The length of the second shortest geodesic.

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Abstract. According to the classical result of J.P. Serre ([S]) any two points on a closed Riemannian manifold can be connected by infinitely many geodesics. The length of a shortest of them trivially does not exceed the diameter of the manifold, d. But how long are the shortest remaining geodesics? In this paper we prove that any two points on a closed n-dimensional Riemannian manifold can be connected by two distinct geodesics of length $\leq 2qd \leq 2nd$, where q is the smallest value of i such that the i-th homotopy group of the manifold is non-trivial.

1. Main result.

Here is the main result of the present paper:

Theorem 1. Let M^n be a closed n-dimensional Riemannian manifold, $q = \min_i\{\pi_i(M^n) \neq 0\}$, and d denote the diameter of M^n. Then for each pair of points $x, y \in M^n$ there exist at least two distinct geodesics connecting x, y of length not exceeding $2qd (\leq 2nd)$.

Observe, that if $x = y$, then the trivial geodesic is the shortest geodesic connecting x and y. In this case our theorem asserts the existence of a geodesic loop based at an arbitrary point x of M^n of length $\leq 2qd$. This result is the main result of the paper [R] of one of the authors. Theorem 1 can be viewed as a generalization of this result. Our proof of Theorem 1 is heavily reliant on methods of [R] that will be reviewed in the next section. Yet here we adopt a different perspective (and a different terminology) than in the exposition of these methods in [R].

2. Filling of cages.

Let us begin by introducing the following definitions and notations.

Definition 1. Let x, y be two points in M^n. An *m-cage* c based at x, y is a collection of m paths $c_1, \ldots c_m$ from x to y. (For every i c_i is a continuous map of $[0, 1]$ into M^n.) The space of all m-cages based at x and y $C_{x,y,m}$ can be identified with the mth power of the space of paths from x to y. For every L $C^L_{x,y,m}$ denotes the space of all m-cages based at x, y such that the length of each of the m paths forming the cage is at most L. $C^{L,\bar{L}}_{x,y,m}$ denotes the space of all m-cages $c = (c_1, \ldots, c_m)$ based at x, y such that the length of c_1 does not exceed $\bar{L}$, and the length of c_i for every $= 2, 3, \ldots, m$ does not exceed L.

Let $\sigma^m = [v_0, v_1, \ldots, v_m]$ be the standard m-dimensional simplex with edges of length one. (Here $v_0, v_1, \ldots v_m$ are its vertices.) As usual, we use the notation $C(\sigma^m, M^n)$ for the space of continuous maps from σ^m to M^n. Of course, this space can be identified with the space of continuous maps of the m-dimensional ball into M^n.

Definition 2. Let x, y be two points in M^n, $L, \bar{L}$ be two positive numbers, and N be a positive integer. A *coherent N-filling* of m-cages based at x, y from $C^{L,\bar{L}}_{x,y,m}$ is a collection of continuous maps $\phi_m : C^{L,\bar{L}}_{x,y,m} \longrightarrow C(\sigma^m, M^n)$ for all $m = 1, 2, 3 \ldots, N$ with the following properties:

1. For every m and m-cage c $\phi_m(c)$ maps the $(m-1)$-dimensional face $[v_1, \ldots, v_m]$ of σ^m into y;

2. For every m and every m-cage $c = (c_1, \ldots, c_m)$ $\phi_m(c)$ maps each of m 1-dimensional simplices $[v_0, v_i]$ by the map c_i. (Here we identify $[v_0, v_i]$ with $[0, 1]$.) In particular, v_0 is mapped into x, and for every 1-cage c $\phi_1(c) = c$.

3. (Coherence) For every $m = 2, 3, 4, \ldots, N$, every m-cage c and every $i = 1, 2 \ldots, m$ the restriction of ϕ_m on the $(m-1)$-dimensional face $[v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m]$ of σ_m coincides with $\phi_{m-1}(c_{(i)})$, where $c_{(i)}$ denotes the $(m-1)$-cage $(c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_m)$.

