ON (CO)HOMOLOGY OF 2-TYPES AND CROSSED MODULES

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1 Introduction

A connected CW-complex X is called 2-type if $\pi_i X = 0$ for $i \neq 1, 2$. In [8] Ellis investigated the homology of 2-types by means of crossed modules, which are algebraic models for 2-types (see [1] and [11] for this and more general results). On the other hand in [4] were defined cohomologies of internal categories in categories of groups with operations. Since internal categories in the category of groups are equivalent to crossed modules (see [11]) the approach of [4] gives also a cohomology theory for crossed modules. This theory was further developed in [5], where cohomologically trivial crossed modules are characterized.

The motivation for the present paper was an investigation of a possible relation between above mentioned two cohomologies and an application of arguments analogous of those given in [14] in order to obtain more information on (co)homology of 2-types.

The relation between cohomologies from [4] and [8] is given in the part i) of Theorem 5.3. For any crossed module (T, G, ∂) Ellis constructed a spectral sequence with abutment the (co)homology of (T, G, ∂) in the sense of [8] (see Corollary 3 of [8] or the first part of Theorem 5.3). It turns out that E_2^{*1} is isomorphic to the cohomology theory considered in [4]. The result of [4] shows that $E_2^{p1} = 0$ provided $p > 2$. For crossed modules with trivial kernel it was proved in [13] in slightly different context that $E^2_{pq} = 0$ provided $p > q$. Moreover, it was proved in [14] a general result, which implies that for any crossed module $E^2_{pq} = 0$ and $E_2^{pq} = 0$ as soon as $p > q$. In the present paper we give a simpler proof of this particular fact. We also improve some results of Ellis [8]. Among other results obtained in this paper we mention a new proof of Theorem 8 of [8], which describes the homology of free crossed modules in dimensions ≤ 4. The original proof used the theory of quadratic complex of Baues [1]. Our proof is based on a theorem of Ratcliffe [15] and uses a standard argument with Serre spectral sequence. Our method works for projective crossed modules and gives an information on homology in dimensions ≤ 6.

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2 Preliminaries on crossed modules

Let G be a group. A *G-group* is a group T together with a homomorphism $\xi : G \to Aut(T)$. For $g \in G$ and $t \in T$ we put ${}^{g}t := \xi(g)(t)$. A *G-crossed module* is a G-group T together with group homomorphism $\partial : T \to G$ satisfying:

$$\partial({}^{g}t) = g\partial(t)g^{-1} \text{ and } {}^{\partial t}s = tst^{-1},\ \ g \in G, t, s \in T.$$

Under the crossed module we shall mean a triple (T, G, ∂), where G is a group and $\partial : T \to G$ is a G-crossed module. It follows from the definition that the image $Im(\partial)$ is a normal subgroup of G, and the kernel $Ker(\partial)$ is in the center of T. Moreover the action of G on T induces an action of Q on $Ker(\partial)$, where $Q = Coker\ \partial$.

A *homomorphism* of crossed modules is a pair of homomorphisms $(\phi, f) : (T, G, \partial) \to (T', \partial', G')$ such that

$$f \circ \partial = \partial' \circ \phi, \quad \phi({}^{g}t) =^{f(g)} \phi(t), g \in G, t \in T.$$

We let **XMod** be the the category of crossed modules. For a group Q and for a Q-module A one denotes by $\mathcal{X}ext(Q, A)$ the category of short exact sequences

$$0 \to A \to T \xrightarrow{\partial} G \to Q \to 0,$$

where (T, G, ∂) is a crossed module and the action of Q on A induced from the crossed module structure coincides with the prescribed one. The morphisms in $\mathcal{X}ext(Q, A)$ are commutative diagrams

$$\begin{array}{ccccccccccc}
0 & \to & A & \to & T & \xrightarrow{\partial} & G & \to & Q & \to & 0 \\
 & & id\downarrow & & \phi\downarrow & & f\downarrow & & id\downarrow & & \\
0 & \to & A & \to & T' & \xrightarrow{\partial'} & G' & \to & Q & \to & 0,
\end{array}$$

where (ϕ, f) is a morphism of crossed modules $(T, G, \partial) \to (T', G', \partial)$. We let $Xext(Q, A)$ be the class of the connected components of the category $\mathcal{X}ext(Q, A)$.

