(z)\tilde{b}(z)^2 + 2\tilde{b}(z)', \tilde{a}(z)$) generate an $\widehat{\mathrm{sl}}_2$-action on $\mathcal{O}^{ch}_{U_0}$ (resp., on $\mathcal{O}^{ch}_{U_1}$), and under this action, $\mathcal{O}^{ch}_{U_i}$, $i = 1, 2$, become the restricted Wakimoto module with zero highest weight. (Restricted here means that the level is critical, and the Sugawara operators act by zero.) It follows from (5.23) that the $\widehat{\mathrm{sl}}_2$-action comes from $\Gamma(X; \mathcal{O}^{ch}_X)$ and, therefore, $\mathcal{O}^{ch}_{U_0} \cap \mathcal{O}^{ch}_{U_1}$ is also an $\widehat{\mathrm{sl}}_2$-module. It follows from [FF1] or [M] that each $\mathcal{O}^{ch}_{U_i}$ contains a unique proper submodule which is isomorphic to the irreducible vacuum representation. To complete the proof, it remains to show that $\mathcal{O}^{ch}_{U_0} \neq \mathcal{O}^{ch}_{U_0} \cap \mathcal{O}^{ch}_{U_1}$, and this is obvious. $\triangle$

5.8. In fact, the first cohomology space $H^1(X; \mathcal{O}^{ch}_X)$ is also isomorphic to the same irreducible $\widehat{\mathrm{sl}}_2$-module.

To prove this, let us compute the Euler character

$$ch(X; \mathcal{O}^{ch}_X) = \sum_{N=0}^{\infty} \chi(X; \mathcal{O}^{ch(N)}_X) \cdot q^N$$

in two different ways. First, by definition

$$ch(X; \mathcal{O}^{ch}_X) = ch(\Gamma(X; \mathcal{O}^{ch}_X)) - ch(H^1(X; \mathcal{O}^{ch}_X))$$

By Theorem 5.7 and [M],

$$ch(\Gamma(X; \mathcal{O}^{ch}_X)) = (1-q)^{-1} \prod_{N=1}^{\infty} (1-q^N)^{-2}$$

On the other hand, formulas (5.23) imply that $\mathcal{O}^{ch}_X$ carries a filtration F such that the image of a_{-n} (resp., b_{-n}) ($n \geq 1$) in Gr^F is a vector field (resp., a 1-form). It follows that each monomial $a_{-n_1} \cdot \ldots \cdot a_{-n_r} b_{-m_1} \cdot \ldots \cdot b_{-m_s}$ contributes $2s - 2r + 1$ in $\chi(X; \mathcal{O}^{ch(N)}_X)$, where $N = \sum n_i + \sum m_j$. Therefore,

$$ch(X; \mathcal{O}^{ch}_X) = \prod_{N=1}^{\infty} (1-q^N)^{-2},$$

hence

$$ch(H^1(X; \mathcal{O}^{ch}_X)) = q \cdot ch(\Gamma(X; \mathcal{O}^{ch}_X))$$

In other words, $H^1(X; \mathcal{O}^{ch}_X)$ has the same (up to the shift by q) character as $\Gamma(X; \mathcal{O}^{ch}_X)$. Again by [M], these two spaces are isomorphic as $\widehat{\mathfrak{g}}$-modules.

C. FLAG MANIFOLDS

5.9. Let G be a simple algebraic group, $B \subset G$ a Borel subgroup, $N \subset B$ the maximal nilpotent subgroup. The manifold N is isomorphic to the affine space, and is a $(\mathfrak{g}, B)$-scheme, where B acts by conjugation. Consider the Heisenberg vertex algebra V associated with the affine space N. According to [FF2], V admits a structure of a $\widehat{\mathfrak{g}}$-module (Wakimoto module); in particular, V is a $(\mathfrak{g}, B)$-module. Note that $x \in \mathfrak{g}$ acts on V as $\int X(z)$ for some $X \in V$.

