International Conference on

Arakelov Theory

Differential Geometric Methods in Arithmetic

Bonn, June 1-7, 1994

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn
Germany

Orginal

MPI / 94-50

Announcement of an International Conference on

Arakelov Theory
Differential Geometric Methods in Arithmetic

Bonn, June 1-7, 1994

At the center of the *Special Activity on Arakelov Theory* held at the *Max-Planck-Institut für Mathematik* in Bonn from May to July 1994 we will organize an *International Conference on Arakelov Theory* from June 1 through 7, 1994. It will take the place and time of the traditional Bonn *Mathematische Arbeitstagung*, which will not be held in 1994.

In the last 10 years, Arakelov theory has developed rapidly and branched out in several directions. The foundations of a general theory have been laid, including the so-called Arithmetic Riemann-Roch Theorem, and significant arithmetic applications of the methods have been found. For example, it is fair to say that the ideas and concepts of Arakelov theory led to all three proofs of the Mordell conjecture (by Faltings, Vojta, Bombieri). Yet, the theory is far from complete, and direct applications are only beginning to emerge. The aim of the conference (as well as of the whole MPI activity) is to bring together the experts in this field, as well as interested non-experts, to allow direct exchange of information, and to present a coherent picture of the current state of the theory.

The Bonn conference is also the last part of a series of activities related to *Arakelov Theory*, organized under the auspices of the *Programme for International Cooperation in Mathematics and its Applications* (PICMA). The central goal of PICMA is to promote mathematical research in developing countries. The first major meeting of this series took place at ICTP (Trieste, Italy), August 31 - September 11, 1992, and the second one was held at Ain Shams University (Cairo, Egypt), September 4-15, 1993. There have been additional local workshops preparing for parts of the project - in China, Iran and Turkey. At Bonn, there will be opportunities for reports on the work in these countries. Participants of previous PICMA activities are encouraged to try and attend the Bonn conference, although funds for financial support are unfortunately rather limited.

Since the goal of the conference is to get a coherent picture of Arakelov theory, and in view of the fact that the audience will not just consist of the inner circle of experts, some of the talks will be of a more synthetic and general nature. The list of speakers will probably include: J. M. Bismut, S. Bloch, J. I. Burgos, R. Elkik, G. Faltings, H. Gillet, V. Maillot, L. Moret-Bailly, A. Parshin, C. Soulé, L. Szpiro, Q. Tian, I. H. Tsai, E. Ullmo, P. Vojta, G. Wüstholz, S. Zhang.

If you wish to attend the conference, please copy the attached form and return your copy with the necessary data (as soon as possible, since hotel space is rather short!). There is also a possibility of requesting financial support, but our resources are fairly restricted. Applications will be treated on the basis of need and availability of funds.

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Scientific Programme

All lectures of the conference (except on June 3) take place at Großer Hörsaal, Mathematisches Institut der Universität, Wegeler Str. 10, Bonn.

The talks by Bismut, Köhler and Soulé are designed to give an overview of the subject (esp. higher dimensional Arakelov theory), and are of a more expository nature.

Tea and coffee will be served between lectures, at 11:15 and 17:00, except on June 1.

Wednesday, 1 June 1994

16:15	*Opening*	
16:30–17:30	C. Soulé	Arithmetic intersection theory
17:45–18:45	K. Köhler	Analytic torsion and Quillen metric

Thursday, 2 June 1994

10:15–11:15	E. Ullmo	Integral points on arithmetic surfaces
11:45–12.45	J. Jorgenson	An analytic discriminant for polarized, algebraic $K3$-surfaces
16:00–17:00	C. Soulé	Chern classes and heights
17:30–18:30	S. Zhang	The height and reduction of a semistable variety

Friday, 3 June 1994 — *Excursion day: lectures take place at Mathematisches Institut der Universität Köln, Weyertal 86–90, Köln.*

16:00–17:00	L. Szpiro	Algebraic geometry over $\overline{\mathbf{Q}}$
17:30–18:30	G. Wüstholz	On Faltings's product theorem

Saturday, 4 June 1994

10:15–11:15	K. Köhler	Analytic torsion on Hermitian symmetric spaces
11:45–12.45	V. Maillot	An arithmetic Schubert calculus
16:00–17:00	J.I. Burgos	Arithmetic Chow rings and Deligne-Beilinson cohomology
17:30–18:30	P. Vojta	Integral points on open subvarieties of semiabelian varieties

Please turn over.......

Sunday, 5 June 1994

10:15–11:15	J.M. Bismut	Complex immersions and Quillen metrics
11:45–12:45	W. Gubler	Heights of subvarieties
15:30–17:00	G. Frey / E. Kani	Curves of genus 2 and the height conjecture for elliptic curves
17:30–18:30	R. Rumely	Existence of asymptotics for volumes of adelic metrized line bundles

Monday, 6 June 1994

10:15–11:15	C. Soulé	Arithmetic Riemann-Roch
11:45–12:45	I.H. Tsai	Connections on determinant line bundles
16:00–17:00	L. Moret-Bailly	Integral points on algebraic stacks
17:30–18:30	A. Smirnov	Some absolute constructions

Tuesday, 7 June 1994

10:15–11:15	A. Abbes	The "arithmetic" Hilbert-Samuel theorem
11:45–12:45	K. Künnemann	On the arithmetic analogues of the standard conjectures
16:00–17:00	J.B. Bost	Stability and heights of arithmetic varieties
17:30–18:30	A. Reznikov	The Bloch Conjecture

SOCIAL EVENTS

Concert
Thursday, 2 June 1994, at the Festsaal der Universität, Am Hof 1, Bonn.

20:00–21:45 Piano recital, Gülsin Onay
(Chopin, Medtner, Mussorgski)

Excursion day: Friday, 3 June 1994 (see separate information sheet)

09:48 departure from Bonn Main Station to Brühl: tour of Schloß Augustusburg. Continuing to Köln (Cologne): possibility of visiting the cathedral, as well as a church from the Romanesque period. After the lectures, back to Köln Altstadt. Returning to Bonn by train later at night.

Reception
Sunday, 5 June 1994, at the Festsaal der Universität, Am Hof 1, Bonn.

19:30–22:00 Rector's reception — cocktails and buffet

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International Conference on
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Piano Recital

GÜLSİN ONAY

Festsaal der Universität, Am Hof 1, Bonn, 2 June 1994, 20:00 h

Frédéric Chopin (1810–1849)	Variations brillantes sur le Rondeau favori 'Je vends des scapulaires' de 'Ludovic' de Hérold et Halévy, B-flat major *op. 12* (1833)
	Ballade A-flat major *op. 47, no. 3* (1841)
	Nocturne f-sharp minor, *op. 48, no. 2* (1841)
	Scherzo b minor *op. 20* (1832)
Nikolai Medtner (1880–1951)	Tema con variazioni c-sharp minor *op. 55, no. 1* (1933)

* * *

Modest Mussorgski (1839–1881)	Pictures at an exhibition (1874) Promenade — Gnomus — The old castle — Tuileries (children in play quarreling) — Bydlo — Ballet of the chicken in their eggs — Samuel Goldenberg and Schmuyle — Promenade — The market of Limoges — Con mortuis in lingua morta — Catacombs (Sepulcrum Romanum) — The hut of Baba-Yaga — The grand gate of Kiev

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İstanbul-born *Gülsin Onay* was first taught by her pianist mother. Her exceptional musical talent was soon recognised and she gave her first public concert at the age of six on Turkish radio. A scholarship from the Turkish government enabled her to study at the Paris *Conservatoire* where her teachers included Nadia Boulanger, Monique Haas and Pierre Sancan and where at just 16 she was awarded the coveted *1er Prix du Piano*. Later she studied with Bernhard Ebert in Hannover. *Gülsin Onay* is a prizewinner of several international competitions, including the *Marguerite Long — Jacques Thibaud* competition in Paris and the *Ferrucio Busoni* competition in Italy.

A truly international artist, *Gülsin Onay* has performed in 30 countries on five continents to date. She has appeared with world class orchestras such as the the Warsaw Philharmonic, *Staatskapelle Dresden*, Dresden Philharmony, Radio Symphony Orchestra Berlin, the Salzburg Mozarteum Orchestra, the Tokyo Symphony, the Japan Philharmonic Orchestra, the Bavarian, Danish, Austrian and Finnish Radio Symphony Orchestras; Academy of London, Basel Sinfonietta, and the Sinfonia Varsovia.

Gülsin Onay is a popular guest at prestigious international Festivals, such as the Warsaw Autumn, Styrian Autumn, Berlin Festival, Mozartfest Würzburg, Schleswig-Holstein Festival, İstanbul Festival.

She has recorded a wide range of solo repertoire including Schubert, Chopin, Debussy, Ravel, Franck, Bartók, as well as piano concertos by Tchaikovsky, Hubert Stuppner and A. Adnan Saygun.

Gülsin Onay's repertoire encompasses most of the European classical and romantic composers, but she is also a highly regarded interpreter of 20th century music, including that of the Turkish composer Ahmed Adnan Saygun whose work she regularly performs and has recorded.

NOTES

Mussorgski's *Pictures at an exhibition* are so well-known, and immediately expressive, that they do not seem to need any commentary. In the first part of the programme, however, three very well-known works of Chopin are surrounded by two variation works which are almost never played in concert. At the moment, there is not even a recording of the Medtner variations available on the market. (*Gülsin Onay* recently finished recording a CD with six variation works by different composers which will also include both variations performed in the recital.)

Chopin's Ludovic variations were perfectly characterized in a critical review that Robert Schumann wrote in his own Music Magazine *"Neue Zeitschrift für Musik"* in 1836, and where he compared *Variations Brillantes* by eight different composers. He regrets the general lack of depth and honesty in this kind of brilliant music. But he also puts Chopin's composition above the others by comparing Chopin to a great actor who draws applause from the public even when he just happens to walk across the open stage, helping somebody carry something: Chopin can never completely betray his true genius, even in a work which is not as deeply conceived and not as painstakingly perfected as his other compositions.

Incidentally, it is precisely this lack of the last finishing touches in Chopin's usual style which produces in some of the Ludovic variations certain passsages that strike us today as distinctly 'Schumannian' .....

Nikolai Medtner's variations from 1933 are the exact opposite of *Variations Brillantes* — they are serious, quite complicated music, written in our century, for the modern grand piano, by an excellent russian pianist who, like Rachmaninoff (both belong to the same generation), left Russia after it had turned into the Soviet Union.

Unlike Rachmaninoff, Medtner never enjoyed big public success. This may be partly due to his somewhat introverted style of writing, which lead him into intricate harmonic developments and unusual rhythms. But such features do of course occur in the works of other, very successful composers of the same period, for instance Stravinski or Bartók.

Indeed: it may be said that Medtner stood in the way of his own success because of the decidedly anti-modern image he liked to give himself. His music is wrapped into overall forms which reminded many music critics of Brahms. And in his opposition to Ravel and Stravinski (in 1935, during his Paris exile, before settling in London) he even went as far as to publish a polemical little book against these modern composers, entitled *The muse and modernism*.

Thus the many wonderful new—and thereby modern—things that Medtner develops in his pianistic style are wrapped in traditional structures. This makes it an extremely rewarding intellectual adventure to hear his music as what it really is: the work of an eminent composer of our 20th century.

Medtner was a lonely man; but since he was one of the biggest masters of world piano literature, there is no reason to condemn his work to the same fate that he had during his lifetime. The short variations performed by *Gülsin Onay* may give some idea of the richness which is still waiting to be discovered in Medtner's *œuvre*.

Excursion, Friday, 3. June 1994

09.48 **Bonn main station** (platform 2) - **train to Brühl** (arrival 10.01; you have to use the ticket the whole day through, so please keep it).

10.05 - 11.55 Visit of **Schloß Augustusburg** (opposite the Brühl station). There will be two guided one-hour tours (one English, one German) at approx. 10.05 and 10.25. Possibility to have a walk in the beautiful park.

12.01 **Brühl station - train to Cologne** (arrival 12.18).

Arranged programme at Köln (Cologne):

13.00 - 15.00 Four guided tours of the **Kölner Dom** have been arranged: two 13.00 - 14.00 (one English, one German), two 14.00 - 15.00 (both English). Meeting at the main entrance.

14.15 - 15.15 Two guided one-hour tours of **St. Gereon** have been arranged (one English, one German).

For those who want to see Dom and St. Gereon: after the Dom tour 13.00 - 14.00 it is possible to walk to St. Gereon in 15 minutes*.
After leaving St. Gereon, you should immediately walk southwards to Neumarkt*.

How to get to the Mathematisches Institut (Department of Mathematics):

either 15.19 departure from **Köln main station** (platform 6, sections D - E) by **train (direction Bonn)**. Get off at **Köln-Süd** (15.26, second stop), leave the station at the north end (via the other platform) onto Zülpicher Straße, turn left and walk through the campus of **Universität zu Köln** (University of Cologne) to the Mathematisches Institut*.

or 15.30 departure from **Neumarkt** (north side of the middle oval) by **bus** (line 136 or 146 leaving every 5 minutes). After about 8 minutes get off at stop '**Hildegardis-Krankenhaus**' (first stop after a large park area), enter the street Weyertal (across the Pedestrian crossing) and reach the Mathematisches Institut after 300 metres on the left*.

In case you want to come directly from Bonn to the afternoon lectures at the Mathematisches Institut Köln, a convenient train leaves Bonn main station at 15.12 and arrives at **Köln-Süd** 15.31. Then follow the map.

Programme at the Mathematisches Institut (Weyertal 86-90):
16.00 - 17.00 Lecture by L. Szpiro 'Algebraic geometry over $\bar{\mathbb{Q}}$'
17.00 - 17.30 Tea
17.30 - 18.30 Lecture by G. Wüstholz 'On Faltings's product theorem'

End of the day

18.45 A bus will take us from the Mathematisches Institut to the **Kölner Altstadt**.

22.20 **Köln main station** (platform 6 B - C) - **train to Bonn** (arrival 22.48).
If you don't use this train, you have to care for yourself. Trains you can use in this case (E and N trains) leave Köln 19.19, 20.19, 21.19, 22.57 and 0.02 (last train!).

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* see the copied map; a guiding person for this group will also be provided

Schloß Augustusburg

Brühl's Rokoko palace was the summer residence of the Cologne archbishops and electoral princes, and in present times is used for state receptions by the German Government. It is on the UNESCO list of cultural monuments and famous for the staircase by Balthasar Neumann and the beautiful palace gardens.

Kölner Dom

This gothic style cathedral, started in 1248 and completed 1880, is Köln's symbol. It impresses by its architecture, precious art treasures (like the medieval choire stall (Chorgestühl, 1320), the shrine of the Holy Three Kings (Drei-Königen-Schrein, 1200) and a glass window cycle from the 16th century), and by various superlatives (like the world's largest bell). The tower platform (95 m, accessible by a staircase with 509 steps) offers a nice view of Köln.

St. Gereon

Köln's oldest church, founded in the 4th century over the grave of the martyr St. Gereon, was completed as a romanesque church in 1227. It hides an outstanding piece of architecture, the boldly constructed decagonal (ten-sided) crossing cupola, which is sometimes compared with the famous cupolas of Hagia Sofia or the Florence cathedral.

Other places of interest which you may explore on your own

- **Wallraf-Richartz-Museum/Museum Ludwig** (between Kölner Dom and Rhein river): two art museums of high repute.

- **Römisch-Germanisches Museum** (south of Kölner Dom): depicts the Roman history of Köln and the Rhein river valley.

- **Hohe Straße/Schildergasse** (connecting Kölner Dom and Neumarkt): a pedestrian zone, Köln's shopping area number one.

- **Rathaus** (city hall in the centre of the Altstadt) with **Praetorium** (underneath; excavation of the seat of the Roman governor) and **Mikwe** (on the forecourt; Jewish cultural bath).

International Conference on
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List of Participants

A. Abbes (Orsay)
V. Acharya (Pune)
P. Arias (MPI)
A. Assadi (Univ. of Wisconsin)
D. Baeumer (MPI)
C. Bejan (Univ. of Targoviste)
R. Berndt (Hamburg)
E. Bifet (MPI)
M. Bilhan (METU Ankara)
J. M. Bismut (Orsay)
A. Borel (IAS, Princeton)
D. Borenstein (Columbia Univ.)
J.-B. Bost (IHES)
V. Brinzanescu (Bucharest)
J. Burgos (Barcelona)
M. Caibar (Univ. of Targoviste)
F. Catanese (Pisa)
A. Chambert-Loir (E.N.S. & Univ. Paris VI)
A. Dabrowski (MPI)
N. Dan (Ecole Normale Supérieure Paris)
A. Dancer (MPI)
R. Del Angel (Univ.Autón.Metrop. Mexico)
H. Elsherbeny (Ain Shams Univ. Cairo)
R. Erné (Univ. de Rennes)
H. Esnault (Essen)
B. Fantechi (Univ.di Trento, Povo)
I. Fesenko (St. Petersburg State Univ.)
T. Fimmel (Köln)
M. Flach (Heidelberg)
M. Fontaine (Bonn)
R. Freitas (Lille)
G. Frey (Essen)
D. Fulea (Mannheim)
C. Gasbarri (Orsay)
L. Göttsche (MPI)
V. Gritsenko (Steklov Inst., St. Petersburg)
F. Grunewald (Düsseldorf)
J. Guardia (Barcelona)
W. Gubler (ETH Zürich)
U. Hamenstädt (MPI)
G. Harder (Bonn)
B. A. Hedi (Tunis)
R. Hill (MPI)
G. Höhn (MPI)
D. Huybrechts (MPI)
A. V. Ivanov (MPI)
J. Jahnel (Göttingen)
U. Jannsen (Köln)
J. Jorgenson (Yale University)
K. Joshi (Tata Inst. Bombay)
Ch. Kaiser (Bonn)
H. Kanarek (Essen)
M. Kaneko (Kyoto Inst. of Technology)
E. Kani (Queens University)
R. Kaufmann (MPI)
I. Kausz (Köln)
E. Kleinert (Hamburg)
T. Kleinjung (Bonn)
K. Köhler (MPI)
W. Kohnen (MPI)
M. Kontsevich (MPI)
J. Kramer (ETH Zürich)
K. Künnemann (Münster)
A. Künzle (MPI)
S. Lang (Yale)
N. Lauritzen (Aarhus)
M. Lippert (Bonn)
W. Lütkebohmert (Ulm)
V. Maillot (Ecole Norm. Supérieure Paris)
Y. Manin (MPI)
T. Maszcyk (Warsaw)
V. Metha (Tata Inst., Bombay)
P. Mikkelsen (Columbia Univ.)
L. Moret-Bailly (Univ. de Rennes 1)
C. Mortici (Univ. of Targoviste)
W. Müller (Bonn)
S. Nayatani (MPI)
G. Negyesi (Szeged University)
J. Nekovář (Prag)
T. Ooe (Mannheim)
N. K. Pandey (Rewa)
Petridis (MPI)
J. Pfeiffer (Bonn)
S. Piunikin (MIT)
D. Portelli (Trieste)
P. Pragacz (MPI)
M. Puschnigg (Heidelberg)
A. Rajaei (Princeton)
W. Raskind (Univ. of South. California)
A. Rasteghar (Princeton)
D. Reed (Oxford)
A. Reznikov (Jerusalem)

M. Richartz (Bonn)
D. Roessler (Paris-Nord)
A. Rosenberg (MPI)
M. Rosellen (Ecole Polytechn. Palaiseau)
K. Rubin (Ohio St. Univ.)
R. Rumely (Univ. of Georgia)
N. Schappacher (Strasbourg)
A. Schmidt (Heidelberg)
A. Schmitz-Tewes (Bonn)
P. Schneider (Köln)
Ch. Schoen (Duke U. Durham)
A. J. Scholl (Durham England)
M. Schröder (MPI)
W. K. Seiler (Mannheim)
S. Sigg (Bonn)
A. Smirnov (Steklov Math. Inst.)
Y. Soibelman (Inst. Adv. Studies Princeton)
C. Soulé (IHES)
J. Spies (ETH Zürich)
V. Srinivas (Bombay)
O. Šuch (Princeton Univ.)
A. Steffens (MPI)
B. Steinert (Bonn)
R. Szöke (Eötvös Lor. Univ. Hungary)
L. Szpiro (Orsay)
H. Tamvakis (Chicago)
S.-L. Tan (MPI)
I.-H. Tsai (Taiwan)
E. Ullmo (Orsay)
A. Venkov (Steklov Inst., St. Petersburg)
B. Vettel (Düsseldorf)
E. Viehweg (Essen)
S. Vishik (MPI)
P. Vojta (Univ. of Calif. Berkeley)
M. Weisfeld (Duke Univ.)
R. Weissauer (Mannheim)
L. Weng (MPI)
A. Werner (Münster)
G. Wiesend (Erlangen)
G. Wüstholz (ETH Zürich)
M. Xu (Academ. Sinica Peking)
Ch. Yogananda (Bangalore)
K. Yokogawa (MPI)
S. Zhang (Princeton)
K. Zuo (Kaiserslautern)

Titel: Arithmetic Intersection Theory

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Joint work with H. GILLET.

X : regular quasi-projective flat scheme / $\mathbb{Z}$

$X(\mathbb{C})$: its complex points. $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ complex conjugation.

$D^{pp}(X)$ = real currents T of type (p,p) on $X(\mathbb{C})$ such that $F_\infty^* T = (-1)^p T$

$\cup$

$A^{pp}(X)$ = real C^∞ forms α of type (p,p) on $X(\mathbb{C})$ such that $F_\infty^* \alpha = (-1)^p \alpha$.

Definitions:

a) An algebraic cycle of codimension p on X is a finite formal sum $Z = \sum_\alpha n_\alpha Y_\alpha$, $n_\alpha \in \mathbb{Z}$,

$Y_\alpha \subset X$ closed irreducible, $\operatorname{cod}_X Y_\alpha = p$

b) A Green current for Z is $g \in D^{p-1,p-1}(X)$ s.t.

$dd^c g + \delta_Z = \omega$ lies in $A^{pp}(X)$, where $\delta_Z :=$ current of integration on $Z(\mathbb{C})$,

i.e., if η is C^∞ form with compact support,

$$\int_{X(\mathbb{C})} \delta_Z \, \eta := \sum_\alpha n_\alpha \int_{Y'_\alpha} \eta$$

where Y'_α is any resolution of singularities of $Y_\alpha(\mathbb{C})$.

$d = \partial + \bar\partial \qquad d^c = \frac{\partial - \bar\partial}{4\pi i} \qquad dd^c = \frac{\bar\partial \partial}{2\pi i}$.

Examples:

1) $\bar{L} = (L, h)$: hermitian line bundle on X, i.e.

L: invertible $\mathcal{O}_X$-module

h: C^∞ metric on the corresponding holomorphic line bundle $L_\mathbb{C}$, with $F_\infty^* h = h$. (let $\|s\|^2 = h(s,s)$)

Choose s: rational section of L on X.

Then $Z = \operatorname{div}(s)$ is an algebraic cycle of codimension one on X and $-\log\|s\|^2$ is L^1 on $X(\mathbb{C})$, hence defines an element in $D^{00}(X)$.

Poincaré – Lelong equation:

$$dd^c(-\log\|s\|^2) + \delta_{\operatorname{div}(s)} = c_1(L_\mathbb{C}, h) \in A^{11}(X)$$

where $c_1(L_{\mathbb{C}}, h)$ is the first Chern form.

2) $X = \mathbb{P}^d_{\mathbb{Z}}$ Homogeneous coordinates $x_0, \ldots, x_d$.

$Y = \{(x.) \in \mathbb{P}^d_{\mathbb{Z}} \ / \ x_0 = \ldots = x_{p-1} = 0\}$

$\tau = \log(|x_0|^2 + \ldots + |x_d|^2)$ $\alpha = dd^c \tau$

$\sigma = \log(|x_0|^2 + \ldots + |x_{p-1}|^2)$ $\beta = dd^c \sigma$

$$\Lambda := (\tau - \sigma) \sum_{i=0}^{p-1} \alpha^i \beta^{p-1-i} \quad \text{on } X(\mathbb{C}) - Y(\mathbb{C}).$$

Theorem (H. Levine): Λ is L^1 on $X(\mathbb{C}) - Y(\mathbb{C})$. The current it defines on $X(\mathbb{C})$ is such that

$$dd^c \Lambda + \delta_Y = \alpha^p$$

In general, if $p \geq 0$, let

$$\widehat{CH}^p(X) := \frac{\text{pairs } (Z, g) \begin{cases} Z \text{ alg. cycle of cod. } p \\ g \text{ Green current for } Z \end{cases}}{\text{subgroup generated by } (0, \partial u + \bar\partial v) \text{ and } (\mathrm{div}(f), -\log|f|^2),}$$

Where: u and v are currents, $\partial u + \bar\partial v \in D^{p-1,p-1}(X)$;

$f \in k(Y)^*$, $Y \subset X$, $\mathrm{cod}_X Y = p-1$,

$$(-\log|f|^2)(\eta) := -\int_{Y(\mathbb{C})} \log|f|^2 \, \eta$$ for all C^∞ forms η with compact support on $X(\mathbb{C})$.

Theorem 1: There are exact sequences

$$CH^{p,p-1}(X) \xrightarrow{\rho} \widetilde{A^{p-1,p-1}(X)} \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{z} CH^p(X) \to 0$$

and a morphism

$$\omega : \widehat{CH}^p(X) \to A^{pp}(X)$$

such that $\omega \circ a = dd^c$.

Here $CH^p(X)$ is the algebraic Chow group, $z(Z,g) = Z$

$\omega(Z,g) = dd^c g + \delta_Z$,

$$\widetilde{A^{p-1,p-1}(X)} := A^{p-1,p-1}(X) \,/\, \operatorname{Im}\partial + \operatorname{Im}\bar\partial,$$

$a(\text{class of } \eta) := \text{class of } (0,\eta)$,

ρ is Beilinson regulator map.

Theorem 2: There exists an associative commutative bilinear pairing

$$\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \to \widehat{CH}^{p+q}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

such that ω and z are ring homomorphisms.

One can avoid tensoring with $\mathbb{Q}$ in the following cases:

a) p or $q \leq 1$;

b) X is smooth over S-integers in a number field;

c) any scheme Y of finite type over $\mathbb{Z}$ with $\dim(Y) \leq \dim(X)$ has a resolution of singularities.

