

VANISHING CYCLES IN HOLOMORPHIC FOLIATIONS BY CURVES AND FOLIATED SHELLS

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ABSTRACT. The purpose of this paper is the study of vanishing cycles of holomorphic foliations by curves on compact complex manifolds. The main result consists in showing that a vanishing cycle comes together with a much richer complex geometric object - we call this object a *foliated shell*.

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1. INTRODUCTION

1.1. Vanishing cycles, compact leaves and simultaneous uniformization. Let \mathcal{L} be a holomorphic foliation by curves on a compact complex manifold X . For the sake of clarity and simplicity of exposition we describe our results in this Introduction assuming \mathcal{L} to be smooth. In the main body of the paper this assumption will be removed (as well, as the assumption of compactivity of X will be replaced by the *disk-convexity*).

Take a point $z \in X$ and denote by \mathcal{L}_z the leaf of \mathcal{L} passing through z . A cycle in \mathcal{L}_z is, by definition, a closed path (a loop) $\gamma : [0, 1] \rightarrow \mathcal{L}_z$. A cycle $\gamma \subset \mathcal{L}_z$ is called a *vanishing cycle* if the following two conditions hold:

- γ is not homotopic to zero in \mathcal{L}_z ;
- there exist a sequence of points $z_n \rightarrow z$ and a sequence of loops $\gamma_n : [0, 1] \rightarrow \mathcal{L}_{z_n}$ such that γ_n uniformly converge to γ and each γ_n is homotopic to zero in \mathcal{L}_{z_n} .

Classically vanishing cycles became the object of study in foliation theory since the seminal paper of Novikov [N], where he used them to produce a compact leaf in every smooth foliation by surfaces on \mathbb{S}^3 , see also [H].

Apart of the question of existence of compact leaves vanishing cycles come into a play as obstructions to the simultaneous uniformization of leaves. Following Il'yashenko, see §2 in [Iy2], take a smooth complex hypersurface D in X transversal to the leaves of \mathcal{L} .

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Such D will be called simply a *transversal* in the sequel. Set $\mathcal{L}_D = \bigcup_{z \in D} \mathcal{L}_z$ and call this open subset of X a *cylinder* of \mathcal{L} over the transversal D .

Let $\tilde{\mathcal{L}}_D = \bigcup_{z \in D} \tilde{\mathcal{L}}_z$ be the union of the universal coverings of the leaves \mathcal{L}_z equipped with the natural topology, see Section 3. Let's call $\tilde{\mathcal{L}}_D$ the *universal covering cylinder* (or, simply the *covering cylinder* if no misunderstanding can occur) of \mathcal{L} over D . It is clear (see Section 3 for more details) that a leaf $\mathcal{L}_z \subset \mathcal{L}_D$ containing a vanishing cycle exists if and only if the natural topology of $\tilde{\mathcal{L}}_D$ is not separable (*i.e.*, is not Hausdorff). Separability of $\tilde{\mathcal{L}}_D$ means that the leaves of \mathcal{L} which cut D can be simultaneously uniformized. Therefore a vanishing cycle in some leaf $\mathcal{L}_z \subset \mathcal{L}_D$ is an obstruction to such simultaneous uniformization. \mathcal{L} is called *uniformizable* if for any transversal D the cylinder \mathcal{L}_D can be uniformized. Therefore \mathcal{L} is uniformizable if and only if it doesn't contain a vanishing cycle in any of its leaves. This explains one more reason for the interest in studying of vanishing cycles.

1.2. Vanishing cycles and foliated shells. The main goal of this paper is to show that in the case when the foliation by surfaces is a holomorphic foliation by complex curves a vanishing cycle generates a very rich complex geometric object - a *foliated shell*.

Let $P = \{z = (z_1, z_2) \in \mathbb{C}^2 : \max\{|z_1|, |z_2|\} \leq 1\}$ be the unit bicylinder in \mathbb{C}^2 and $B = \{z = (z_1, z_2) \in \mathbb{C}^2 : \max\{|z_1|, |z_2|\} = 1\}$ its boundary. For some $0 < \varepsilon < 1$, set $B^\varepsilon = \{z \in \mathbb{C}^2 : 1 - \varepsilon < \max\{|z_1|, |z_2|\} < 1 + \varepsilon\}$ a shell around B . Denote by $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$ the canonical projection $\pi(z) = z_1$ onto the first coordinate of \mathbb{C}^2 . Note that B^ε is foliated by π over the disk $\Delta_{1+\varepsilon}$ of radius $1 + \varepsilon$ (Δ_r denotes the disk of radius $r > 0$ in \mathbb{C}). Denote this foliation by \mathcal{L}^π . Its leaves $\mathcal{L}_{z_1}^\pi := \pi^{-1}(z_1)$ are disks $\Delta_{1+\varepsilon}$ if $1 - \varepsilon < |z_1| < 1 + \varepsilon$ and are annuli $A_{1-\varepsilon, 1+\varepsilon} := \Delta_{1+\varepsilon} \setminus \bar{\Delta}_{1-\varepsilon}$ if $|z_1| < 1 - \varepsilon$.