For a simplicial group G_* we let N_*G_* be the Moore normalization of G_* (see [12]), which is a chain complex, whose n-dimensional component is given by

$$N_nG_* = \bigcap_{i>0} Ker(\partial_i^n : G_n \to G_{n-1}),\ \ n > 0$$

while the boundary map $d : N_nG_* \to N_{n-1}G_*$ is the restriction of ∂_0 on N_nG_*. By [11] we know that the Moore normalization yields an equivalence of categories $N_* : \mathbf{SG}_{\leq 1} \to$ **XMod**. Here $\mathbf{SG}_{\leq 1}$ is the category of simplicial groups of length one.

Let (T, G, ∂) be a crossed module. Since G acts on T, it acts also on T^n by ${}^g(t_1, \cdots, t_n) = ({}^g t_1, \cdots, {}^g t_n)$. Hence we can take the semi-direct product $G_n = T^n \rtimes G$. Here $G_0 = G$. Let $N_*^{-1}(T, G, \partial)$ be a simplicial group, which is given as follows

$$N_n^{-1}(T, G, \partial) = G_n, \ n \geq 0,$$

$$\partial_i(t_1, \cdots, t_n, g) = (t_1, \cdots, \hat{t}_i, \cdots, t_n, g), \ 1 \leq i \leq n,$$
$$\partial_0(t_1, \cdots, t_n, g) = (t_2 t_1^{-1}, \cdots, t_n t_1^{-1}, \partial t_1 g),$$
$$s_i(t_1, \cdots, t_n, g) = (t_1, \cdots, t_{i-1}, 0, t_i, \cdots, t_n, g).$$

One easily checks that $N_*(N_*^{-1}(T, G, \partial)) \cong (T, G, \partial)$. Thus $N_*^{-1} : \mathbf{XMod} \to \mathbf{SG}_{\leq 1}$ is also an equivalence of categories and for any $G_* \in \mathbf{SG}_{\leq 1}$ one has an isomorphism $N_*^{-1}(N_* G_*) \cong G_*$

Now for a crossed module (T, G, ∂) we can define the classifying space $B(T, G, \partial)$ to be the classifying space of the simplicial group $N_*^{-1}(T, G, \partial)$. Hence $\pi_i B(T, G, \partial) = 0, \ i > 1$ and

$$\pi_1 B(T, G, \partial) \cong Coker\ \partial, \ \pi_2 B(T, G, \partial) \cong Ker\ \partial.$$

By abuse of notation we shall write

$$\pi_1(T, G, \partial) = Coker\ \partial, \ \pi_2(T, G, \partial) = Ker\ \partial, \ \pi_i(T, G, \partial) = \ 0, \ i > 2.$$

Thus $B(T, G, \partial)$ is a two-type, meaning a path-connected CW-complex X with $\pi_i X = 0$ for $i \geq 3$. Therefore the homotopy type of $B(T, G, \partial)$ is completely determined by π_1, π_2 and the corresponding k-invariant of $B(T, G, \partial)$ which is an element in $H^3(\pi_1, \pi_2)$. We let $k(T, G, \partial)$ be this element. So one gets a map

$$k : Xext(Q, A) \to H^3(Q, A).$$

According to a result of Loday [10] this map is a bijection.

A morphism of crossed modules $(\phi, f) : (T, G, \partial) \to (T', \partial', G')$ is said to be a *weak equivalence* if it induces isomorphisms $\pi_1(T, G, \partial) \cong \pi_1(T', \partial', G')$ and $\pi_2(T, G, \partial) \cong \pi_2(T', \partial', G')$. It is well-known that there is an equivalence of categories

$$\text{Ho } (2-\text{types}) \ \to \text{Ho (crossed modules)}.$$

between corresponding homotopy categories (see for example [11], [1]). Here Ho $(2-\text{types})$ (resp. Ho (crossed modules)) is the localization of the category of 2-types (resp. crossed modules) with respect of weak equivalences.

