Discrete Symmetries and multi-Poisson structures of 1+1 integrable systems

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Discrete Symmetries and multi-Poisson structures of $1+1$ integrable systems

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Abstract

The pairs of Hamiltonians operators for $(1+1)$ integrable systems are constructed in explicit form. Only one assumption of invariance of equations of such systems with respect to discrete transformation was used.

1 Introduction

Roughly speaking, the most characteristic properties of $(1+1)$ completely integrable systems can be described as follows. Every such system has infinite number of involutive conservation laws, which can be written as

$$\frac{\partial I_0^n}{\partial t} + \frac{\partial I_1^n}{\partial x} = 0. \tag{1.1}$$

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Each time component I_0^n can be considered as a Hamiltonian density, generating with the help of the corresponding Poisson brackets a completely integrable equation. The totality of these equations form a so-called hierarchy of integrable equations. To find the explicit form of a hierarchy it is necessary to know all conserved quantities I_0^n. There are two main methods to find conserved quantities. One of them based on the inverse scattering method, consists of concerning a recurrence relation allowing to get consequently all conserved quantities, starting from a few initial ones. The other method uses the fact that any completely integrable equation, can be written using different Poisson brackets and different Hamiltonians. The different Poisson brackets are constructed with the help of the corresponding Hamiltonian operators. Knowing two Hamiltonian operators, forming a Hamiltonian pair [1],[2] we can find all conserved quantities, starting again from a few initial ones.

These facts were known for a long time. However, there were no simple constructive methods for finding integrals and Hamiltonian pairs. The problem of finding of Hamiltonian operators changed radically after discovering the fact that all the most important properties of completely integrable systems, including their explicit solutions, are the direct consequence of their discrete symmetry. It became clear that the requirement of the invariance of a Hamiltonian operator with respect to the discrete symmetry is a powerful method of constructing the corresponding hierarchy of Hamiltonian structures.

In papers [4] the invariance condition was used to formulate a constructive procedure to find coefficients of the expansion of a Hamiltonian operator over powers of the total space derivative operator D. The starting point of the calculations was there the fixation of the maximal positive degree of the operator D. The proposed method allowed, in principle, to find successfully all the required coefficients. It was essential to know the behaviour of the conserved quantities under the discrete transformation.

The goal of present paper as a consequence of analyze of the results of [4], to understand the general structure of Hamiltonian operators, their connection with conserved quantities and obtain explicit expressions for them in the case of $(1+1)$ integrable systems.

2 Discrete transformation of integrable systems

All integrable systems under consideration are invariant with respect to discrete local invertible transformation describing by the substitution

$$\tilde{u} = \phi(u, u', u'', ..., u^{(r)}) \equiv \phi(u) \tag{2.1}$$

where u is s-dimensional vector function, u',u",.. its derivatives of the corresponding order with respect to "space" coordinates.

The property of invertibility means that equality (2.1) can be resolved and "old" function u may be written locally in terms of new functions $\tilde{u}$ and its derivatives.

Frechet derivative $\phi'(u)$ corresponding to substitution (2.1) is the $s \times s$ matrix operator defined as

$$\phi'(u) = \phi_u + \phi_{u'} D + \phi_{u''} D^2 + ... \tag{2.2}$$

where D^m is operator of m-time differentiation with respect to corresponding $u''^{..'}$ space coordinates. The reader can find more detail information about this object in [1].

Let us consider the equation

$$F_n(\phi(u) = \phi'(u) F_n(u) \tag{2.3}$$

which in some other different notations was considered firstly in [3]. where $F_n(u)$ is unknown s-component vector function, each component of which depends on u and its derivatives up to n-order.

The equation (2.3) possesses one obvious trivial solution $F_n(u) = u'$ for each substitution. In order to prove this it is sufficient to differentiate the equation (2.1) once with respect to some of its space coordinate.

If the equation (2.2) possesses some other solution (for a given $\phi(u)$) except of the trivial one, then we will call such substitution as the integrable substitution or mapping.