Definition 1.1. *The pair $(B^\varepsilon, \mathcal{L}^\pi)$ will be called the standard foliated shell.*

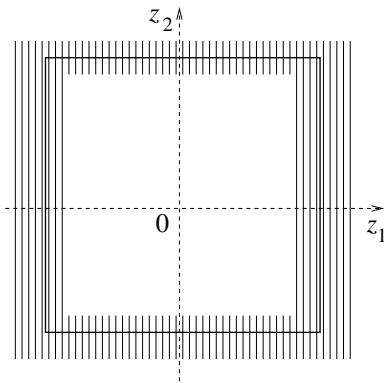


FIGURE 1. The *standard foliated shell* is foliated by disks and annuli over the disk $\Delta_{1+\varepsilon}$.

Let $h : (B^\varepsilon, \mathcal{L}^\pi) \rightarrow (X, \mathcal{L})$ be a foliated holomorphic immersion of the standard foliated shell into (X, \mathcal{L}) (an immersion between two foliated manifolds is called *foliated* if it sends leaves to leaves). Denote by Σ the image of the boundary B under h .

Definition 1.2. The image $h(B^\varepsilon)$ is called a *foliated shell* in (X, \mathcal{L}) if:

- 1) immersion h is a *generic injection*, i.e., is such that for all $z_1 \in \Delta_{1+\varepsilon}$ except of a finite set the restriction $h|_{\mathcal{L}_{z_1}^\pi} : \{z_1\} \times A_{1-\varepsilon, 1+\varepsilon} \rightarrow X^0$ is an imbedding;
- 2) Σ is not homologous to zero in X .

Ruffly speaking the condition (1) means that h is (much) better than simply an immersion. The main point is of course the condition (2). It is very strong and our corollaries will demonstrate this.

Example 1.1. The reader should think about the Hopf surface $H^2 = \mathbb{C}^2 \setminus \{0\} / z \sim 2z$. The same “vertical foliation” \mathcal{L}^π is invariant under the action $z \sim 2z$ and therefore projects to a foliation \mathcal{L} on H^2 . Let $h : \mathbb{C}^2 \setminus \{0\} \rightarrow H^2$ be the canonical projection. It obviously induces a “foliated inclusion” $h : (B^\varepsilon, \mathcal{L}^\pi) \rightarrow (H^2, \mathcal{L})$. $\Sigma = h(B)$ is of course not homologous to zero in H^2 .

Let ω be a $(1,1)$ -form on X . ω is called *pluriclosed* if $dd^c\omega = 0$. Sometimes one calls such ω also *dd^c-closed*. Recall that $d^c := \frac{i}{2}(\bar{\partial} - \partial)$ and therefore $dd^c = i\partial\bar{\partial}$. We call a form ω a *taming form* for \mathcal{L} if $\omega|_{\mathcal{L}} > 0$ (see Section 2 for the more precise definition). Foliations admitting a pluriclosed taming forms we shall call *pluritamed*. Our first result is the following:

Theorem 1. Let \mathcal{L} be a holomorphic foliation by curves on a compact complex manifold X which admits a pluriclosed taming form, and let D be a transversal to \mathcal{L} in X . Then the following statements are equivalent:

- i) Some leaf $\mathcal{L}_{z^0} \subset \mathcal{L}_D$ contains a vanishing cycle.
- ii) The cylinder \mathcal{L}_D contains a foliated shell.

Remark 1.1. (a) Statement (ii) means that the mapping $h : B^\varepsilon \rightarrow X$, which “supports” the foliated shell in X , actually takes values in the cylinder \mathcal{L}_D (but $h(B)$ is not homologous to zero in the whole of X !).

(b) A transversal D is irrelevant in this theorem: if \mathcal{L}_{z^0} contains a vanishing cycle then (ii) is true for every transversal $D \ni z^0$.

(c) Recall that a two-dimensional shell in a complex manifold X is a holomorphic image Σ of (a neighborhood of) B such that Σ is not homologous to zero in X . Such shells can exist only on non-Kähler X by the Hartogs-type extension theorem for Kähler manifolds, see [Iv2] (and therefore foliations on Kähler manifolds don’t have vanishing cycles). We want to stress here that X may contain a two-dimensional shell, but it may not be a *foliated shell* for the given foliation \mathcal{L} . A simple example is the elliptic fibration on the same Hopf surface H^2 . This fibration doesn’t admit a foliated shell, while H^2 itself does contain a two-dimensional shell.