To explain the meaning of conditions 1, 2 collapse $[v_1, \ldots, v_m]$ into a point. Identify this point with the North pole of a ball D^m and identify v_0 with the South pole of this ball. The m edges $[v_0, v_i]$ become m meridians on the sphere bounding this ball. The meaning of conditions 1, 2 is that we assign to each m-cage a map of the m-ball into M^n, so that the m meridians on the boundary sphere are mapped into the m-cage.

Proposition 2. Let $L, \bar{L}$ be positive real numbers such that $\bar{L} \geq L$, and N be a positive integer. Let x, y, z be any three points of M^n such that the distance between any two of them does not exceed L. Assume that there exists exactly one geodesic between x and y of length $\leq \bar{L} + (2N-3)L$. (If $x = y$, then this geodesic is the trivial geodesic.) Then there exists a coherent N-filling of m-cages based at x, z from $C^{L,\bar{L}}_{x,z,m}$.

Proof: We present a proof by induction. The induction is with respect to N. Its base corresponds to the case $N = 1$. In this case $\phi_1(c) = c$. (Recall that each 1-cage is, by definition, a path in M^n, i.e. a continuous map of $\sigma^1 = [0, 1]$ into M^n.) The proof of the induction step will be based on the following lemma:

Lemma 1. Let $\bar{L}, L$ be positive numbers. Assume that x, y, z are three points in M^n such that all distances between them do not exceed L. Assume that there exists only one geodesic between x and y of length $\leq \max\{\bar{L}, L\} + L$. Then any two paths γ_1, γ_2 starting at x and ending at z such that the length of γ_1 is $\leq \bar{L}$ and the length of γ_2 is $\leq L$ can be connected by a path homotopy that passes only through paths of length $\leq \bar{L} + 2L$. This path homotopy depends continuously on γ_1 and γ_2.

Proof of Lemma 1. Let σ be the unique shortest geodesic from x to y , τ be one of the shortest geodesics from z to y (see Fig. 1).

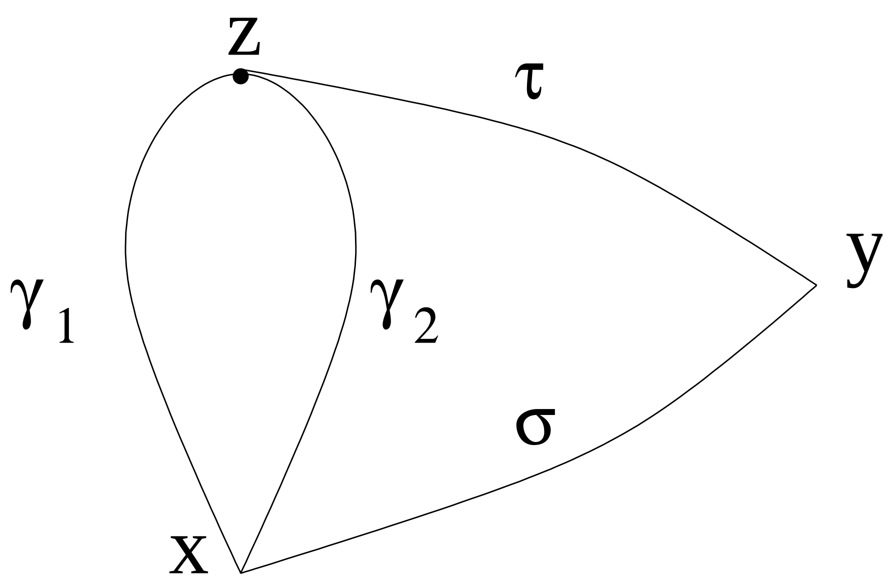

Figure 1.

Every path from x to y of length $\leq 2L$ or $\leq \bar{L}+L$ can be connected to σ by a length non-increasing homotopy. (Otherwise, there will be a second geodesic of length $\leq \max\{\bar{L}, L\}+L$.) Moreover, we can choose a specific length non-increasing homotopy, e.g. the Birkhoff curve-shortening process with fixed endpoints. (See [C] for a detailed description of the Birkhoff curve-shortening process.) This homotopy will depend continuously on the initial path. In particular, this homotopy can be used to deform $\gamma_i \circ \tau$ to σ, $(i=1,2)$, as well as to deform σ back to $\gamma_i \circ \tau$.