Consequently, considered as an affine space, V admits a structure of a $(\mathfrak{g}, B)$-scheme. Let M be the algebra of functions on V. Proceeding as in 3.9, with $K = B$, $\hat{X} = G$, and $X = G/B$, we get the sheaf of ind-schemes

$$U \mapsto \mathrm{Spec}(H_\nabla(\Delta(M)))$$

on X. The sheaf of its $\mathbb{C}$-points is called the *chiral structure sheaf* of X and denoted by $\mathcal{O}_X^{ch}$.

5.10. If $G = SL(n)$ then the sheaf $\mathcal{O}_X^{ch}$ admits a more explicit construction, using charts and gluing functions. In this case $X = GL(n)/(B \times \mathbb{C}^*)$. The Weyl group is identified with the symmetric group S_n and can be realized as the subgroup of $GL(n)$ consisting of permutation matrices. One checks that in terms of the Lie algebra $\mathrm{gl}(n)$, the simple permutation r_i (interchanging i and $i+1$) can be written as follows

$$r_i = \exp(\pi\sqrt{-1}E_{ii})\exp(E_{i+1,i})\exp(-E_{i,i+1})\exp(E_{i+1,i}) \tag{5.24}$$

where E_{ij} $(1 \leq i, j \leq n)$ form the standard base of $\mathrm{gl}(n)$.

The manifold X is covered by $|S| = n!$ charts, the chart associated with an element $w \in S_n$ being $U_w = wNw_0B$, where $N \subset B$ is the unipotent subgroup consisting of all upper-triangular matrices and $w_0 \in S_n$ is the element of maximal length. Let us identify U_w with N using the bijection $n \mapsto wnw_0B$. Under this identification, if $x \in U_{w_1} \cap U_{w_2}$, then the change from the coordinates determined by U_{w_1} to the ones determined by U_{w_2}, is given by $x \mapsto w_2^{-1}w_1x$.

Each U_w may be identified with the affine space $\mathbb{C}^{n(n-1)/2}$. To define $\mathcal{O}_X^{ch}$, we declare that $\mathcal{O}_X^{ch}\big|_{U_w} = \mathcal{O}_{U_w}^{ch}$ where the last sheaf is defined in 3.4. Now we have to glue these sheaves over the pairwise intersections in a consistent manner.

Let V denote the vertex algebra $V_{n(n-1)/2} = \Gamma(U_w; \mathcal{O}_{U_w}^{ch})$. First, we extend the $\widehat{\mathrm{sl}}(n)$-module structure on V to a $\widehat{\mathrm{gl}}(n)$-module structure. For that, in addition to the formulae in [FF1], p. 279, define

$$E_{ii}(z) = -\sum_{j>i} b^{ij}a^{ij}(z) + \sum_{j<i} b^{ji}a^{ji}(z) \tag{5.25}$$

It is easily checked that in this way we indeed get an action of $\widehat{\mathrm{gl}}(n)$ on V.

For any $A \in \mathrm{End}(V)$, introduce the formal exponent $\exp(tA) : V \longrightarrow V \otimes \mathbb{C}[[t]]$,

$$\exp(tA)(v) = \sum_{i=0}^{\infty} \frac{A^i(v)}{i!}t^i \tag{5.26}$$

Working over $\mathbb{C}[[t]]$, we can easily compose such maps. Motivated by (5.24), set

$$r_i(t) = \exp(t\pi\sqrt{-1}\int E_{ii}(z))\cdot$$

$$\cdot\exp(t\int E_{i+1,i}(z))\exp(-t\int E_{i,i+1}(z))\exp(t\int E_{i+1,i}(z)) \tag{5.27}$$

Thus, $r_i(t)$ is a map $V \longrightarrow V[[z, z^{-1}]][[t]]$. Note that $V[[t]]$ is naturally a vertex algebra. It contains the vertex subalgebra "generated by functions, rational in t", that is,

$$V(t) = R(N \times \mathbb{C}) \otimes_A V \subset V[[t]] \tag{5.28}$$

Here $A = \Gamma(N; \mathcal{O}_N)$, and $R(N \times \mathbb{C})$ denotes the ring of rational functions on $N \times \mathbb{C}$, regular at $N \times \{0\}$.

5.11. Lemma. (a) $r_i(t)V \subset V(t)$. *Further,* $r_i(t)V$ *is generated by functions well-defined for any value of* t.