(d) In fact in the process of the proof of Theorem 1 we establish the following useful characterization of shells:

Proposition 1. Let w be a *dd^c-closed taming form* for \mathcal{L} . A holomorphic foliated immersion $h : B^\varepsilon \rightarrow X$ represents a *foliated shell* if and only if it is a *generic injection* and

$$\int_B d^c(h^*\omega) \neq 0. \quad (1.1)$$

I.e. not only $h(B)$ is not homologous to zero in X but, moreover, the distinguished closed 3-form $d^c\omega$ should not vanish on $h(B)$.

In particular we obtain the following

Corollary 1. *If the taming form ω of the foliation \mathcal{L} is d -closed then \mathcal{L} has no vanishing cycles.*

For foliations on Stein manifolds this result was proved by Ilyahsneko in [Iy2], for compact Kähler ones by Brunella in [Br2].

(e) The boundary B is topologically the three-dimensional sphere \mathbb{S}^3 . It is not difficult to produce algebraic (and therefore Kähler) manifolds with nontrivial π_3 , but none of them contains a shell. The reason is that a shell is a global *pseudoconvex* object in the complex manifold X and not simply an element of $\pi_3(X)$.

(f) The meaning of the Theorem 1 is that a topological property of (X, \mathcal{L}) to contain a vanishing cycle is equivalent to a complex geometric (even analytic) property to contain a foliated shell.

1.3. Pluriexact foliations. Now let us clarify our assumption on a taming form ω to be pluriclosed. We shall prove the following:

Proposition 2. *Let \mathcal{L} be a holomorphic foliation by curves on a compact complex manifold X . Then the following two assertions are equivalent:*

- i) (X, \mathcal{L}) doesn't admits a pluriclosed taming form.
- ii) There exists a non-trivial, positive, dd^c -exact $(1, 1)$ - current T on X directed by \mathcal{L} .

A foliated manifold (X, \mathcal{L}, ω) admitting a non-trivial, positive, dd^c -exact, bidimension $(1, 1)$ current T tangent to (or, directed by) \mathcal{L} we shall call *pluriexact*. Remark that such current T is closed and therefore in a standard way, see [Go], we obtain the following:

Corollary 2. *A pluriexact holomorphic foliation on curves on a compact complex manifold admits a transversal measure.*

Via the aforementioned duality the characterization result of the Theorem 1 shows that the class of all holomorphic foliations by curves on compact complex manifolds splits naturally into the following three non-intersecting subclasses: the class \mathcal{S} of *shelled foliations*, the class \mathcal{U} of *uniformizable foliations*, and the class \mathcal{E} of *pluriexact foliations*. A *shelled foliation* or a *foliation with shells* is a foliation on a compact manifold which contains foliated shells.

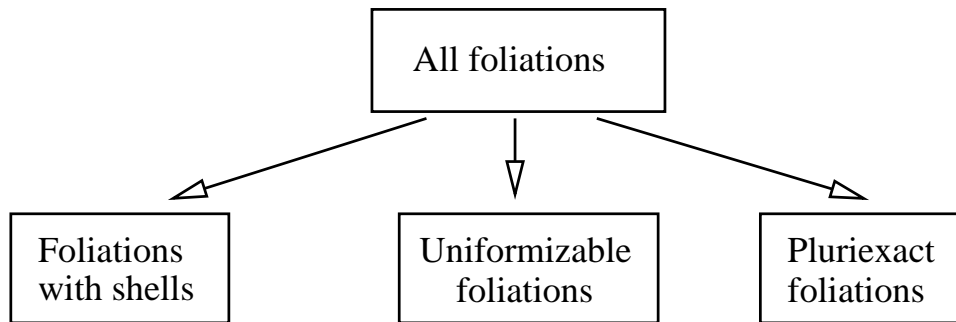


FIGURE 2

Note that in the definition of classes \mathcal{S} and \mathcal{U} we require both a pluriclosed taming form and a shell/or absence of shells. The point is that a foliated shell is of real importance only in the presence of such a taming form. In that case it turns out to be a dominating object in (X, \mathcal{L}) . As well as the uniformizability condition on \mathcal{L} implies more information

about this foliation provided \mathcal{L} admits a pluritaming form. While in the absence of such a form a dd^c -exact current tangent to \mathcal{L} is (eventually) of much greater importance.

We see the future development of the subject as the study of each of these classes separately, eventually with the very different tools, and \mathcal{E} to be certainly further subdivided. This paper is almost entirely devoted to the class \mathcal{S} .