3 On homology of crossed modules

Let (T, G, ∂) be a crossed module and let M be a $\pi_1(T, G, \partial)$-module. According to [8] we define the *homology* $H_*((T, G, \partial), M)$ *and cohomology* $H^*((T, G, \partial), M))$ *of* (T, G, ∂) *with coefficients in* M to be the homology and cohomology of $B(T, G, \partial)$ with coefficients in the local system corresponding to M. If $(\phi, f) : (T, G, \partial) \to (T', \partial', G')$ is a weak equivalence then it induces an isomorphism $H_*((T, G, \partial), M) \to H_*((T', \partial', G'), M)$. Of course the similar statement is true in cohomological framework. Let us note that if $T = 0$, then $\pi_1 = G$ and $B(T, G, \partial) = K(G, 1)$. Hence in this case one obtains the classical homology and cohomology of groups. The second extreme case is when $G = 1$. In this case one has $B(T, G, \partial) \cong K(\pi_2, 2)$. The homology of Eilenberg - Mac Lane spaces $K(A, 2)$ are well understood thanks to the work of Eilenberg and Mac Lane, Serre and Cartan. It is well known that

$$H_1K(A, 2) = 0 = H_3K(A, 2),\ \ H_2K(A, 2) \cong A, \quad H_4K(A, 2) \cong \Gamma^2 A.$$

Here $\Gamma^2 A$ is the Whitehead's universal quadratic functor, which is the degree two part of the divided power algebra $\Gamma^* A$ generated by A [7]. Here we assume that elements of A have degree two. Moreover if A is a torsion free abelian group, then $H_*K(A, 2) \cong \Gamma^* A$. Hence in this case $H_iK(A, 2) = 0$ for all odd i. For general abelian groups one can find in [7] that $H_6K(A, 2) \cong \Gamma^3 A$ and $H_5K(A, 2) \cong R(A)$. Here $R : Ab \to Ab$ is the first simplicial derived functor of the functor Γ^2. It is known that $R(A \oplus B) \cong R(A) \oplus R(B) \oplus Tor(A, B)$. Moreover R preserves all filtered colimits, vanishes on torsion free groups and $R(\mathbf{Z}/n) = \mathbf{Z}/(2, n)$. Here $(2, n)$ is the ideal in $\mathbf{Z}$ generated by 2 and n.

If the action of π_1 on π_2 is trivial and $k(T, G, \partial) = 0$, then $B(T, G, \partial)$ has the same homotopy type as the product $K(\pi_1, 1) \times K(\pi_2, 2)$ and hence the Künneth theorem describes the (co)homology of (T, G, ∂) in terms of (co)homology of π_1 and $K(\pi_2, 2)$. Such situation arises when (T, G, ∂) is abelian. We recall that (T, G, ∂) is abelian if both G and T are abelian groups and the action of G on T is trivial. In this case the k-invariant is trivial because it lies in the image of

$$Ext^2_{\mathbf{Z}}(\pi_1, \pi_2) \to Xext(\pi_1, \pi_2) \cong H^3(\pi_1, \pi_2)$$

and $Ext^2_{\mathbf{Z}} = 0$. This simple observation shows that the homology of abelian crossed modules $H_*(T, G, \partial)$ completely can be described in terms of π_1 and π_2. This improves Theorem 24 of [8].