Theorem 3: i) Any map $f : Y \to X$ induces

$f^* : \widehat{CH}^p(X) \to \widehat{CH}^p(Y)$ with $f^*(xy) = f^*(x) f^*(y)$

ii) Any map $f : X \to Y$ which is proper and such that $f : X(\mathbb{C}) \to Y(\mathbb{C})$ is smooth induces

$$f_* : \widehat{CH}^p(X) \longrightarrow \widehat{CH}^{p+\dim(Y)-\dim(X)}(Y),$$

such that $f_*(x\, f^*(y)) = f_*(x)\, y$.

Example: $\widehat{CH}^p(\operatorname{Spec} \mathbb{Z}) = \begin{cases} \mathbb{Z} & p = 0 \\ \mathbb{R} & p = 1 \\ 0 & p > 1 \end{cases}$

where $\widehat{\deg} : \widehat{CH}^1(\operatorname{Spec} \mathbb{Z}) \xrightarrow{\sim} \mathbb{R}$ maps $(0, 2\lambda)$ to λ for any $\lambda \in \mathbb{R}$.

If $d = \dim(X/\mathbb{Z})$ we get

$$\widehat{CH}^p(X) \otimes \widehat{CH}^{d+1-p}(X) \to \widehat{CH}^{d+1}(X) \xrightarrow{f_*} \widehat{CH}^1(\mathbb{Z}) \overset{\widehat{\deg}}{\underset{\sim}{\to}} \mathbb{R}$$

$$(x \otimes y) \longmapsto x \cdot y$$

i.e. arithmetic intersection numbers.

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2) "Lectures on Arakelov Geometry", Soulé, Abramovich, Burnol, Kramer, Cambridge Studies in advanced Mathematics 33 + references therein.

Titel: Analytic torsion and Quillen metric
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Let $\begin{matrix}(E,h)\\ \downarrow\\ Z\end{matrix}$ denote a holomorphic hermitian vector bundle over a compact hermitian manifold. The analytic torsion τ is a positiv real number associated to the spectrum of the Kodaira Laplacian acting on forms over Z with coefficients in E. The Quillen metric is a certain metric on the determinant of the cohomology $H^{0,*}(Z,E)$.

The results described in this note where developped by Bismut, Gillet and Soulé in [BGS].

1) Bott-Chern secondary classes

The data $\begin{matrix}(E,h)\\ \downarrow\\ Z\end{matrix}$ determines uniquely a holomorphic hermitian connection ∇^E. Its curvature $\Omega^E := (\nabla^E)^2 \in \Lambda^{1,1} T^*Z \otimes \operatorname{End} E$ is a (1,1)-form with coefficients in End E. Let $\phi : \operatorname{End}(\mathbb{C}^{\dim E}) \to \mathbb{C}$ be a polynomial or a formal power series and assume that ϕ is invariant under conjugation with $GL_{\dim E}(\mathbb{C})$, e.g.

$$ch(A) = \operatorname{Tr} e^A \quad \text{or} \quad Td(A) = \det \frac{A}{1-e^{-A}}.$$

Then $\phi(E,h) := \phi\left(\frac{-\Omega^E}{2\pi i}\right)$ is a well-defined sum of (p,p)-forms on Z. By Chern-Weil theory, $\phi(E,h)$ is closed and its cohomology class in $H^*(Z,\mathbb{C}) \cong d\text{-closed forms}/\operatorname{im} d$ is independent of the metrics. Furthermore, for a map $f: X \to Z$ the pull-back bundle f^*E over X has the ϕ-class $\quad \phi(f^*E) = f^*\phi(E)$.

Consider now exact sequences of holom. vector bundles

$$\mathcal{E} : 0 \to \overline{E}' \to \overline{E} \to \overline{E}'' \to 0 \qquad (*)$$

where E', E, E'' are equipped with arbitrary hermitian metrics.

Th. [BGS]: The following axioms determine uniquely secondary classes $\tilde{\phi}(\mathcal{E}) \in \bigoplus_p \tilde{A}^{p,p}(Z)$:

i) $\frac{\partial\bar\partial}{2\pi i}\tilde\phi(\mathcal{E}) = \phi(\bar E) - \phi(\bar E' \oplus \bar E'')$ for any sequence (*),

ii) $\tilde\phi(f^*\mathcal{E}) \equiv f^*\tilde\phi(\mathcal{E})$ for $f: X \to Z$,

iii) $\mathcal{E}$ splits metrically $\Rightarrow \tilde\phi(\mathcal{E}) \equiv 0$.

Construction for the sequence $\mathcal{E}: 0 \to (E,h) \to (E,h') \to 0$:

Choose $\begin{matrix}(E,\tilde h)\\ \downarrow \\ Z\times\mathbb{P}^1\mathbb{C}\end{matrix}$ s.t. $\tilde h_{|Z\times\{0\}} = h$ and $\tilde h_{|Z\times\{\infty\}} = h'$, set

$\tilde\phi(\mathcal{E}) :\equiv -\int_{\mathbb{P}^1\mathbb{C}} \phi(E,\tilde h)\log|z|^2$ (z coordinate on $\mathbb{P}^1\mathbb{C}$).

Then

$$\frac{\partial\bar\partial}{2\pi i}\tilde\phi(E,h,h') = -\int_{\mathbb{P}^1\mathbb{C}}\phi(E,\tilde h)\underbrace{\frac{\partial\bar\partial}{2\pi i}\log|z|^2}_{=\delta_\infty-\delta_0} = \phi(E,h)-\phi(E,h').$$

In the general case, one associates to $\mathcal{E}$ a sequence

$$\tilde{\mathcal{E}}: 0 \to E'(1) \to E\oplus E'(1)/E' \to E'' \to 0$$
$$\downarrow$$
$$Z\times\mathbb{P}^1\mathbb{C}$$

where E' is mapped to $E'(1) = E'\otimes\mathcal{O}(1)$ via a section which vanishes at $\{\infty\}$.

Example: $\tilde c_1(L,h,h') = \log\frac{h}{h'}$ for any line bundle L.

II) Construction of the Ray-Singer torsion and the Quillen metric

Motivation: Consider the complex $(\Gamma^\infty(Z,\Lambda T^{*0,1}Z\otimes E),\bar\partial)$ equipped with the L^2-metric $\|\alpha\|^2_{L^2} = (2\pi)^{-\dim Z}\int_Z\langle\alpha\wedge *\alpha\rangle_h$. We would like to construct the determinant of this complex; the determinant of a finite vector space V is defined as $\Lambda^{\max}V$.

Consider a complex of finite-dim. hermitian vector spaces

$$C_0 \xrightarrow{\bar\partial} C_1 \to \dots \to C_m \quad \text{with Laplacian } \square := (\bar\partial+\bar\partial^*)^2.$$

Then each C_q splits as $C_q = \ker\square_q \oplus \mathrm{im}\,\bar\partial_{q-1}\oplus\mathrm{im}\,\bar\partial^*_{q+1}$ and $\ker\square_q \cong H^q(C_\cdot,\bar\partial)$. Then one verifies easily that

$$\det C_\cdot := \det C_0 \otimes \det C_1^* \otimes \det C_2 \dots \cong \det H^0\otimes\det H^{1*}\otimes\det H^2\otimes\dots =: \det H^\cdot \quad (**)$$

$\det H^\cdot$ has a canonical metric $\|\cdot\|^2_{|H^\cdot}$ induced by restricting the metric on $C^\bullet_\cdot$ on $H^\cdot \subseteq C^\cdot_\cdot$. But with this metric, the isomorphism is <u>not</u> an isometry. It gets isometric by taking instead

$$\det(C_\cdot, \|\cdot\|^2) \cong (\det H^\cdot, \|\cdot\|^2_{|\det H^\cdot} \cdot \frac{(\det'\Box_0)^0 (\det'\Box_2)^2 \cdots}{(\det'\Box_1)^1 (\det'\Box_3)^3 \cdots})$$

($\det'$ denotes the product of the non-zero eigenvalues). The factor may be written as

$$\exp\left(\frac{\partial}{\partial s}\Big|_{s=0} \sum_{q\geq 1} (-1)^{q+1} q \sum_{\substack{\lambda \in \operatorname{Spec}\Box_q \\ \lambda \neq 0}} \lambda^{-s}\right).$$

In the infinite dim. situation, for $(\Gamma(\Lambda T^{*0,q} Z \otimes E), \bar\partial)$, Ray and Singer defined the zeta function

$$\zeta_{RS}(s) := \sum_{q\geq 1} (-1)^{q+1} q \sum_{\substack{\lambda \in \operatorname{Spec}\Box_q \\ \lambda \neq 0}} \lambda^{-s} \quad \text{for } \operatorname{Re} s > \dim Z \text{ [RS]}.$$

This zeta function has a meromorphic extension to the complex plane which is holomorphic at $s=0$.

<u>Def.</u>: The <u>Ray-Singer analytic torsion</u> is defined as $e^{\zeta'(0)} =: \tau$.

The <u>Quillen metric</u> is the hermitian metric

$$h_Q := \|\cdot\|^2_{L^2} \cdot \tau \quad \text{on the line } \det H^{0,*}(Z,E).$$

III) Properties of the Quillen metric for fibrations

Consider now a fibration $\begin{smallmatrix} M \\ \downarrow \\ B \end{smallmatrix}$ of complex manifolds which is proper holomorphic with fibre Z; let E be a holom. herm. vector bundle over M and choose a smooth varying Kähler metric g^{TZ} on the fibres. Assume that locally on B there exists a Kähler metric on M (which need not to be compatible with g^{TZ})

$$\begin{matrix} E \\ \downarrow \\ M \\ \pi \downarrow Z \\ B \end{matrix}$$

The <u>Knudsen-Mumford determinant</u> $\lambda_{KM} := \det R^\cdot \pi_* E$ is a holom. line bundle on B with fibres $\det H^{0,*}(Z, E_{|Z})$.

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Th. [BGS]: The Quillen metric h_Q induces a smooth Herm. metric on λ_{KM}.

This does not hold in general for the L^2-metric $\|\cdot\|^2_{L^2}$!

Now, the classical Grothendieck-Riemann-Roch th. states that

$$ch(R^\cdot \pi_* E) \equiv \int_Z Td(TZ)\, ch(E) \quad (\text{mod im } d).$$

In particular,

$$c_1(\lambda_{KM}, h_Q) \equiv \left[\int_Z Td(TZ, g^{TZ})\, ch(E,h)\right]^{(2)} (\text{mod im } d).$$

Th. [BGS]: $c_1(\lambda_{KM}, h_Q) = \left[\int_Z Td(TZ, g^{TZ})\, ch(E,h)\right]^{(2)}$ as forms on B.

Thus, one has a strong refinement of the classical Riemann-Roch.

Consider now two pairs of metrics $(g^{TZ}, h), (g^{TZ\prime}, h')$ on Z and E. These induce different Quillen metrics h_Q and h'_Q, whose difference is given by the following theorem:

Th. [BGS]:

$$\widetilde{c}_1(\lambda_{KM}, h, h') = \left[\int_Z (\widetilde{Td}(TZ, g^{TZ}, g^{TZ\prime})\, ch(E,h) + Td(TZ, g^{TZ\prime})\, \widetilde{ch}(E,h,h'))\right]^{(0)}.$$

The last two theorems may be extended to the entire Chern character of $R^\cdot \pi_* E$ if the fibration is Kähler and if the dimensions of $H^{0,*}(Z, E|_Z)$ are constant [BK]. In this case, one considers the so-called analytic torsion forms $T \in \bigoplus_p A^{p,p}(B)$, $T^{(0)} = \tau$.

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Titel: Integral points on arithmetic surfaces
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Let K be a number field, $\mathcal{O}_K$ his ring of integers. Let $F_i(X_1, \dots, X_n)$ a family of polynomials with coefficients in $\mathcal{O}_K$. An integral point of this family will be a solution $\underline{x} = (x_1, \dots, x_n) \in \mathcal{O}_{\bar{K}}^n$ such that $F_i(x_1, \dots, x_n) = 0 \quad \forall i \in [1, m]$.

Geometrically let $X \xrightarrow{f} \mathrm{Spec}(\mathcal{O}_K)$ be a finite type scheme, an integral point is just a point of $X(\mathcal{O}_{\bar{K}})$.

Theorem (Rumely)

If X is separated, irreducible and geometrically irreducible then $X(\mathcal{O}_{\bar{K}}) \neq \emptyset \iff f$ surjective.

We discuss in this talk some bounds of the degree and the height of an integral point on an arithmetic surface.

Question Let $X \to \mathrm{Spec}(\mathcal{O}_K)$ be an arithmetic surface $P \in X_K(K)$ and E_P the corresponding section. Let $X_0 = X - E_P$

1) Find $Q \in X_0(\mathcal{O}_{\bar{K}})$ with bounded height and bounded degree.

2) Find $A(X, K)$, $B(X, K)$ constants independant of P and $Q \in X_0(\mathcal{O}_{\bar{K}})$ such that $[K(Q) : K] < A(X,K)$ and $h(Q) < B(X,K)$.

Example 1

Let $X = \mathbb{P}^1_{\mathcal{O}_K}$ and L be the Hilbert class field of K.
There exits $Q \in X_0(\mathcal{O}_L) = (X - E_P)(\mathcal{O}_L)$ such that

$$h(Q) \leq h(P) + \log\left[\exp(Nc - 2h(P)) + N D_K^{1/2}\left(\frac{2}{\pi}\right)^{\frac{N}{2}}\right]$$

- h is the logarithmic height on $\mathbb{P}^1_{\bar{K}}$: If $R = (\alpha, \beta)$

$$h(R) = \log\left(\prod_{v \in \mathcal{S}_{K(R)}} \max(|\alpha|_v, |\beta|_v)\right)^{\frac{1}{[K(R):\mathbb{Q}]}}$$

- $N = [L:\mathbb{Q}] = |\mathrm{Pic}\,\mathcal{O}_K|\,[K:\mathbb{Q}] = r_1 + 2r_2$
- D_K is the discriminant of K
- Let $\Gamma \subset H = \{\underline{x} = (x_1, \dots, x_n) \in \mathbb{R}^{r_1+r_2} \mid \sum_{i=1}^{r_1+r_2} x_i = 0\} \subset \mathbb{R}^{r_1+r_2}$

be the lattice of units, $c = \inf\{t \in \mathbb{R}_+$, such that Γ is spanned by sections of sup-norm less than $t\}$

Example 2

Let X_K be a curve of genus g and $\pi : X_K \longrightarrow \mathbb{P}^1_K$ a morphism of degree d. Chose a model $\tilde{X}_{\mathcal{O}_K}$ such that the following diagram is commutative

$$\begin{array}{ccc} X_K & \longrightarrow & X_{\mathcal{O}_K} \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{P}^1_K & \longrightarrow & \mathbb{P}^1_{\mathcal{O}_K} \end{array}$$

Let $P \in X_K(K)$ and $X_0 = (X_{\mathcal{O}_K} - E_P)$.
There exits Q in $X_0(\mathcal{O}_{\bar{K}})$ such that

1) $[K(Q):\mathbb{Q}] \leq d \cdot |\mathrm{Pic}\,\mathcal{O}_K|\,[K:\mathbb{Q}]$
2) $h(Q) \leq h(P) + \log\left[\exp[Nc - 2h(P)] + N D_K^{1/2}\left(\frac{2}{\pi}\right)^{\frac{N}{2}}\right]$

Where the height $h(R)$ of a point in $X_K(\bar{K})$ is $h(\pi(R))$.

Example 3

Let $X \downarrow^{\pi} \mathrm{spec}(\mathcal{O}_K)$ be an arithmetic surface of genus 1 with good reduction everywhere.

Let $P_0 \in X_K(\bar{K})$ be a torsion point with non prime order.

Then for each $P \in X_K(k)$, $Q = P + P_0$ is an integral point of $X_0(\mathcal{O}_{\bar{K}}) = (X - E_P)(\mathcal{O}_{\bar{K}})$.

(So $[\bar{K}(Q):\mathbb{Q}]$ is bounded and $h_{N.T}(P) = h_{N.T}(Q)$)

Remark

If you can solve the 2° question for an arithmetic surface of genus ≥ 2, you have Mordell conjecture.

Proof: Let $P_1, \dots, P_r$ the finite number of points in $X_K(\bar{K})$ such that $\begin{cases} [\bar{K}(P_i):K] \leq A(X,K) \\ h(P_i) \leq B(X,K) \end{cases}$. Let $D_1, \dots, D_r$ be the associated divisors.

By Siegel's theorem $X_i(\mathcal{O}_K) = (X - D_i)(\mathcal{O}_K)$ is finite.

Every point $P \in X_K(k)$ is in $X_i(\mathcal{O}_K)$ for some $i \in [1, r]$,

so $X_K(K)$ is finite.

Theorem

Let $X \to \mathrm{Spec}(\mathcal{O}_K)$ be an arithmetic surface equipped with Arakelov's metric at infty. Let L a positive hermitian line bundle ($(L.L) > 0$, $(L.\mathcal{O}_X(D)) > 0$ for integral D).

For each $P \in X_K(k)$, there exists $Q \in (X - E_P)(\mathcal{O}_{\bar{K}})$ such that $h_L(Q) \leq \dfrac{(L.L)}{\deg(L)}$.

Titel: An analytic discriminant for polarized algebraic K3 surfaces

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Seite: 1

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It is natural to ask for specific "evaluations" of analytic torsion in terms of data, in one form or another, which describes the geometric situation under consideration. For example, if $X = \mathbb{C}/\langle 1,\tau \rangle$ denotes a marked elliptic curve, equipped with flat metric. Then Ray-Singer (Ann. Math 98) proved that the determinant of the Laplacian $\det^*\Delta$ can be evaluated using Kronecker's limit formula. In fact, they proved

$$\frac{\det^*\Delta}{\mathrm{vol}(X)\,\mathrm{Im}(\tau)} = c\,|\Delta(\tau)|^{1/6} \qquad (1)$$

for some universal constant c (depending on various normalizations). Further, one can show that (1) is independent of the scale of the flat metric on X, hence $\det^*\Delta/\mathrm{vol}\,X$ is an invariant of the elliptic curve, and (1) is to be viewed as a function on the upper half plane, thought of as the space of marked elliptic curves together with a family $\{dz\}$ of holomorphically varying holomorphic one-forms.

If X is a compact Riemann surface of genus $g \geq 2$, equipped with its natural hyperbolic metric, then the spectrum of the Laplacian Δ acting on smooth functions can be studied by using the Selberg trace formula.

With this tool, Sarnak (CMP 110) proved the relation

$$\det(\Delta + s(s-1)) = (G(s))^{g-1} Z_X(s) \qquad (2)$$

where $G(s)$ is a universal function and $Z_X(s)$ is the Selberg zeta function, which is defined in the half-plane $\mathrm{Re}(s) > 1$ via an Euler product involving the lengths of primitive closed geodesics. The factor in the left-hand-side of (2) is the zeta-regularized product of the sequence $\{\lambda_n + s(s-1)\}$ where $\{\lambda_n\}$ is the sequence of eigenvalues of the Laplacian Δ.

One question to consider is the following: Can other zeta functions be expressed as a zeta-regularized product take over the set of its zeros and poles, even in situations when there is no existing knowledge of an operator, such as a Laplacian? This question has been considered by many authors, with a very general theorem given by Jorgenson-Lang (Math. Ann. 297).

As a generalization of (1), the case of polarized algebraic K3 surfaces has been considered by Jorgenson-Todorov (MPI preprint). In work due to Piatetski-Shapiro-Shafarevich (Math USSR Izv. 5) and Kulikov (Math USSR Izv. 11), it is shown that the moduli space of polarized algebraic K3 surfaces of degree d with a canonical basis of $H_2(X, \mathbb{Z})$ can be realized through

the period map as a <u>Zariski open subset</u> of the symmetric
space associated to the group SO(2,19). The points
not represented by marked, polarized K3 surfaces can
be viewed as a type of degeneration of the metric or of
the surface (see Kobayashi - Todorov (Tôhoku Math 39)
and Kobayashi (Advanced studies in MATH 18-II)). Yau
proved that associated to every polarized, algebraic K3
surface there is, up to scale, a natural Ricci flat
Kähler metric (Comm. Pure. Appl. Math 31). The analogue of
the family of forms $\{dz\}$ on the moduli of marked elliptic
curves can be constructed using the Torelli theorem
and Todorov's theorem (Izv. Akad USSR 10) on the
monodromy operator on $H_2(X_t, \mathbb{Z})$ for semi-stable families
of polarized K3 surfaces. With this family of forms
$\{\omega\}$, Tian and Todorov prove that $\log(\|\omega\|^2_{L^2})$ is
a potential for the canonical Weil-Petersson metric on
$\mathcal{M}^d_{mpa}$, the moduli space of marked, polarized, algebraic
K3 surfaces of degree d (Todorov - CMP 126). The
Quillen metric on the determinant line $\det \mathcal{H}(\Theta)$ also
gives a potential for the Weil-Petersson metric (see
Todorov, CMP 126; Bismut - Gillet - Soulé, CMP 115,
and Fujiki - Schumacher, RIMS 26). Combining all
this information, one can generalize (1) to $\mathcal{M}^d_{mpa}$.
Namely, there exists a holomorphic function F on $\mathcal{M}^d_{mpa}$

Such that

$$\frac{\exp(-T_\mu'(0))}{\mathrm{Vol}_\mu(X)\,(\|\omega\|^2_{L^2})} = |F| \qquad (4)$$

where $-T_\mu'(0)$ is the analytic torsion relative to the Calabi-Yau metric. In the case X is a Kummer surface with a polarization constructed from an even theta divisor on the associated abelian surface, then F can be related to classical modular forms (Jorgenson-Todorov: in preparation).

In (4), it was important to obtain two potentials of the Weil-Petersson metric (one from the period map and one from the analytic torsion). An extension of the above results to general Calabi-Yau varieties is under investigation, as are the arithmetic properties of F, which we call the discriminant of the polarized K3 surface.

Titel: Chern classes and heights

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1] Chern classes:

Joint work with H. GILLET.

X: regular quasi-projective flat scheme $/\mathbb{Z}$.

$\overline{E} = (E, h) =$ hermitian vector bundle on X, i.e.

$E :=$ locally free coherent $\mathcal{O}_X$-module

$h := C^\infty$ hermitian metric on the corresponding holomorphic vector bundle, i.e., for all $x \in X(\mathbb{C})$, an hermitian scalar product

$$E_x \otimes E_x \longrightarrow \mathbb{C}$$

whose entries in any local trivialization are C^∞.

We also assume $F_\infty^* h = h$.

Theorem 1: The following properties define uniquely a theory of Chern classes

$$\hat{c}_p(\overline{E}) \in \widehat{CH}^p(X):$$

1) $\hat{c}_0 = 1_{\not{}}$,

2) If $\overline{L}$ = ~~...~~ hermitian line bundle,

$\hat{c}_p(\bar{L}) = 0$ if $p \geq 2$,

$\hat{c}_1(\bar{L}) =$ class of $(\mathrm{div}(s), -\log\|s\|^2)$ for any rational section $s \not\equiv 0$ of L on X.

3) For any $f: Y \to X$,

$f^* \hat{c}_p(\bar{E}) = \hat{c}_p(f^* \bar{E})$

4) Give $\bar{E} = (E, h)$ and $\bar{F} = (F, h')$ on X, let

$\bar{E} \oplus \bar{F} = (E \oplus F, h \overset{\perp}{\oplus} h')$, $h \overset{\perp}{\oplus} h'$: orthogonal direct sum.

Then

$$\hat{c}_p(\bar{E} \oplus \bar{F}) = \sum_{i+j=p} \hat{c}_i(\bar{E})\, \hat{c}_j(\bar{F})$$

4) $\bar{L}$ = hermitian line bundle, $\bar{E}$ = h.v.b. of rank n,

$$\hat{c}_p(\bar{E} \otimes \bar{L}) = \sum_{i=0}^{p} \binom{n-i}{i} \hat{c}_i(\bar{E})\, \hat{c}_1(\bar{L})^{p-i}$$

5) $\omega(\hat{c}_p(\bar{E})) = c_p(\bar{E}_{\mathbb{C}}) \in A^{pp}(X)$, the p-th Chern form of $(E_{\mathbb{C}}, h)$ i.e. the coefficient of t^p in $\det\left(Id_{E_{\mathbb{C}}} - \frac{t}{2\pi i} \nabla_h^2\right)$, ∇_h : holomorphic hermitian connection of $(E_{\mathbb{C}}, h)$.

Similarly one can define a Chern character class

$\widehat{ch}(\overline{E}) \in \bigoplus_{p \geq 0} \widehat{CH}^p(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and a Todd class

$\widehat{Td}(\overline{E}) \in \bigoplus_{p \geq 0} \widehat{CH}^p(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

A consequence of the axioms is the following

Proposition:

Given $\mathcal{E} : 0 \to \overline{S} \to \overline{E} \to \overline{Q} \to 0$ an exact sequence of algebraic vector bundles on X, equipped with arbitrary hermitian metrics, then

$\widehat{c}_p(\overline{S} \oplus \overline{Q}) - \widehat{c}_p(\overline{E}) = a(\widetilde{c}_p(\mathcal{E}))$,

where $\widetilde{c}_p(\mathcal{E}) \in \widetilde{A^{p-1,p-1}(X)}$ is the Bott-Chern secondary characteristic class of $\mathcal{E}$.

2) Projective heights:

Joint work with J.-B. BOST and H. GILLET.

$\mathbb{P}^N$ = N-dimensional projective space over $\mathbb{Z}$

$\cup$

Y any irreducible closed subvariety, $\dim(Y) = p$.

If g_Y is any Green current for Y in $\mathbb{P}^N$, and $x \in \widehat{CH}^p(\mathbb{P}^N)$, the following real number does not depend on g_Y:

$$\widehat{\deg}(x \mid Y) := x \cdot y - \frac{1}{2} \int_{\mathbb{P}^N(\mathbb{C})} g_Y \, \omega(x)$$

(more generally $\widehat{\deg}(x \mid Y)$ makes sense for $Y \subset X$, X regular quasi-projective flat / $\mathbb{Z}$, $\dim Y = p$, $x \in \widehat{CH}^p(X)$, Y proper over $\mathbb{Z}$).