1.4. Imbedded cycles and imbedded shells. Note that our foliated shells are, after all, an *immersed* objects in X (even if they are “generic injections”). It would be definitely preferable to have really an *imbedded* ones. However, let us stress at this point that not all foliations in \mathcal{S} contain an *imbedded* foliated shell, *i.e.*, such that $h : B^\varepsilon \rightarrow X$ is an imbedding. The reason is that the underlying manifold X may not contain an imbedded two-dimensional shell at all.

Example 1.2. Let, for example, $H^2/(z \sim -z)$ be the quotient of our Hopf surface by the antipodal involution. The vertical foliation \mathcal{L}^π described in the Example 1.1 is stable under this involution and we obtain a foliated manifold $(H^2/\mathbb{Z}_2, \mathcal{L}/\mathbb{Z}_2)$. The standard foliated shell immerses to H^2/\mathbb{Z}_2 and B maps onto the quotient B/\mathbb{Z}_2 which is topologically a lens space. *I.e.*, we have here an immersed foliated shell. Due to a result of Kato, see [K1], would $H^2/(z \sim -z)$ contained an imbedded shell, it would be a deformation of a blown-up primary Hopf surface, *i.e.*, its fundamental group would be \mathbb{Z} . And this is not the case, because $\pi_1(H^2/\mathbb{Z}_2) = \mathbb{Z} \times \mathbb{Z}_2$.

Nevertheless one can find an *imbedded* foliated shell in a shelled foliation $(X, \mathcal{L}) \in \mathcal{S}$ provided that

- the vanishing cycle γ_0 is *imbedded* into its leaf \mathcal{L}_{z^0} ;
- the shell itself is allowed to have somewhat more complicated topology.

Let us more carefully explain what does it mean that $\gamma_0 \subset \mathcal{L}_{z^0}$ is imbedded. Let d be the order of the holonomy of \mathcal{L} along the imbedded loop γ_0 . It should be finite, otherwise γ_0 cannot be approximated by the loops γ_n in the nearby leaves which are homotopic to zero. But then for a generic nearby leaf \mathcal{L}_{z_n} the nearby loop $\gamma_n \subset \mathcal{L}_{z_n}$ will approximate $d \cdot \gamma_0$ (not just γ_0 !) Therefore in the definition of an *imbedded vanishing cycle* one should specify that $\gamma_n \rightarrow d \cdot \gamma_0$ where $d \geq 1$ is the order of the holonomy of \mathcal{L} along γ_0 .

Now let us turn to the topology of shells. Recall that a cyclic surface quotient is a normal complex space $\mathcal{X}^{l,d}$ which is the quotient of \mathbb{C}^2 by the finite group $\Gamma_{l,d}$ of transformations given by $(z_1, z_2) \rightarrow (e^{\frac{2\pi il}{d}} z_1, e^{\frac{2\pi i}{d}} z_2)$. Here $1 \leq l < d$ is relatively prime with d . This action preserves the vertical foliation on \mathbb{C}^2 and therefore $\mathcal{X}^{l,d}$ is equipped with the “vertical” foliation to, which we denote by \mathcal{L}^π again. Note that the standard “vertical” projection $\pi : \mathcal{X}^{l,d} \rightarrow \mathbb{C} / \langle e^{\frac{2\pi il}{d}} \rangle = \mathbb{C}$ is well defined and its fibers are still the leaves of our vertical foliation. Take some smoothly bounded domain $G \Subset \Delta$ such that $\partial G \not\equiv 0$ but $G \ni 0$ and consider the domain $P = \bigcup_{z \in G} \Delta_z \subset \mathcal{X}^{l,d}$ (here $\Delta_z := \{z\} \times \Delta$). Remark that the boundary B of P lies in the smooth part of $\mathcal{X}^{l,d}$. For some $\varepsilon > 0$ denote by B^ε the ε -neighborhood of B .

Definition 1.3. A *foliated cyclic shell* in (X, \mathcal{L}) is a *foliated holomorphic immersion* $h : (B^\varepsilon, \mathcal{L}^\pi) \rightarrow (X, \mathcal{L})$ such that:

- 1) h is a generic injection;
- 2) $\Sigma := h(B)$ is not homologous to zero in X .

With this notion at hand we can state the following:

Theorem 2. *Let (X, \mathcal{L}, ω) be a pluritamed foliation on a compact complex manifold X and let D be a transversal in X . Then the following conditions are equivalent:*

- 1) *Some leaf $\mathcal{L}_{z^0} \subset \mathcal{L}_D$ contains an imbedded vanishing cycle.*
- 2) *The cylinder \mathcal{L}_D contains an imbedded foliated cyclic shell.*

We should point out that the topology of cyclic shell as we define it can be quite complicated. It is not just a lens space, i.e., is not simply a quotient of \mathbb{S}^3 by a free action of a finite group.