In general the classifying space of a crossed module (T, G, ∂) fits in a fibration sequence

$$K(\pi_2, 2) \to \mid B(T, G, \partial) \mid \to K(\pi_1, 1).$$

Therefore there exists a spectral sequences

$$E^2_{pq} = H_q(\pi_1, H_q(K(\pi_2, 2), M)) \Longrightarrow H_{p+q}((T, G, \partial), M) \tag{3.1}$$

$$E_2^{pq} = H^q(\pi_1, H^q(K(\pi_2, 2), M)) \Longrightarrow H^{p+q}((T, G, \partial), M).$$

Since the space $K(\pi_2, 2)$ is 1-connected one can use the universal coefficient theorem to get isomorphisms

$$E^2_{*0} = H_*(\pi_1, M), \quad E^2_{*1} = 0, \quad E^2_{*2} = H_*(\pi_1, M \otimes \pi_2),$$
$$E^2_{*3} = H_*(\pi_1, Tor_1^{\mathbf{Z}}(M, \pi_2), \quad E^2_{*4} = H_*(\pi_1, M \otimes \Gamma^2 \pi_2)$$

and exact sequences

$$\cdots \to H_*(\pi_1, M \otimes R(\pi_2)) \to E^2_{*5} \to H_*(\pi_1, Tor_1^{\mathbf{Z}}(M, \Gamma^2 \pi_2)) \to$$
$$H_{*-1}(\pi_1, M \otimes R(\pi_2)) \to E^2_{*-1,5} \to H_{*-1}(\pi_1, Tor_1^{\mathbf{Z}}(M, \Gamma^2 \pi_2)) \to \cdots$$

and

$$\cdots \to H_*(\pi_1, M \otimes \Gamma^3(\pi_2)) \to E^2_{*6} \to H_*(\pi_1, Tor_1^{\mathbf{Z}}(M, R(\pi_2))) \to$$
$$H_{*-1}(\pi_1, M \otimes \Gamma^3(\pi_2)) \to E^2_{*-1,6} \to H_{*-1}(\pi_1, Tor_1^{\mathbf{Z}}(M, R(\pi_2))) \to \cdots$$

It is well-known that the first nontrivial differential

$$d_3 : E^2_{p0} = H_p(\pi_1, M) \to H_{p-3}(\pi_1, M \otimes \pi_2) = E^2_{p-3,2}$$

is equal to $(-) \cap k$, where $k \in H^3(\pi_1, \pi_2)$ is the k-invariant of (T, G, ∂) (see Theorem 10.4 of [9]). Of course the similar results hold for cohomology as well. For instance the differential $d^3 : H^{p-3}(\pi_1, Hom(\pi_2, M)) \to H^p(\pi_1, M)$ fits in the commutative diagram

$$\begin{array}{ccc} H^{p-3}(\pi_1, Hom(\pi_2, M)) & \xrightarrow{\cup\, k} & H^p(\pi_1, \pi_2 \otimes Hom(\pi_2, M)) \\ d^3 \searrow & & \swarrow ev_* \\ & H^p(\pi_1, M) & \end{array}$$

where $ev : \pi_2 \otimes Hom(\pi_2, M) \to M$ is given by $x \otimes f \mapsto f(x)$.

4 On homology of projective crossed modules

Let G be a fixed group and $\delta : Z \to G$ be a function from a set Z to G. Consider the category of crossed modules (T, G, ∂) together with maps $\alpha : Z \to T$ satisfying $\partial\alpha = \delta$. One can prove that this category has an initial object, which is called *free crossed module* generated by δ (see [15] and references given there). A G-crossed module (T, G, ∂) is called projective if it is a projective object in the category of G-crossed modules (see [15]). Clearly any free crossed module is projective. A theorem of Ratcliffe [15] claims that (T, G, ∂) is projective iff T_{ab} is projective as $\pi_1 = Coker\partial$-module and $\partial_* : H_2T \to H_2N$ is trivial. Here $N = Im\partial$. The aim of this section is to prove the following theorem.