Define:

<u>Faltings height:</u> $h_F(Y) := \widehat{\deg}\left(\hat{c}_1(\overline{\mathcal{O}(1)})^p \mid Y\right)$

where $\mathcal{O}(1)$ is equipped with dual metric to $\mathcal{O}(-1)$, and $\mathcal{O}(-1)$ has metric induced by trivial metric from $\mathbb{C}^{N+1}$ via the canonical exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^N}^{N+1} \to Q \to 0.$$

On the other hand, let $\overline{Q}$ be equipped with the induced metric and $\mathbb{P}^{N-p} = \{x_0 = \dots = x_{p-1} = 0\}$.

Then (Gillet-S.)

$\hat{c}_p(\overline{Q})$ = class of $(\mathbb{P}^{N-p}, \Lambda)$

Λ: Levine form.

Define the projective height of Y to be

$h(Y) := \widehat{\deg}(\widehat{c}_p(\overline{Q}) \mid Y)$.

let $\sigma_p = \frac{1}{2} \sum_{n=1}^{p} \sum_{k=1}^{m} \frac{1}{k}$ if $p \geq 1$, $\sigma_p = 0$ if $p \leq 0$.

Theorem 2 :

Y closed irreducible, $\dim(Y) = p$.

a) If $p \geq 1$, $h(Y) = h_F(Y) - \sigma_{p-1} \deg(Y_{\mathbb{Q}})$,

where $\deg(Y_{\mathbb{Q}})$ = algebraic degree in $\mathbb{P}^N_{\mathbb{Q}}$.

b) $h(Y) \geq 0$ with equality iff Y is a standard linear subspace $\tau(\mathbb{P}^{p-1})$, $\tau \in \mathfrak{S}_{N+1}$.

Theorem 3 : $Y \subset \mathbb{P}^N$ $Z \subset \mathbb{P}^N$

$\dim Y = p$, $\dim Z = q$, $p+q \geq N+1$, $Y_{\mathbb{Q}}$ meets $Z_{\mathbb{Q}}$ properly.

Then

$$h(Y.Z) \leq \deg(Y_{\mathbb{Q}})\, h(Z) + h(Y) \deg(Z_{\mathbb{Q}}) + \left(N+1 - \frac{p+q}{2}\right) \log(2) \deg(Y_{\mathbb{Q}}) \deg(Z_{\mathbb{Q}}).$$

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Titel: Heights and Reductions of Semistable Varieties

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Seite: 1

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The following three results are presented in my talk:

1. We show that the analogue of "Deligne - Mumford" semistable reduction" is Poincare metric for a curve defined over complex numbers.

For a normed field K, a curve C over K, an ample line bundle $\mathcal{L}$ on C, we defined the object in the category $Pic(K)\otimes_{\mathbb{Z}}\mathbb{Q}$

$$Q(\mathcal{L}) := \frac{1}{d}\langle \mathcal{L}, \Omega\rangle - \frac{g-1}{d^2}\langle \mathcal{L},\mathcal{L}\rangle,$$

where $d = \deg \mathcal{L}$, Ω = canonical bundle of C, $\langle\ ,\ \rangle$ is the Deligne - Pairing. Then a "positive norm" on $\mathcal{L}$ induces a norm on Ω by its "curvature" and therefore defines a norm $Q(\|\ \|)$ on $Q(\mathcal{L})$. Then we show that

<u>Theorem</u>:

(1) If K is nonarchimedean with a valuation ring R, and $\|\ \|$ is a norm on $\mathcal{L}$ induced by a model $(X_R, \mathcal{L}_R)$ of $(X,\mathcal{L})$ on Spec R such that the embedding $X_R \hookrightarrow \mathbb{P}(\Gamma(X_R, \mathcal{L}_R^{\otimes n})$ is Chow - semistable

for $n \gg 0$, i.e. $(X_R, \mathcal{L}_R)$ is asymptotically Chow-semistable, then $\| \|$ is a maximal point for the functional Q.

(ii) If $K = \mathbb{C}$ is archimedean, $\| \|$ is a metric on $\mathcal{L}$ which is harmonic with respect to the Poincare metric, then $\| \|$ is the unique maximal point of the functional Q.

By a theorem of Mumford, when K is nonarchimedean, the metric on ω induced by "Deligne-Mumford semistable reduction" is a maximal point of the functional Q from "metrics on ω" to "metrics on $Q(X, \omega)$". This shows the analogy by letting $\mathcal{L} = \omega$ in (ii) and (i).

2. For a semistable variety $X \hookrightarrow \mathbb{P}^N_{\bar{\mathbb{Q}}}$, we construct a height $\hat{h}(X)$ which is invariant under $SL(N+1, \bar{\mathbb{Q}})$ and we show that $\hat{h}(X)$ is bounded below by an explicit constant and is associated to an "ample metrized line bundle" on the quotient Chow variety.

For an embedding $X \hookrightarrow \mathbb{P}^N_{\bar{\mathbb{Q}}}$, a hermitian vector bundle on $\operatorname{Spec} O_K$ with an isomorphism $\mathcal{E}_{\bar{\mathbb{Q}}} \cong \bar{\mathbb{Q}}^{N+1}$, one defines the Faltings type height

$$h_{\mathcal{E}}(X) = \left[\frac{c_1(\mathcal{L}_{\mathcal{E}})^{\dim X+1}}{(\dim X+1)\deg X} - \frac{\deg \mathcal{E}}{N+1}\right] / [K:\mathbb{Q}],$$

where $\overline{X}$ = Zariski closure of X in $\mathbb{P}(\mathcal{E})$,

$\mathcal{L}_{\mathcal{E}} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_{\overline{X}}$.

<u>Theorem</u>. (i) X is semistable $\Leftrightarrow$ $h_{\mathcal{E}}(X)$ is bounded below as a function of $\mathcal{E}$. ~~of and~~ ~~an~~

(ii) If X is semistable then

$$h_{\mathcal{E}}(X) \geq -\frac{1}{2(N+1)} \sum_{n=1}^{N} \sum_{m=1}^{n} \frac{1}{m}$$

(iii) $\hat{h}(X) = \inf_{\mathcal{E}} h_{\mathcal{E}}(X)$ is a height function on the quotient chow variety $Ch_{N,d,n}//SL(N+1)$ where $d = \deg X$, $n = \dim X$, associated to an "ample metrized line bundle".

<u>Remark</u> a.) Part "$\Rightarrow$" of (i) is due to Cornalba-Harris in functional case and Bost.

b) Soulé gives a bound for $h_{\mathcal{E}}(X)$ in terms of successive minima of $\mathbb{P}(\mathcal{E})$, which in ~~turn~~ gives a lower bound depending on the definition field of $\mathcal{E}$.

our proof is based on Soulé's argument.

3. We construct canonical Fubini-Study metrics for semistable varieties $X \hookrightarrow \mathbb{CP}^N$.

<u>Definition</u>: Let $\mathcal{L}$ be an very ample line bundle on a complex variety. $V \hookrightarrow \Gamma(X,\mathcal{L})$ be a very ample linear system. ~~$X \hookrightarrow \mathbb{P}(X,V)$~~ $X \hookrightarrow \mathbb{P}(V)$. A positive metric $\|\;\|$ on $\mathcal{L}$ is critical w.r.t V if for an orthonormal basis $s_0, \ldots, s_N$ with respect to inner product: $\langle s_i, s_j \rangle_{L^2} := \frac{1}{\deg X} \int_X \langle s_i, s_j \rangle c_1(\mathcal{L})^n$

$$= \delta_{ij},$$

one has that the distorsion function is 1:

$$b_{\|\;\|}(x) := \frac{1}{N+1} \sum_{i=0}^{N} \|s_i\|^2(x) = 1.$$

<u>Theorem</u>: (i) If $\mathcal{L}$ has a critical metric w.r.t V then $X \hookrightarrow \mathbb{P}(V)$ is semistable

(ii) If $X \hookrightarrow \mathbb{P}(V)$ is stable then there exists a unique critical metric on $\mathcal{L}$ w.r.t to V, up to constant multiple and automorphism of pair $(X, \mathcal{L})$.

Autor: Shouwu Zhang Seite: 5

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ALGEBRAIC GEOMETRY OVER $\overline{\mathbb{Q}}$

LUCIEN SZPIRO

In 1978 I gave a conference at the ENS in Paris with title "Faisceaux arithmétiques coherents". The object was to introduce the mathematical public to what I called "Arakelov theory". To justify this introduction I explained what could be done with the idea of Arakelov and Parshin: "Put metrics at infinity on vector bundles and you will have geometric intuition of compact varieties to help you". I also explained that my seminar [Sz 1] was an open book of conjectures once you knew the translation of:effective divisor, Kodaira-vanishing theorem, bounded families, Hodge index etc. ... Needless to say I did not raised enthousiasm at this point!

I present here a somehow detailed plan of what has been done on this program.

1. Faisceaux arithmétiques cohérents.

1-1 **Heights** the local-global equality defining the height of a point $\mathbf{x} \in \mathbb{P}^n(K)$ is: (noting $L = \mathcal{O}(1)$)

$$(\star) h(\mathbf{x}) = \frac{1}{K:\mathbb{Q}} \log \frac{\mathrm{vol}(\mathcal{O}_K)}{\mathrm{vol}(L/E_{\mathbf{x}})} = \frac{1}{K:\mathbb{R}} \log \frac{\prod_v \sup_i |x_i|v}{N(\sum x_i \mathcal{O}_K)}$$

where $E_{\mathbf{x}}$ is the section of $\mathbb{P}^n_{\mathcal{O}_K} \to \operatorname{Spec} \mathcal{O}_K$ corresponding to the point $\mathbf{x}$. This formula teaches you many things:

(1) It is a **Riemann-Roch** theorem in dimension one analogous to $\chi(L) = \deg L - g + 1$ on a curve.

(2) **The height is the "intersection"** of a scheme of dimension 1 ($E_{\mathbf{x}}$) with a cycle of codim 1 $c_1(L)$, whence L is metrized. The fundamental theorem on heights is typical of the type of result one is able to get on $\overline{\mathbb{Q}}$:

Theorem 1(Northcott's theorem):.
Given $d \in \mathbb{N}$ and $A \in \mathbb{R}_+$ the set

$$\{\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{R}}) | h(\mathbf{x}) \leq A;\ deg[K(\mathbf{x}) : \mathbb{Q}] \leq d\}$$

is finite.

As a corollary, once you know $\hat{h}$ (the **Neron-Tate height** on an abelian variety A) you get the finitness of the torsion of $A(K)$ because ($\hat{h}(P) = 0 \Leftrightarrow P$ torsion). Also you get: [weak (Mordell-Weil) $\Rightarrow$ (**Mordell-Weil** ($A(K)$ finite type over $\mathbb{Z}$)].

1-2 **Basic theorems of algebraic number theory** It is good time now to prove in the language of metrized line bundles on Spec$\mathcal{O}_K$ the following classics: $cl(\mathcal{O}/K)$ is finite, $|d_K| > 1$, Dirichlet units'theorem, fix d_K there is only a finite number of K possible. In fact this last statement is not the useful one. One needs-and one proves - the following

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Theorem 2:. *Given $n \in \mathbb{N}$ and a finite set of primes $\mathfrak{p}_1 - \mathfrak{p}_r$ in $\mathcal{O}_K$ the set of number fields L such that $[L : K] \leq n$ and $\mathcal{O}_L$ is ramified over $\mathcal{O}_K$ only over the $\mathfrak{p}_i$ is finite.*

The proof of this **is not** "arakelovian" you have to bound the wild vanification (something false in equal charater $p > 0$) This is typical: arakelovian methods gives you finitness of certain objects in $\overline{\mathbb{Q}}$, then the arithmetic of things allows you to work over a given number field. In particular theorem 2 is what you need to prove weak Mordell-Weil.

1-3 **The paper of Arakelov** In 1972 Arakelov introduces an intersection theory on arithmetic surfaces with the following properties:

(i) **Adjunction** formula (with a "grain de sel": when the divisor is not a section).

(ii) The theory extends **Neron-Tate** pairing on divisors of degree zero.

Z One should remark at this point that an arithmetic surface $X \to \text{Spec}\mathcal{O}_K$ is analogous to a surface fibered over a compact curve but with **no fixed part in the jacobian**. It is clear: for the Neron-Tate height is zero on the fixed part and not only on torsion points.

(iii) The **admissible metrics** he introduces are of close arithmetic content, one can see that in two occasions (at least)

a) the case of **elliptic curves** has been worked out quite completly in [Sz 2] after a start in [F 1].

b) the **self intersection** $(\omega_{X_{\mathcal{O}_K}} \cdot \omega_{X_{\mathcal{O}_K}})$ is a new invariant for curves of genus at least 2. It is exploited below (2-1)

1-4 **Cohomologie des faisceaux cohérents**

We all know that a theorem of **Riemann-Roch** computing the volume for the Quillen metric of Rf_*E for a metrized line bundle on a generically smooth $X \to \text{Spec}\mathcal{O}_K$ has been proposed by Gillet-Soulé or Faltings. One should note that it is not clear to anyone that the two versions of Riemann-Roch coincide. The work of Gillet-Soulé has forced the beautiful long paper of Bismut-Lebeau in analysis. To get down to business one needs another paper in analysis by Bismut Vasserot, and one finally gets:

Theorem 3 (The arithmetic Hilbert-Samuel theorem).
Let L be a positively metrized ample line bundle on $f : X \to \text{Spec}\mathcal{O}_K$,

$$if n \gg 0 \; then \; - \log vol_{L^2}(f_* L^{\otimes n}) = \frac{n^{d+1}}{(d+1)!}(L \cdots L) + o(n^{d+1})$$

This is the definition of $(L \cdots L)$ $(d+1)$ times. (d is the dimension of the generic fiber). The following corollaries where indicated in my talk in 78:

Corollary 1 (existence theorem for small sections).
If $(L \cdots L) > 0$ and $n \gg 0$ there exists $s \in f_* L^{\otimes n}$ such that

$$||s||_{L^2_{,\sigma}} \leq 1 \, \forall \sigma \in \infty$$

Proof. Minkowski famous theorem on lattice points gives a term in $n^d \log n$ (d comes from Riemann-Roch on the genetic fiber)

Corollary 2. *If H is numerically ample and $(L_{/H} \cdot L_{/H} \cdots L_{/H})$ (d times) is zero then $(L \cdot L \cdots L) \leq 0(d+1$ times). (**index theorem** most striking when $d = 1$: $(L \cdot H = 0 \to (L \cdot L) \leq 0)$)*

1-5 **More applications of** Corollary 1 of arithmetic Hilbert-Samuel.

The existence of small sections is crucial in many cases we list a few below: (At this point it is noticeable to quote the paper of **Abbes and Bouche** which gives in less than 30 pages a self contained proof of theorem 3 ([A-B])).

(i) $(\omega_{X O_K} \cdot \omega_{X O_K}) \geq 0 \quad g \geq 1$ (Faltings [F 1])
(ii) second proof of Mordell conjecture (**Vojta** [V])
(iii) **Miyaoka** $c_1^2 \leq 4c_2$ for stables vector bundles on curves $g \geq 2$
(iv) My proof of Bogomolov conjecture generalizing Raynaud's theorem [Sz 2]
(v) **Shouwu Zhang**'s Nakai Moisheson is criterium of ampelness [Zh 1]

1-6 **The degree of a subvariety is the self-intersection.** The problem is non generically smooth subvarieties (of $\mathbb{P}^n$ for example or ...) of course one wants to put $(0(1) \cdots 0(1) \cdot 0(1))$ number of times equal to dim of generic fiber plus one as a definition!! This is fine when the generic fiber is smooth. Faltings finds a way in [F 2], Gillet-Soulé catched up later The more natural way is to get:

Theorem 3′(arithmetic Hilbert Samuel for any variety). *$-\log vol_{L^2} f_\star L^{\otimes n} = \frac{n^{d+1}}{(d+1)!}(L \cdots L) + o(n^{d+1})$ when the L^2 norm is computed on the smooth locus of the reduced variety.*

This is deduced of theorem 3 by S. Zhang in [Zh 2] using resolution of singularities).

The next usefull result is:

Theorem 4. *The degree of the cut out of a small hyperplane section gives a subvariety of small degree.*

(This can be made precise of coures (cf.[F2]). This statement with corollary 1 of theorem 3 is what is needed of the theory in the proof by Faltings of his famous "product theorem". Then, with that in hands, and Vojta-Siegel idea, one gets on to prove Lang's conjecture on abelian varieties:

Theorem 5. *$X \hookrightarrow A$ abelien varieties $X(K) \subset \bigcup_{i=1 \cdots n} B_i(K)$ where B_i are translate of sub-abelian varieties.*

1-7 **Grothendieck-Riemann-Roch?**

For a morphism $X \xrightarrow{f} Y$ over Spec$\mathcal{O}_K$ a few cases have been done:

(i) Max Noether formula $12X(\mathcal{O}_X) = c_1 + 4c_2$ (Faltings [F1])
(ii) Max Noether formula on $\mathcal{M}_g$ (L.Moret-Bailly [M-B 1])
(iii) Functional equation of θ functions (L. Moret-Bailly [M-B 2])
(iv) $c_1(Rf_\star)$ for f local complete intersection (**Lin Weng** [W])
(v) $c_1(Rf_\star)$ for Macaulay schemes (R. Elkik ([E 1], [E 2])

There is clearly work to be done if one likes Riemann-Roch theorems.

2. A la recherche de petits points: Numerical properties of the relative dualizing.

To try to progress and get (conjecturally so far) effectivity statements I have proposed in [Sz 1] to look for small points. The reason is that the **Parshin-Kodaira constructivon** tells you that one is interested in bounding $(\omega_{X/\mathcal{O}_K})^2$ and, the following lemma I have proved:

Lemma 5.

$$(\omega_{X/\mathcal{O}_K} \cdot \omega_{X/\mathcal{O}_K}) \leq (-E_P^2)(2g-2)(2g)$$

where X_K is a curve of genus $g \geq L$ and E_P the section of $X \to Spec\mathcal{O}_K$ given by a rational point $P \in X_K(K)$.

2-1 $\omega_{X/\mathcal{O}_K}$ **is big** (i.e. points are not too small).

Theorem 6. *Let X be an arithmetic surface of genus $g \geq 2$ and $X_K \hookrightarrow A$ an embedding of X_K in a polarised abelian variety, than there exists $\varepsilon > 0$ such that $\{P \in X_K(\overline{\mathbb{Q}}) | \hat{h}(P) \leq \varepsilon\}$ is finite exept may be if P "divides" $\omega_{X/\mathcal{O}_K}$. When X is smooth over $\mathcal{O}_K$ then $\omega^2_{X/\mathcal{O}_K} > 0$ is equivalent to the full finitness statement above.*

This result which generalized **Raynaud's** famous theorem on torsion points it was conjectured by **Bogomolov**. The exeption case is very interesting. It gives an arithmetic meaning of an Arakelov invariant. To prove that $\omega^2_{X/\mathcal{O}_K} = 0$ implies there is an infinite sequence of points $x_n \in X(\overline{\mathbb{Q}})$ with $\hat{h}(x_n) \to 0$, I needed to prove that the linear system $f_*\omega^{\otimes n}_{X/\mathcal{O}_K}$ has no fixed part. This has been proved by Kim ([K]). A more general theorem has been obtained by **S. Zhang** [Z 1].

Theorem 7 (Nakai Moisheson theorem for arithmetic surface). *If L is numerically ample(i.e. $(L \cdot L > 0$ $L \cdot D > 0$ for any effective $D)$ then $L^{\otimes n}$ is generated by its sections smaller than one for the L^2 norm, when $n \gg 0$.*

2-2 **When is** $(\omega_{X/\mathcal{O}_K} \cdot \omega_{X/\mathcal{O}_K}) > 0$

S. Zhang [Zh 2] has proved the following:

Theorem 8. *If $X \longrightarrow Spec\mathcal{O}_K$ is semi-stable and* **not smooth** *of genus $g \geq 2$ then $(\omega_{X/\mathcal{O}_K} \cdot \omega_{X/\mathcal{O}_K}) > 0$.*

For the smooth case not everthing is known. The following authors have made examples of $(\omega_{X/\mathcal{O}_K} \cdot \omega_{X/\mathcal{O}_K}) > 0$ for smooth fibrations. (Burnol), (S. Zhang), (Mestre, Bost, Moret-Bailly) in genus 2

2-3 $\omega_{X/\mathcal{O}_K}$ **must be small**

To prove such a statement I have in [Sz 1] a programm in two points

(i) - Kodaira vanishing

(ii) - Kodaira Spencer class.

In fact (i) has essentially been solved by **Miyaoka** 10 years ago

Theorem 9 (Miyaoka). *If E is a stable vector bundle on an arithmetic surface then $c_1^2 \leq 4c_2$.*

This has been written up by **Moriwaki**. The vanishing theorem is then deduced by the Mumford-Reider method. C. Soulé has had the good taste of believing in Miyaoka all these years. He then has published the proof of the following [S 1]:

Corollary 10. *If L is a numericllary ample like bundle on an arithmetic surface and $s \in R^1 f_* L^{\otimes -1}$, then $||s||_{L^2} \geq e(\log e = 1)$.*

2-4 **The instrisic small points conjecture.** I have made numerous variations on [Sz 1]. The following is a version I rather like: Let X_K be a curve of genus $g \geq 2$ over a number field K_0. then there exists constants $A(n)$ and $B(n)$ depending only on the curve over K_0 and on the integer n such that : if $(K : \mathbb{Q}) \leq n$, for every $P \in X(K) \exists P' \in X(\overline{\mathbb{Q}})$ such that

(i) - $E_{P'}^2 \leq [K(P') : \mathbb{Q}](A(n) \log D_K + B(n))$

(ii) $(E_P \cdot E_{P'}) \leq [KP, P') : \mathbb{Q}](A(n) \log D_K + B(n))$ this imply a strong effective Mordell which if proved for only **one curve** implies the (a,b,c) conjecture (or the conjecture of discriminant for elliptic curves) -a result of **L. Moret-Bailly** and myself following **A. Parshin** for modular curves-. In this direction note the theorem of **E. Ullmo**:

Theorem 11. *For every point $P \in X(K) \exists Q \in X(\overline{\mathbb{Q}})$ such that $(E_P \cdot E_Q)_{\text{finite}} = \emptyset$ and height (Q) is bounded.*

Of course remains the problem of $(E_P \cdot E_Q)_\infty$!!

3. What is our subject.

3-1 **Integral models or adelic studies**

The difficulties of looking at geometrical model integral over Spec$\mathcal{O}_K$ has lead many authors to try to go back to Weil'adelic point of view. In this direction S. Zhang [Zh 2], Rumely, Soulé-Gillet-Bloch have developped an adelic Arakelov theory. In [Zh 2] S. Zhang uses his theory to prove theorem 8.

3-2 **What metrics to choose**

Each problem seems to have its metric

i) Faltings put $\int \omega \wedge \overline{\omega}$ on $\overset{\max}{\wedge} \Omega_A^1$ for A an abelian variety with the success we know. It leads him to ask himself the purely arithmetic question of evaluating the discriminant of the kernel of an isogeny. This is a good example of the following philosophy:

α) determine a statement on $\overline{\mathbb{Q}}$ using a metric (here prove that the modular height has logarithmic singularities and satisfies Northcott theorem)

β) Then use arithmetic (here Raynaud $(p, \ldots, p)$ theorem and Grothendieck-Illusie "Barsotti-Tate tronqués")

ii) S. Zhang these days seems to prefer the Poincaré metric to Arakelov's original one for it extends on the closure of the Deligne-Mumford compartification of $\mathcal{M}_g$

iii) In defense of admissible Arakelov metrics I will quote:

α) it extends Neron-Tate (hence theorem 6)

β) $\omega_{X/\mathcal{O}_K}^2$ is a height with log singularities on $\mathcal{M}_g$ as the work of **Jorgenson** on Faltings δ function shows

γ) I proved in [Sz 2] $12\deg_{\text{Arakelov}}(\omega) = \log D_{\min}$ for an elliptic curve

iv) Soulé, Gillet, Bismut have choosen the Quillen metric fo it is the one that gives then a Riemann-Roch theorem!

3-3 Galois action

It is my conviction, that to really prove statements on a number field (or better on $\mathbb{Q}$) and not on $\overline{\mathbb{Q}}$ (as an inductive limit) one has to put not only metrics at infinity but to take into account the Galois action.

Faltings [F 3] is, like I note above, a perfect example.

To give a chance to the opposite opinion I will note two things:

α) my conjecture 2-4 about intisic small points is purely on $\overline{\mathbb{Q}}$ exept that D_K the discriminant is there.

β) E. Ullmo has proved the following:

Theorem 12. *Let X be an semi-state elliptic curve, $d = \frac{1}{12}log(D_{\min})$ then for every $\varepsilon > 0$ there exists only a finite number of* **torsion points** *P in $X(\overline{\mathbb{Q}})$ such that $-\phi_P^2/[K(P):K] \leq d - \varepsilon$. Here ϕ_P is the $\mathbb{Q}$ divisor that makes $E_P - E_0 + \phi_P$ purely of degree zero.*

This is a clever corollary of [Zh 1] and [Sz 2]. If $d > 0$ (i.e. if X **does not** have good reduction). Theorem 12 implies that **the set of torsion points which always pass by the connected component of the Néron model is finite over $\overline{\mathbb{Q}}$.** This could be obtained by Galois considention using Serre's theorem on the irreducibility of the image of Galois in $Gl_2(\mathbb{F})$ for $l \gg 0$. Here the proof is global and arakelovian !!

3-4 Conversation

Overheard at Max-Planck-Institute for Mathematics in Bonn June 94:

K-theorist: Let z be a complex number.

Analyst: We should consider $|z|$.

Together: Ah! We just have proved a theorem in arithmetic.