We emphasize once more that equation (2.2) contain two unknown functions $\phi(u)$ and $F_n(u)$ and only for narrow class of integrable substitutions it possesses nontrivial solutions for $F_n(u)$ function.

It is possible to connect the equation (system) of evolution type with each nontrivial solution of (2.2)

$$u_t = F_n(u) \tag{2.4}$$

which is obviously invariant with respect to substitution $u \to \phi(u)$. (It is easy to see that equation $u_t = u'$ is indeed invariant with respect to arbitrary substitution.)

Let us now compare the equation (2.3) with definition of linear representation $T(g)$ of some group (for definiteness Lee group)

$$\Phi(gx) = T(g)\Phi(x) \tag{2.5}$$

where g is the group element,$T(g)$ is the group operator for some representation, $\Phi(x)$ the basis of the corresponding representation space.

The obvious correspondence takes place after comparison (2.5) with (2.2)

$$\Phi(x) \to F_n(u), \quad T(g) \to \phi'(u)$$

If this correspondence has a deep group theoretical foundation then it is possible to anticipate that the different solutions of the equation (2.2) are connected by some linear transformation

$$F_n(u) = H^{n,n'} F_n(u) \tag{2.6}$$

Indeed all solutions (with different n) of the equation (2.2) (from this point of view) are the basis vectors of some group representation with the group operator $\phi'(u)$ and in the case of its irreducibility all possible basics of this representation are connected by linear transformation.

The exactly same situation takes place in the theory of $(1+1)$ integrable systems. Operators $H^{n,n'}$ in this theory are known as Hamiltonian operators of different degree. We will show below how to construct these operators using only discrete symmetry requirement.

3 General structure of Hamiltonian operators

The solutions of following two equations will be important for further consideration

$$\phi'(u)H(u)\phi'(u)^{-1} = H(\phi(u)), \quad \phi'(u)J(u)\phi'(u)^T = J(\phi(u)) \tag{3.1}$$

where $\phi'(u)^T = \phi_u^T - D\phi_{u'}^T + D^2\phi_{u''}^T - ...$, and $H(u)$, $J(u)$ are unknown $s \times s$ matrix operators, the matrix elements of which are polynomial of some finite order with respect to operator of differentiation D (of its positive and negative degrees).

From (3.1) and (2.2) it follows immediately that if $F_n(u)$ is some solution of main equation (2.2) then $H^p(u)F_n(u)$ (p is arbitrary natural number) will be some other solution of the same equation.

The solution of the second equation (3.1) under additional condition of its skew symmetry may be connected (interpreted) as a Poisson structure which is invariant with respect to transformation of discrete symmetry. Skew symmetrical operators $J(u)$ are known as Hamiltonian ones. Two different solutions of the second equation (3.1) (if it is possible to find them), for instance $J_1(u)$ and $J_2(u)$ in combination $J_1J_2^{-1}$ satisfy the first equation (3.1). Operator $J_1J_2^{-1}J_1(u)$ is again the solution of the second equation (3.1) and so on. This is the way how in the theory of integrable systems usually Hamiltonian operators arise. It is necessary from independent assumption to find two different Poisson structures and after this fulfil describing above operation. In the problem of construction of Hamiltonian operators for integrable systems the equations (3.1) was firstly used in [4].

According to [4] it is possible to expect that (some partial) solutions of equations (3.1) may be represented in the form

$$J(u) = \sum_{\alpha,\beta} a_\alpha D^{-1} b_\beta + \sum_i A_i D^i$$

where a_α,b_β are some s-dimensional column (line) vectors, A_i are some $s \times s$ matrixes the components and matrixes elements of which are the functions of u and its derivatives. A more detail information about the properties of integro-differential operators of such kind reader can find in [6].