Now we must to explain when the existence of a vanishing cycle in some leaf \mathcal{L}_z of (X, \mathcal{L}) implies the existence of an *imbedded* one (in the same leaf). It occurs to depend on “integral” properties of our distinguished 3-form $d^c\omega$:

Theorem 3. *Let (X, \mathcal{L}, ω) be a pluritamed compact foliated manifold. Suppose additionally that:*

- i) *ω is a metric form;*
- ii) *$d^c\omega \in H^3(X, \mathbb{Z})$.*

If some leaf \mathcal{L}_{z^0} of (X, \mathcal{L}) contains a vanishing cycle then it contains also an imbedded vanishing cycle.

Let’s say a few more words about foliations with shells. First we remark that shells do come in families. Intuitively speaking we want to say that if our foliated manifold (X, \mathcal{L}) contains a foliated shell then it breaks into a complex $(\dim_{\mathbb{C}} X - 2)$ - parameter family of “foliated universes” each containing a foliated shell. More precisely, the following is true:

Proposition 3. *Let \mathcal{L} be a holomorphic foliation by curves on a compact manifold X of complex dimension $n \geq 3$ which admits a pluriclosed taming form. Suppose that (X, \mathcal{L}) contains a foliated shell $h : (B^\varepsilon, \pi) \rightarrow (X, \mathcal{L})$ (imbedded or immersed). Then there exists a smooth family $\{h_\lambda\}_{\lambda \in \Delta^{n-2}}$ of foliated shells containing h and transversal to \mathcal{L} in the sense that:*

- $h_0 = h$;
- $D_\lambda h_0(T_0 \Delta^{n-2}) \cap D_z h_0(T_z B^\varepsilon) = \{0\}$ for every $z \in B^\varepsilon$;

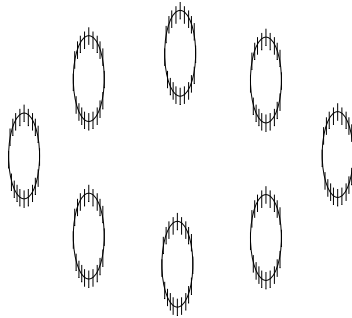


FIGURE 3. Shells persist transversally to \mathcal{L} : if there exists a foliated shell in (X, \mathcal{L}) then it is not disappearing: one can move it transversally to \mathcal{L} .

Such families of shells clearly come out in our proofs of Theorem 1 and 2. If a two-dimensional “foliated universe” is, moreover, compact then it can be listed explicitly. Namely, the remarkable result of Kato in [K2] (but even more the “pseudoconvex surgery” invented there) allows us to describe all possible pairs (X, \mathcal{L}) with compact complex surface X which contain vanishing cycles:

Corollary 3. *Let X be a compact complex surface and \mathcal{L} a (singular) holomorphic foliation by curves such that some leaf \mathcal{L}_{z_0} of \mathcal{L} contains a vanishing cycle γ_0 . Then X is:*

- i) *either a modification of a Hopf surface and \mathcal{L}_{z_0} is an elliptic curve (i.e., a torus);*
- ii) *or, modification of a Kato surface and the closure of \mathcal{L}_{z_0} is a rational curve.*

In both cases the foliated shell in question is either spherical or a (holomorphic) quotient of the standard \mathbb{S}^3 by $\Gamma_{l,d}$ for some l, d .

For the definition of a foliated spherical shell see Subsection 4.1 (in fact it means that as the boundary B one can take the standard sphere $\mathbb{S}^3 \subset \mathbb{C}^2$). This Corollary is the only formulation which we state for singular foliations in this Introduction. The reason is that the case ii) appears only for a singular \mathcal{L} .

1.5. Uniformizable foliations. For uniformizable foliations tamed by a pluriclosed form we expect more or less the same results as for foliations on compact Kähler (or algebraic) manifolds. Let's state some typical results.

Corollary 4. *Let \mathcal{L} be a holomorphic foliation by curves on a compact complex manifold X which admits a plurinegative taming form. Suppose that \mathcal{L} contains a leaf whose universal cover is $\mathbb{C}\mathbb{P}^1$. Then the universal cover of every leaf is $\mathbb{C}\mathbb{P}^1$ and, moreover, \mathcal{L} is a rational quasi-fibration.*

For the foliations on Kähler manifolds this result is proved in [Br2]. In Section 5 we prove also the following version of the Reeb stability theorem:

Proposition 4. *Let \mathcal{L} be a holomorphic foliation on by curves on a compact complex manifold X admitting a dd^c -negative taming form.*

- i) *If \mathcal{L} has a compact leaf with finite holonomy then all leaves of \mathcal{L} are compact with finite holonomy.*
- ii) *If every leaf of \mathcal{L} is compact then every leaf has finite holonomy. In that case there is an upper bound on volumes of leaves and the leaf space is Hausdorff.*

For the foliations on Kähler manifolds this result is well known, see [Ga, P].