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Titel: Analytic torsion for Hermitian symmetric spaces
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The Ray-Singer analytic torsion is a positive real number associated to the spectrum of the Kodaira-Laplacian on Hermitian vector bundles over compact Hermitian manifolds [9]. Using the torsion as part of a direct image, Bismut, Lebeau, Gillet and Soulé were able to prove an arithmetic Grothendieck-Riemann-Roch theorem relating the determinant of the direct image in the K-theory to the direct image in the Arakelov Chow ring. One important step in the proof of the theorem was its explicit verification for the canonical projection of the projective spaces to Spec **Z** by Gillet, Soulé and Zagier [4]. In particular, the Gillet-Soulé R-genus, a rather complicated characteristic class occuring in the theorem was determined this way. The discovery of the same genus in a completely different calculation of secondary characteristic classes associated to short exact sequences by Bismut gave further evidence for the theorem.

In [6], an equivariant version of the analytic torsion was introduced and calculated for rotations with isolated fixed points of complex projective spaces. The result led Bismut to conjecture an equivariant arithmetic Grothendieck-Riemann-Roch formula [1]. Redoing his calculations concerning short exact sequences, he found an equivariant characteristic class R which equals the Gillet-Soulé R-genus in the non-equivariant case and the function R^{rot} in the case of isolated fixed points. In [2], he was able to show the compatibility of his conjecture with immersions.

In this script, we give the value of equivariant torsion for all equivariant vector bundles over any compact Hermitian symmetric space G/K with respect to the action of any $g \in G$ as calculated in [8]. The result is of interest also in the non-equivariant case: The torsion was known only for very few manifolds; the projective spaces, the elliptic curves and the tori of dimension > 2 (for which it is zero for elementary reasons). Also, Wirsching [11] found a complicated algorithm for the determination of the torsion of complex Grassmannians $G(p, n)$, which allowed him to calculate it for $G(2, 4)$, $G(2, 5)$ and $G(2, 6)$. Thus, our results extend largely the known examples for the torsion.

Remarkably enough, a similar calculation in the context of flat vector bundles over odd-dimensional Riemannian manifolds provides the diffeomorphy classification of some locally symmetric spaces of the compact type [7].

Let M be a compact n-dimensional Kähler manifold with holomorphic tangent bundle TM. Consider a hermitian holomorphic vector bundle E on M and let $\bar\partial$ and $\bar\partial^*$ denote the associated Dolbeault operator and its formal adjoint. Let $\Box_q := (\bar\partial + \bar\partial^*)^2$ be the Kodaira-Laplacian acting on $\Gamma(\Lambda^q T^{*0,1}M \otimes E)$. We denote by $\mathrm{Eig}_\lambda(\Box_q)$ the eigenspace of $\Box_q$ corresponding to an eigenvalue λ. Consider a holomorphic isometry g of M which induces a holomorphic isometry

g^* of E. Then the equivariant analytic torsion is defined via the zeta function

$$Z_g(s) := \sum_{q>0}(-1)^{q+1} q \sum_{\substack{\lambda \in \operatorname{Spec}\Box_q \\ \lambda \neq 0}} \lambda^{-s} \operatorname{Tr} g^*_{|\operatorname{Eig}_\lambda(\Box_q)}$$

for $\Re s \gg 0$. Classically, this zeta function has a meromorphic continuation to the complex plane which is holomorphic at zero. The equivariant analytic torsion is defined as $\tau_g := \exp(Z_g'(0))$. This gives for $g = \operatorname{Id}_M$ the ordinary analytic torsion τ of Ray and Singer [9].

Let G/K be a compact hermitian symmetric space, equipped with any G-invariant metric $\langle \cdot, \cdot \rangle_\circ$. We may assume G to be compact and semi-simple. Let $T \subseteq K$ denote a fixed maximal torus. Let Θ be a system of positive roots of K (with respect to some ordering) and let Ψ denote the set of roots of an invariant complex structure in the sense of [3]. Then $\Theta \cup \Psi =: \Delta^+$ is a system of positive roots of G for a suitable ordering, which we fix [3, 13.7].

Let ρ_G denote the half sum of the positive roots of G and let W_G be its Weyl group. As usual, we define $(\alpha, \rho_G) := 2\langle \alpha, \rho_G \rangle_\circ / \|\alpha\|_\circ^2$ for any weight α. For any weight b, the (virtual) character χ_b evaluated at $t \in T$ is given via the Weyl character formula by

$$\chi_b(t) = \frac{\sum_{w \in W_G} \det(w)\, e^{2\pi i w b(t)}}{\sum_{w \in W_G} \det(w)\, e^{2\pi i w \rho_G(t)}}.$$

This extends to all of G by setting χ_b to be invariant under the adjoint action. Let V be an irreducible K-representation with highest weight Λ and let $E := (G \times V)/K$ denote the associated G-invariant holomorphic vector bundle on G/K. The metric $\langle \cdot, \cdot \rangle_\circ$ on $\mathfrak{g}$ induces a hermitian metric on E. Using similar methods as in [6], one may reduce the problem of determining $Z_g(s)$ to a problem in finite-dimensional representation theory. This way one gets our key result

Theorem 1 *The zeta function Z associated to the vector bundle E over G/K is given by*

$$Z(s) = -2^s \sum_{\substack{\alpha \in \Psi \\ k>0}} \langle k\alpha, k\alpha + 2\rho_G + 2\Lambda \rangle_\circ^{-s} \chi_{\rho_G + \Lambda + k\alpha}.$$

Let for $\phi \in \mathbf{R}$ and $s > 2$

$$\zeta_L(s, \phi) = \sum_{k>0} \frac{e^{ik\phi}}{k^s}$$

denote the Lerch zeta function. Let $P : \mathbf{Z} \to \mathbf{C}$ be a function of the form

$$P(k) = \sum_{j=0}^{m} c_j k^{n_j} e^{ik\phi_j}$$

with $m \in \mathbf{N}_0$, $n_j \in \mathbf{N}_0$, $c_j \in \mathbf{C}$, $\phi_j \in \mathbf{R}$ for all j. Set $P^{\text{odd}}(k) := (P(k) - P(-k))/2$. We define analogously to [4, 2.3.4]

$$\zeta P := \sum_{j=0}^{m} c_j \zeta_L(-n_j, \phi_j), \qquad \zeta' P := \sum_{j=0}^{m} c_j \zeta'_L(-n_j, \phi_j)$$

$$\tilde{\zeta} P := \sum_{j=0}^{m} c_j \zeta_L(-n_j, \phi_j) \sum_{\ell=1}^{n_j} \frac{1}{\ell} \quad \text{and} \quad P^{*}(p) := - \sum_{\substack{j=0 \\ \phi_j \equiv 0 \bmod 2\pi}}^{m} c_j \frac{p^{n_j+1}}{4(n_j+1)} \sum_{\ell=1}^{n_j} \frac{1}{\ell}.$$

Then theorem 1 implies by some calculus on zeta functions

Theorem 2 *The logarithm of the equivariant torsion of E on G/K is given by*

$$Z'(0) = 2\zeta' \sum_{\alpha \in \Psi} \chi^{\text{odd}}_{\rho_G + \Lambda + k\alpha} - 2 \sum_{\alpha \in \Psi} \chi^{*}_{\rho_G + \Lambda - k\alpha}((\alpha, \rho_G + \Lambda))$$
$$+ \sum_{\alpha \in \Psi} \sum_{k=1}^{(\alpha, \rho_G + \Lambda)} \chi_{\rho_G + \Lambda - k\alpha} \log k + \sum_{\alpha \in \Psi} \left(\frac{1}{2} - \zeta \chi^{\text{odd}}_{\rho_G + \Lambda + k\alpha} \right) \log \frac{\|\alpha\|_{\circ}^2}{2}.$$

One can show that the polynomial degree in k of $\sum_{\Psi} \chi_{\rho_G + \Lambda + k\alpha}(g)$ for any $g \in G$ is at most the dimension of the fixed point set of the action of g on G/K. In particular, it is less or equal $\#\Psi$. The torsion behaves additively under direct sum of vector bundles, thus this result gives the torsion for any homogeneous vector bundle.

Remark: If the decomposition of the space G/K in its irreducible components does not contain one of the spaces $\mathbf{SO}(p+2)/\mathbf{SO}(p) \times \mathbf{SO}(2)$ ($p \geq 3$) or $\mathbf{Sp}(n)/\mathbf{U}(n)$ ($n \geq 2$), one may choose the metric $\langle \cdot, \cdot \rangle_{\circ}$ in such a way that $\log \|\alpha\|_{\circ}^2/2 = 0$ for all $\alpha \in \Psi$. Thus the corresponding term in theorem 2 vanishes.

We shall now compare the result with Bismut's conjecture of an equivariant Riemann-Roch formula. Consider again a compact Kähler manifold M and a vector bundle E acted on by g and let M_g denote the fixed point set. Let N be the normal bundle of the imbedding $M_g \hookrightarrow M$. Let $\gamma^N_{|x}$ (resp. $\gamma^E_{|x}$) denote the infinitesimal action of g at $x \in M_g$. Let Ω^{TM_g}, Ω^N and Ω^E denote the curvatures of the corresponding bundles with respect to the hermitian holomorphic connection. Define the function Td on square matrices A as Td(A):=det A/(1-exp(-A)).

Definition 1 *Let* $\mathrm{Td}_g(TM)$ *and* $\mathrm{ch}_g(TM)$ *denote the following differential forms on* M_g*:*

$$\mathrm{Td}_g(TM) := \mathrm{Td}\left(\frac{-\Omega^{TM_g}}{2\pi i} \right) \det \left(1 - (\gamma^N)^{-1} \exp \frac{\Omega^N}{2\pi i} \right)^{-1}$$

and

$$\mathrm{ch}_g(TM) := \mathrm{Tr}\,\gamma^E \exp \frac{-\Omega^E_{|M_g}}{2\pi i}.$$

Assume now for simplicity that E is the trivial line bundle. In [1], Bismut introduced the equivariant R-genus $R_g(TM)$. Using this genus we may reformulate theorem 2 as follows:

Theorem 3 *The logarithm of the torsion is given by the equation*

$$\log \tau_g(G/K) - \log \mathrm{vol}_\circ(G/K) + \sum_{\Psi} \left(\frac{1}{2} + \zeta\chi^{\mathrm{odd}}_{\rho_G+k\alpha}(g)\right) \log \frac{\|\alpha\|^2_\circ}{2}$$
$$= \int_{(G/K)_g} \mathrm{Td}_g(T(G/K))R_g(T(G/K)) - \overline{\zeta}\sum_{\Psi} \chi^{\mathrm{odd}}_{\rho_G+k\alpha}(g) - 2\sum_{\Psi} \chi_{\rho_G-k\alpha}(g)^* \left((\alpha,\rho_G)\right).$$

Using the R-genus, Bismut formulated a conjectural equivariant arithmetic Grothendieck-Riemann-Roch theorem [1]. Suppose that M is given by $\mathcal{M}\otimes\mathbf{C}$ for a flat regular scheme $\pi : \mathcal{M} \to \mathrm{Spec}\,\mathbf{Z}$ and that E stems from an algebraic vector bundle $\mathcal{E}$ over $\mathcal{M}$. Let $\sum(-1)^q R^q\pi_*\mathcal{E}$ denote the direct image of $\mathcal{E}$ under π. We equip the associated complex vector space with a hermitian metric via Hodge theory. Bismut's conjecture implies that the equivariant torsion verifies the equation

$$\log \tau_g(M,E) + \hat{c}^1_g\left(\sum_{q\geq 0}(-1)^q R^q\pi_*\mathcal{E}\right) = \pi_*\left(\widehat{\mathrm{Td}}_g(T\mathcal{M})\widehat{\mathrm{ch}}_g(\mathcal{E})\right)^{(1)}$$
$$+ \int_{(G/K)_g} \mathrm{Td}_g(T(G/K))R_g(T(G/K))\mathrm{ch}_g(E) \quad (1)$$

(We identify the first arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathrm{Spec}\,\mathbf{Z})$ with $\mathbf{R}$). Here $\hat{c}^1_g$, $\widehat{\mathrm{Td}}_g$ and $\widehat{\mathrm{ch}}_g$ denote certain equivariant arithmetic characteristic classes which are only defined in a non-equivariant situation up to now (see [10]). Bismut [2] has proven that this formula is compatible with the behaviour of the equivariant torsion under immersions and changes of the occuring metrics. In the non-equivariant case, equation (1) has been proven by Gillet, Soulé, Bismut and Lebeau [5]. In our case, the $\hat{c}^1_g$ term in (1) should be independent of g. By the definition of $\hat{c}^1$, it should equal minus the logarithm of the norm of the element $1 \in H^0(G/K)$, thus $-\log \mathrm{vol}_\circ(G/K)$. Hence, theorem 3 fits very well with Bismut's conjecture.

We consider now the case $g = \mathrm{Id}$. For this action, the equivariant torsion equals the original Ray-Singer torsion. The values of the characters $\chi_{\rho_G+k\alpha}$ at zero are given by the Weyl dimension formula

$$\chi_{\rho_G+k\alpha}(0) = \dim V_{\rho_G+k\alpha} = \prod_{\beta\in\Delta^+}\left(1 + k\frac{\langle\beta,\alpha\rangle}{\langle\beta,\rho_G\rangle}\right).$$

In particular, the first term in theorem 2 is given by ζ' applied to the odd part of the polynomial

$$\sum_{\alpha\in\Psi} \chi_{\rho_G+k\alpha}(0) = \sum_{\alpha\in\Psi} \prod_{\beta\in\Delta^+} \left(1 + k\frac{\langle\beta,\alpha\rangle}{\langle\beta,\rho_G\rangle}\right).$$

At a first sight, this looks like a polynomial of degree $\#\Delta^+$, but it has in fact degree $\leq \#\Psi$, thus all higher degree terms cancel. By combining theorem 3 with the arithmetic Riemann-Roch theorem, we get the following formula:

Theorem 4 *The direct image of the arithmetic Todd class is given by*

$$\begin{aligned}\left(\pi_*\widehat{\mathrm{Td}}(T\mathcal{M})\right)^{(1)} &= \sum_{\Psi} \left(\frac{1}{2} + \zeta\,(\dim V_{\rho_G+k\alpha})^{\mathrm{odd}}\right) \log\frac{\|\alpha\|_0^2}{2} \\ &+ \bar{\zeta}\sum_{\Psi}(\dim V_{\rho_G+k\alpha})^{\mathrm{odd}} + 2\sum_{\Psi}(\dim V_{\rho_G+k\alpha})^{*}\left((\alpha,\rho_G)\right).\end{aligned}$$

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Titel: An arithmetic Schubert Calculus

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Let $G(p,n)$ be the Grassmannian over Spec $\mathbb{Z}$ which parametrize for any field k the p-planes in k^n. For example, $\mathbb{P}_n = G(1,n+1)$. We considere $\bar{E}$ (resp. $\bar{S}$, resp. $\bar{Q}$) the trivial bundle (resp. the universal bundle, resp. the universal quotient bundle) equipped with the trivial metric (resp. the induced metric) resp. the quotient metric). We note $c_i(\bar{S})$ (resp. $c_i(\bar{Q})$) the i-th Chern form of $\bar{S}$ (resp. $\bar{Q}$). We denote by $\mathcal{H}^*(\bar{G}(p,n))$ the ring of real harmonic forms on $G(p,n)(\mathbb{C})$ with respect to the kähler form $\omega_G = c_1(\bar{Q})$. Let $CH^*(\bar{G}(p,n))$ be the Chow-Arakelov ring of $G(p,n)$ and denote by "a" the canonical morphism from $\mathcal{H}^*(\bar{G}(p,n))$ to $\widehat{CH}^*(\bar{G}(p,n))$. ([GS1], p 165). Denote by $\hat{c}(\bar{S})$ (resp. $\hat{c}(\bar{Q})$) the total arithmetic Chern class of $\bar{S}$ (resp. $\bar{Q}$) ([GS2], p 197) and let $\tilde{c}(\bar{\mathcal{E}})$ be the Bott-Chern classe ([BC], p 97-85) ([GS2], p 167) associated with the short exact sequence:

$$\bar{\mathcal{E}} : 0 \to \bar{S} \to \bar{E} \to \bar{Q} \to 0.$$

let introduce two rings:

$A = \mathbb{Z}[\hat{a}_1, \ldots, \hat{a}_{n-p}, \hat{b}_1, \ldots, \hat{b}_p]$,

$B = \mathbb{R}[a_1, \ldots, a_{n-p}, b_1, \ldots, b_p, c_2, \ldots c_n]$.

Denote by $\omega : A \to B$ the ring morphism defined by $\omega(\hat{a}_i) = a_i$ and $\omega(\hat{b}_j) = b_j$. We endow $A \oplus B$ with the product $(\hat{\alpha} \oplus \alpha)(\hat{\beta} \oplus \beta) = \hat{\alpha}\hat{\beta} \oplus (\omega(\hat{\alpha})\beta + \alpha\omega(\hat{\beta}))$. Let $(\mathfrak{R})$ be the following relations:

$$\sum_{i=0}^{l} \hat{a}_i \hat{b}_{l-i} = c_l \quad \text{and} \quad \sum_{i=0}^{l} a_i b_{l-i} = 0 \quad \text{for } l \geq 1$$

$$c_{l+1} = \mathcal{H}_l \sum_{\substack{r+s=l \\ 0 \leq s \leq n-p \\ 0 \leq r \leq p}} r\, a_s b_r \qquad \text{for } (n-1) \geq l \geq 1$$

(Here $\hat{\alpha} = \hat{\alpha} \oplus 0$, $a_i b_j = 0 \oplus a_i b_j$, $c_l = 0 \oplus c_l$ and $\mathcal{H}_l = \sum_{j=1}^{l} 1/j$).

<u>Theorem 1</u>: There exists a unique isomorphism of ring

$$\varphi : A \oplus B/(\mathfrak{R}) \longrightarrow CH^*(\overline{G}(p,n))$$

such that:

$$\varphi(\hat{a}_i) = \hat{c}_i(\overline{Q}) \quad \varphi(a_i) = a(c_i(\overline{Q}))$$
$$\varphi(\hat{b}_j) = \hat{c}_j(\overline{S}) \quad \varphi(b_j) = a(c_j(\overline{S}))$$
$$\varphi(c_l) = a(\tilde{c}_l(\overline{\mathcal{E}})).$$

Sketch of the proof: The existence of such a morphism is a consequence of the relation $\hat{c}(\overline{S})\hat{c}(\overline{Q}) - 1 = a(\tilde{c}(\overline{\mathcal{E}}))$ and of the following result:

<u>Theorem 2</u>: The Bott-Chern class of $\overline{\mathcal{E}}$ is given by <u>the</u> following formula:

$$\tilde{c}(\overline{\mathcal{E}}) = \sum_{\substack{r \geq 0 \\ s \geq 0}} r\, H_{r+s}\, c_s(\overline{Q})\, c_r(\overline{S}).$$

It is a direct consequence of some formulas in ([BC], 4.18)

Using a cellular decomposition of $G(p,n)$, we prove that $CH^{p,p-1}(G(p,n)) = 0$. We get then the following exact sequence.

$$0 \to \mathcal{H}^{p-1,p-1}(\bar{G}(p,n)) \xrightarrow{a} \widehat{CH}^p(\bar{G}(p,n)) \xrightarrow{\zeta} CH^p(G(p,n)) \to 0$$

We conclude by using some classical results about cohomology and Chow Groups of Grassmannians. □

The equality

$$\hat{c}(\bar{S}) \cdot \hat{c}(\bar{Q}) = 1 + a(\hat{c}(\bar{\mathcal{E}}))$$

may be written:

$$\hat{c}(\bar{S}) \cdot c(\bar{Q})\left(1 - a(\tilde{c}(\bar{\mathcal{E}}))\right) = 1.$$

Let $\hat{c}(\bar{Q} - \bar{\mathcal{E}}) = \hat{c}(\bar{Q})\left(1 - a(\hat{c}(\bar{\mathcal{E}}))\right)$. Using the formalism of Schur Functions ([Fu], ch. 14.5), we are able to produce an "arithmetic Littlewood-Richardson product formula". Let $\mathcal{I}(p,n)$ (resp. $\mathcal{C}(p,n)$) the set of p-partitions λ such that $(n-p) \geq \lambda_1$ (resp. $\lambda_1 > (n-p)$). For λ a p-partition, we define:

$$c_\lambda(\bar{Q}) = \det\left(c(\bar{Q})_{\lambda_i + j - i}\right)$$

and

$$\hat{c}_\lambda(\bar{Q} - \bar{\mathcal{E}}) = \det\left(\hat{c}(\bar{Q} - \bar{\mathcal{E}})_{\lambda_i + j - i}\right).$$

Let α, β, λ three partitions such that $|\lambda| = |\alpha| + |\beta| - 1$. Denote by $\widehat{N}^{\lambda}_{\alpha,\beta}$ the "arithmetic Littlewood-Richardson coefficient" attached to (α, β, λ).

$$\widehat{N}^{\lambda}_{\alpha,\beta} \doteq \sum_{\substack{\mu \in \mathcal{C}(p,n) \\ |\mu| = |\alpha| + |\beta|}} N^{\mu}_{\alpha,\beta} \sum_{\substack{r+s+t+1 = \mu_i + j \\ r,s,t,j \geq 0 \\ p-1 \geq j}} (-1)^{r+j+1}\, r\, H_{r+s}\, N^{S(r,s,t,\mu)}_{\lambda,\lambda(r,s,t,\mu,j)}$$

where:

$$\delta(r,s,t,\mu) = (\mu_2+s+t+1, \mu_2+t+1, (\mu_2+1)^{(r)}, \mu_2, \mu_3, \dots, \mu_p)$$

$$\gamma(r,s,t,\mu,j) = (\mu_2+t+1, \mu_2+1, (\mu_2)^{(r)}, (1)^{(j)}).$$

(Here the $N^{\delta}_{\lambda,\gamma}$ are the classical Littlewood-Richardson numbers).

<u>Theorem 3</u>: 1) Let $t \in \{0, \dots, p(n-p)+1\}$. Each element of $CH^t(\overline{G}(p,n))$ can be written in a unique way

$$x = \sum_{\substack{\lambda \in \mathcal{P}(p,n) \\ |\lambda| = t}} k_\lambda \hat{c}_\lambda(\overline{Q}-\overline{\mathcal{E}}) + \sum_{\substack{\lambda \in \mathcal{P}(p,n) \\ |\lambda| = t-1}} a(f_\lambda c_\lambda(\overline{Q}))$$

where $k_\lambda \in \mathbb{Z}$ and $f_\lambda \in \mathbb{R}$.

2) Let $(\lambda,\mu) \in \mathcal{P}(p,n)^2$, then:

$$\hat{c}_\lambda(\overline{Q}-\overline{\mathcal{E}}) \cdot \hat{c}_\mu(\overline{Q}-\overline{\mathcal{E}}) = \sum_{\substack{\nu \in \mathcal{P}(p,n) \\ |\nu| = |\lambda|+|\mu|}} N^{\nu}_{\lambda,\mu} \hat{c}_\nu(\overline{Q}-\overline{\mathcal{E}}) + \sum_{\substack{\nu \in \mathcal{P}(p,n) \\ |\nu| = |\lambda|+|\mu|-1}} a(\tilde{N}^{\nu}_{\lambda,\mu} c_\nu(\overline{Q}))$$

$$\hat{c}_\lambda(\overline{Q}-\overline{\mathcal{E}}) \cdot a(c_\mu(\overline{Q})) = \sum_{\substack{\nu \in \mathcal{P}(p,n) \\ |\nu| = |\lambda|+|\mu|}} a(N^{\nu}_{\lambda,\mu} c_\nu(\overline{Q})),$$

$$a(c_\lambda(\overline{Q})) \cdot a(c_\mu(\overline{Q})) = 0.$$

We deduce from this a formula for the height of the image of the Plücker embedding of $G(p,n)$

<u>Theorem 4</u>: $h(\varphi(G(p,n))) =$

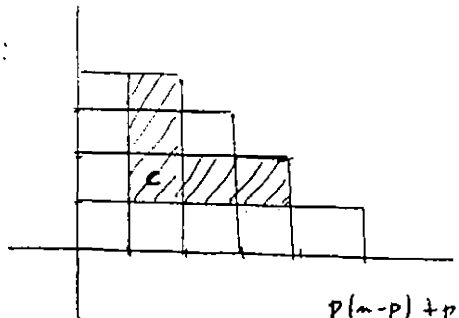
$$\frac{1}{2} \sum_{\substack{\lambda \in \mathcal{C}(p,n) \\ |\lambda| = p(n-p)+1}} S(\lambda) \sum_{\substack{r+s+t+1 = \lambda_1+j \\ r,s,t,j \geq 0 \\ p-1 \geq j}} (-1)^{r+j+1} r \, \mathcal{H}_{r+s} \, N^{\delta(r,s,t,\lambda)}_{\lambda,\gamma(r,s,t,\lambda,j)}$$

Here η_0 is the partition $(\underbrace{n-p, \dots, n-p}_{p \text{ times}})$ and

$$S(\lambda) = \frac{|\lambda|!}{\prod_{c \in [\lambda]} L(c)}$$

Example:

(λ):



$L(c) = 5$

Corollary: $h(\mathcal{G}(G(p,n))) \in \sum_{j=1}^{p(n-p)+p-1} \frac{1}{2j}\mathbb{Z} \subset \mathbb{Q}$

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Titel: ARITHMETIC CHOW RINGS & DELIGNE BEILINSON COHOMOLOGY

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The pourpose of this talk is to give a formalism to deal with the Arquimedian part of arithmetic Chow rings.

Idea: to replace the equation

$$dd^c g + \delta_Y = \omega$$

by a cohomological condition.

Let Y be a codimension p algebraic cycle
g a Green form of log. type for Y
$[g]$ the associated current.

We may think that $dd^c = d_\wp$ is the differential of some complex. Then

$$Res_{d_\wp}(g) = [d_\wp g] - d_\wp[g] = \delta_Y$$

This implies that the pair (ω, g) represents the class of Y on some cohomology group "H_Y"(X).

1) <u>The cohomology of logarithmic forms</u>

Let X be a ~~~~ compact Kähler manifold; $D \subset X$ a div normal crossing divisor (DNC), write $V = X - D$ and let $j: V \to X$ the inclusion. Let us denote by $\mathcal{E}_X^*$ the sheaf of C^∞ differential forms on X.