Now one can construct the desired path homotopy from γ_1 to γ_2 as follows: $\gamma_1 \longrightarrow \gamma_1 \circ \tau \circ \tau^{-1} \longrightarrow \sigma \circ \tau^{-1} \longrightarrow \gamma_2 \circ \tau \circ \tau^{-1} \longrightarrow \gamma_2$. Here arrows denote homotopies. For any path p p^{-1} denotes p traversed in the opposite direction. All homotopies depend continuously on γ_1 and γ_2. For each of these homotopies the length of paths during the homotopy does not exceed the maximum of lengths of paths at its begining and its end. Lemma 1 is now proven. QED

To prove the induction step assume that for some m the maps $\phi_1, \ldots, \phi_{m-1}$ were already constructed. We need to construct ϕ_m. Let $c=(c_1, c_2, \ldots, c_m)$ be an m-net. We need to map $\sigma^m=[s_0, \ldots, s_m]$ to M^n. Because of the coherence condition a map $\psi_m(c)$ defined as the restriction of ϕ_m to $\partial\sigma^m$ into M^n is already prescribed. By virtue of the induction assumption ψ_m is a continuous function of c. We need only to contract the (map of the) sphere $\psi_m(c)$ to a point so that the contracting homotopy depends continuously on c. To achieve this goal note that according to Lemma 1 there exists a path homotopy between c_1 and c_2 that passes only through paths c_t, $t \in [1,2]$, of length $\leq \tilde{L}=\bar{L}+2L$. (Here we use the fact that for every $m \geq 2$ $(2m-3)L \geq L$. So, the assumption of Lemma 1 about the non-existence of a second short geodesic between x and y follows from a similar assumption in Proposition 2.) Consider a 1-parametric family of m-cages $c^{(t)}=(c_t, c_2, \ldots, c_m)$. So,

$c^{(1)} = c$ and $c^{(2)} = (c_2, c_2, c_3, \ldots, c_m)$. For each t $c^{(t)} \in C^{L,\tilde{L}}_{x,y,m}$. By virtue of the induction assumption there exists a coherent filling of all $(m-1)$-subcages of $c^{(t)}$ obtained by removal of one of m strands $c_i^{(t)}$, and for every t the resulting m maps of $(m-1)$-dimensional simplices can be "glued" to each other into a map $\psi_m(c^{(t)}) : \partial\sigma^m \longrightarrow M^n$. Of course, here it is important that $\tilde{L} + (2(m-1)-3)L = \bar{L} + (2m-3)L$, and so the required assumption about the non-existence of a second geodesic between x and y of length $\leq \tilde{L}+(2(m-1)-3)L$ holds. Thus, one obtains a homotopy $\psi_m(c^{(t)})$ between $\psi_m(c)$ and $\psi_m(c^{(2)})$.

It remains to show that $\psi_m(c^{(2)})$ is canonically and, therefore, continuously contractible. (Here we are concerned about the continuity of the contracting homotopy as a function of c.) Note that the boundary of σ^m consists of $(m+1)$ $(m-1)$-dimensional simplices. The maps ϕ_{m-1} and, thereby, ψ_m map two of these faces, namely, faces corresponding to two copies of the $(m-1)$-cage $(c_2, c_3, \ldots, c_m)$ in an identical way. Together these two cells form a "folded" map of S^{m-1} to M^n that factors through the projection of S^{m-1} to the disc D^{m-1}. This map is obviously canonically contractible. In order to construct a homotopy of $\psi_m(c^{(2)})$ to this "folded" map we need to "eliminate" the remaining $(m-1)$ maps of $(m-1)$-dimensional faces of σ^m. But one of these maps is constant, and the remaining $(m-2)$ maps correspond to $(m-1)$-cages of the form $(c_2, c_2, \ldots)$. Therefore each of these maps is similarly "folded" and can be connected by a canonical homotopy (over its image) to a map of the corresponding face which is a composition of the projection of the considered face to one of its codimension one faces and $\phi_{m-2}((c_2, c_3, \ldots, c_{i-1}, c_{i+1}, \ldots, c_m))$ for an appropriate i. These homotopies eliminate the remaining $m-1$ faces, as desired. QED.