Take a cycle γ on some leaf \mathcal{L}_z^0 . Following [LP], see also [Ly1], we define in Subsection 5.3 the domain of preservation of the homotopy class $[\gamma]$ and prove the following:

Proposition 5. *Let \mathcal{L} be a holomorphic foliation by curves on a compact complex manifold X admitting a pluriclosed taming form. Then:*

- 1) *either the domain of preservation Ω_γ is Hausdorff (and therefore is a complex manifold) for every loop γ ,*
- 2) *or (X, \mathcal{L}) contains a foliated shell.*

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2. PLURINEGATIVE TAMING FORMS AND FOLIATED IMMERSIONS

In this Section we shall prove the principal technical tool of the present paper - Theorem 2.2. From one side we need to make our estimates from [Iv3] “unparametric”. This task with respect to Kähler case of [Iv2] was fulfilled in [Br2]. From another side we need to learn to work with taming forms. These forms are positive only along directions tangent to \mathcal{L} in TX , and this is the source of additional difficulties.

2.1. Generalized Hartogs figures. Let us start with some definitions. By a *foliated manifold* in this paper we shall understand a pair (X, \mathcal{L}) , where X is a complex manifold (separable and countable at infinity) and \mathcal{L} is a singular holomorphic foliation by curves on X . One of the ways to define such \mathcal{L} is the following. Take a sufficiently fine open covering $\{\Omega_\alpha\}$ of X . Then \mathcal{L} will be defined by a holomorphic vector fields $v_\alpha \in \mathcal{O}(\Omega_\alpha, TX)$ which are related on a non-empty intersections $\Omega_{\alpha,\beta} := \Omega_\alpha \cap \Omega_\beta$ as $v_\alpha = h_{\alpha,\beta} v_\beta$. Here $h_{\alpha,\beta} \in \mathcal{O}^*(\Omega_{\alpha,\beta})$. After contracting the common factors one immediately sees that the singular set $\mathcal{L}^{\text{sing}}$ of \mathcal{L} , which is defined as $\mathcal{L}^{\text{sing}} = \{z : v_\alpha(z) = 0\}$, is an analytic subset of X of codimension at least two. Set $X^0 := X \setminus \mathcal{L}^{\text{sing}}$. The leaves of \mathcal{L} are, in the first approximation, defined as the leaves of the smooth foliation $\mathcal{L}^0 := \mathcal{L}|_{X^0}$, i.e., they are entirely off the singular set of \mathcal{L} . Then, depending on the someone goals, one adds to them some “ends”. We shall do that in the following Section.

A particular class of foliated manifolds are fibrations by curves, i.e., triples (W, π, V) where W is a complex manifold of dimension $\dim V + 1$ and $\pi : W \rightarrow V$ is a surjective holomorphic submersion with connected fibers. A holomorphic mapping $f : (X, \mathcal{L}) \rightarrow (X', \mathcal{L}')$ is said to be a foliated immersion if it is an immersion and sends fibers to fibers. In the case of fibrations, i.e., if $f : (W, \pi, V) \rightarrow (W', \pi', V')$, one can be more precise: there exists a holomorphic map $f_v : V \rightarrow V'$ such that for all $z \in V$ one has $f(W_z) \subset W'_{f_v(z)}$. Dimension of W' might be bigger then that of W . If $V' = V$ one often supposes also that W_z goes to W'_z for all $z \in V$. This will be clear from the context.

Definition 2.1. *A generalized Hartogs figure is a quadruple (W, π, U, V) , where W and V are connected complex manifolds, U an open subset of V and $\pi : W \rightarrow V$ is a holomorphic submersion such that:*

- i) for all $z \in V \setminus U$ the fiber $W_z = \pi^{-1}(z)$ is diffeomorphic to an annulus;
- ii) for $z \in U$ the fiber W_z is diffeomorphic to a disk.

Generalized Hartogs figures are foliated manifolds (even fibrations) of a special type: they are concave in the most naïve and clear sense. But, attention, they are very far from the standard Hartogs figures. We shall be required to say more about this in a while. Manifold W has a distinguished part of the boundary formed by the outer boundaries $\partial_0 W_z$ of annuli W_z . We shall suppose that W is smooth up to this part of its boundary and denote it by $\partial_0 W$, i.e., $\partial_0 W = \cup_{z \in V} \partial_0 W_z$. Projection π is also supposed to be smooth up to $\partial_0 W$ and therefore $\pi : \partial W_0 \rightarrow V$ is a circle fibration. For $z \in U$ the outer boundary $\partial_0 W_z$ is actually the boundary of the disk W_z .