<u>Def</u>. The sheaf of C^∞ forms with logarithmic singularities along D: $\mathcal{E}_X^*(\log D)$ is the sub $\mathcal{E}_X^*$-algebra of $j_*\mathcal{E}_V^*$ generated locally by the sections.

$$dz_i/z_i\ ,\ d\bar{z}_i/\bar{z}_i\ ,\ \log \|z_i\|^2 \quad i=1,\dots,k$$

where $z_1\cdots z_k=0$ is a local equation for D.

This complex has a real structure: $\mathcal{E}^*_{X,\mathbb{R}}(\log D)$, a bigrading:

$$\mathcal{E}^m_X(\log D)=\bigoplus_{p+q=m}\mathcal{E}^{p,q}_X(\log D)$$

a Hodge filtration:

$$F^p\mathcal{E}^*_X(\log D)=\bigoplus_{p'\geq p}\mathcal{E}^{p',*}_X(\log D)$$

and a weight filtration W.

<u>Theorem</u> ([B1]): The pair

$$((\mathcal{E}^*_{X,\mathbb{R}}(\log D),W),(\mathcal{E}^*_X(\log D),W,F))$$

is a cohomological mixed Hodge complex which induces in $H^*(V,\mathbb{C})$ the Deligne mixed Hodge structure.

Let X be a quasi-proj. complex manifold. Let us write

$$I=\left\{\begin{array}{l}\bar{X}_\alpha \text{ smooth compactifications}\\ \text{of } X, \text{ with } D_\alpha=\bar{X}_\alpha-X \text{ a DNC}\end{array}\right\}$$

<u>Def.</u>

$$E^*_{\log}(X)=\varinjlim_{I}\Gamma(\bar{X}_\alpha,\mathcal{E}^*_{\bar{X}_\alpha}(\log D_\alpha))\subset E^*(V)=\Gamma(V,\mathcal{E}^*_X)$$

2) <u>Deligne-Beilinson cohomology.</u>

We shall write $\boxed{A^*=E^*_{\log}(X)}$.

Notation: $A^*_{\mathbb{R}}$ the real part, $A^*_{\mathbb{R}}(p)=(2\pi i)^pA_{\mathbb{R}}\subset A^*$

The Deligne-Beilinson complex is

$$A^{\bullet}_{\mathbb{R}}(p)_{\mathcal{D}} = s\left(A^{\bullet}_{\mathbb{R}}(p) \oplus F^{p}A^{\bullet} \xrightarrow{u} A^{\bullet}\right),$$

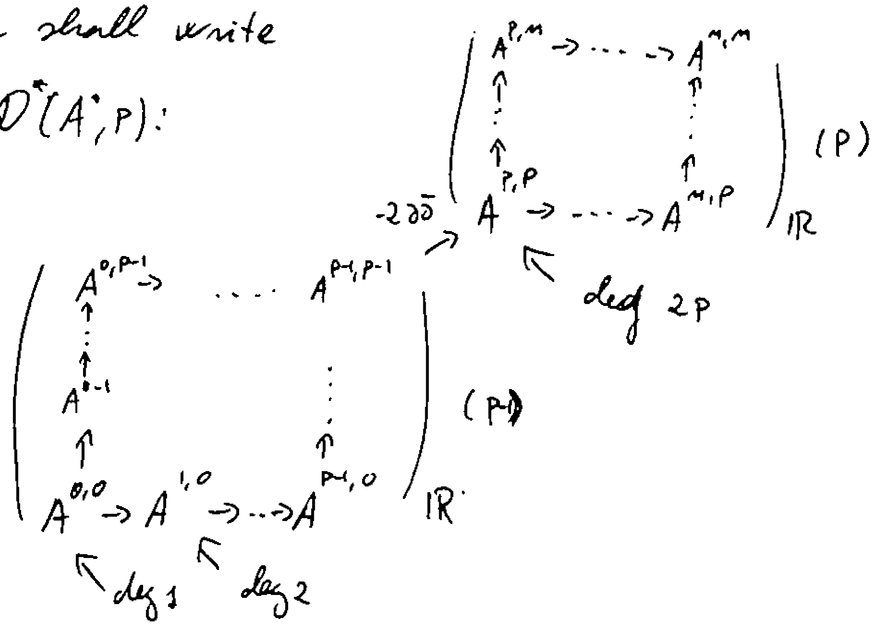
where $u(a,b) = b-a$ and $s(\ )$ denotes the simple of a morphism of complexes.

__Corollary__: $H^{*}(E^{*}_{\log}(X)(p)_{\mathcal{D}}) = H^{*}_{\mathcal{D}}(X, \mathbb{R}(p))$.

Now we want to simplify this complex ([W], [B3]).

We shall write

$\mathcal{D}^{*}(A^{\bullet}, p)$:



__Proposition__. There are homotopy equivalences

$$A_{\mathbb{R}}(p)_{\mathcal{D}} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathcal{D}(A^{\bullet}, p).$$

__Corollary__: $H^{\bullet}(\mathcal{D}^{\bullet}(E^{\bullet}_{\log}(X), p)) = H^{\bullet}_{\mathcal{D}}(X, \mathbb{R}(p))$

Notation: $\boxed{\mathcal{D}(X,p) = \mathcal{D}(E^{*}_{\log}(X), p)}$.

3) Truncated relative cohomology groups.

Let $f: A^\cdot \to B^\cdot$ be a morphism of complexes. Then we write $ZA^\cdot = \{x \in A \mid dx = 0\}$, $\tilde{B}^\cdot = B^\cdot / \operatorname{Im} d$.

Def: $\hat{H}^m(A^\cdot, B^\cdot) = \{(a, \tilde{b}) \in ZA^m \oplus \tilde{B}^{m-1} \mid f(a) = db\}$

we have maps:

$cl: \hat{H}^m(A^\cdot, B^\cdot) \to H^m(A^\cdot, B^\cdot)$ $\quad (a, \tilde{b}) \mapsto \{(a,b)\}$ (cohomology class)

$\omega: \hat{H}^m(A^\cdot, B^\cdot) \to ZA^m$ $\quad (a, \tilde{b}) \mapsto a$

$a: \tilde{A}^{m-1} (= A^{m-1}/\operatorname{Im} d) \to \hat{H}^m(A, B)$ $\quad \tilde{a} \mapsto (da, f(a)^\sim)$

$b: H^{m-1}(B) \to \hat{H}^m(A, B)$ $\quad \{b\} \mapsto (0, \tilde{b})$

and exact sequences

$$H^{m-1}(A, B) \to \tilde{A}^{m-1} \xrightarrow{a} \hat{H}^m(A^\cdot, B^\cdot) \xrightarrow{cl} H^m(A, B) \to 0$$

$$0 \to H^{m-1}(B) \xrightarrow{b} \hat{H}^m(A^\cdot B^\cdot) \xrightarrow{\omega} ZA^m \to H^m(B)$$

Example:

X quasi-projective complex manifold, F_∞ an antilinear involution. y a cod p real algebraic cycle. $Y = \operatorname{supp} y$, $V = X - Y$.

Put $A = D^\cdot(X_{\mathbb{R}}, p) = \{x \in D(X, p) \mid F_\infty^* x = \bar{x}\}$

$B = D^\cdot(V_{\mathbb{R}}, p) = \{x \in D(V, p) \mid F_\infty^* x = \bar{x}\}$.

Then we have a surjective map

$$\hat{H}^{2p}(A, B) \to H^{2p}_{\mathcal{D}, Y}(X, \mathbb{R}(p))$$

and a class $\rho(y) \in H^{2p}_{\mathcal{D}, Y}(X, \mathbb{R}(p))$.

Def. The space of Green forms for y is

$$GE(y) = \{g \in \hat{H}^{2p}(A^\cdot, B^\cdot) \mid cl(g) = \rho(y)\}$$

Theorem ([B2], [B3], [G-S]) If X is projective, then there is an isomorphism

$$GE(y) \to \left\{ \begin{matrix} \text{green currents} \\ \text{for } y \end{matrix} \right\} \Big/ \operatorname{Im}\partial + \operatorname{Im}\bar{\partial}.$$

Using these Green forms we can define the $*$-product, direct image and inverse images which are compatible with the corresponding operations for Green currents. Moreover we can define Arithmetic Chow groups, which in the projective case coincide with the groups defined by Gillet and Soulé. In the quasi-projective case they satisfy:

1) $CH^{p,p-1}(X) \to \tilde{E}_{\log}^{p-1,p-1}(X_{\mathbb{R}}) \to \widehat{CH}^p(X) \to CH^p(X) \to 0$

2) $CH^{p,p-1}(X) \to H_{\mathcal{D}}^{2p-1}(X_{\mathbb{R}}, \mathbb{R}(p)) \to \widehat{CH}^p(X) \to CH^p(X) \oplus Z E_{\log}^{p,p}(X_{\mathbb{R}}) \to$
$\to H_{\mathcal{D}}^{2p}(X_{\mathbb{R}}, \mathbb{R}(p)) \to 0$

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Titel: Integral Points on Open Subvarieties of Semiabelian Varieties

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Recall that a semiabelian variety is a group variety A such that, after suitable base change, there exists an abelian variety A_0 and an exact sequence

$$0 \to \mathbf{G}_{\mathrm{m}}^{\mu} \to A \xrightarrow{\rho} A_0 \to 0.$$

Let k be a number field and let S be a finite set of places of k containing all archimedean places. Let R be the localization of the ring of integers of k away from places in S.

Let X be a quasi-projective variety over k. Then a **model for** X **over** R is an integral scheme, surjective and quasi-projective over $\operatorname{Spec} R$, together with an isomorphism of the generic fiber over k with X. An **integral point** of $\mathscr{X}$ is an element of $\mathscr{X}(R)$.

Let us begin by stating four theorems.

Theorem 1 (Faltings). *Let X be a subvariety of an abelian variety A/k. Then there exists a finite collection $B_1 \dots, B_n$ of translated abelian subvarieties of A such that $B_i \subseteq X$ for all i and*

$$X(k) = \bigcup B_i(k).$$

Theorem 2 (Vojta). *Let X be a closed subvariety of a semiabelian variety A. Let $\mathscr{X}$ be a model for X over R. Then there exists a finite collection $\mathscr{B}_1, \dots, \mathscr{B}_n$ of subschemes of $\mathscr{X}$ whose generic fibers (over k) are translated semiabelian subvarieties of A, such that*

$$\mathscr{X}(R) = \bigcup \mathscr{B}_i(R).$$

Theorem 2 is the obvious generalization of Theorem 1 to the case of semiabelian varieties. Indeed, in the notation of Theorem 2, if A is an abelian variety then we can take $\mathscr{X}$ and $\mathscr{B}_1, \dots, \mathscr{B}_n$ proper over $\operatorname{Spec} R$ and then the valuative criterion of properness implies that

$$\mathscr{X}(R) = X(k) \qquad \text{and} \qquad \mathscr{B}_i(R) = B_i(k).$$

Theorem 3 (Faltings). *Let D be an ample effective divisor on an abelian variety A and let $\mathscr{U}$ be a model for $A \setminus D$. Then $\mathscr{U}(R)$ is finite.*

Again, one would like to find a Theorem 4 that generalizes this to the semiabelian case. Here, however, the assertion that $\mathscr{U}(R)$ is finite is too strong, as the following examples illustrate.

Example. Let $A = \mathbb{G}_m^2$ and let D be the diagonal on A. Completing A in the obvious way to $(\mathbb{P}^1)^2$, it follows that the closure of D is ample. Yet

$$A \setminus D \cong (\mathbb{G}_m \setminus \{1\}) \times \mathbb{G}_m,$$

so it may have infinitely many integral points (depending on the model).

Of course in this case there is a nontrivial group action on A. The following example does not have this property.

Example. Let E be an elliptic curve and let $A = \mathbb{G}_m \times E$. Let f be a nonzero rational function on E with a pole at a rational point P. Let D be the closure in $\mathbb{G}_m \times E$ of the graph of the rational function f. Then $A \setminus D$ has no nontrivial positive dimensional group action, yet it contains the nontrivial translated subgroup $\mathbb{G}_m \times \{P\}$.

A correct formulation of Theorem 4 therefore needs a weaker conclusion:

Theorem 4. *Let D be an effective divisor on a semiabelian variety A, and let $\mathscr{U}$ be a model for $A \setminus D$. Then there is a finite collection $\mathscr{B}_1, \dots, \mathscr{B}_n$ of subschemes of $\mathscr{U}$ whose generic fibers (over k) are translated semiabelian subvarieties of A, such that*

$$\mathscr{U}(R) = \bigcup \mathscr{B}_i(R).$$

Theorem 4 implies Theorem 3 by induction on dimension, since the restriction of D to a B_i is still ample.

Theorems 1–3 were conjectures of Serge Lang.

The conclusions of Theorems 2 and 4 are very similar. This suggests that they can be combined to a single theorem:

Theorem. *Let X be a closed subvariety of a semiabelian variety A, let D be an effective divisor on X, and let $\mathscr{U}$ be a model for $X \setminus D$. Then there is a finite collection $\mathscr{B}_1, \dots, \mathscr{B}_n$ of subschemes of $\mathscr{U}$ whose generic fibers (over k) are translated semiabelian subvarieties of A, such that*

$$\mathscr{U}(R) = \bigcup \mathscr{B}_i(R).$$

The proof of this, given Theorems 2 and 4, is very short:

Proof. First, let $\mathscr{X}$ be some model for X. Then

$$\mathscr{U}(R) \subseteq \mathscr{X}(R) = \bigcup \mathscr{B}_i(R)$$

by Theorem 2; hence applying Theorem 4 to the $\mathscr{B}_i \cap \mathscr{U}$ gives

$$\mathscr{U}(R) = \bigcup (\mathscr{B}_i \cap \mathscr{U})(R) = \bigcup_i \bigcup_j \mathscr{B}_{ij}(R).$$

□

Time does not permit a full proof of Theorem 4, but it is possible to state the main approximation theorem, after a few definitions.

Definition. Let X/k be a variety. A **generalized Weil function** is a function

$$g\colon \coprod_v U(\bar{k}_v) \to \mathbf{R},$$

where U is a nonempty Zariski-open subset of X, such that there exists a proper birational morphism $\Phi\colon X^* \to X$ such that $g \circ \Phi$ extends to a Weil function for some divisor D^* on X^*.

Definition. Let Y be a proper closed subset of a variety X/k. Then a **logarithmic distance function for** Y is a generalized Weil function g on X such that, if $\Phi\colon X^* \to X$ and D^* are as above, then D^* is effective and $\operatorname{Supp} D^* = \Phi^{-1}(Y)$.

Note that this is not really minus the logarithm of the distance to Y, especially near singularities.

Definition. Let G be a variety. A **compactification of** G is a proper variety $\bar{G}$ with an open immersion $G \hookrightarrow \bar{G}$. A compactification $\bar{G}$ of a group variety G is **translation invariant** if the group law $G \times G \to G$ extends to a morphism $G \times \bar{G} \to \bar{G}$.

Remark. A translation invariant compactification $\bar{G}$ of $\mathbb{G}_m^\mu$ determines a translation invariant compactification $\bar{A}$ of A such that ρ extends to a morphism $\bar{\rho}\colon \bar{A} \to A_0$ with fibers isomorphic to $\bar{G}$.

Theorem (Main Approximation Theorem). *Let A be a semiabelian variety, $\bar{A}$ a translation invariant compactification of A as in the remark, L an ample line sheaf on $\bar{A}$, h_L a height function relative to L, λ_w a generalized Weil function on $\bar{A}$ at a place $w \in S$, and $\lambda_{\infty,w}$ a logarithmic distance function for $\bar{A} \setminus A$. Then there do not exist $\kappa > 0$ and $\mathscr{S} \subseteq A(R)$ such that (1)*

$$\lambda_w(P) \ge \kappa h_L(P)$$

for all $P \in \mathscr{S}$; and (2) for all $\eta > 0$ the inequality

$$\lambda_{\infty,w}(P) \le \eta h_L(P)$$

holds for infinitely many $P \in \mathscr{S}$.

The proof that the Main Approximation Theorem implies Theorem 4 is an application of the theory of toric varieties; see **[V 2]** for details.

The proof of this theorem is an application of Arakelov theory and Faltings' Product Theorem to Thue's method; details will appear in **[V 1]** and **[V 2]**. We sketch some of the main ideas here.

First, we may assume that $\bar{A}$ is the translation invariant compactification of A associated to the compactification $\mathbb{G}_{\mathrm{m}}^{\mu} \hookrightarrow (\mathbf{P}^1)^{\mu}$. Let L_0 be an ample symmetric divisor on A_0, and let L_1 be the divisor $\bar{A} \setminus A$. Then

$$L := \rho^* L_0 + L_1$$

is ample. We may also assume that g_w is a Weil function associated to an effective divisor D.

Given rational $\epsilon > 0$ and $\mathbf{s} \in \mathbf{Z}_+^m$, let

$$L_{\epsilon,\mathbf{s}} = \sum_{i=1}^{m-1} (s_{i+1} \cdot \mathrm{pr}_{i+1} - s_i \cdot \mathrm{pr}_i)^* \rho^* L_0 + \sum_{i=1}^{m-1} (s_{i+1}^2 \cdot \mathrm{pr}_{i+1} - s_i^2 \cdot \mathrm{pr}_i)^* L_1 + \epsilon \sum_{i=1}^{m} \mathrm{pr}_i^* L$$

and

$$\begin{aligned} M_{\epsilon,\mathbf{s}} = & \sum_{i=1}^{m-1} (s_{i+1} \cdot \mathrm{pr}_{i+1} - s_i \cdot \mathrm{pr}_i)^* \rho^* L_0 \\ & + s_1^2 \,\mathrm{pr}_1^* L_1 + 2 s_2^2 \,\mathrm{pr}_2^* L_1 + \cdots + 2 s_{m-1}^2 \,\mathrm{pr}_{m-1}^* L_1 + s_m^2 \,\mathrm{pr}_m^* L_1 \\ & + \epsilon \sum_{i=1}^{m} \mathrm{pr}_i^* L \\ = {} & L_{\epsilon,\mathbf{s}} + \text{an effective divisor.} \end{aligned}$$

Note that $L_{\epsilon,\mathbf{s}}$ is not defined on $\bar{A}$, because the $s_{i+1}^2 \cdot \mathrm{pr}_{i+1} - s_i^2 \cdot \mathrm{pr}_i$ do not give morphisms $\bar{A}^m \to \bar{A}$. Instead a certain blowing-up of $\bar{A}^m$ is needed. Also, these are $\mathbf{Q}$-divisors, but they will always be multiplied by an integer which will cancel the denominators.

The key step in the proof, then, is to construct a small section of

$$(*) \qquad \Gamma(\mathcal{Y}, dM_{\epsilon,\mathbf{s}}) \cap \Gamma(W, dL_{\epsilon,\mathbf{s}}) \cap \Gamma_\delta\Big(\prod \bar{X}_i, dM_{\epsilon,\mathbf{s}}\Big)$$

for sufficiently large and divisible d. Here X_i are closed subvarieties of A gotten from the product theorem, $\bar{X}_i$ are the closures of X_i in $\bar{A}$, W is a blowing-up of $\prod \bar{X}_i$ so that $dL_{\epsilon,\mathbf{s}}$ is defined as a line sheaf, and $\mathcal{Y}$ is a model for $\prod \bar{X}_i$. The subscript δ in the third term above refers to the submodule of sections with index $\geq \delta$ at $(D \cap X_1) \times \cdots \times (D \cap X_m)$. Likewise, the second term is determined by an appropriate index-like condition.

To construct this section, we first prove bounds on the rank and Arakelov degree of $\Gamma(\mathcal{Y}, dM_{\epsilon,\mathbf{s}})$. We then successively restrict to eventually get the metrized R-module (*), keeping bounds on the degree at each step. Each step involves derivatives, which are defined as follows.

Let X/k be a smooth variety, let $D_1, \ldots, D_r$ be smooth irreducible divisors on X which meet transversally, and let $\xi \in \Gamma(X, \mathscr{L})$ be a section of a line sheaf. Then given $i_1, \ldots, i_r \in \mathbf{N}^r$, we may define a "derivative"

$$D_{i_1,\ldots,i_r}\xi(D_1 \cap \cdots \cap D_r) \in \Gamma(D_1 \cap \cdots \cap D_r, \mathscr{L} \otimes \mathscr{O}(-i_1 D_1 - \cdots - i_r D_r).$$

It is well defined if it represents a leading term; *i.e.*, if all derivatives for smaller $(i_1, \ldots, i_r)$ vanish.

Of course, in order to control the degrees of the above modules, we need to find these derivatives in the Arakelov setting. This leads to two questions:

(1). How much do the metrics of this derivative increase over the metrics of ξ?
(2). If ξ is integral (over some model of X and $D_1, \ldots, D_r$), then how large will the denominators of the derivatives be?

These are actually two versions of the same question, for archimedean and finite places, respectively. The answer to (1) is obtained by complex analysis via the Cauchy integral formula, and the answer to (2) is obtained similarly, in the context of p-adic analysis. Thus in this case the complex analysis comes first, and provides intuition for the finite places.

This is the reverse of the usual situation in applications of Arakelov theory: one usually proves something first in the function field case; if it is done right (*e.g.*, does not involve differentiation with respect to the base curve) then the translation to the number field case is straightforward.

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Titel: Complex immersions and Quillen metrics
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Let $i: Y \hookrightarrow X$ be an embedding of smooth compact complex manifolds. Let η be a holomorphic vector bundle on Y, let

$$(\xi, v): \quad 0 \to \xi_m \xrightarrow{v} \quad \cdots \quad \xrightarrow{v} \xi_0 \to 0$$

be a holomorphic complex of vector bundles on X, which together with an augmentation map $r: \xi_0|_Y \to \eta$, provides a resolution of $i_* \mathcal{O}_Y(\eta)$.

Set

$$\lambda(\eta) = (\det H(Y, \eta))^{-1}.$$

Similarly, we define the $\lambda(\xi_i)$'s. Set

$$\lambda(\xi) = \bigotimes_{i=0}^{m} (\lambda(\xi_i))^{(-1)^i}.$$

Then the complex lines $\lambda(\eta)$ and $\lambda(\xi)$ are canonically isomorphic.

Let g^{TX} be a Kähler metric on TX, let $g^{TY}, g^{N_{Y/X}}$ be the induced metrics on $TY, N_{Y/X}$. Let $g^{\xi} = \bigoplus_{i=0}^{m} g^{\xi_i}$, g^{η} be Hermitian metrics on $\xi =$

$\bigoplus_{i\geq 0} \xi_i$, η. We assume that the metric g^{ξ} verifies the compatibility assumption (A) with respect to $g^{N_{Y/X}}, g^{\eta}$.

Namely using the local uniqueness of resolutions, we find that

$$(1) \quad H((\xi, v)_{|Y}) \simeq \Lambda N^*_{Y/X} \otimes \eta.$$

Now by finite dimensional Hodge theory, $H((\xi, v)_{|Y})$ can be identified to a smooth subbundle of $\xi_{|Y}$. Let $g^{H((\xi,v)_{|Y})}$ be the induced metric on $H(\xi, v)_{|Y}$. We say that (A) is verified if (1) is an isometry. It can be easily shown that given $g^{N_{Y/X}}, g^{\eta}$, metrics g^{ξ} always exist such that (A) holds [4].

Let $\| \ \|_{\lambda(\xi)}$, $\| \ \|_{\lambda(\eta)}$ be the corresponding Quillen metrics on $\lambda(\xi)$, $\lambda(\eta)$.

Theorem 1 (Bismut-Lebeau [3]) The following identity holds.

$$(2) \quad \operatorname{Log}\left(\frac{\| \ \|_{\lambda(\xi)}}{\| \ \|_{\lambda(\eta)}}\right)^2 = -\int_X Td(TX, g^{TX})\, T(\xi, g^{\xi}) + \int_Y \frac{\widetilde{Td}(TY, TX_{|Y}, g^{TX_{|Y}})}{Td(N_{Y/X}, g^{N_{Y/X}})}\, ch(\eta)$$

$$-\int_X \mathrm{Td}(TX)\, R(TX)\, \mathrm{ch}(\xi) + \int_Y \mathrm{Td}(TY)\, R(TY)\, \mathrm{ch}(\eta).$$

where:

• $T(\xi, g^{\xi})$ is a current on X whose construction will be briefly described, such that

(3) $$\frac{\bar{\partial}\partial}{2i\pi} T(\xi, g^{\xi}) = \mathrm{Td}^{-1}(N_{Y/X}, g^{N_{Y/X}})\, \mathrm{ch}(\eta, g^{\eta})\, \delta_Y - \mathrm{ch}(\xi, g^{\xi})$$

• $\widetilde{\mathrm{Td}}(TY, TX|_Y, g^{TX|_Y})$ is the Bott-Chern class associated to the exact sequence

$$0 \longrightarrow TY \longrightarrow TX|_Y \longrightarrow N_{Y/X} \longrightarrow 0.$$

• R is the additive genus associated to the Gillet-Soulé power series

(4) $$R(x) = \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \left(2\frac{\zeta'(-n)}{\zeta(-n)} + \sum_{1}^{n} \frac{1}{j} \right) \frac{\zeta(-n)\, x^n}{n!}$$

($\zeta(s)$ is the Riemann zeta function).

By using a result of Bismut-Gillet-Soulé [B], where the right-hand side of (2) is explicitly calculated in terms of arithmetic characteristic classes, we get the following result.

Theorem 2: Assume that $i : Y \hookrightarrow X$ is an embedding of proper arithmetic variety over $\mathrm{Spec}(\mathbb{Z})$. Assume that $X(\mathbb{C})$, $Y(\mathbb{C})$ are smooth. Take F_∞ invariant metrics on the considered vector bundles. Then

$$(5) \quad \mathrm{Log}\left(\frac{\| \ \|_{\lambda(\xi)}}{\| \ \|_{\lambda(\eta)}}\right) = \widehat{\deg}\left(\widehat{Td}^A(TX)\, \widehat{ch}(\xi)\right) - \widehat{\deg}\left(\widehat{Td}^A(TY)\, \widehat{ch}(\eta)\right).$$

In (5), $\widehat{Td}^A$ is the modified Todd genus of Gillet-Soulé.