Theorem 4.1 *Let G be a free group and let (T, G, ∂) be a projective G-crossed module. Let $\pi_1 = Coker\partial$ and $\pi_2 = Ker\partial$. Let M be a π_1-module. Then there are isomorphisms*

$$H_3((T,G,\partial),M) = 0, \quad H_4((T,G,\partial),M) \cong H_0(\pi_1, \Gamma^2\pi_2 \otimes M)$$

$$H_5((T,G,\partial),M) \cong H_1(\pi_1, \Gamma^2\pi_2 \otimes M)$$

and one has an exact sequence:

$$H_3(\pi_1, \Gamma^2\pi_2 \otimes M) \to H_0(\pi_1, \Gamma^3\pi_2 \otimes M) \to$$
$$\to H_6((T,G,\partial),M) \to H_2(\pi_1, \Gamma^2\pi_2 \otimes M) \to 0.$$

Remark. The results on H_3 and H_4 were obtained by Ellis (see Theorem 8 of [8]). His proof uses results of [1] on quadratic complexes. We leave the task to the interested reader to make the trivial reformulation for the corresponding result for cohomology groups.

Proof. Let $N = Im\partial$. Then one has short exact sequences

$$0 \to \pi_2 \to T \to N \to 0$$

and

$$0 \to N \to G \to \pi_1 \to 0.$$

Since N is free and the first short exact sequence is central, it splits and hence induces short split exact sequences of abelian groups

$$0 \to \pi_2 \to T_{ab} \to N_{ab} \to 0,$$

$$0 \to \pi_2 \otimes M \to T_{ab} \otimes M \to N_{ab} \otimes N \to 0.$$

However, these are in general nontrivial extensions in the category of π_1-modules. The class of the first short exact sequence in $Ext^1_{\mathbf{Z}[\pi_1]}(N_{ab}, \pi_2)$ is denoted by k_1. Since G is free, the second short exact sequence yields the following short exact sequence

$$0 \to N_{ab} \to G/[N,G] \to \pi_1 \to 0.$$

Denote its corresponding class in $H^2(\pi_1, N_{ab})$ by k_2. Clearly $k = k_1 \cup k_2$. Since G is free it follows from the cup product reduced theorem [16] that

$$\cap\, k_2 : H_n(\pi_1, M) \to H_{n-2}(\pi_1, M \otimes N_{ab})$$

is an isomorphism if $n \geq 3$ (this fact is also a consequence of the Hochschild-Serre spectral sequence for the following short exact sequence of groups $1 \to N \to G \to \pi_1 \to 1$ and the fact that homologies of free groups vanishes in dimension ≥ 2). By Theorem of Ratcliffe T_{ab} is a projective π_1-module. Hence $T_{ab} \otimes M$ is also a projective π_1-module and

$$\cap\, k_1 : H_n(\pi_1, M \otimes N_{ab}) \to H_{n-1}(\pi_1, M \otimes \pi_2)$$

is an isomorphism if $n \geq 2$ and monomorphism for $m = 1$. Then follows that the differential

$$d_3 = \cap\, k : H_n(\pi_1, M) \to H_{n-3}(\pi_1, M \otimes \pi_2)$$

of the Serre spectral sequence

$$E^2_{pq} = H_p(\pi_1, H_q(K(\pi_2, 2), M)) \Longrightarrow H_{p+q}((T, G, \partial), M)$$

is an isomorphism for $n \geq 4$ and is a monomorphism for $n = 3$. Since π_2 is a subgroup of a projective $\mathbf{Z}[\pi_1]$-module, it is free as an abelian group and by the arguments from Section 3 we have $E^2_{pq} = 0$ for odd q and $E^2_{pq} = H_p(\pi_, M \otimes \Gamma^n \pi_2)$ for $q = 2n$, $n \geq 0$ and result follows.

5 Another spectral sequence

In this section we consider the spectral sequence constructed in Corollary 3 of [8] which converges to $H_*(T, G, \partial)$. It is a consequence of a general result of [14] that $E^2_{pq} = 0$ for $p > q$. We give here more direct proof of this fact, which allows also to get more informations on E^2.