Recall that the *standard Hartogs figure* is the open subset of \mathbb{C}^{n+1} of the form

$$H_\varepsilon = (\Delta_{1+\varepsilon}^n \times A_{1-\varepsilon, 1+\varepsilon}) \cup (\Delta_\varepsilon^n \times \Delta_{1+\varepsilon}) \quad (2.1)$$

for some $\varepsilon > 0$, see Fig. 4. H_ε likewise carries our “vertical foliation” \mathcal{L}^π . This time the leaves \mathcal{L}_z^π are disks $\Delta_{1+\varepsilon}$ if $\|z'\| < \varepsilon$ and annuli for $\varepsilon \leq \|z'\| < 1 + \varepsilon$. Here $z' = (z_1, \dots, z_n)$

and $\|\cdot\|$ is the polydisk-norm in \mathbb{C}^n . Remark now that $(H_\varepsilon, \mathcal{L}^\pi)$ fits, of course, into the Definition 2.1 with $V = \Delta_{1+\varepsilon}^n$, $U = \Delta_\varepsilon^n$ and π being the restriction of the canonical “vertical” projection $\mathbb{C}^{n+1} \rightarrow \mathbb{C}^n$ onto H_ε . Remark furthermore that the standard foliated shell is also a generalized Hartogs figure. Namely it can be viewed as $(B^\varepsilon, \pi, A_{1-\varepsilon, 1+\varepsilon}, \Delta_{1+\varepsilon})$.

Definition 2.2. *If $U = \emptyset$ we call (W, π, \emptyset, V) trivial, if $U = V$ we call (W, π, V, V) complete and in the latter case often denote it as (W, π, V) .*

The trivial figure W is diffeomorphic to the product of V with an annulus and complete figure to the product with a disk. The standard Hartogs figure is never trivial by definition, i.e., it is commonly accepted that always $\varepsilon > 0$. Let D be a non-empty open subset of V . Set $W|_D = \pi^{-1}(D)$ and consider it also as a generalized Hartogs figure $(W|_D, \pi|_D, D \cap U, D)$ - a subfigure of (W, π, U, V) . Moreover, if $S \subset V$ is a submanifold of V one can consider the restriction $(W|_S, \pi|_S, S \cap U, S)$ and it is again a generalized Hartogs figure. In the sequel we shall often avoid the word “generalized” and call our figures simply Hartogs figures, specifying over what V they are considered.

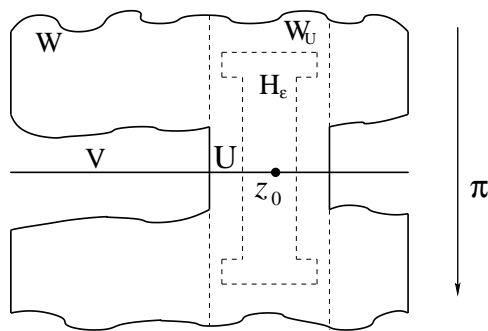


FIGURE 4. This is the standard Hartogs figure imbedded into a generalized Hartogs figure W constructed in [CI]. Every attempt to imbed H_ε into this W will look like on the picture: if the fiber over the origin in H_ε is mapped to a fiber over some point $z_0 \in U$ then the image of H_ε will never leave $W|_U$.

Remark 2.1. (a) The necessity of considering generalized Hartogs figures in foliation theory comes from the example constructed in [CI]. This example is explained on the Figure 4. Namely the following can happen. There exists a Hartogs figure W over a disk (i.e., both $U \subset V$ are disks in \mathbb{C}) with the following property. Whenever a holomorphic foliated imbedding $h : (z_1, z_2) \rightarrow (h_1(z_1), h_2(z_1, z_2))$ of H_ε into W is given such that $h_1(0) = z^0 \in U$ then necessarily $h_1(\Delta_{1+\varepsilon}) \subset U$ (whatever $\varepsilon > 0$ is). This fact is implicit in Lemma 1 of [CI]. It follows from the explicit form of the holomorphic function f_1 constructed there: its “pole set” (in some sense) is precisely $V \setminus U$.

(b) Another point is why one needs Hartogs figures at all? This will be clear from our proofs.

2.2. Extension after a reparametrization. The following notion comes back to [Ti], see also [Bl]. Let $f : A_{1-\varepsilon, 1} \rightarrow X$ be a holomorphic immersion.