Some evidence for Theorem 1

If $Y \hookrightarrow X$ are themselves the fibers over a base S of a Kähler fibration, then by the curvature Theorem of [1],

$$(6) \quad \frac{\bar{\partial}\partial}{2i\pi} \log\left(\frac{\| \ \|_{\lambda(\xi)}}{\| \ \|_{\lambda(\eta)}}\right)^2 = \left\{\int_X Td(TX, g^{TX})\, ch(\xi, g^{\xi}) - \int_Y Td(TY, g^{TY})\, ch(\eta, g^{\eta})\right\}^{(2)}.$$

The right-hand side of (2) verifies (6). From (6), we find it is not too surprising that the left-hand side of (2) is given by a local formula.

Let ∇^ξ be the holomorphic Hermitian connection on ξ. Let v^* be the adjoint of v. Set $V = v + v^*$. For $T \geq 0$, set $C_T = \nabla^\xi + \sqrt{T}\, V$. Then $V \in \mathrm{End}^{odd}(\xi)$, and C_T is a superconnection in the sense of Quillen. Also the curvature C_T^2 is a section of $\Lambda(T^*_{\mathbb{R}}X) \hat{\otimes} \mathrm{End}(\xi)$. For $T \geq 0$, set $\alpha_T = \varphi \mathrm{Tr}_s[\exp(-C_T^2)]$. By [6], α_T is a closed form, and $[\alpha_T) = ch(\xi)$.

Theorem 3 ([4]): As $T \to +\infty$,

$$(7) \quad \alpha_T \longrightarrow \alpha_\infty = \frac{ch(\eta, g^\eta)}{Td(N_{Y/X}, g^{N_{Y/X}})}\, \delta_Y .$$

Set $\beta_T = \varphi \mathrm{Tr}_s[N_H \exp(-C_T^2)]$, where N_H is the number operator of ξ. Then

$$(7) \quad \frac{\partial \alpha_T}{\partial T} = \frac{\bar{\partial}\partial}{2i\pi} \frac{\beta_T}{T} .$$

Definition [5] Set $T(\xi, g^\xi) = \left\{ \frac{1}{\Gamma(s)} \int_0^{+\infty} T^{s-1} (\beta_T - \beta_\infty)\, dT \right\}'(0)$.

$T(\xi, g^\xi)$ is exactly the current appearing in (2) and verifies (3).

In the talk, I showed how the right-hand side of (2) could be partly interpreted as a renormalized analytic torsion of the infinite dimensional complex

$$0 \to \Omega(X, \xi_m) \xrightarrow{v} \cdots \xrightarrow{v} \Omega(X, \xi_0) \to 0 .$$

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Titel: Heights of Subvarieties
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§1 Height of Points

Let X be a projective variety over the number field K. Choose an embedding $\varphi: X \to \mathbb{P}^n$. On $\mathbb{P}^n(\bar{K})$, we have the usual absolute Weil height h. For $P \in X(\bar{K})$, let $h_\varphi(P) := h \circ \varphi(P)$.

Basic question: How h_φ depends on φ?

Theorem (Weil) h_φ does only depend on the isomorphism class of $\varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$ up to bounded functions.

By a theorem of Serre, any $\mathcal{L} \in Pic(X)$ is difference of two very ample ones. This leads to

Height machine The height induces a homomorphism:

$$h: Pic(X) \longrightarrow \mathbb{R}^{X(\bar{K})} / O(1).$$

This is an important tool in diophantine geometry. In the following, Weil's theorem will be extended to height of subvarieties. As a corollary, we shall get Néron-Tate heights for subvarieties. Most of the text is covered in [Gu].

§2 Weil's Theorem for Subvarieties

Setup: X proper variety over K, $t \in \mathbb{N}$, $M_0, \dots, M_t \in Pic(X)$ basepoint-free.

How to define $h_{M_0,\dots,M_t}$ on $Z_t(X)$ (t-dim. cycles) and what is the error term ?

Example $X = \mathbb{P}^n$, $t = n-1$, $M_j = \mathcal{O}(1)$ $(j=0,\dots,t)$. Let F be a homogeneous polynomial of degree d with coefficients in K. Then the height of $\operatorname{div}(F)$ is more or less the height of the coefficient vector. By a linear change of variables, we get

$$h'(\operatorname{div}(f)) = h(\operatorname{div}(f)) + O(\deg(f)).$$

In general, we define the height of a cycle by push forward to multi-projective space and use the Faltings height there:

The error term will be $O(\delta)$ where

$$\delta(Z) := \underbrace{\sum_{j=0}^{t} d_{M_0,\dots,M_{j-1},M_{j+1},\dots,M_t}(Z)}_{\text{multi-degree of } Z \in Z_t(X_{\overline{K}})} .$$

A positive realization $\underline{\varphi}$ of $M_0,\dots,M_t$ is:

- $\overline{P} = \mathbb{P}^{n_0}_{O_K} \times \cdots \times \mathbb{P}^{n_r}_{O_K}$ (any projective arithmetic variety is good enough)
- $\overline{\mathcal{L}}_0,\dots,\overline{\mathcal{L}}_t$ basepoint-free line bundles on $\overline{P}$ with positive metrics.
- $\varphi: X \to P$ morphism over K such that $\varphi^*\mathcal{L}_j \cong M_j$ $(0 \le j \le t)$.

On $\overline{P}$, we have the absolute Faltings height $h_{\overline{\mathcal{L}}_0,\dots,\overline{\mathcal{L}}_t}$. For $Z \in Z_t(X_{\overline{K}})$, let

$$h_{\underline{\varphi}}(Z) := h_{\overline{\mathcal{L}}_0,\dots,\overline{\mathcal{L}}_t}\left(\overline{\varphi_*(Z)}\right).$$

Theorem 1 ([Gu]) $h_{\underline{\varphi}}$ doesn't depend on the choice of the positive realization of $M_0,\dots,M_t$ up to $O(\delta)$.

We write $h_{M_0,\dots,M_t}$ for the resulting class. As a formal consequence, we get:

height machine: Let X be a projective variety over K and deg the degree with respect to an ample line bundle on X. Then the height induces a multilinear map

$$h: \mathrm{Pic}(X)^{t+1} \to \mathbb{R}^{Z_t(X_{\overline{K}})} / O(\deg).$$

For $f: Y \to X$ morphism of projective varieties, we have

$$h_{M_0,\dots,M_t} \circ f_* = h_{f^*M_0,\dots,f^*M_t}.$$

For the proof of Theorem 1, we have to consider

§3 Local Heights

Let $\bar{P}$ be a ~~regular~~ projective arithmetic variety, $\bar{\mathcal{L}}_j$ a metrized line bundle and s_j a non-zero meromorphic section of $\mathcal{L}_j$ for $j = 0,\dots,t$. The generic fibre of $\bar{P}$ is denoted by P, the morphism of structure by $\pi: \bar{P} \to \mathrm{Spec}\, \mathcal{O}_K$.

Let $M_{\overline{K}}$ be the set of absolute values of $\overline{\mathbb{Q}}$ extending the usual absolute values of $\mathbb{Q}$.

For $v \in M_{\overline{K}}$, $v | \infty$, the local height of $Z \in Z_t(P_{\overline{K}})$ is defined by

$$\lambda_{s_0,\dots,s_t}(Z,v) := \frac{1}{2}\left(\log\|s_0\|_v^{-2} * \cdots * \log\|s_t\|_v^{-2} \wedge \delta_{Z_v}\right)(1).$$

If $v \nmid \infty$ with valuation ring R_v, let $\bar{Z}^v$ be the closure of Z in $\bar{P}_{R_v}$. Then

$$\lambda_{s_0,\dots,s_t}(Z,v) := -\log \left| \underbrace{\pi_*(\mathrm{div}\, s_0 \dots \mathrm{div}\, s_t . \bar{Z}^v)}\right|_v$$

to compute over a finite extension of K. Take $|\;|_v$ of a generator of the corresponding fractional ideal.

Note that in both cases, the local heights are only defined if $|Z| \cap |\mathrm{div}\, s_0| \cap \dots \cap |\mathrm{div}\, s_t| = \emptyset$ in the generic fibre P.

We have $M_{\bar{K}} = \varprojlim_L M_L$ where L ranges over all number fields with $K \subset L \subset \bar{K}$. On $M_{\bar{K}}$, we have a positive measure μ given by

$$\mu\{u \in M_{\bar{K}} \mid u|w\} = [K_w : Q_w]/[K:Q]$$

(completions)

for any $w \in M_L$.

Then the Faltings height of $Z \in Z_t(P_{\bar{K}})$ is given by

$$h_{\bar{\mathcal{L}}_0,\dots,\bar{\mathcal{L}}_t}(Z) = \int \lambda_{s_0,\dots,s_t}(Z,v)\, d\mu(v).$$

Here we choose $s_0,\dots,s_t$ such that the local heights are well-defined. By the product formula, the Faltings height doesn't depend on the choice of $s_0,\dots,s_t$.

§ 4 Proof of Theorem 1

We use the same assumptions as in §3. Let $Y \in Z_t(\bar{P})$ (t = relative dimension $/\mathbb{Z}$), then

$$Y = Y_K + \sum_{v \neq \infty} Y_v$$

(Y_K horizontal, Y_v in fibre over v)

Lemma 2

$$\lambda_{s_0,\dots,s_t}(Y,v) - \lambda_{s_0,\dots,s_{t-1}}(\operatorname{div}(s_t|Y)) = -\begin{cases} \int_{Y_v} \log \|s_t\|_v \, c_1(\mathcal{L}_0, \|\ \|_v) \wedge \dots \wedge c_1(\mathcal{L}_{t-1}, \|\ \|_v) & v \mid \infty \\ \log |\pi_*(\operatorname{div} s_0 \dots \operatorname{div} s_{t-1} . (\operatorname{div} s_t . Y)_v)|_v & v \nmid \infty \end{cases}$$

The proof is an easy consequence of the definitions. Now we consider transformations of local heights:

Let X be an O_K-variety, $(\bar{P}; \bar{\mathcal{L}}_0,\dots,\bar{\mathcal{L}}_t; s_0,\dots,s_t)$ as in §3. Let $(\bar{P}'; \bar{\mathcal{L}}_0',\dots,\bar{\mathcal{L}}_t'; s_0',\dots,s_t')$ be another choice and let $\bar{\varphi}: \bar{X} \to \bar{P}$, $\bar{\psi}: \bar{X} \to \bar{P}'$ be O_K-morphisms such that, for $j = 0,\dots,t$, we have

$$\varphi^* \mathcal{L}_j = \psi^* \mathcal{L}_j' \quad \text{and} \quad \varphi^* s_j = \psi^* s_j' \quad \text{(both on generic fibre).}$$

Let $\rho_j^v := \psi^* \| \; \|_{j,v} / \varphi^* \| \; \|_{j,v}$ $(v | \infty)$ and $E_j^v := \operatorname{div} \bar{\psi}^* s_j' - \operatorname{div} \bar{\varphi}^* s_j$. Then the following <u>transformation formula</u> is true for $Z \in Z_t(X)$:

$$\lambda_{s_0,\ldots,s_t}(\bar{\varphi}_* Z, v) - \lambda_{s_0',\ldots,s_t'}(\bar{\varphi}_* Z, v)$$

$$= \begin{cases} \sum_{j=0}^{t} \int_{Z_v} c_1(\psi^* L_0, \| \; \|_v) \wedge \ldots \wedge c_1(\psi^* L_{j-1}, \| \; \|_v) \log \rho_j^v \, c_1(\varphi^* L_{j+1}, \| \; \|_v) \wedge \ldots \wedge c_1(\varphi^* L_t, \| \; \|_v), & v | \infty \\ \sum_{j=0}^{t} \log | \Pi_* (\operatorname{div} \bar{\psi}^* s_0 \ldots \operatorname{div} \bar{\psi}^* s_{j-1} . E_j^v . \operatorname{div} \bar{\varphi}^* s_{j+1} \ldots \operatorname{div} \bar{\varphi}^* s_t . Z^v) |_v, & v \nmid \infty \end{cases}$$

Proof of Theorem 1: It's easy to reduce the problem to the setup above. Using $\log \rho_j^v = O(1)$ and $-\Pi^* \operatorname{div}(a) \leq E_j^v \leq \Pi^* \operatorname{div}(a)$ for some $a \in O_K$, we get the claim. □

Proof of transformation formula: Let $\Lambda_{\underline{s},\underline{s}'}$ be the right hand side of the claim. By Stokes and commutativity of intersection, we get $\Lambda_{\underline{s},\underline{s}'} + \Lambda_{\underline{s}',\underline{s}''} = \Lambda_{\underline{s},\underline{s}''}$. By passing to $\bar{P} \times \bar{P}'$, we may assume $\bar{P} = \bar{P}'$, $\bar{\varphi} = \bar{\psi}$, $\bar{L}_j = \bar{L}_j'$ (as $s_j = s_j'$ ... $t-1$). Using Lemma 2, we get

$$\lambda_{s_0,\ldots,s_t}(\bar{\varphi}_* Z, v) = \lambda_{s_0,\ldots,s_{t-1}}(\bar{\varphi}_* (\operatorname{div} \bar{\varphi}^* s_t . Z)_K) \begin{cases} -\int_{Z_v} \log \| \varphi^* s_t \|_v \, c_1(\varphi^* L_0, \| \; \|_v) \wedge \ldots \wedge c_1(\varphi^* L_{t-1}, \| \; \|_v) \\ -\log | \Pi_* (\operatorname{div} \bar{\varphi}^* s_0 \ldots \operatorname{div} \bar{\varphi}^* s_{t-1} . (\operatorname{div} \bar{\varphi}^* s_t . Z)_v) |_v \end{cases}$$

Replacing s_t by s_t', we get the claim. □

<u>§5 Néron-Tate heights</u>

Let A be an abelian variety over K and $M_0, \ldots, M_t \in \operatorname{Pic}(A)$. We have to define a canonical function $\hat{h}_{M_0,\ldots,M_t}$ in the height class. Beilinson, Bloch have given a definition if at least two are odd. Philippon handled the case $M_0 = \ldots = M_t$ very ample and even. A different approach is given by Kramer.

Theorem ([Gu]) If $M_0, \ldots, M_t$ are even (resp. t bundles even and one odd), then there is a unique function $\hat{h}_{M_0,\ldots,M_t}$ in the height class $h_{M_0,\ldots,M_t}$ with

$$\hat{h} \circ [m]_* = m^{2t+2} \cdot \hat{h} \quad (\text{resp. } m^{2t+1} \cdot \hat{h}) \qquad \forall m \in \mathbb{Z}.$$

The proof is a formal consequence of the height machine and Tate's limit argument.

<u>Final remark</u> All above remains valid for fields with product formula.

<u>Reference</u> [Gu] Gubler, W.: Höhentheorie (mit einem App. von J. Kramer). Math. Ann. 298, 427-455 (1994).

Titel: Curves of genus 2 and the height conjecture for elliptic curves I
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The following lecture is closely related to joint work with E. Kani.
We'll discuss the height conjecture for elliptic curves and related conjectures and then we'll explain how this height conjecture would follow from a conjecture about the self intersection number of the canonical sheaf of arithmetical surfaces whose generic fibers are curves of genus 2.

1.) First we have to introduce some notation.
Let K be a global field, i.e. K is either a finite number field or a function field of one variable over a perfect field of characteristic $p_0 \geq 0$. Let $g(K)$ be the genus of K (which is equal to $\frac{1}{2}\log|\Delta_{K/\mathbb{Q}}|$ in the number field case. Let Σ_K be the set of places of K, $\Sigma_{K,f}$

the set of non archimedean places.

Let $C|k$ be a curve of genus $g \geq 1$. We always assume that C has semistable reduction at all $\wp \in \Sigma_{k,f}$.

Let $\mathcal{C}$ be the minimal model of $C|k$ which we endow with its natural structure as arithmetical surface over the base B determined by Σ_k with projection map $\pi: \mathcal{C} \to B$.

The conductor N_C of C is defined by

$$N_C = \prod_{\substack{\wp \in \Sigma_{k,f} \\ \mathcal{C}_\wp \text{ is singular}}} \wp \qquad (\mathcal{C}_\wp = \text{fiber of } \mathcal{C} \text{ over } \wp)$$

Let J_C be the Jacobian of C, $\mathcal{J}_\mathcal{C}$ its Néron model.

The conductor of J_C is

$$N_{J_C} = \prod_{\substack{\wp \in \Sigma_{k,f} \\ \mathcal{J}_{\mathcal{C},\wp} \text{ is not an abelian variety}}} \wp$$

One knows that $N_{J_C} \mid N_\mathcal{C}$, and so $\deg J_\mathcal{C} \leq \deg N_C$.

Let $\omega_{\mathcal{C}/B}$ be the relative canonical sheaf of $\mathcal{C}/B$.

<u>Definition</u>. The Faltings height of C is

$$h(C) = \deg \pi_*(\omega_{\mathcal{C}/B}).$$

Noether's formula states:

$$12\, h(C) = \omega_{e/B}^2 + \sum_{\mathfrak{q} \in \Sigma_K} \delta_{\mathfrak{q}} \quad , \text{ where}$$

$\delta_{\mathfrak{q}}$ = # singular points in $C_{\mathfrak{q}}$ for $\mathfrak{q} \in \Sigma_{K,f}$, and $\delta_{\mathfrak{q}}$ in the archimedean case will be discussed more closely in the next talk.

2.) The height conjecture for elliptic curves over K states: (H_E): There are constants $c = c(g(K))$, $d = d(g(K))$ such that for all semistable elliptic curves E/K one has: $h(E) \leq c \deg N_E + d$.

The following conjectures are true if (H_E) holds:

Conjecture (R_E): For all elliptic curves E_0/K there is a number $m = m(E_0, g(K))$ such that for $E' \in \mathfrak{M}_{E_0} = \{ E/K ;$ there is a K-isomorphism $\alpha : E_{0,n} \to E_n$ for some n not dividing $30 \cdot p_0 \}$ we have: $h(E') \leq m$.

Conjecture $(R_{E,2})$ is derived from (R_E) by requiring that n is even, and Conjecture $(\bar{R}_E)$ is derived from (R_E) by requiring that $\Delta_\alpha = \{ (P, \alpha P) ; P \in E_{0,n} \}$ is isotropic w.r.to the Weil pairing on $E_{0,n} \times E_n$.

Remark: If K is a number field it follows that

$\mathcal{M}_{E_0}$ is finite.

It is not difficult to prove

Proposition 1: (H_E) implies (R_E), and so $(R_{E,2})$ and $(\overline{R_E})$.

For the next conjecture we fix a finite set of places S of K and denote by O_S^* the group of S-units of K.

For $n \in \mathbb{N}$ and $a,b,c \in O_S^*$ we define

$\mathcal{L}_{n,(a,b,c)} = \{ (z_1,z_2,z_3) \in K^3 ; a z_1^n - b z_2^n = c z_3^n \} / \sim$ $(\mathbb{P}^2 / K)$.

($\sim$ is equivalence in $\mathbb{P}^2 / K$)

Conjecture (F_{as}) ("asymptotic Fermat conjecture"):

For fixed S there is a number $k = k(S, g(K))$ such that for $(z_1,z_2,z_3) \subset \bigcup_{\substack{a,b,c \in O_S^* \\ n \geq 5}} \mathcal{L}_{n,(a,b,c)}$

we have: $h_{\mathbb{P}^2}(z_1;z_2,z_3) \leq k$.

Proposition 2: (H_E) implies (F_{as}), and (H_{O_S}) implies $(R_{E,2})$.

For $K = \mathbb{Q}$ define $\mathcal{L}^{mod}_{n,(a,b,c)} = \{ (z_1,z_2,z_3) \in \mathcal{L}_{n,(a,b,c)} ;$ such that $y^2 = X(X - a z_1^n)(X - b z_2^n)$ is modular$\}$.

Proposition 3: $\bigcup_{\substack{n \geq 5 \\ a,b,c \in \mathbb{Z}_S^*}} \mathcal{L}^{mod}_{n,(a,b,c)}$ is finite if $(R_{E,2})$ is true.

Let E_1, E_2 be two elliptic curves over K, and
$\alpha : E_{1,n} \to E_{2,n}$ a G_K-isomorphism $(n \geq 2)$ and
that $\Delta_\alpha = \{(P, \alpha P) ; P \in E_{1,n}\}$ is isotropic in $E_{1,n} \times E_{2,n}$.

Then $E_1 \times E_2 / \Delta_\alpha$ is principally polarized, and the polarization Θ is a curve C of genus 2 if and only if there is no isogeny $\eta : E_1 \to E_2$ with $\deg(\eta) = k(n-k)$ $(0 < k < n)$ and $\eta_{|E_{1,n}} \circ k = \alpha$.

Using this result we get

<u>Proposition 2</u>: Assume that (P_E) is wrong. Then there is an elliptic curve E_0 / K such that for infinitely many primes p there is a curve C_p of genus 2 and $f_p : C_p \to E$ defined over K and of degree p. The Galois group of the normal closure $\tilde{C}_p$ of C_p / E is equal to S_p, the fixed field of A_p is a curve of genus 2 over which $\tilde{C}_p$ is an unramified geometric extension.

Next we'll relate $(\underline{H_E})$ with height conjectures for curves of genus 2. We fix n and an elliptic curve E_0 / K (for instance assume that E_0 has good reduction everywhere), let E be an elliptic curve. If necessary we replace K by $K(E_{0,n}, E_n)$ and cover both E_0 and E by a curve C of genus 2 with Jacobian

$J_C = E_0 \times E / \Delta_\alpha$ with $\alpha : E_{0,n} \to E_n$.

Obviously we have:

1.) $\deg(N_{J_C}) = \deg(N_E) + \deg(N_{E_0}) = \deg N_E + c_1(E_0)$

2.) $h(C) = h(J_C) = h(E) + c_2(E_{0,n})$.

Hence to bound $h(E)$ in terms of $\deg(N_E)$ it is enough to bound $h(C)$ in terms of $\deg(N_{\mathcal{F}C})$.

We state the following "height conjectures" for (s.s) curves C of genus g / K where K is a global field of char. 0:

Conjecture $H_C(g)$: There exist constants $c_g = c_g(g(K), g)$ and $d_g = d_g(g(K), g)$ such that for all semistable curve C/K of genus g we have

$$h(C) \leq c_g \deg N_C + d_g.$$

Conjecture $(H_{\mathcal{F}}(g))$: Replace N_C by $N_{\mathcal{F}C}$ in $(H_C(g))$

So $(H_{\mathcal{F}}(g))$ implies $(H_C(g))$.

Noether's formula motivates the following variants:

Conjecture $(S_C(g))$: There exist constants c_g, d_g, such that for all semistable curves C/K of genus g one has: $\omega^2_{C/B} \leq c_g \deg N_C + d_g$.

Again we get conjecture $(S_{\mathcal{F}}(g))$ by replacing N_C by $N_{\mathcal{F}C}$.

Clearly $(S_{\mathcal{F}}(g))$ implies $(S_C(g))$, and by work of Parshin one knows that $S_C(g)$ for g sufficiently large implies $(H_E) = H_{\mathcal{F}}(1)$.

Theorem. $(S_{\underline{7}}(2))$ implies $(H_{\underline{7}}(1)) = (H_{\underline{E}})$.

By the considerations made above it is clear that $(H_{\underline{7}}(2))$ implies $(H_{\underline{7}}(1))$, and so the crucial step in the proof of the theorem is to discuss how $(H_{\underline{7}}(2))$ is related to $(S_{\underline{7}}(2))$, and this discussion is done in the lecture given by E. Kani.

Reference: G. Frey, E. Kani: Curves of genus 2 covering elliptic curves and an arithmetical application, in "Arithmetic Algebraic Geometry", PM 89, 1991.

Titel: Curves of genus 2 and the Height Conjecture for Elliptic Curves II
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This lecture is a report of joint work with G. Frey and is a continuation of his lecture. The main result is that (a slight sharpening of) the conjectural Bogomolov-Miyaoka-Yau inequality ($BMY^+(2)$) for arithmetic surfaces of genus 2 implies the height conjecture for elliptic curves. More precisely, we prove (see Frey's talk for the notation):

<u>Th</u>.1: $S_J(2) \Rightarrow H_E$.

This represents a partial sharpening of a well-known result of Parshin who showed that the usual B-M-Y inequality for arithmetic surfaces of large genus implies the height conjecture:

$$S_C(g),\ g \gg 0 \overset{\text{Parshin}}{\Longrightarrow} H_E .$$

The main ingredients of the proof of Th.1 are the following:

① <u>The construction of curves of genus</u> 2 <u>of type</u> <u>(E_1, E_2, N)</u>:

Given two elliptic curves E_1, E_2 and an integer $N \geq 2$, there exists a curve C of genus 2 which admits (minimal) morphisms $f_i : C \to E_i$ of degree N.

② An arithmetic analogue of the Freitag–Cornalba–Harris relation on $\overline{M}_2$ (= moduli stack of stable genus 2 curves):

(*) $\lambda^{\otimes 10} = \delta \otimes \delta_1$ (in $Pic(\overline{M}_2)$)

Here $\lambda = \det R^{\cdot}\pi_* \omega_{X/\overline{M}_2}$ (where $\pi : X \to \overline{M}_2$ denotes the universal curve), $\delta = \mathcal{L}(\Delta)$, $\delta_i = \mathcal{L}(\Delta_i)$ where $\Delta = \overline{M} \setminus M_2$ (M_2 = moduli stack of smooth genus 2 curves) and $\Delta = \Delta_0 \cup \Delta_1$ ($Int(\Delta_1)$ = {union of two elliptic curves meeting at one point}).

[In the preliminary version of Th. 1 presented at the Texel Conference in 1989 (Progress in Math. 89 (1991)), we prove a rough formula approximating the arithmetic analogue of (*). At the same time, J.-B. Bost (letter to Mazur, 1989) proved the exact arithmetic analogue of (*).]

③ Lower bounds on $\delta_1(C) = \sum_v \delta_{1,v}(C) \deg(v)$:

Let C/K (= number field) be a smooth projective curve of genus 2 whose associated (minimal) arithmetic surface $\mathcal{C}$ is semi-stable.