3. Filling of $(m, \epsilon)-$umbrellas.

Let $\sigma^{m-1} = [s_1, \ldots, s_m]$ denote the standard $(m-1)$-dimensional simplex such that the lenghs of all its edges are equal to 1, s_* denote the center of σ^{m-1}.

Definition 3. An $(m, r)-$umbrella based at x, y consists of a singular $(m-1)$-simplex $\rho : \sigma^{m-1} \longrightarrow M^n$, a point $x \in M^n$ and m paths in M^n connecting x with images of the vertices of σ^{m-1} under ρ so that $y = \rho(s_*)$, the image of ρ is contained in a metric ball of radius r in M^n centered at y, and the length of the image of every straight line segment in σ^{m-1} under ρ is less than ϵ. (See Fig. 2).

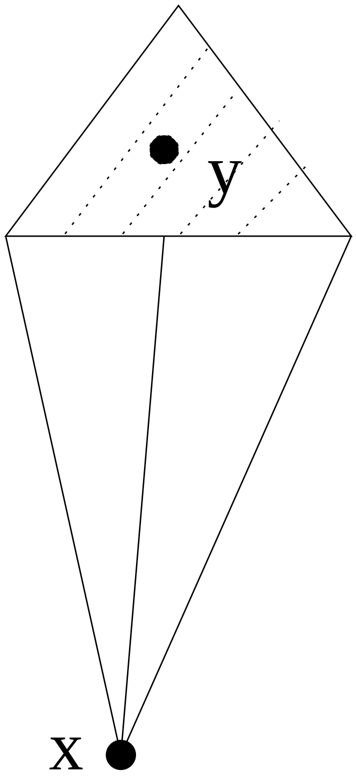

Figure 2

This notion generalizes the notion of m-cages that can be considered as $(m,0)-$umbrellas (with constant ρ). The goal of this section is to generalize the notion of coherent filling for $(m,r)-$umbrellas and to extend Theorem 1 to $(m,\epsilon)-$umbrellas for small positive ϵ. Denote the space of all $(m,r)-$umbrellas based at x, y by $U_{m,x,y}$ and its subspace formed by all umbrellas where the length of the paths connecting x with the first vertex of ρ does not exceed $\bar{L}$, and the lengths of all paths connecting x with the remaining $m-1$ vertices of the singular simplex ρ do not exceed L by $U^{L,\bar{L}}_{m,x,y}$. Each umbrella u can be represented as $(c_1, \ldots, c_m, \rho)$, where c_i are continuous paths from x to the vertices of the singular simplex ρ. It is obvious that 1-umbrellas based at x, y are merely continuous paths starting at x and ending at y.

Definition 4. Let $N \geq 1$ be an integer number. Let $L, \bar{L}$ be positive numbers. A coherent N-filling of $(m,r)-$umbrellas based at x, y from $U^{L,\bar{L}}_{m,x,y}$ is a family of continuous maps $\phi_m : U^{L,\bar{L}}_{m,x,y} \longrightarrow C(\sigma^m, M^n)$, $m = 1, 2, \ldots, N$ such that for every $(m,\epsilon)-$umbrella $u = (c_1, c_2, \ldots, c_m, \rho)$ the following conditions hold:
1) The restriction of $\phi_m(u)$ to the $(m-1)$-dimensional face $[s_1, s_2, \ldots, s_m]$ coincides with ρ;
2) The restrictions of $\phi_m(u)$ to 1-dimensional simplices $[s_0 s_i]$ coincide with c_i for $i = 1, 2 \ldots, m$. In particular, $\phi_1(u) = u$ for all 1-umbrellas u.;
3) (Coherence) For every $i = 1, 2, \ldots, m$ the restriction of $\phi_m(u)$ to $(m-1)$-dimensional

simplex $[s_0, s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_m]$ coinsides with $\phi_{m-1}(u_i)$, where $u_i = (c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_m, \rho_i)$, and ρ_i is the restriction of ρ to the $(m-2)$-dimensional face of the standard simplex σ^{m-1} obtained by exclusion of the ith vertex.