Let G_* be a simplicial group and let M be a $\pi_0(G_*)$-module. Since $\pi_1 BG_* = \pi_0 G_*$ one can consider M as a local system on on the classifying space BG_* of G_*. By abuse of notation we will denote the homology and cohomology of BG_* with coefficient in M by $H_*(G_*, M)$ and $H^*(G_*, M)$. Clearly for $G_* = N_*^{-1}(T, G, \partial)$ one recovers the groups considered in section 3. Here $N_*^{-1} : \mathbf{Xmod} \to \mathbf{SG}_{\leq 1}$ was constructed in Section 2.

We give also an algebraic definition of the same groups. For a group G and G-module A we let $C_*(G, A)$ (resp. $C^*(G, A)$) be the standard chain (resp. cochain) complex for the group homology (resp. cohomology). Now if G_* is a simplicial group and if M is a $\pi_0(G_*)$-module, then one can form the bicomplex $C_*(G_*, M)$ (resp. $C^*(G_*, M)$) whose (n, m)-component is $C_n(G_m, M)$ (resp. $C^n(G_m, M)$). The following fact is well-known.

Lemma 5.1 *Let G_* be a simplicial group and let M be a $\pi_0(G_*)$-module. Then one has the following isomorphisms and spectral sequences*

$$H_*(Tot(C_*(G_*, M))) \cong H_*(G_*, M), \quad H^*(Tot(C^*(G_*, M))) \cong H^*(G_*, M)$$

$$E^1_{pq} = H_q(G_p, M) \Longrightarrow H_{p+q}(G_*, M), \quad E_1^{pq} = H^q(G_p, M) \Longrightarrow H^{p+q}(G_*, M).$$

Proof. If G_* is a simplicial group, then $B(G_*)$ is a diagonal of the bisimplicial set associated to the simplicial set given by

$$[n] \mapsto B(G_n).$$

Now the result is a consequence of the Eilenberg-Zilberg-Cartier theorem [6].

Let us observe that E^1_{*q} is a simplicial abelian group. So we can take the Moore normalization of it to obtain $N_*E^1_{*q}$. Similarly, E_1^{*q} is a cosimplicial abelian group and we still can take the normalization of it to obtain $N^*E_1^{*q}$ (see [2]). Then the classical normalization theorem [12] together with Theorem 5.1 of [14] imply the following

Corollary 5.2 *Let G_* be a simplicial group and let M be a $\pi_0(G_*)$-module. Then there exists a spectral sequence*

$$N_pE^1_{*q} \Longrightarrow H_{p+q}(G_*, M),$$

which has the same E^r_{**} terms as previous one as soon as $r \geq 2$. Moreover if $N_nG_* = 0$ for $n > k$ then $N_pE^1_{*q} = 0 = E^r_{pq}$ provided $p > kq$ and $r \geq 2$. Similarly for cohomology.

Theorem 5.3 *Let (T, G, ∂) be a crossed module and let M be a $\pi_1(T, G, \partial)$-module.*

i) Then there exist spectral sequences

$$E_1^{pq} = H^q(G_p, M) \Longrightarrow H^{p+q}((T, G, \partial), M),$$

$$E^1_{pq} = H_q(G_p, M) \Longrightarrow H_{p+q}((T, G, \partial), M).$$

Here $G_* = N_*^{-1}(T, G, \partial)$ and hence $G_p = T^p \rtimes G$. The cohomology of (T, G, ∂) with coefficients in M in the sense of [4] is isomorphic to E_2^{p1} provided $p > 0$.

ii) One has $N_pE^1_{*q} = 0 = E^r_{pq}$ as soon as $p > q$ and $r \geq 2$. The similar vanishing result holds also for cohomology.

iii) One has isomorphisms

$$N_pE^1_{*p} \cong H_0(G, (T^{ab})^{\otimes p} \otimes M)$$

$$N^pE_1^{*p} \cong Hom_G((T^{ab})^{\otimes p}, M).$$

Here $p \geq 1$ and $T^{ab} = T/[T, T]$.