In order to understand this structure let us consider the action of $\phi'(u)$ operator on solution of main equation (2.2) $F_n(u)$. We obtain

$$\phi'(u)F_n(u) = (\phi'(u)F_n(u)) + c_1D + c_2D^2 + ... = F_n(\phi(u)) + c_1D + c_2D^2 + ..$$

From this consideration it follows immediately that if the operator $J(u)$ choosen in the form

$$J(u) = \sum_{n,n'} F_n(u)D^{-1}F_{n'}(u) + \sum_i A_i(u)D^i$$

then for operator $\phi'(u)J(u)\phi'(u)^T$ we obtain

$$\phi'(u)J(u)\phi'(u)^T = \sum_{n,n'} F_n(\phi(u))D^{-1}F_{n'}(\phi(u)) + \sum_i \tilde{A}_i(u)D^i$$

and the second equation (3.1) is equivalent to equality $\tilde{A}_i(u) = A_i(\phi(u))$.

The above notice allows to obtain comparatively simple explicit expressions for operators J_0, J_1 for construction of which as a rule it is necessary to use except of trivial solutions u' of (2.2) some other simplest one u for example.

In the next section we shall represent the list of integrable substitutions with corresponding expressions for Frechet derivatives, explicit form of operators J_0, J_1 which as was explained before is sufficient to construct the whole hierarchy of equations which are all invariant with respect to transformation of given discrete symmetry. In other words it is possible to say that from the point of view of theory of integrable substitution this is the recurrent method for construction new solutions of equation 2.2) from some initial one.

4 The list of discrete substitutions and Hamiltonian operators corresponding to them

Here we give a list of integrable substitutions [5] together with the corresponding Hamiltonian operators J_0, J_1. It is possible to reconstruct the whole hierarchy of equations with the same discrete symmetry with the help of them. We represent also the explicit form of Frechet operator by help of which all results can be checked without any difficulties. All examples below are connected with two component vector function u where for first component we conserve notation u and the second one will be denoted by v. The simplest system of hierarchy will be represented in the beginning of the corresponding subsection.

It is very interesting that all known examples of integrable substitutions in the case of rational spectral parameter (in terminology of inverse scattering method) are connected with equations of Toda chains (infinite) or some its generalizations. Is it possible to obtain this result and how from condition of resolving of equation (2.2)?

4.1 Nonlinear Schrödinger substitutions.

4.1.1 Nonlinear Schrödinger hierarchy without the derivative

$$\dot{u} + u'' - 2u(uv) = 0 \quad - \dot{v} + v'' - 2v(uv) = 0$$

The direct and inverse substitution in this case have the form

$$\tilde{u} = \frac{1}{v}, \qquad \tilde{v} = v[vu - (\ln v)'']$$

$$v = \frac{1}{\tilde{u}}, \qquad u = \tilde{u}[\tilde{u}\tilde{v} - (\ln \tilde{u})''] \tag{4.1}$$

If it rewrite substitution (4.1) we will obtain the infinite chain of equations for unknown function $x_n = \ln v_n$

$$x_n'' = \exp(x_n - x_{n-1}) - \exp(x_{n+1} - x_n)$$

This is exact infinite Toda chain in its original form.

Frechet matrix of substitution is the following

$$\phi'(u) = \begin{pmatrix} 0 & -\frac{1}{v^2} \\ v^2 & 2(uv) - \frac{(v')^2}{v^2} + \frac{2v'}{v}D - D^2 \end{pmatrix}$$

Keeping in mind the results of the last section one can convinced by direct calculations that the following two operators J_0, J_1

$$J_0(u,v) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad J_1(u,v) = \begin{pmatrix} 2uD^{-1}u & D - 2uD^{-1}v \\ D - 2vD^{-1}u & 2vD^{-1}v \end{pmatrix}$$

satisfy the second equation (3.1).

$F_0 = (u, -v)$ is the simplest nontrivial solution of equation (2.2). All other equations of this hierarchy may be obtained by multi-time application of operator $H = J_1 J_0^{-1}$ to this solution.