(m, ϵ)-umbrellas can be regarded as generalization of cages, where one of the endpoints is being "enlarged" into a small simplex. (If this simplex degenerates into a point, the umbrella becomes a cage.) The next proposition asserts that there exists a generalization of the process of filling of cages described in the proof of Proposition 2 to (m, ϵ)-umbrellas for small ϵ. the idea of the proof of this proposition is very simple. Its main idea is that we can shrink the small simplex ρ in the definition of umbrellas over itself to a point, thus, obtaining a cage, which can be filled as in the proof of Proposition 2.

Proposition 3. Let $L, \bar{L}$ be positive numbers such that $\bar{L} \geq L$. Let x, y, z be any three points in a closed Riemannian manifold M^n such that the distance between any two of these three points does not exceed L. Let N be an integer number greater than 1. For every positive ϵ less than half of the injectivity radius of M^n the following assertion is true: Assume that there exists exactly one geodesic between x and y of length $\leq \bar{L} + (2N-3)L + (2N-2)\epsilon$. Then there exists a coherent N-filling of $(m, \epsilon)-$umbrellas based at x, z from $U^{L,\bar{L}}_{m,x,z}$.

Proof: The proof is inductive with respect to N. It follows the same pattern as the proof of Proposition 2. To prove the base of induction we define $\phi_1(u) = u$ for every 1-umbrella u. Assume now that the theorem holds for $N = m-1$, $(m > 1)$. To prove the theorem for $N = m$ note that conditions 1, 3 imply that we have no choice in construction $\psi_m(u) = \phi_m(u)|_{\partial\sigma^m}$: One of $(m+1)$ $(m-1)$-dimensional faces of σ^m must be mapped using the mapping ρ, and the remaining m $(m-1)$-dimensional faces should be mapped using $\phi_{m-1}(u_i)$. Using a part of the induction assumption we can conclude that ψ_m is a continuous function of u.

It remains only to contract $\psi_m(u)$ by a continuously depending on u homotopy. The idea is to eliminate the simplex ρ by contracting it over its image and then to proceed as in the proof of Proposition 2.

Recall that s_* denotes the center of σ^{m-1}. Fix a contraction h_t of $\sigma^{m-1} = [s_1, s_2, \ldots, s_m]$ to $\{s_*\}$, (h_0 is the identity map, $h_1(\sigma^{m-1}) = \{s_*\}$), such that all points of σ^{m-1} move to s_* along straight lines with a constant speed. This will provide us with a homotopy of umbrellas: If $u = (c_1, \ldots, c_m, \rho)$, then $H_t(u)$ is defined as $(c_{1t}, \ldots, c_{mt}, \rho \circ h_t)$, where for every i c_{it} is the join of c_i with $\rho([s_i h_t(s_i)])$. If u is an $(m, \epsilon)-$umbrella the length of c_{it} does not exceed the sum of the length of c_i and ϵ. For every $t \in [0, 1]$ we can consider $\psi_m(H_t(u)))$. The composition $\psi_m \circ H_t$ will constitute the first stage in a homotopy contracting $\psi_m(u)$.

It remains to contract $\psi_m(H_1(u))$. Note that $H_1(u)$ looks like an m-cage since its $(m-1)$-dimensional simplex is constant. Therefore we can contract the resulting $(m-1)$-dimensional sphere repeating almost verbatim the corresponding step in the proof of Proposition 2.

Namely, we use Lemma 1 to construct a path homotopy c_{1t}, $(t \in [1, 2])$, between $c_{11} = h_1(c_1)$ and $c_{21} = h_1(c_2)$ such that it passes only through paths of length $\leq \bar{L} + 2L + 3\epsilon$.

Let $u_t = (c_t, c_{21}, \ldots, c_{m1}, \rho \circ h_1)$. The next stage of our homotopy contracting $\psi_m(u)$ will consist of $(m-1)$-dimensional spheres $\psi_m(u_t)$, $t \in [1,2]$.

Finally, note that $u_2 = (c_{21}, c_{21}, \ldots, c_{m1}, \rho \circ h_1)$, so that $\phi_m(u_2)$ will be a "folded" $(m-1)$-dimensional sphere that can be canonically contracted over itself exactly as this had been done in the proof of Proposition 2. QED.