Remarks. 1) The spectral sequence is the same as in Corollary 3 of [8].
2) The existence of the spectral sequences as well as part ii) are consequence of Corollary 5.2. Indeed it suffices to take $G_* = N^{-1}(T, G, \partial)$ in Corollary 5.2. However we give also a new proof for ii), which is far simpler then the original proof of the vanishing result in Corollary 5.2 given in [14].

The proof of part ii) and iii) of Theorem 5.3 is based on the notion of a degree of a functor [7]. Since we are in a slightly different situation then *loc. cit.* we give definitions of a cross-effect and of a degree of a functor in our framework. We let $\mathbf{G}_G$ be the category of G-groups. For a functor $F : \mathbf{G}_G \to Ab$ with $F(0) = 0$ one can define the *cross-effect* functors $cr_n F : \mathbf{G}_G^n \to Ab,\ n \geq 0$ as follows. We put $cr_1 F = F$. Then the bifunctor $cr_2 F$ is defined by the short exact sequence

$$0 \to cr_2 F(T_1, T_2) \to F(T_1 \times T_2) \to F(T_1) \times F(T_2) \to 0.$$

In general

$$cr_n F(T_1, \cdots, T_n) =$$
$$Ker(F(T_1 \times \cdots \times T_n) \to \prod_i F(T_1 \times \cdots \times \hat{T_i} \times \cdots \times T_n)).$$

We claim that *if*

$$0 \to F_1 \to F \to F_2 \to 0$$

is a short exact sequence of functors, then

$$0 \to cr_n F_1 \to cr_n F \to cr_n F_2 \to 0$$

is also an exact sequence, $n \geq 1$. The claim is obvious for $n = 1$ and for $n = 2$ it is a consequence of 3×3 lemma. The general case follows from this by induction, because

$$cr_n F(X_1, \cdots, X_n) \cong cr_2(U(X_1, \cdots, X_{n-2}))(X_{n-1}, X_n).$$

Here

$$U(X_1, \cdots, X_{n-2}) : \mathbf{G}_G \to Ab$$

is given by

$$U(X_1, \cdots, X_{n-2})(Y) = cr_{n-1} F(X_1, \cdots, X_{n-2}, Y).$$

Following [7] we will say that a functor F is of *degree* $\leq n$ if $cr_{n+1} F = 0$. Thus if $degF = n$ and F' is a subquotient of F then $degF' \leq n$.

Proof of Theorem 5.3. i) Here we have only to establish relationship between the definition of cohomology in [4] and the one given in Section 3. Comparing the definitions one sees that the cochain complex $K^*((T, G, \partial), M)$ constructed in Section 3 of [4] is isomorphic to the cochain complex $Der(G_*, M)$ and the exact sequence of cosimplicial abelian groups

$$0 \to H^0(G_*, M) \to M \to Der(G_*, M) \to H^1(G_*, M) \to 0$$

shows that one has an isomorphism $H^p K^*((T, G, \partial), M) \cong E_2^{p1}$ provided $p > 0$.

ii) and iii) We consider only homology. The dual argument works for cohomology as well. The case $q=0$ is trivial, because in this case $E^1_{*0}=H_0(G_*,M)$ is a constant cosimplicial abelian group with value $H_0(\pi_1,M)$. So we assume that $q>0$.

Let N be a G-module and let K be a G-group. We can consider N as a trivial K-module. It is a consequence of the Künneth theorem that the functor

$$H_n(-,N):\mathbf{G}_G\to Ab,\ K\mapsto H_n(K,N)$$

is of degree n and the last nontrivial cross-effect is

$$cr_n(H_n(-,N))(K_1,\cdots,K_n)=K_1^{ab}\otimes\cdots\otimes K_n^{ab}\otimes N.$$

For any $p\geq 0$ and $q\geq 1$ we let

$$F_{pq}:\mathbf{G}_G\to Ab$$

be a functor given by $F_{pq}(K)=H_p(G,H_q(K,N))$. Since cr_n is exact and preserves direct sums we see that $deg(F_{pq})\leq q$ and

$$cr_qF_{pq}(K_1,\cdots,K_q)\cong H_p(G,K_1^{ab}\otimes\cdots\otimes K_q^{ab}\otimes N).$$