4.1.2 Modified nonlinear Schrödinger hierarchy

$$\dot{u} + u'' + 2u'(uv) = 0 \quad - \dot{v} + v'' - 2v'(uv) = 0$$

The direct and inverse substitution in this case have the form

$$\tilde{u} = \frac{1}{v}, \qquad \tilde{v} = v[vu - (\ln \frac{v'}{v})']$$

$$v = \frac{1}{\tilde{u}}, \qquad u = \tilde{u}[\tilde{u}\tilde{v} + (\ln \frac{\tilde{u}'}{\tilde{u}})'] \tag{4.2}$$

Having written substitution (4.2) as a result of multi-time application of discrete transformation to a some initial solution we obtain the infinite chain of equations for unknown function $x_n = \ln v_n$

$$x_n'' = x_n'(\exp(x_n - x_{n-1}) - \exp(x_{n+1} - x_n))$$

The reader can find in [5] the explicit solution of this chain under appropriate boundary conditions.

Frechet matrix of substitution is the following

$$\phi'(u) = \begin{pmatrix} 0 & -\frac{1}{v^2} \\ v^2 & 2(uv) - \frac{v''}{v'} + (1 + \frac{v''v}{v^2}D - \frac{v}{v'}D^2 \end{pmatrix}$$

Keeping in mind the results of the last section one can convinced by direct calculations that the following two operators J_0, J_1

$$J_0(u,v) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$J_1(u,v) = \begin{pmatrix} uD^{-1}u' + u'D^{-1}u & D - uv + uD^{-1}v' - u'D^{-1}v \\ D + uv - vD^{-1}u' + v'D^{-1}u & -vD^{-1}v' - v'D^{-1}v \end{pmatrix}$$

satisfy the second equation (3.1).

$F_0 = (u, -v)$ is the simplest nontrivial solution of equation (2.2). All other equations of this hierarchy or, and this is the same, other solutions of equation (3.1) may be obtained by multi-time application of operator $H = J_1 J_0^{-1}$ to this solution.

4.1.3 Nonlinear Schrödinger hierarchy with derivative

$$\dot{u} + u'' - 2(u(uv))' = 0 \quad -\dot{v} + v'' + 2(v(uv))' = 0$$

The direct and inverse substitution in this case have the form

$$\begin{aligned} \tilde{u} = v, \qquad \tilde{v} = u - (\frac{1}{v})' \\ \\ v = \tilde{u}, \qquad u = \tilde{v} + (\frac{1}{\tilde{u}})' \end{aligned} \tag{4.3}$$

Having written substitution (4.3) as a result of multi-time application of discrete transformation to a some initial solution we obtain the infinite chain of equations for unknown function $x_n = v_n$

$$x_n' = x_n^2(x_{n+1} - x_{n-1})$$

The reader can find in [5] the explicit solution of this chain Under appropriate boundary conditions.

Frechet matrix of substitution under consideration is the following

$$\phi'(u) = \begin{pmatrix} 0 & 1 \\ 1 & -2\frac{v'}{v^3} + \frac{1}{v^2}D \end{pmatrix}$$

Keeping in mind the results of the last section one can convince by direct calculations that the following two operators J_0, J_1

$$J_1(u,v) = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}$$

$$J^0 = \begin{pmatrix} -2uD^{-1}u & 2uD^{-1}v + 1 \\ 2vD^{-1}u - 1 & -2vD^{-1}v \end{pmatrix}$$

satisfy the second equation (3.1).

$F_0 = (u, -v)$ is the simplest nontrivial solution of equation (2.2). All other equations of this hierarchy or, and this is the same, other solutions of equation (3.1) may be obtained by multi-time application of operator $H = J_1 J_0^{-1}$ to this solution.

We represent also the expression for J^2 Hamiltonian operator in illustrative purposes

$$J^2 = \begin{pmatrix} -2u^2D - 2u'u + 2u'D^{-1}u' & D^2 - 2uvD - 2u'v + 2u'D^{-1}v' \\ -D^2 - 2uvD - 2v'u + 2v'D^{-1}u' & -2v^2D - 2vv' + 2v'D^{-1}v' \end{pmatrix}$$

4.1.4 Nonlinear Schrödinger hierarch with derivative and nonlinearity of the third degree

The system of Ablovitz-Kaup

$$-\dot{u} + u'' - 2u^2(v' + uv^2) = 0 \quad \dot{v} + v'' + 2v^2(u' - vu^2) = 0 = 0$$

and Lund-Pohlmeyer-Regge equations

$$\dot{u}' - 4u - 2\dot{u}(uv) = 0 \quad \dot{v}' - 4v + 2\dot{v}(uv) = 0$$

possesses the same discrete transformation and so belong to the single hierarchy.