(b) $r_i(1)$ *is well defined (by* (a)*), and determines a map*

$$r_i(1): \ \Gamma(U_w; \mathcal{O}^{ch}_{U_w}) \longrightarrow \Gamma(U_w \cap U_{r_i w}, \mathcal{O}^{ch}_{U_w}) \tag{5.29}$$

As $\int X(z)$, $X \in V$, is a derivation, the map $r_i(t)$ is an embedding of vertex algebras. Therefore, it is enough to compute $r_i(t)$ on generators.

There are two types of generators in V: (i) those coming from the subspace $W \subset V$, canonically isomorphic to $\mathfrak{g} = sl(n)$, such that the fields $X(z)$ $(X \in W)$ generate the action of $\widehat{\mathfrak{g}}$; (ii) those coming from $\mathbb{C}[b^{ij}_n]\mathbf{1} \subset V$. Obviously, the endomorphisms $\int E_{ij}(z)$ $(i \neq j)$, act on W as E_{ij} on $sl(n)$; in particluar, this action is nilpotent, and $r_i(t)$ is polynomial in t. As for $\mathbb{C}[b^{ij}_n]\mathbf{1}$, this subspace is identified with the symmetric algebra $S^\bullet(\Omega^1(U_w))$. For any $X \in \mathrm{gl}(n) \subset \Gamma(X; \mathcal{T}_X)$, the element $\int X(z)$ acts on this space as the Lie derivative along X. Consequently, $\exp(\int X(z))$ maps $S^\bullet(\Omega^1(U_w))$ into $S^\bullet(\Omega^1(\exp(X) \cdot U_w))$. $\triangle$

Repeated application of this lemma gives for any $v = r_{i_1} \cdot \ldots r_{i_k} \in S_n$ the map

$$v(1) = r_{i_1} \cdot \ldots \cdot r_{i_k}: \ \Gamma(U_w; \mathcal{O}^{ch}_{U_w}) \longrightarrow \Gamma(U_w \cap U_{v \cdot w}; \mathcal{O}^{ch}_{U_w}) \tag{5.30}$$

Finally, to complete our construction of $\mathcal{O}^{ch}_X$, glue the sheaves $\mathcal{O}_{U_{w_1}}$ and $\mathcal{O}_{U_{w_2}}$ using the maps

$$\Gamma(U_{w_1}; \mathcal{O}^{ch}_{U_{w_1}}) \longrightarrow \Gamma(U_{w_1} \cap U_{w_2}; \mathcal{O}^{ch}_{U_{w_2}}) \longleftarrow \Gamma(U_{w_2}; \mathcal{O}^{ch}_{U_{w_2}}) \tag{5.31}$$

where the first (resp. second) arrow is $w_2 w_1^{-1}(1)$ (resp., $w_1 w_2^{-1}(1)$). Since the gluing maps are induced by the action of S_n, they are transitive, and the sheaf $\mathcal{O}^{ch}_X$ is well defined.

5.12. Example. Let $\mathfrak{g} = sl(2)$. We have $N = \mathbb{A}^1 \subset X = \mathbb{P}^1$; b is the coordinate on N such that generators of gl(2) act as the following vector fields: $E_{21} \mapsto -\partial_b$, $E_{12} \mapsto b^2 \partial_b$, $E_{11} \mapsto b\partial_b$.

One easily calculates that

$$\exp(E_{21}): \ b \mapsto b - 1; \ \exp(-E_{12}): \ b \mapsto b/(b+1); \ \exp(\pi\sqrt{-1}E_{11}): \ b \mapsto -b$$

The simple reflection is

$$r = \exp(\pi\sqrt{-1}E_{11}) \exp(E_{21}) \exp(-E_{12}) \exp(E_{21}): \ b \mapsto b^{-1} \tag{5.32}$$

To do the chiral analogue of this computation, recall (cf. [W]) that $E_{21}(z) = -a(z)$, $E_{12}(z) = b^2a(z)+2b(z)'$, $E_{11}(z) = ba(z)$. A direct computation using Wick theorem yields

$$r(1)a_{-1}\mathbf{1} = (b^2a_{-1} + 2b_{-1})\mathbf{1} \tag{5.33}$$

which coincides with (5.23b). Computation of $r(1)b\mathbf{1}$ does not differ from the classical one, see (5.32).