We need also a functor $F_q:\mathbf{G}_G\to Ab$ given by

$$F_q(K):=Ker(H_q(K\rtimes G,N)\to H_q(G,N)).$$

It is clear that $F_q(0)=0$ and by the Hochschild-Serre spectral sequence one has the spectral sequence of functors

$$E^2_{pq}=F_{pq}\Longrightarrow F_{p+q},\quad q>0.$$

Let us fix an integer n. Since the correspondence $F\mapsto cr_nF$ gives an exact functor we have aslo the spectral sequence $E^2_{pq}=cr_nF_{pq}\Longrightarrow cr_nF_{p+q}$. Since $degF_{pq}\leq q$ we see that $E^2_{pq}=0$ for $q<n$ and $E^2_{0n}=H_0(G,cr_nH_n)$. Therefore $degF_n\leq n$ and $cr_nF_n=cr_nF_{0n}=H_0(G,cr_nH_n)$.

Now we take $N=M$. One observs that the constructions of $G_n=T^n\rtimes G$ and $\partial_i,i>0$ use only the action of G on T and not the homomorphism $\partial:T\to G$, which appears only in the definition of ∂_0. Hence $N_pE^1_{*q}$ depends only on T considered as G-module and not on ∂. Comparing the definitions it is obvious that for any crossed module (T,G,∂) one has the following isomorphism $N_pE^1_{*q}(T,G,\partial)\cong cr_pF_q(T,\cdots,T)$. This completes the proof.

Let us recall that for any crossed module (T,G,∂) there is a weak homotopic equivalence $(T',G',\partial')\to(T,G,\partial)$ with free group G' (see for example page 11 in [8]). Our next result is an immediate consequence of Theorem 5.3.

Corollary 5.4 *For a crossed module (T, G, ∂) one has an exact sequence*

$$H_2G \to H_2(T, G, \partial) \to H_0(G, T^{ab}) \to G^{ab} \to H_1(T, G, \partial) \to 0.$$

Moreover if G is a free group then

$$H_2(T, G, \partial) \cong Ker(T/[T, G] \to G^{ab})$$

and there exists an exact sequence

$$H_3(T \rtimes G) \to H_4(T, G, \partial) \to T^{ab} \otimes_G T^{ab} \xrightarrow{\alpha} H_2(T \rtimes G) \to H_3(T, G, \partial) \to 0.$$

The similar result is true also for cohomology.

Remarks. 1) The isomorphism $H_2(T, G, \partial) \cong Ker(T/[T, G] \to G^{ab})$ for free G was proved by Ellis (see Theorem 6 of [8]) using different methods.

2) The homomorphism $\alpha : T^{ab} \otimes_G T^{ab} \to H_2(T \rtimes G)$ is the restriction of $d_0 : H_2(T^2 \rtimes G) \to H_2(T \rtimes G)$ on the subgroup $T^{ab} \otimes_G T^{ab} \subset H_2(T^2 \rtimes G)$ and hence in terms of Brown-Loday tensor product [3] it is given by $\bar{x} \otimes \bar{y} \to (x^{-1}, \partial x) \otimes (y, 1)$. Here $x, y \in T$ and $\bar{x}$ denotes the class of x in T^{ab}.

Corollary 5.5 *Let (T, G, ∂) be a crossed module. Choise any crossed module (T', G', ∂') which is weakly equivalent to (T, G, ∂), such that G' is a free group. Then there is an isomorphism*

$$H^2((T, G, \partial), A) \cong \mathbf{H}^1((T', G', \partial'), A),$$

where the right hand side is an internal category cohomology corresponding to the crossed module (T', G', ∂') defined in [4].

The proof is the direct consequence of Theorem 3.1 of [4] and Theorem 6 of [8] (or the cohomological version of the previous Corollary).

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