For a place v of K put

$\delta_{1,v} = \#\{P \in \mathcal{C}_v \otimes \overline{k(v)} : P$ disconnects the fibre $\mathcal{C}_v \otimes \overline{k(v)}\}$
if v is a finite place of K,

$\delta_{1,v} = 4 \log\|H\| + 4\log(2\pi)$, if v is archimedean,

where $\log\|H\|$ is as in Bost's article (CRAS 305 (1987))
i.e. $\log\|H\| = \frac{1}{2}\int_{Pic_1(C)} \log\|\vartheta_{Pic(C)}\| d\mu$.
ϑ-function

If v is finite, then clearly $\delta_{1,v} \geq 0$, but for the archimedean places lower bounds are harder to obtain. In the case of those curves which we are interested in, we have:

Th. 2: Let C be a curve of genus 2 of type (E_1, E_2, N). If $j(E_1) = 1728$ and $N=2$ then

(1) $\delta_{1,v}(C) \geq -\frac{1}{12}\delta_v(E_2) + c_{1,v}$, $\forall v$

(with $c_{1,v} = 0$ for v finite), hence

(2) $\delta_1(C) \geq -h(E_2) + c_1[K:\mathbb{Q}]$

where $h(E_2)$ denotes the Faltings modular height.

Titel: Existence of asymptotics for volumes of adelic metrized line bundles
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In this lecture we prove the existence of the arithmetic degree of an adelic metrized line bundle, under very weak conditions on the metrics. Our setting is as follows:

K = global field

X/K = projective, geometrically irreducible variety

$\mathcal{L}$ = ample line bundle on X. $d = \dim(X)$.

For each place v of K, let $\|\ \|_v$ be a metric on the geometric fibres of $\mathcal{L}_v = K_v \otimes \mathcal{L}$, that is, a function such that for each $x \in X(\mathbb{C}_v)$, each local section f at x, and each $a \in K_v$,

$$\|a \cdot f(x)\|_v = |a|_v \cdot \|f(x)\|_v .$$

We assume the metrics are galois-stable, eg. each continuous automorphism $\sigma \in \mathrm{Gal}(\mathbb{C}_v/K_v)$ satisfies $\|(\sigma f)(\sigma x)\|_v = \|f(x)\|_v$, but we make no hypotheses about smoothness, positivity, or even point by point continuity. The metrics may be 0 on certain fibres.

Let $R = \bigoplus_{n=0}^{\infty} \Gamma(\mathcal{L}^{\otimes n})$; suppose $g_1, \ldots, g_M$ is a set of generators for R as a K-algebra. In addition to galois stability, we assume the metrics are bounded in the following sense. For $f \in \Gamma(\mathcal{L}^{\otimes n})$, write $\|f\|_v = \sup_{x \in X(\mathbb{C}_v)} \|f(x)\|_v$.

① For each v, there is a constant $B_v < \infty$ such that $\|g_i\|_v \le B_v$ for all i;

② For all but finitely many v, $B_v = 1$.

For each n, put

$$F_v(n) = \{ f \in K_v \otimes \Gamma(\mathcal{L}^{\otimes n}) : \|f\|_v \le 1 \},$$

and, writing $\mathbb{A}$ for the adele ring of K, put

$$\mathbb{F}_{\mathbb{A}}(n) = \Big(\prod_v F_v(n)\Big) \cap (\mathbb{A} \otimes \Gamma(\mathcal{L}^{\otimes n})) .$$

Let vol denote a Haar measure on $\mathbb{A}\otimes\Gamma(\mathcal{L}^{\otimes n})$; write $\text{covol}_{\mathbb{A}}(n)$ for the volume of a fundamental domain for $\mathbb{A}\otimes\Gamma(\mathcal{L}^{\otimes n})/\Gamma(\mathcal{L}^{\otimes n})$. Write $\bar{\mathcal{L}}=\{\mathcal{L},\{\|\ \|_v\}\}$.

<u>Theorem</u>. In this setting, the following limit exists:

$$\deg(\bar{\mathcal{L}}) = \lim_{n\to\infty} \frac{-(d+1)!}{n^{d+1}} \log(\text{vol}(\mathbb{F}_{\mathbb{A}}(n))/\text{covol}_{\mathbb{A}}(n)).$$

The idea for the proof is to first define local degrees $\deg_v(\bar{\mathcal{L}})$ and show they exist, and then to show that

$$\deg(\bar{\mathcal{L}}) = \sum_v \deg_v(\bar{\mathcal{L}})\cdot\log(Nv)$$

where the numbers Nv are chosen as in the product formula. The construction of the local degrees is based on a constraction which V.P. Zaharjuta used to show the existence of the Fekete-Leja transfinite diameter on $\mathbb{C}^N$. The main fact which it rests upon is the existence of a K-basis for $R=\bigoplus_{n\geq 0}\Gamma(\mathcal{L}^{\otimes n})$ with properties much like those of the standard monomials, for $\mathbb{P}^N$. This basis is constructed by an inductive procedure, cutting X by well-chosen hyperplanes, and abutting in the case of curves.

The work described above is joint work with C.F. Lau and R. Varley.

<u>References</u>.

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[Z] Transfinite diameter, Chebyshev constants, and capacity for compacta in $\mathbb{C}^N$, by V.P. Zaharjuta. Math USSR Sbornik 25 (1975), 350-364.

Titel: Arithmetic Riemann-Roch
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X, Y: regular quasi-projective flat schemes / $\mathbb{Z}$.
$f: X \to Y$ projective morphism such that
$f: X(\mathbb{C}) \to Y(\mathbb{C})$ is smooth i.e.

$Tf_{\mathbb{C}} = \ker(TX(\mathbb{C}) \to f^* TY(\mathbb{C}))$ is an
holomorphic vector bundle on $X(\mathbb{C})$.
Choose an hermitian metric h_f on $Tf_{\mathbb{C}}$ such that
$F_\infty^*(h_f) = h_f$. For any $y \in Y(\mathbb{C})$ let
$X_y = f^{-1}(y)$. The tangent bundle to this
smooth projective complex manifold is $TX_y = Tf_{\mathbb{C}} | X_y$.
Let $d = \dim(X_y)$ and $\omega_0 \in A^{11}(X_y)$ the
normalized Kähler form of h_f. In any local
holomorphic coordinates $(z_1, \ldots, z_d)$ it is given by

$$\omega_0 = \sum_{\alpha,\beta=1}^{d} \frac{i}{2\pi} h_f\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta}\right) dz_\alpha \, d\bar{z}_\beta .$$

We assume that, for all $y \in Y(\mathbb{C})$ this form ω_0 is <u>closed</u>.
Now let $\bar{E} = (E, h)$ be an hermitian
vector bundle on X. For any $y \in Y(\mathbb{C})$ we let

$$A^{0q}(X_y, E_{\mathbb{C}}) := C^\infty(X_y, \Lambda^q(\overline{TX_y^*}) \otimes E_{\mathbb{C}})$$

denote the space of C^∞ forms of type $(0,q)$ with coefficients in $E_{\mathbb{C}}$ and

$\bar\partial : A^{0q} \longrightarrow A^{0,q+1}$

the Cauchy-Riemann operator. Given two sections s and t in $A^{0q}(X_y, E_{\mathbb{C}})$ the metrics h_f on TX_y^* and h on $E_{\mathbb{C}}$ induce a pointwise scalar product $\langle s(x), t(x) \rangle$, $x \in X_y$. We let

$$\langle s, t \rangle_{L^2} := \int_{X_y} \langle s(x), t(x) \rangle \frac{\omega_0^d}{d!}$$

and we denote by $\bar\partial^* : A^{0,q+1} \longrightarrow A^{0,q}$ the adjoint of $\bar\partial$:

$$\langle s, \bar\partial t \rangle_{L^2} = \langle \bar\partial^* s, t \rangle_{L^2}.$$

The Laplace operator $\Box_q = \bar\partial \bar\partial^* + \bar\partial^* \bar\partial$ on A^{0q} has a zeta function

$$\zeta_q(s) = \mathrm{Tr}\left(\Box_q^{-s} \mid \mathrm{Ker}(\Box_q)^\perp \right)$$

converging if $\mathrm{Re}(s) > d$, with a meromorphic continuation to the whole complex plane and such that 0 is not a pole, so that its derivative $\zeta_q'(0)$ is well defined.

The Knudsen – Mumford – Grothendieck line bundle

$$\lambda(E) = \det Rf_*(E)$$

on Y can be equipped with the Quillen metric, defined as follows. If $y \in Y(\mathbb{C})$, there are canonical isomorphisms

$$\lambda(E)_y \simeq \bigotimes_{q \geq 0} \Lambda^{max} H^q(X_y, E_{\mathbb{C}})^{(-1)^q} \simeq \bigotimes_{q \geq 0} \Lambda^{max}(\mathrm{Ker}\, \square_q)^{(-1)^q},$$

so the L^2-metric on harmonic forms defines a metric h_{L^2} on the line $\lambda(E)_y$. The <u>Quillen metric</u> is

$$h_Q = h_{L^2} \exp\Big(\sum_{q \geq 0} (-1)^{q+1} q\, \zeta_q'(0)\Big).$$

This gives a C^∞ metric <u>on $\lambda(E)$</u> (Bismut - Gillet - S.).

Consider a factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & P \\ & f \searrow \quad \swarrow g & \\ & Y & \end{array}$$

where $g : P \to Y$ is a projective bundle and i a closed regular immersion. Choose (arbitrary) metrics on the normal bundle N of X in P and on the relative tangent space Tg of g.
On $X(\mathbb{C})$ we have an exact sequence

$$\xi : 0 \to \overline{Tf_{\mathbb{C}}} \to i^* \overline{Tg_{\mathbb{C}}} \to \overline{N_{\mathbb{C}}} \to 0,$$

hence a Bott-Chern secondary characteristic class $\widetilde{Td}(\mathcal{E}) \in \bigoplus_{p\geq 0} \widetilde{A^{pp}(X)}$ such that

$$dd^c\, \widetilde{Td}(\mathcal{E}) = Td(Tf_{\mathbb{C}})\, Td(\overline{N}_{\mathbb{C}}) - Td(i^* \overline{Tg}_{\mathbb{C}}).$$

The element

$$\widehat{Td}(f, h_f) := \widehat{Td}(i^* \overline{Tg})\, \widehat{Td}(\overline{N})^{-1} + a(\widetilde{Td}(\mathcal{E})\, Td(\overline{N}_{\mathbb{C}})^{-1})$$

in $\bigoplus_{p\geq 0} \widehat{CH}^p(X) \otimes \mathbb{Q}$ depends on f and h_f, but not on the choice of the factorization of f.

On the other hand, let

$$R(Tf_{\mathbb{C}}) \in \bigoplus_{p\geq 0} H^{p,p}(X(\mathbb{C}), \mathbb{R})$$

be the value on $Tf_{\mathbb{C}}$ of the additive characteristic class attached to the genus

$$R(x) := \sum_{\substack{m \text{ odd} \\ m\geq 1}} \left(2\zeta'(-m) + \left(1 + \frac{1}{2} + \cdots + \frac{1}{m}\right)\zeta(-m)\right) \frac{x^m}{m!}$$

(where $\zeta(s)$ is Riemann zeta). We let

$$Td^A(f, h_f) = \widehat{Td}(f, h_f)\,(1 - a(R(Tf_{\mathbb{C}})))$$

in $\bigoplus_{p\geq 0} \widehat{CH}^p(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Theorem 1 (Gillet-S.):

The following equality holds in $\widehat{CH}^1(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$:

$$\hat{c}_1(\lambda(E), h_Q) = f_*\left(\widehat{ch}(E,h)\, Td^A(f,h_f)\right)^{(1)},$$

where $\alpha^{(1)}$ is the component of degree one of $\alpha \in \bigoplus_{p \geq 0} \widehat{CH}^p(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Our proof relies on several papers, including the work of Bismut-Lebeau on immersions and Quillen metrics.

We also have results when X (or Y) admits singular special fibers over $\mathbb{Z}$. An arithmetic Riemann-Roch in higher degrees was given by G. Faltings.

Let us mention one application (using a weaker version of Theorem 1 for curves, first shown by Deligne).

Theorem 2 (Miyaoka): Assume $d = 1$, $Y = \mathrm{Spec}(\mathbb{Z})$ and $rk(E) = r$. Then, if the metric h is Hermite-Einstein,

$$\widehat{\deg}\, \hat{c}_2(E,h) \leq \frac{2r}{r-1}\, \widehat{\deg}\left(\hat{c}_1(E,h)^2\right) \in \mathbb{R}.$$

This result was given another proof and generalized by Moriwaki.

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Gillet, S.: "Lectures on A.T."... "An Arithmetic Riemann-Roch Theorem", Inv. 110, 1992, pp. 473-543
Miyaoka: "Bogomolov inequality on arithmetic surfaces", talk at the Oberwolfach conf. on "Arithmetical Alg. Geom.", July 1988.
C. Soulé: A vanishing theorem on arithmetic surfaces, Inv. volume dedicated to A. Borel, 1994.

Titel: Connections on determinant line bundles
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Seite: 1
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Let $W \to \Delta = \{|z|<1\}$ be a holomorphic family of smooth compact Riemann surfaces, and E a holomorphic vector bundle over W. Given smooth Hermitian connections ∇_T, ∇^E on TW/Δ, E, resp., Beilinson-Schechtman constructed a smooth connection ∇_{BS} on the determinant line bundle λ_E associated with E, [BS].

The first half of the talk is to outline the theory of Beilinson-Schechtman, which involves the Atiyah algebra $\mathcal{A}_E$ of the bundle E, its $\omega_{W/\Delta}$-extension ${}^{tr}\mathcal{A}_E^{+}$, and a canonical $\bar{\partial}_E$-parametrix.

In the second part of the talk we shall brief on the relationship between ∇_{BS} and ∇_Q, the Quillen connection, and on results for the case of degenerating families of Riemann surfaces. More precisely,

Theorem 1 ([TT1]). In the preceding notation, assume that ∇_T, ∇^E are metric-connections. Let ∇_Q be the associated Quillen connection. Then $\nabla_{BS} = \nabla_Q$.

By considering the extension of Beilinson-Schechtman theory with logarithmic singularity, we show

<u>Theorem 2</u> ([TT2]). Let $W \xrightarrow{\pi} \Delta$ be a degenerating family of Riemann surfaces such that $\pi^{-1}(0)$ is the only singular fiber ~~with~~ whose singularities are ordinary double points Σ, δ denoting the cardinality of Σ. Let ∇_ω, ∇_E be smooth connections on $\omega_{W/\Delta}$, E. Then ∇_{BS}, well-defined over $\Delta \setminus \{0\}$, can be extended as an L^1-connection on Δ. Furthermore, the distributional derivative of ∇_{BS} is

$$\bar\partial \nabla_{BS} = \left(\int_{X/S} Td(T_{W/\Delta}, \nabla_{\omega^{-1}})\, ch(E, \nabla_E) \right)^{(1,1)} + \frac{\mathrm{rank}(E)}{12}\, \delta .$$

A consequence of Theorem 1 and Theorem 2 is ~~the~~ an "algebraic" proof of the following corollary:

<u>Corollary 3.</u> In the situation of Theorem 2, let σ be a section of λ_E. Supposed that ∇_ω, ∇_E are metric-connections. Then with respect to

the associated Quillen metric $\| \ \|_Q$, we have

$$\|\sigma\|_Q(z) \sim |z|^{\frac{\mathrm{rank}(\xi)}{12}\cdot\delta}$$

Theorem 2 for the Quillen connection ∇_Q was first proved by Bismut-Bost [BB], and Corollary 3 was first proved in [BB] by estimates on analytic torsion with heat kernels.

References:

[BB] Bismut, J.M., Bost, J.B.: Fibrés déterminants, métriques de Quillen et dégénérescence des courbes. Acta Math. 165 (1990), 1–103.

[BS] Beilinson, A., Schechtman, V., Determinant bundles and Virasoro algebras, CMP, 118 (1988); 651–701.

[TT1] Tong, Y.-L., Tsai, I-H., An identification of the connections of Quillen and Beilinson–Schechtman, 15-9 (1994), 443–457.

[TT2] Tong, Y.-L., Tsai, I-H., Curvature of determinant bundles for degenerating families, preprint.

intersection of L and note it $(L)^{n+1}$.

Using the first Minkowski theorem we find the next criterion for existance of section of $L^{\otimes n}$ with sup. norm less than 1 in each infinite place

Corollary

if $(L)^{n+1} > 0$ then $H^0(X, L^n) \cap B_n \neq 0$

for $n >> 0$

Remarks

1) In the case of arithmetic surface and when the metrics of L are admissible, the self-intersection introduced in the theorem coincide with the Arakelov self-intersection. So the Arakelov intersection (not only the Arakelov self-intersection) can be defined by this theorem.

2) In higher dimension the self-intersection given by the theorem coincide with the

self-intersection obtained from the theory of intersection of Gillet and Soulé and of Elkik. The corollary was proved in this case by Gillet and Soulé using their Riemann-Roch theorem (the first version) and analytic results of Bismut and Vasserot.
The idea of proving the "arithmetic" Hilbert-Samuel theorem without using the Riemann-Roch theorem was proposed to me by my advisor L. Szpiro.
3) S. Zhang gives a generalisation of the theorem for schemes with generic fiber not smooth.

It's easy to give an equivalent version of the theorem using the L^2-norms. To prove the L^2-version, we construct additive volumes and compare them to the L^2-volumes.

The "Arithmetic" Hilbert-Samuel theorem is a joined work with T. Bouche from Institut Fourier / Grenoble.

I think L. Szpiro for his generous help.

Titel: On the arithmetic analogues of the standard conjectures
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Let X be a regular scheme which is projective and flat over $\mathrm{Spec}(\mathbb{Z})$. We assume that X is of pure absolute dimension $n+1$. We denote by

$$L_{(\mathcal{L},\|.\|)} : \widehat{CH}^p(X) \longrightarrow \widehat{CH}^{p+1}(X)$$
$$\alpha \longmapsto \alpha . \hat{c}_1(\mathcal{L},\|.\|)$$

the Lefschetz operator associated with the hermitian line bundle $(\mathcal{L},\|.\|)$ on X. We will discuss the following arithmetic analogues of Grothendieck's standard conjectures which are stated by Gillet and Soulé in [G-S].

<u>Conjecture 1</u>: Let $\mathcal{L}$ be an ample line bundle on X. Then one can choose a positive hermitian metric $\|.\|$ on $\mathcal{L}$ such that, for $2p \leq n+1$,

i) $L^{n+1-2p}_{(\mathcal{L},\|.\|)} : \widehat{CH}^p(X)_{\mathbb{R}} \xrightarrow{\sim} CH^{n+1-p}(X)_{\mathbb{R}}$

is an isomorphism,

ii) If $x \in \widehat{CH}^p(X)_{\mathbb{R}}$, $x \neq 0$, and $L^{n+2-2p}_{(\mathcal{L},\|.\|)}(x) = 0$ then

$$(-1)^p \, \widehat{\deg}\left(x . L^{n+1-2p}_{(\mathcal{L},\|.\|)} x\right) > 0 .$$

The conjecture holds for arbitrary X and $p=0$, and for arithmetic surfaces with $n=p=1$ by the work of Faltings, Hriljac, and Gillet-Soulé [G-S]. In codimension one, part ii) of the conjecture was shown by Moriwaki [M]. For Grassmanniens the conjecture holds also (joint work with V. Maillot). Furthermore, we have the following reduction principle. A positive metric $\|.\|$ determines a Kähler form $\omega = c_1(\mathcal{L}\|.\|)$ on $X(\mathbb{C})$. Then it is sufficient to show that the statements i) and ii) hold for the Arakelov Chow group $CH^p(\overline{X})_{\mathbb{R}}$ of the Arakelov variety $\overline{X}=(X,\omega)$. This is mainly a consequence of the following result from Kähler geometry. Let M be a compact Kähler manifold and denote the complex valued (p,q)-forms on M by $A^{p,q}(M)$. There is a hard Lefschetz and a Hodge index theorem for the groups

$$A^{p,q}(M) \big/ \operatorname{Ker}(dd^c)$$

which measure the difference between the Arakelov Chow groups and the arithmetic Chow groups.

This reduction applies in particular to the following situation. Let A/S be an abelian scheme of relative dimension g where $S = \mathrm{Spec}(\mathcal{O}_F)$ and F is a number field. $A(\mathbb{C})$ is a disjoint union of complex abelian varieties. On each of these, we fix a translation invariant Kählermetric h and consider the Arakelov variety $\overline{A} = (A, h)$. The Arakelov Chow group decomposes

$$CH^p(\overline{A})_{\mathbb{Q}} = \bigoplus_{i=0}^{2g} \underbrace{\{\alpha \in CH^p(\overline{A})_{\mathbb{Q}} \mid \mathrm{mult}(m)^*(\alpha) = m^i\alpha \ \forall m\}}_{=:\ CH^p(R^i(A/S))_{\mathbb{Q}}}$$

We have a relative version of the first part of Conjecture 1, namely an isomorphism

$$(*) \quad L^{g-i}_{(\mathcal{L}, \|.\|_{cube})} : CH^p(R^i(\overline{A/S}))_{\mathbb{Q}} \xrightarrow{\ \simeq\ } CH^{p+g-i}(R^{2g-i}(\overline{A/S}))_{\mathbb{Q}}$$

for $i \in \{0, \dots, g\}$ where $\mathcal{L}$ is a rigidified ample symmetric line bundle on A equipped with a canonical hermitian metric $\|.\|_{cube}$.

We have the following conjectural description of the eigenspace decomposition above.

Conjecture 2:

$$\bigoplus_{i \le 2p-k} CH^p(R^i(\overline{A/S}))_{\mathbb{Q}} = \begin{cases} CH^p(\overline{A})_{\mathbb{Q}} & k=0, \\ \operatorname{Ker}(CH^p(\overline{A})_{\mathbb{R}} \xrightarrow{d} H^{p,p}(A)) & k=1, \\ \operatorname{Im}(H^{p-1,p-1}(A) \xrightarrow{a} CH^p(\overline{A})_{\mathbb{Q}}) & k=2, \\ (0) & k>2. \end{cases}$$

Another conjecture, stated by Beilinson, says:

Conjecture 3:

$$H^{p,p}(A) = \operatorname{Im}(CH^p(\overline{A})_{\mathbb{R}} \xrightarrow{d_{\mathbb{R}}} H^{p,p}(A)) \oplus \operatorname{Im}(CH^{p+1,p}(A)_{\mathbb{R}} \xrightarrow{\rho_{\mathbb{R}}} H^{p,p}(A))$$

Now assume that Conjecture 2 and Conjecture 3 hold and let $\mathcal{L}$ be a symmetric ample line bundle on A. Then the hard Lefschetz statement of Conjecture 1 is a consequence of the relative hard Lefschetz theorem (*).

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Titel: Stability and heights of arithmetic varieties
Autor: J.-B. Bost
Adresse: IHES, Bures-sur-Yvette, France.

In this talk, I describe a construction of natural heights one some classes of polarized ~~ab~~ varieties over $\overline{\mathbb{Q}}$, based on higher dimensional Arakelov geometry and on geometric invariant theory.

1. Normalized heights

1.1 Let K be a number field, $\pi : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ a projective arithmetic variety of relative dimension d, and $\overline{\mathcal{L}}$ a hermitian line bundle over $\mathcal{X}$.

Consider the following conditions:

(H1) $\overline{\mathcal{L}}$ is relatively semi-positive, namely:

- for any prime $\wp$ of $\mathcal{O}_K$, $\mathcal{L}$ is nef over $\mathcal{X}_{\mathbb{F}_\wp}$
- $c_1(\overline{\mathcal{L}}) \geq 0$ over $\mathcal{X}(\mathbb{C})$.

(H2) $\mathcal{L}_K$ is big (i.e. $\delta := c_1(\mathcal{L}_K)^d > 0$) and generated by its global sections over $\mathcal{X}_K$.

When they are satisfied, we can define

$$h_{norm}(\mathcal{X}, \overline{\mathcal{L}}) := \frac{1}{[K:\mathbb{Q}]} \left(\frac{\widehat{\deg}(\hat{c}_1(\overline{\mathcal{L}})|\mathcal{X})}{(d+1) \quad c_1(\mathcal{L}_K)^d} - \frac{\widehat{\deg}\, \pi_* \overline{\mathcal{L}}}{rk\, \pi_* \mathcal{L}} \right)$$

where $\pi_* \overline{\mathcal{L}}$ denotes $\pi_* \mathcal{L}$ equipped with the L^2-metric $(\| \; \|_\sigma)_{\sigma : K \hookrightarrow \mathbb{C}}$ defined by

$$\| s \|^2_\sigma = \int_{\mathcal{X}_\sigma(\mathbb{C})} \| s \|^2_{\overline{\mathcal{L}}} \; \frac{c_1(\overline{\mathcal{L}})^d}{\delta} .$$

Moreover, according to (H2), the linear system of sections of $\mathcal{L}_K$ defines a generically finite map $i : \mathcal{X}_K \to \mathbb{P}(H^0(\mathcal{X}_K, \mathcal{L}_K))$.

Theorem 1. For any $(\mathcal{X}, \overline{\mathcal{L}})$ such that (H 1-2) are satisfied and $i_* \mathcal{X}_K$ is Chow semistable, we have

$$h_{norm}(\mathcal{X}, \overline{\mathcal{L}}) \geq C(d, \delta)$$

for some effective constant $C(d, \delta)$ depending only on d and δ.

1.2 For any $(d, \delta) \in \mathbb{N}^{*2}$, let $\mathcal{V}_{st}(d, \delta)$ be the set of isomorphism classes of pairs (X, L), where X is a smooth algebraic variety over $\overline{\mathbb{Q}}$ and L a very ample line bundle over X such that, embedded in $\mathbb{P}(H^0(X, L))$, X is Chow stable. For any such (X, L), let

$$h_{norm}(X, L) := \inf\left\{ h_{norm}(\mathcal{X}, \overline{\mathcal{L}}),\ (\mathcal{X}, \overline{\mathcal{L}}) \text{ model of } (X, L) \text{ such that } \overline{\mathcal{L}} \text{ is relatively } 1/2\text{-positive} \right\}$$

$$(\geq C(d, \delta) \text{ by Theorem 1}).$$

Theorem 2. h_{norm} is a height on $\mathcal{V}_{st}(d, \delta)$; namely, for any $A \in \mathbb{R}$, the set

$$\left\{ [(X, L)] \in \mathcal{V}_{st}(d, \delta) \;\middle|\; \begin{array}{l}(X, L) \text{ may be defined over a number} \\ \text{field of degree at most } A \text{ and } h_{norm}(X, L) \leq A\end{array} \right\}$$

is finite.