The direct and inverse discrete substitution in this case are the following

$$\tilde{u} = \frac{1}{v' + uv^2}, \quad \tilde{v} = -(v' + uv^2)' + \frac{(v' + uv^2)^2}{v}$$

$$v = -\frac{1}{\tilde{u}' - \tilde{v}(\tilde{u})^2}, \quad u = -(\tilde{u}' - \tilde{v}(\tilde{u})^2)' + \frac{(\tilde{u}' - \tilde{v}(\tilde{u})^2)^2}{\tilde{u}} \tag{4.4}$$

The discrete transformation in the form of infinite chain in this case coincides with infinite Lotky-Volterra system

$$N_j' = N_i(N_{j+1} - N_{j-1})$$

where $N_{2i} = \frac{v_{2i}}{(v'_{2i-1} + u_{2i-1}v^2_{2i-1})}, N_{2i-1} = -\frac{(v'_{2i-1} + u_{2i-1}v^2_{2i-1})}{v_{2i}}$

It is not difficult to check that after introduction of the new unknown pair of functions $q = u, r = v' + uv^2$ or $q = u' - vu^2, r = v$ in both cases q, r satisfy the usual nonlinear Schrödinger equation from the first subsection. This fact allows to construct the corresponding Hamiltonian operators using the results of first subsection.

For inverse Frechet operator we obtain immediately

$$(\phi')^{-1} = \begin{pmatrix} -\frac{1}{(\tilde{u}v)^2} & -\frac{2}{\tilde{u}v} - D \\ 0 & v^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D - 2\tilde{u}\tilde{v} & -(\tilde{u})^2 \end{pmatrix}$$

And after this we obtain for Hamiltonian operators

$$J_{-1} = \begin{pmatrix} 0 & (D - 2uv)^{-1} \\ (D + 2uv)^{-1} & 0 \end{pmatrix} \tag{4.5}$$

$$J_0 = \begin{pmatrix} -2uD^{-1}u & +1 + 2uD^{-1}v \\ -1 + 2vD^{-1}u & -2vD^{-1}v \end{pmatrix} \tag{4.6}$$

4.2 XYZ-hierarchy in classical region.

The system of equations describing the Heisenberg unhomogeneous ferromagnetic in classical region, the equation of Landau-Lifshitz (L-L), is the single known (at least to authors) example of integrable systems with elliptic spectral parameter. We will pay some more attention to this case.

In vector notations L-L system has the form

$$\dot{(\vec{S})} = \vec{S} \times S'' + \vec{S} \times \vec{(JS)}$$

$$\vec{S} = (S_1, S_2, S_3), \quad (\vec{S})^2 = 1, \quad J = diag(J_1, J_2, J_3)$$

Under the steriographic projection

$$u = \frac{S_1 + iS_2}{1 + S_3} \qquad v = \frac{S_1 - iS_2}{1 + S_3}$$

and exchanging $-it \to t$ it became a system of equations:

$$-\dot{u} = u'' - 2v\frac{(u')^2 + R(u)}{1 + uv} + \frac{1}{2}\frac{\partial}{\partial u}R(u) \equiv u^{(2)} = 0$$

$$\dot{v} = v'' - 2u\frac{(v')^2 + R(v)}{1 + uv} + \frac{1}{2}\frac{\partial}{\partial v}R(v) \equiv v^{(2)} = 0 \qquad (4.7)$$

where $R(x) = \alpha x^4 + \gamma x^2 + \alpha$ $\frac{\partial R}{\partial x} = 4\alpha x^3 + 2\gamma x = 2\frac{R+\alpha(x^4-1)}{x}$ $\alpha = \frac{J_2-J_1}{4}$ $\gamma = \frac{J_1+J_2}{2} - J_3$ The system (4.7) is invariant under transformation $u \to U, v \to V$:

$$\tilde{u} = \frac{1}{v} \qquad \frac{1}{1 + \tilde{v}\tilde{u}} - \frac{1}{1 + uv} = \frac{vv'' - (v')^2 + \alpha(v^4 - 1)}{(v')^2 + R(v)} \qquad (4.8)$$

which is the discrete substitution for this system. The inverse substitution to (4.7) is the following

$$v = \frac{1}{\tilde{u}} \qquad \frac{1}{1 + vu} - \frac{1}{1 + \tilde{u}\tilde{v}} = \frac{\tilde{u}\tilde{u}'' - (\tilde{u}')^2 + \alpha(\tilde{u}^4 - 1)}{(\tilde{u}')^2 + R(\tilde{u})} \qquad (4.9)$$

In the form of infinite chain substitution (4.8) may be rewritten in two equivalent forms

$$\frac{1}{1 + \frac{v_{n+1}}{v_n}} - \frac{1}{1 + \frac{v_n}{v_{n-1}}} = \frac{v_n v_n'' - (v_n')^2 + \alpha(v_n^4 - 1)}{(v_n')^2 + R(v_n)}$$

or

$$\frac{1}{v_{n+1}+v_n}+\frac{1}{v_n+v_{n-1}}=\frac{v_n''+R_{v_n}(v_n)}{(v_n')^2+R(v_n)}$$

Reader can find the corresponding solution in [7],[8].

4.2.1 Nondegenerate case

In the case of arbitrary α, γ the main equation (2.3) does not possesses the solution F_0 the components of which have dependence only upon on u, v but not of their derivatives, as solution $F_0 = (u, -v)$ of the previous subsections. Due to this reason the second equation (3.1) in the case of XYZ hierarchy possesses the solutions J_n only of the even order but in the degenerated cases solution of this kind takes place. And as corollary in the degenerate cases there take place the Hamiltonian operators of odd and even degree.

We have

$$J_0 = (1+uv)^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The matrix elements of Hamiltonians operators in what follows will be sufficiently complicated to write them in the single matrix form. So we will introduce notations

$$J_s = \begin{pmatrix} a_s & b_s \\ -b_s^T & d_s \end{pmatrix}$$

and represent the explicit expressions for a_s, b_s, d_s in all other cases.

So we obtain

$$a_2 = 2u^{(2)}D^{-1}u' + 2u'D^{-1}u^{(2)}$$

where notations $u^{(2)}, v^{(2)}$ are introduced in (4.7) and explicit form of which is the following

$$u^{(2)} = u'' - \frac{2v(u')^2 + 2\alpha v + \gamma vu^2 - \gamma u - 2\alpha u^3}{1+uv}$$

$$v^{(2)} = -v'' + \frac{2u(v')^2 + 2\alpha u + \gamma uv^2 - \gamma v - 2\alpha v^3}{1+uv}$$

Next

$$b_2 = (1+uv)^2D^2 + 4uv'(1+uv)D + (p'+p^2+2H)(1+uv)^2 + 2u^{(2)}D^{-1}v' + 2u'D^{-1}v^{(2)}$$

where $p = \frac{2uv'}{1+uv}$ the impulse and $H = \frac{2(u'v'+\alpha(u^2+v^2)-\gamma uv)}{(1+uv)^2}$ hamiltonian of L-L equation. And at last

$$d_2 = 2v^{(2)}D^{-1}v' + 2v'D^{-1}v^{(2)}$$

In vector notations the explicit expression for J_2 Hamiltonian operator was obtained firstly in [9].

4.2.2 Degenerate cases

In the partial cases $\alpha = 0, \gamma = \pm 2\alpha$ the symmetry of L-L equation is changed by the jump (it became more wide). The main equation (2.3) possesses solutions which do not depend on derivatives on u, v and as a consequence it is possible to find solution of the equation (3.1) also for odd n.