5.13. Theorem. *The space $\Gamma(X;\mathcal{O}_X^{ch})$ admits a natural structure of a $\widehat{\mathfrak{g}}$-module at the critical level, such that it is a submodule of the restricted Wakimoto module, and its conformal weight zero component is* 1*-dimensional.*

Proof is not much different from that of Theorem 5.7. The space of sections $W = \Gamma(N;\mathcal{O}_X^{ch})$ over the big cell is the restricted Wakimoto module with the zero highest weight. Restricted here again means that the center of $U(\widehat{\mathfrak{g}})_{loc}$ acts trivially. As the action of B on V preserves $\widehat{\mathfrak{g}} \subset \mathrm{End}(V)$, $\widehat{\mathfrak{g}}$ is spanned by the Fourier modes of fields associated with certain elements of $\Gamma(X;\mathcal{O}_X^{ch})$. Therefore, $\Gamma(X;\mathcal{O}_X^{ch})$ is a $\widehat{\mathfrak{g}}$-module at the critical level, and there arises a $\widehat{\mathfrak{g}}$-morphism

$$\Gamma(X;\mathcal{O}_X^{ch}) \longrightarrow W \tag{5.34}$$

By construction, the sheaf $\mathcal{O}_X$ admits a canonical filtration whose associated quotients are free $\mathcal{O}_X$-modules of finite rank. Therefore, the map (5.34) is an injection. By construction, the conformal weight zero component of $\Gamma(X;\mathcal{O}_X^{ch})$ is equal to $\Gamma(X;\mathcal{O}_X) = \mathbb{C}$. $\triangle$

This theorem is less precise than 5.7, the reason being that the representation-theoretic result of [FF1], [M] mentioned in 5.7 is only available in the $sl(2)$-case. However, Theorem 5.1 of [FF1] makes it plausible that this representation-theoretic statement carries over to any $\mathfrak{g}$, and hence that $\Gamma(X;\mathcal{O}_X^{ch})$ is in fact the irreducible vacuum $\widehat{\mathfrak{g}}$-module.

5.14. Note that the whole sheaf $\mathcal{O}_X^{ch}$ admits a structure of a $\widehat{\mathfrak{g}}$-module at the critical level.

5.15. Localization for non-zero highest weight. For any integral weight of $\mathfrak{g}$, there exists a twisted analogue of $\mathcal{O}_X^{ch}$, to be denoted by $\mathcal{L}_\lambda^{ch}$. Its construction repeats word for word 5.9, except that the action of $(\mathfrak{g}, B)$ on V is to be twisted by λ, see [W, FF1]. If λ is a regular dominant weight, then Theorems 5.7 and 5.13, along with their proofs, generalize in the obvious way. For example, the word "vacuum" in the formulation of Theorem 5.7 is to be replaced with "highest weight λ". Likewise, the claim that "conformal weight zero component is 1-dimensional" should be replaced with the following: "conformal weight zero component is isomorphic to $\Gamma(X;\mathcal{L}_\lambda)$", where $\mathcal{L}_\lambda$ is the standard line bundle on X.

One may want to regard $\mathcal{L}_\lambda^{ch}$ as a sheaf of modules over a sheaf of vertex algebras, just as $\mathcal{O}_X^{ch}$ is a sheaf of modules over itself. From this point of view, $\mathcal{O}_X^{ch}$ is a chiral analogue of the sheaf $\mathcal{D}_X$ of differential operators on X. Thus, one cannot expect $\mathcal{L}_\lambda^{ch}$ to be a module over $\mathcal{O}_X^{ch}$. We believe (and have checked this for $\mathfrak{g} = sl(2)$) that one can define a sheaf of vertex algebras $\mathcal{O}_\lambda^{ch}$ which acts on $\mathcal{L}_\lambda^{ch}$ and is locally isomorphic to $\mathcal{O}_X^{ch}$. Therefore, this sheaf can be regarded as a chiral partner of the sheaf $\mathcal{D}_\lambda$ of twisted differential operators on X.

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