1.3 Theorems 1 and 2 are elaborations of the results in [B], which were motivated by the results of Cornalba and Harris [C-H] in the geometric case. Other developments and variants of [B] have been obtained by Soulé [S] and Zhang [Z].

2. Applications

To get "concrete" results from Theorems 1 and 2, we use them together with the arithmetic Riemann-Roch Theorem, and some information on analytic torsion.

2.1 Curves

Let X be a smooth projective curve of genus $g \geq 2$ over $\overline{\mathbb{Q}}$. Let K be a number field on which X may be defined and has stable reduction, and let $\mathcal{X}$ be the regular semi-stable model of X over $\mathcal{O}_K$. For any prime $\wp$ of $\mathcal{O}_K$, let

$$\delta_\wp = \# \{\text{singular points of } \mathcal{X}(\overline{\mathbb{F}}_\wp)\}$$

__Theorem 3.__ There exists a continuous exhaustion function $\psi : \mathcal{M}_g(\mathbb{C}) \longrightarrow \mathbb{R}$ such that

$$(*) \quad h_{norm}(X, \omega_X) \leq \frac{2g+1}{g(g-1)} \left[h(\operatorname{Jac} X) - \frac{[K:\mathbb{Q}]^{-1}}{(8+\frac{4}{g})} \left(\sum_\wp \delta_\wp \log N\wp + \sum_{\sigma: K \to \mathbb{C}} \psi([X_\sigma]) \right) \right]$$

More precisely, if $\delta = 0$ is a local equation for the divisor of singular curves in $\overline{\mathcal{M}_g}(\mathbb{C})$, we have

$$\psi(z) \geq -\log|\delta(z)| + o(\log|\delta(z)|).$$

The proof of Theorem 3 is based on Noether's formula for arithmetic surfaces and the asymptotics on the analytic torsion of degenerating curves established in [B-B].

__Corollary.__ The RHS of $(*)$ is bounded below on $\mathcal{M}_g(\overline{\mathbb{Q}})$ and defines a height on $\mathcal{M}_g(\overline{\mathbb{Q}}) - \{\text{hyperelliptic curves}\}$.

2.2 Abelian varieties.

We shall denote $h(A)$ the Faltings height of an abelian variety defined over $\overline{\mathbb{Q}}$.

Theorem 4. Let A be an abelian variety over $\overline{\mathbb{Q}}$, of dimension g, and let L be a very ample line bundle over A. Then

$$h_{norm}(A, L) \leq \frac{1}{2} h(A) + C(g, c_1(L)^g),$$

for some constant $C(g, c_1(L)^g)$ depending only on g and $c_1(L)^g$.

Theorem 4 follows from the arithmetic R.R. and the computation of the analytic torsion of ample line bundles on abelian varieties, and, when A has not good reduction, from results of Moret-Bailly [M-B]. Together with Theorem 2, it implies that the Faltings height is an height!

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[Z] S. Zhang - Talk at this conference.

Titel: THE BLOCH CONJECTURE

Autor: Alexander Reznikov

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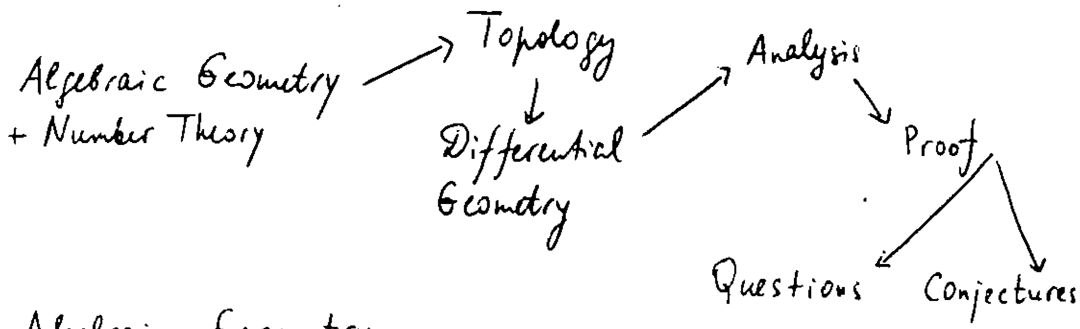
<u>Theorem</u> Let $X/\mathbb{C}$ be smooth projective. Let $\rho : \pi_1(X) \to SL(n, \mathbb{C})$ be a representation and let E_ρ be the corresponding flat bundle with canonical holomorphic structure. Then for all $i \geq 2$, $c_i(E_\rho) \in H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i))$ is torsion.

This is the Bloch conjecture (1977).

Scheme of the proof:



A. Algebraic Geometry.

The Deligne cohomology group is an extension

$$0 \to \mathcal{J}^i(X) \to H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i)) \to H^{2i}(X, \mathbb{Z}(i)) \overset{2\pi\sqrt{-1}}{\approx} \approx H^{2i}(X, \mathbb{Z})$$

Any element z in the Chow group $Ch^i(X)$ has a classical cycle class image in $H^{2i}(X, \mathbb{Z})$. This was extended by Deligne et al to a map $Ch^i(X) \to H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i))$.

If $E \to X$ is a holomorphic bundle, we first look at at its Chern class in the Chow group and then apply the map above to get an element in $H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i))$. Its image in $H^{2i}(X, \mathbb{Z})$ is just the topological Chern class, which is torsion, if E is flat (Chern-Weil).

On the other hand, for any smooth (real) manifold X and a flat bundle E there exists a secondary class of Cheeger-Chern-Simons $Ch_i(E) \in H^{2i-1}(X, \mathbb{C}/\mathbb{Z})$.

Theorem (Bloch, Soulé, Gillet-Soulé). In the situation of the theorem above, the image of $Ch(E)$ in $H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i))$ is $c_i(E)$.

So we seek to prove that $Ch_i(E)$ is torsion.

Rigidity of $Ch(E_\rho)$. On any connected component of the representation variety $Hom(\pi_1(X), SL(n, \mathbb{C}))$, $Ch_i(E)$ is constant.

Thus we can assume that ρ is defined over a number field F, or even over some $\mathcal{O}_S \subset F$.

Decompose $\mathbb{C}$ as $\mathbb{R} \oplus i\mathbb{R}$ and correspondingly $Ch_i(E_\rho) = \sqrt{-1}\,Vol(\rho) + ChS(\rho)$, where $Vol(\rho) \in H^{2i-1}(X, \mathbb{R})$ and $ChS(\rho) \in H^{2i-1}(X, \mathbb{R}/\mathbb{Z})$. Then $Vol(\rho)$ is called the Borel regulator and $ChS(\rho)$ is called the Chern-Simons regulator.

Universal classes There exist universal classes $Vol \in H^{2i-1}(BSL^{\delta}(n, \mathbb{C}), \mathbb{R})$ and $ChS \in H^{2i-1}(BSL^{\delta}(n, \mathbb{C}), \mathbb{R}/\mathbb{Z})$

such that $\mathrm{Vol}(\rho) = \Psi_\rho^*(\mathrm{Vol})$ and $\mathrm{ChS}(\rho) = \Psi_\rho^*(\mathrm{ChS})$, where $\Psi_\rho : X \to BSL^\delta(n, \mathbb{C})$ is a continious map, inducing ρ.

<u>FUNDAMENTAL FORMULA</u> For fixed F and $\mathcal{O}_S \subset F$, and any $\rho : \pi_1(X) \to SL(n, \mathcal{O}_S)$, there exist real numbers a_j and a natural number M, depending only on F and $\mathcal{O}_S$, such that

$$\mathrm{ChS}(\rho) \equiv \sum a_i \,\mathrm{Vol}(\sigma_i \circ \rho) \pmod{\tfrac{1}{M}\mathbb{Z}}$$

Here $\{\sigma_i\}$ is a maximal set of nonconjugate embeddings of F into $\mathbb{C}$.

This uses the Borel's study of stable cohomology of arithmetic groups.

<u>FUNDAMENTAL LEMMA</u> For X Kähler and any ρ,

$\mathrm{Vol}(\rho) = 0$.

B. <u>Topology</u>. Let f be a cohomology theory. A functor z : Spaces $\to$ Abelian Groups is called a cocycle functor (theory), if there is a natural transformation $z \to f$. We say that z is infinitisemal, if $U \mapsto z(U)$ is a sheaf.
Let X and Y be CW-complexes and let $\rho : \pi_1(X) \to \mathrm{Homeo}(Y)$ be a representation. Let $z \in z(Y)$ be ρ-invariant. Form a flat bundle

$$Y \to \widetilde{X} \times_{\pi_1(X)} Y \xrightarrow{\pi} X.$$

For $U \subset X$ simply-connected, trivialize $\pi^{-1}(U) \simeq U \times Y$

and pull back z onto $\pi^{-1}(U)$. Because z is $\rho(\pi_1(X))$ invariant and $\mathfrak{z}$ is infinitisemal, these cocycles fit into a global cocycle in $\mathfrak{z}(\tilde{X} \times_{\pi_1(X)} Y)$. Its class in $\mathfrak{f}(\tilde{X} \times_{\pi_1(X)} Y)$ is called the regulator class $r(z, \rho)$.

Examples. 1. Let $Y = SL(n, \mathbb{C})$, $\rho : \pi_1(X) \longrightarrow SL(n, \mathbb{C})$ and consider the left translation action. Take z a left-invariant closed k-form on Y, whose values on the Lie algebra are given by $X_1, \cdots X_k \longmapsto \mathrm{Alt}\,\mathrm{Tr}\, X_1 \cdots X_k$ (k odd). Then $r(z, \rho) \in H^k(\tilde{X} \times_{\pi_1(X)} SL(n, \mathbb{C}))$. Suppose the principal bundle $\tilde{X} \times_{\pi_1(X)} SL(n, \mathbb{C})$ is trivial topologically, i.e. admits a section s. Then $s^* z = Ch(E_\rho)$.

2. Let X be a homology sphere, and let $Y = GL(n, \mathbb{C}^\infty(M))$ where M is a manifold. Then a $\rho : \pi_1(X) \to GL(n, C^\infty(M))$ defines an element in $K^{alg.}(M)$. Applying our construction, one gets a Bloch-Beilinson regulator map

$$Ker(K_*^{alg} \to K_*^{top}) \longrightarrow H^{*-1-2s}(X, \mathbb{C}/\mathbb{Z}).$$

This uses Vogel-Haussman theory.

3. Let X is as above and let $Y = \mathcal{D}iff(S^2)$. Then one shows that the ChS class extends from $H^3(SL(2, \mathbb{C}), \mathbb{R}/\mathbb{Z})$ to $H^3(\mathcal{D}iff(S^2), \mathbb{R}/\mathbb{Z})$.

C. Geometry & Analysis.

Let M, N be compact Riemannian manifolds. A map $f : M \to N$ has energy $E(f)$ defined by

$$\int_M \|Df\|^2 \, dVol_M$$

We say that f is harmonic, if $\delta\ (f) = 0$.

Theorem (Eells-Sampson). If $K(N) \leq 0$, then any class in $[M, N]$ contains unique harmonic map.

Theorem (Sampson, following Bochner-Siu). If M is Kähler, and N is locally homogeneous of noncompact type, and if $f: M \to N$ is harmonic, then $Df_x^{1,0}: T^{1,0}M \to$ $\to T_{f(x)} N \otimes \mathbb{C}$ has abelian image, and the same for $Df_x^{0,1}$.

D. Proof of the Theorem.

Everything was reduced to showing that $Vol(\rho) = 0$ for any $\rho: \pi_1(X) \to SL(n, \mathbb{C})$. Form a flat bundle $SL(n, \mathbb{C})/SU(n) \to \tilde{X} \times_{\pi_1(X)} (SL(n, \mathbb{C})/SU(n)) \to X$, and contruct a regulator $r(z, \rho)$ where z is (unique up to a multiplier) invariant $(2i-1)$-form on SL/SU. The theory of regulators above shows that for any section s of the bundle, $s^* r(z, \rho) = Vol(E_\rho)$. The Corlette-Donaldson extension of Eells-Sampson theorem suggest a harmonic section s. Using (the extension of) the Sampson degeneration result, one shows that $s^* z$ is pointwise zero, Q.E.D.

E. Questions and Problems.

I. Is the Theorem true for a bundle with holomorphic connection? (The values of c_i are countable by Esnault-Srinivas)

II. How to gain the unifying approach to regulators in Algebraic K-theory of Frechet-Lie groups, starting with [3] and MK-theory of Karoubi and Soulé?

III. Develop the connection between arithmetics of secondary classes and rational points in representation varieties [2], [3.1], [3.2].

IV. Find an étale setting for Chern-Simons in spirit of Quillen, Grothendieck.

V. Let X be a Riemann surface and $Y = G/K$ a symmetric space of noncompact type. A theory for the equation $\tau f = \gamma \cdot [f_x, f_y]$ is suggested in [3]. Study a "polyCMC" maps of a Kähler X to Y, along the lines of polyharmonic theory. Derive applications to Simpson's theory.

VI. Find a Lefschetz hyperplane theorem for Simpson's theorem, using Witten's Morse theory via quantum deformations of de Rham complex.

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Titel: Integral points on algebraic stacks Seite 1

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Let K be a number field, $\overline{K}$ an algebraic closure of K, and let R be some localization of the ring of integers of K (such as a ring of S-integers). We denote by Max (R) the set of (nonzero) maximal ideals of R, identified with a subset of $M_K = \{\text{all places of } K\}$. As usual, K_v (resp. R_v) denotes the v-completion of K (resp. R) for $v \in M_K$ (resp. $v \in \mathrm{Max}(R)$).

Let X be an algebraic R-stack (in the sense of Artin [A]). We assume that X is of finite type, flat and surjective over Spec (R) and that X_K is geometrically irreducible over K.

Then X has a point over $\overline{R}$, the integral closure of R in $\overline{K}$.

In fact we prove more : let Σ be a finite subset of M_K ; we assume (mainly for convenience) that $\Sigma \cap \mathrm{Max}\ (R) = \emptyset$, and (this is essential) that $\Sigma \cup \mathrm{Max}\ (R) \neq M_K$. For each $v \in \Sigma$, fix a nonempty open subset Ω_v of $X(K_v)$. Here "open" means that:

(i) Ω_v is a set of objects of $X(K_v)$, stable under isomorphisms ;

(ii) for each K_v-scheme of finite type S and each object x of $X(S)$, the set of points $s \in S(K_v)$ such that the induced object $s^*(x)$ is in Ω_v is open in $S(K_v)$ for the v-topology.

Denote by $K^{\Sigma} \subset \overline{K}$ the maximal extension of K which is totally split over each $v \in \Sigma$, and by R^{Σ} the integral closure of R in K^{Σ}. If $x \in X(K^{\Sigma})$ and $v \in \Sigma$, we say, by abuse of language, that $x \in \Omega_v$ if, for every place w of K^{Σ} above v, the image of x in $X(K_v)$ via $K^{\Sigma} \longrightarrow (K^{\Sigma})_w \xrightarrow{\sim} K_v$ belongs to Ω_v.

Theorem. With the above notations and assumptions, there exists an object $x \in X(R^{\Sigma})$ which is in Ω_v, in the above sense, for all $v \in \Sigma$.

This theorem generalizes results in [S], [C-R], [R], and [MB] ; in fact the proof proceeds by reduction to the case where X is a scheme, treated in [MB].

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Titel: Some absolute constructions

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1. Subject of the talk

The Arakelov theory allows to "compactificy" schemes and to construct an intersection theory on corresponding spaces. However, many important construction of algebraic geometry are volens-nolens only relative (over Spec $\mathbb{Z}$) in that theory. For example, there exist no differentials on Spec $\mathbb{Z}$ and no 2-dimensional space Spec $\mathbb{Z}$ × Spec $\mathbb{Z}$. It seems, that a reason is there exists no F_1 - a "common" part of all the fields $\mathbb{F}_p$. Nevertheless, there are some objects and some constructions over F_1. Those are a subject of the talk.

2. Objects over F_1

ⓐ We start from the point:

a vector space over F_1 = a set.

Then $GL_n(F_1) = \mathrm{Aut}\{1,\dots,n\} = S_n$ - the symmetric group.

ⓑ Let G be a semi-simple algebraic group.

We can define $G(F_1) = W_G$ - the Weyl group of G. In the case $G = SL_n$ that gives just S_n.

ⓒ According to Kapranov we can put:

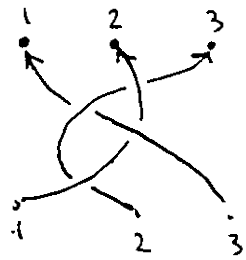
$GL_n(F_1[t,t^{-1}]) = B_n$ - the Artin group of braids.

We have analogues of two following maps:

(i) $\det: GL_n(F_1[t,t^{-1}]) \to t^{\mathbb{Z}}$;

(ii) $GL_n(F_1[t,t^{-1}]) \xrightarrow{\text{specialisation } t=1} GL_n(F_1)$.

(i) $B_n \to B_n/[B_n, B_n] \cong \mathbb{Z} \cong t^{\mathbb{Z}}$ - determinant;

(ii) 

$t=1$ The specialisation is the standart map from B_n to S_n.

(d) K-theory of F_1.

$S_\infty = \varinjlim S_n$ is a quasi-perfect group; so we can apply the Quillen +-construction and to put

$$K_*(F_1) = \pi_*(BS_\infty^+).$$

Theorem (Quillen) $K_*(F_1) = \pi_*^{st}(S^0)$, where S^0 is the 0-dimensional sphere.

(e) Varieties.

One can hope that some very "rigid" (at least without modules) varieties are defined over F_1. The first candidate to be existing is $\mathbb{P}^1$.

Concerning a discussion of $\mathbb{P}^1/F_1$ - see below. Here we mention only that any rational number $f \in \mathbb{Q}$ gives a rational map: $f: \operatorname{Spec} \mathbb{Z} \to \mathbb{P}^1$. If we take into account only the ramification over the three following points of $\mathbb{P}^1(F_1)$: $\infty, 0, 1$, then we get an inequality (instead of the Hurwitz genus formula). This conjectural inequality coincides with the well-know Masser-Oesterlé ABC-conjecture:

$$\forall \varepsilon > 0 \ \exists \text{ const s.t. } \forall A, B \in \mathbb{N};\ (A,B)=1$$

$$C \leqslant \text{const rad}(ABC)^{1+\varepsilon}, \text{ where}$$

$C = A+B$, rad - is the square-free part.

(3) Extensions

We consider "extensions" of F_1 as "fields of constants" of algebraic number fields, i.e. consisting of these element's which are "integer" at all the places, including the archimedian those. So we put:

$F_N = \{0\} \cup \mu_N$, where μ_N is the group of all the roots of unity of the degree N.

Now we can introduce the "field" F_N into constructions for F_1. For example:

a vector space over F_N = a set V, pointed by 0, with an action of F_N (as a monoid) such, that the action μ_N on $\tilde{V} = V - \{0\}$ is free.

$$GL_n(F_N) = \mathrm{Aut}_{F_N}\left(\{0\} \cup \bigcup_1^n \mu_N\right).$$

<u>Remark</u> In fact, the groups $GL_n(F_N)$ were used by Harris, Segal [1] and Soulé [2] to construct non-trivial elements in $K_*(\mathbb{Z})$. Main tool is a possibility to split a natural map on K-theories by using the following diagram:

$$K_*(\mathbb{Z}[\mu_{p-1}]) \xleftarrow{\;p\;} K_*(F_{p-1}) \xrightarrow{\;i\;} K_*(\mathbb{F}_p),$$

where i arises from the natural map: $GL_n(F_{p-1}) \subset GL_n(\mathbb{F}_p)$. The map i is quite available to be controlled.

(a) $\mathbb{P}^1/F_N$ (see [3]).

Define $\mathbb{P}^1/F_N = \{0\} \cup \{\infty\} \cup \mu/\mathrm{gal}(\mathbb{Q}(\mu_N))$, where $\mu = \varinjlim \mu_N$.

So, in the case $N=2$: $\mathbb{P}^1/F_2 = \{0\} \cup \{\infty\} \cup \mu/\mathrm{gal}\,\mathbb{Q}$.

For $f \in \mathbb{Q}$ we have a "rational" map $f: \operatorname{Spec} \mathbb{Z} \to \mathbb{P}^1/\mathbb{F}_2$.

$$f(p) = \begin{cases} 0, \text{ if } v_p(f) > 0; \quad \infty, \text{ if } v_p(f) < 0 \\ \{\varepsilon \in \mu \mid \text{(i) order of } \varepsilon \text{ is prime to } p; \\ \quad \text{(ii) } \exists \text{ a prime divisor } \Pi \text{ in } \mathbb{Q}(\varepsilon) \text{ s.t. } \\ \quad \Pi \mid p \text{ and } f \equiv \varepsilon \bmod \Pi. \end{cases}$$

One can define $\deg f$, ramification indexes etc. $\mathbb{P}^1/\mathbb{F}_2$ has four "rational" points: ∞, 0, $\{+1\}$, $\{-1\}$. If we evaluate the ramification f over those points as in the Hurwitz formula, we can state the

<u>Conjecture</u> $\forall \varepsilon > 0$ $\exists$ const s.t. if $m, n \in \mathbb{N}$, $(m,n) = 1$, then (here $m > n$, $f = \frac{m}{n}$).

$$\delta_\infty + \delta_0 + \delta_1 + \delta_{-1} \le 2 + \varepsilon + \frac{\text{const}}{\log m},$$

where

$$\delta_0 = \frac{\log m_1}{\log m}, \quad \delta_\infty = \frac{\log n_1 + \log(m/n) - 1}{\log m}, \quad \delta_1 = \frac{\log(m-n)_1}{\log m},$$

$\delta_{-1} = \dfrac{\log k_1}{\log m}$, $k = (m+n)/2^{v_2(m+n)}$ and for $A \in \mathbb{N}$ $A_1 = A/\operatorname{rad} A$.

Without of δ_{-1} it is exactly the ABC-conjecture.

Let $f = \frac{m}{n}$, $f \neq 0, \pm 1$, $a_p = v_p(f^p - f)$, $S = \{p \mid a_p > 1\}$. Applying of the above method to $\mathbb{Q}(\mu_N)$ allows to state:

<u>Conjecture</u> $\forall \varepsilon > 0$ $\exists$ const s.t. $\forall f$

$$\sum_{p \in S} \log\left(\frac{1}{1-p^{-1}}\right) + \sum_{p \in S} (a_p - 2) \log p \le (2+\varepsilon) \log m + \text{const}$$

(8) Determinants and the Hilbert symbol

For $\mathbb{F}_N$-vector spaces one can define maps, basises, dim, dual spaces, $\oplus$, $\otimes$, $\wedge$. For example

$V_1 \otimes V_2 = \{0\} \cup \tilde{V}_1 \times \tilde{V}_2 / \mu_N$, where μ_N acts by antidiagonal way.

$\Lambda^k V = \{0\} \cup (V^{\otimes k} - Z)/S_k$, where $Z = \{v_1 \otimes \ldots \otimes v_k \mid \exists i,j,\varepsilon \ v_i = \varepsilon v_j\}$; $\dim V = \# \tilde{V}/\mu_N$.

Lemma Let $0 \to A \to B \to C \to 0$ be an exact sequence of abelian groups, which are F_N-vector spaces (of course, that sequence isn't exact in sense of F_N-spaces). Then (i) $\dim B \equiv \dim A + \dim C \mod N$ (ii) $\det B = \det A \cdot \det C$

Further we assume that $N=2$. Let $V \supset A, B$ be abelian groups. Let A and B are commensurable (i.e. $A \cap B$ has a finite odd index in A and in B).

Put $[A,B] = \dim(A/A\cap B) - \dim(B/A\cap B) \mod 2$

$$(A|B) = \det(A/A\cap B)\det^{-1}(B/A\cap B).$$

Now let $V = \mathbb{Q}_p$, $a, b \in \mathbb{Q}_p$, p - odd. There are two natural isomorphisms of 1-dimensional spaces:

$$(\mathbb{Z}_p | a\mathbb{Z}_p)(a\mathbb{Z}_p | ab\mathbb{Z}_p) \overset{b}{\underset{a}{\rightleftarrows}} (\mathbb{Z}_p | b\mathbb{Z}_p)(b\mathbb{Z}_p | ab\mathbb{Z}_p)$$

$$\cong (\mathbb{Z}_p : ab\mathbb{Z}_p) \cong$$

Put $\{a,b\}$ = the quotient of two those isomorphisms.

Theorem $\{a,b\}\cdot(-1)^{[\mathbb{Z}_p, a\mathbb{Z}_p][\mathbb{Z}_p, b\mathbb{Z}_p]} = \left(\frac{a,b}{p}\right)$ - the Hilbert symbol

Remark The construction of the Hilbert symbol is similiar to construction of the tame symbol in [4] and is done by suggestion of Parshin. An origin was a remark by Kapranov that the Jacobi symbol $\left(\frac{a}{b}\right)$ is an analogue of the resultans of two polinomials.

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Let K be a number field and $\mathcal{O}_K$ its ring of integers. let $X \longrightarrow \operatorname{spec} \mathcal{O}_K$ be a projective flat scheme of relative dimension r with a smooth generic fiber. let L be an ample line bundle on X with a hermitien positive metric on L_σ (for all embedding $\sigma : K \hookrightarrow \mathbb{C}$) stable under conjugation.

The canonical embedding

$$H^\circ(X, L^n) \hookrightarrow H^\circ(X, L^n) \underset{\mathbb{Z}}{\otimes} \mathbb{R} = W_n \simeq \underset{\sigma}{\oplus} H^\circ(X, L^n) \underset{\sigma}{\otimes} K_\sigma$$

makes $H^\circ(X, L^n)$ a lattice in W_n.

We note B_n the unit ball of W_n with its sup. norm.

<u>Theorem</u>

The quantity

$$\frac{(r+1)!}{n^{r+1}} \left(-\log \operatorname{vol}\left(H^\circ(X, L^n)\right) + \log \operatorname{vol}(B_n) \right)$$

tends to a finite limit. we will call it self-