In the case $\gamma = 2\alpha$ for components of J_1 Hamiltonian operator we obtain

$$a_1 = 2u'D^{-1}u' + 2\alpha(u^2+1)D^{-1}(u^2+1)$$

$$b_1 = (1+uv)^2D + 2uv'(1+uv) + 2u'D^{-1}v' + 2\alpha(u^2+1)D^{-1}(1+v^2)$$
$$d_1 = 2v'D^{-1}v' + 2\alpha(v^2+1)D^{-1}(v^2+1)$$

Ii is not difficult to check that independent on derivatives solution of (2.3) has the form $F_0 = (u^2+1, v^2+1)$.

In the case $\gamma = -2\alpha$ the corresponding values are the follows

$$a_1 = 2u'D^{-1}u' + 2\alpha(u^2-1)D^{-1}(u^2-1)$$

$$b_1 = (1+uv)^2D + 2uv'(1+uv) + 2u'D^{-1}v' - 2\alpha(u^2-1)D^{-1}(v^2-1)$$
$$d_1 = 2v'D^{-1}v' + 2\alpha(v^2-1)D^{-1}(v^2-1)$$

$F_0 = (u^2-1, -v^2+1)$.

And at last in the case $\alpha = 0$

$$a_1 = 2u'D^{-1}u' + 2\gamma uD^{-1}u$$

$$b_1 = (1+uv)^2D + 2uv'(1+uv) + 2u'D^{-1}v' - 2\gamma uD^{-1}v$$
$$d_1 = 2v'D^{-1}v' + 2\gamma vD^{-1}v$$

$F_0 = (u, -v)$.

In vector notations this result was obtained in paper [?].

5 Matrix hierarchies

In order to complete our previous results,we are going to present here the generalization for the case of noncommutative unknown functions. We represent our results in brief form for the hierarchies of Nonlinear matrix Schrödinger equation and Nonlinear Schrödinger one with derivative. In spite of our generalization for the structure of invariant Hamiltonian operators,we were to use the primary scheme (3) in order to find the correct expressions in noncommutative case. Though it is possible to guess such operators.

5.1 Nonlinear matrix Schrödinger hierarchy

$$\dot{u} = -u_{xx} + 2(uvu) \quad -\dot{v} = -v_{xx} + 2(vuv)$$

We consider variables u and v as nonsingular square matrices of N^2 elements. The generalization of the corresponding direct and inverse substitutions takes the form

$$\tilde{u} = v^{-1} \quad \tilde{v} = vuv - v_{xx} + v_x v^{-1} v_x$$

$$(5.1)$$

$$v = \tilde{u}^{-1} \quad u = \tilde{u}\tilde{v}\tilde{u} - \tilde{u}_{xx} + \tilde{u}_x \tilde{u}^{-1} \tilde{u}_x$$

If we rewrite substitution (5.1) we will obtain the infinite matrix Toda chain of equations for unknown function $w_n = v$

$$(w_{nx} w_n^{-1})_x = w_n w_{n-1}^{-1} - w_{n+1} w_n^{-1}$$

Frechet derivative have the following form

$$\varphi' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

α,β,δ,γ are the matrix operators with the following components

$$\alpha_{ij}^{kl} = 0 \quad \beta_{ij}^{kl} = -(v^{-1})_{ki}(v^{-1})_{jl} \quad \gamma_{ij}^{kl} = v_{ki} v_{jl}$$

$$\delta_{ij}^{kl} = \delta_{ki}(uv)_{jl}+(vu)_{ki}\delta_{jl}-(v_xv^{-1})_{ki}(v^{-1}v_x)_{jl}+[\delta_{ki}(v^{-1}v_x)_{jl}+(v_xv^{-1})_{ki}\delta_{jl}]D_x-\delta_{ki}\delta_{jl}D_x^2$$

Each of the indexes k,l,i,j runs from 1 to N.