4. Proof of Theorem 1.

We are going to prove the theorem by contradiction. Assume that there exists exactly one geodesic between x and y of length $\leq 2qd$. Therefore there exists $\delta > 0$ such that there exists exactly one geodesic between x and y of length $\leq 2qd + \delta$. (Indeed, otherwise there will be a sequence of geodesics between x and y with lengths strictly decreasing to $2qd$. The Ascoli-Arzela theorem implies that a subsequence of this sequence converges to a geodesic between x and y of length $2qd \geq 2d > d$. Therefore, this geodesic cannot be minimizing and, therefore, is the second geodesic between x and y of length $\leq 2qd$, which contradicts to our assumption.) Let $\epsilon = \min\{\frac{\delta}{2n}, \frac{\mathrm{inj}(M^n)}{2}\}$, where $\mathrm{inj}(M^n)$ denotes the injectivity radius of M^n. Let $f : S^q \longrightarrow M^n$ be a non-contractible map of the q-dimensional sphere into M^n.

We are going to extend f to a map of the $(q+1)$-dimensional disc D^{q+1} thereby reaching the desired contradiction. First, choose a fine smooth triangulation of S^q such that that for every singular simplex $\tau : \sigma^i \longrightarrow S^{q+1}$, ($i \in \{1, \ldots, q+1\}$), the image under $f \circ \tau$ of σ^i is contained in an ϵ-ball centered at the image of the center of σ^i under $f \circ \tau$, and the length of the image of every straight line segment in σ^i under $f \circ \tau$ is less than ϵ.

Triangulate D^{q+1} as the cone over the chosen triangulation of S^{q+1}. Extend f to the 1-skeleton of the triangulation of D^{q+1} by mapping the center of D^{q+1} to x, and every new 1-dimensional simplices into a minimal geodesic between the images of the endpoints of the 1-simplex. (Here one can choose any minimal geodesic, if there is more than one.)

We are going to continue the extension process inductively. Assume that we have already extended f to the i-skeleton of the triangulation of D^{q+1}. In order to extend it to the $(i+1)$-skeleton observe that every new $(i+1)$-dimensional simplex is a cone over a i-dimensional simplex τ of the chosen triangulation of S^q. Consider a $(i+1, \epsilon)-$umbrella based at x and the image of the center of τ under f, such that $\rho = f \circ \tau$. Take $\bar{L} = L = d$. Apply Theorem 3 to fill this umbrella. The coherence condition implies that the resulting map of the $(i+1)$-dimensional simplex of the triangulation of D^{q+1} extends maps of its faces constructed on the previous steps of the induction.

Once f is extended to the $(q+1)$-skeleton of D^{q+1}, the extension process becomes complete, and we obtain the desired contradiction. QED.

5. Concluding remarks.

In [NR] we made the following conjecture:

Conjecture. For every positive integer k, every closed Riemannian manifold M^n and every pair of points $x, y \in M^n$ there exist k geodesics between x and y in M^n of length $\leq kd$, where d denotes the diameter of M^n.

If $x = y$, then we consider differently oriented geodesic loops based at x as different geodesics between x and y. (Otherwise the conjecture will be false.) This conjecture holds for round spheres. The example of round spheres shows that the conjectured upper bound in terms of k and d is optimal (if true). It is not difficult to demonstrate that the conjecture is true for all non-simply connected closed Riemannian manifolds with torsion-free fundamental groups - see [NR1].

Yet we are not able to prove this conjecture in any other case - even when M^n is a convex surface in R^3, $x = y$ and $k = 2$. (If $x = y$, then the conjecture asserts that the length of the shortest non-trivial geodesic loop based at x does not exceed $2d$). Therefore it makes sense to try to prove the existence of a curvature-free upper bound $c(n)kd$ or even $c(n, k)d$ for the lengths of first k geodesics between x and y.

In the present paper we found such a bound for $k = 2$ for an arbitrary M^n and arbitrary $x, y \in M^n$. Our forthcoming paper [NR2] will contain the only other result in this direction that we are currently able to prove: If $n = 2$ and M^n is diffeomorphic to S^2 then for every k and every pair of points x, y in the Riemannian manifold there exist k distinct geodesics between x and y of length $\leq c(k)d$, where $c(k)$ depends only on k.

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