The Hamiltonian operators for this hierarchy are the following

$$J^0 = \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix}$$

$$T_{ij}^{kl} = \delta_{kj}\delta_{il} \quad k,l,i,j = 1...N$$

$$J^1 = \begin{pmatrix} a & b \\ -b^T & d \end{pmatrix}$$

$$a_{ij}^{kl} = u_{kj}D^{-1}u_{il} + u_{il}D^{-1}u_{kj} \quad b_{ij}^{kl} = \delta_{kj}\delta_{il}D - \delta_{il}u_{ks}D^{-1}v_{sj} - \delta_{kj}u_{sl}D^{-1}v_{is}$$

$$d_{ij}^{kl} = v_{kj}D^{-1}v_{il} + v_{il}D^{-1}v_{kj}$$

k,l,i,j=1...N

One may easily convince by direct calculations that the operators J^0,J^1 satisfy the second equation (3.1)

5.2 Nonlinear matrix Schrödinger hierarchy with derivative

$$\dot{u} = -u_{xx} + 2(uvu)_x \quad \dot{v} = v_{xx} + 2(vuv)_x$$

The generalization of the corresponding direct and inverse substitution (4.3) is the following

$$\tilde{u} = v \quad \tilde{v} = u + v^{-1}v_xv^{-1}$$

$$(5.2)$$

$$v = \tilde{u} \quad u = \tilde{v} - \tilde{u}^{-1}\tilde{u}_x\tilde{u}^{-1}$$

If we rewrite the substitution (5.2) we obtain the infinite Toda chain of equations for unknown function $w_n = v$

$$w_{nx} = w_n(w_{n+1} - w_{n-1})w_n$$

Frechet derivative takes the following form

$$\varphi' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$\alpha_{ij}^{kl} = 0 \quad \beta_{ij}^{kl} = \delta_{ki}\delta_{jl} \quad \gamma_{ij}^{kl} = \delta_{ki}\delta_{jl}$$

$$\delta_{ij}^{kl} = -(v^{-1})_{ki}(v^{-1}v_x v^{-1})_{jl} - (v^{-1}v_x v^{-1})_{ki}(v^{-1})_{jl} + (v^{-1})_{ki}(v^{-1})_{jl}D_x$$

Each of the indexes k,l,i,j runs from 1 to N

Hamiltonian operators J^0 and J^1 for this hierarchy are the following

$$J^0 = \begin{pmatrix} a & b \\ -b^T & d \end{pmatrix}$$

$$a_{ij}^{kl} = -u_{kj}D^{-1}u_{il} - u_{il}D^{-1}u_{kj} \quad b_{ij}^{kl} = \delta_{kj}\delta_{il} + \delta_{il}u_{ks}D^{-1}v_{sj} + \delta_{kj}u_{sl}D^{-1}v_{is}$$

$$d_{ij}^{kl} = -v_{kj}D^{-1}v_{il} - v_{il}D^{-1}v_{kj}$$

k,l,i,j=1...N

$$J^1 = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} D_x$$

$$T_{ij}^{kl} = \delta_{kj}\delta_{il} \quad k, l, i, j = 1...N$$

One may easily verify that the operators J^0 and J^1 satisfy the second equation (3.1)

We see in both cases that the limit $N = 1$ gives us 2 in corresponding operators.

6 Conclusion

The main result of the present paper consists of explicit formulae for Hamiltonian operators which allow to reconstruct all equations for most often used hierarchies of integrable systems in physical applications. We have not included in this list results concerning main chiral field problem. The reason of that is the fact of nonlocality of corresponding substitution and some specific difference of this case in compare with the examples considered in this paper. We hope to return to this problem in recent future.

Methods of construction of Hamitonian operators,which were used in this paper, are typical for group representation theory. The equations (7) from group theoretical point of view are the equations on some invariant kernels. The main equation (2.2) (with given integrable substitution $\varphi(u)$) is the determination of some linear representation of the group of integrable mappimgs as was mentioned above. So independent investigation and construction of the representation theory of this object is equivalent to the theory of integrable systems of the given hierarchy. That is the second and may be more important conclusion which follows from the results of this paper.

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