Definition 2.3. *We say that f extends to Δ after a reparametrization if for some $\delta > 0$ there exists an imbedding $h : A_{1-\delta, 1} \rightarrow A_{1-\varepsilon, 1}$ sending $\partial\Delta$ to $\partial\Delta$ and preserving the canonical orientation of $\partial\Delta$, such that $f \circ h$ holomorphically extends to Δ .*

It is clear that such h , if exists, should be holomorphic. We shall use also the following form of this notion. Let γ be a simple oriented loop on a bordered Riemann surface W which is simply a collar adjacent to γ . Let $f : W \rightarrow X$ be a holomorphic immersion. Suppose that there exist a Riemann surface \tilde{W} which is a bordered disk with boundary $\tilde{\gamma}$ (canonically oriented) and a biholomorphic mapping h from a collar adjacent to $\tilde{\gamma}$ onto W (smooth up to the boundaries) and sending $\tilde{\gamma}$ onto γ preserving orientations such that the composition $f \circ h$ holomorphically extends onto the disk \tilde{W} . Then we shall say that f extends onto the disk \tilde{W} after a reparametrization. If such $\tilde{W}, \tilde{\gamma}$ and h do exist but are not specified we shall say simply that f holomorphically extends onto a disk after a reparametrization.

Remark 2.2. (a) There is one case when the extension of an immersion after a reparametrization may be not unique in the sense that there may not exist an automorphism φ of Δ such one extension is equal to the second composed with φ . Take a function $f(z) = 4z + \sqrt{z^2 - 1}$ and consider it as a holomorphic mapping from a thin annulus around $\partial\Delta(2)$ - the circle of radius 2, into $\mathbb{C}\mathbb{P}^1$. Then f has two extensions after a reparametrization :

1) An injective one. Indeed, f is an imbedding of $\partial\Delta(2)$ into $\mathbb{C}\mathbb{P}^1$ and therefore bounds a disk, say D . Let $r : \Delta(2) \rightarrow D$ be a Riemann mapping (it is biholomorphic in a neighborhoods of the close disks). Set $h = f^{-1} \circ r$ - a reparametrization of $\partial\Delta(2)$. Then $f \circ h = r$ is the extension of f onto $\Delta(2)$ after a reparametrization .

2) A non-injective one. This is given by the formula defining f . It has two ramification points ± 1 and extends onto the union $\Delta(2) \cup \mathbb{C}\mathbb{P}^1$ appropriately glued along the slit $[-1, 1]$. The Riemann surface obtained is again a disk. This second extension is non-injective.

(b) At the same time, if f was a *generic injection* (i.e. injective outside of a finite set) then its extension after reparametrization, which we also require to be a generic injection, is unique (if exists). Uniqueness means here up to a biholomorphic automorphism of the disk.

Now let's turn ourselves to the families of immersions.

Definition 2.4. A holomorphic mapping $f : (W, \pi, V) \rightarrow X$ of a fibration (W, π, V) into a complex manifold X is called *generically injective* if for all $z \in V$ outside of a proper analytic subset $A \subset V$ the restriction $f_z := f|_{W_z}$ is a generic injection.

Note that we do not ask f to be "generically injective" itself but only its restrictions onto "generic" fibers. Actually f may not be even an immersion. However in most cases mappings appearing in this paper will be both immersions and generic injections. We shall also need a corresponding notion for the meromorphic case.

Definition 2.5. A meromorphic mapping $f : W \rightarrow X$ between complex manifolds is a *meromorphic immersion* if it is an immersion outside of its indeterminacy set I_f . If, moreover, (W, π, V) is a holomorphic fibration then a meromorphic mapping f is called *generically injective* if $f|_{W_z}$ is a generic injection for z outside of a proper analytic subset of V .

Here and always in this paper writing $f(C)$ for some meromorphic map and some complex curve C we mean that the restriction $f|_C$ of f onto C is well defined (this means that C is not contained in the indeterminacy set of f) and $f(C)$ is actually $f|_C(C)$. Again we will mostly work with meromorphic maps which are both meromorphic immersions and generic injections.

Let a holomorphic generic injection $f : (W, \pi, U, V) \rightarrow X$ of a generalized Hartogs figure into a complex manifold X be given.

Definition 2.6. *We say that f meromorphically extends onto the complete Hartogs figure $(\tilde{W}, \tilde{\pi}, V)$ over (the same!) V after a reparametrization if there exists a foliated biholomorphism of trivial figures $h : (\partial_0 \tilde{W}, \tilde{\pi}, \emptyset, V) \rightarrow (\partial_0 W, \pi, \emptyset, V)$ (i.e., h is defined in a one-sided neighborhood of $\partial_0 \tilde{W}$ and $h(z)$ tends to $\partial_0 W$ when z tends to $\partial_0 \tilde{W}$) such that $f \circ h$ extends to a generically injective meromorphic map $\tilde{f} : (\tilde{W}, \tilde{\pi}, V) \rightarrow X$.*

Remark 2.3. (a) We say simply that f extends after a reparametrization if the data as in Definition 2.6 do exist (but, may be, are not specified). In that case we often omit tildes over the extended objects, such as W , π and f . I.e. we often say that f extends onto (W, π, V) after a reparametrization.

(b) Remark that if f extends as a meromorphic map being a generic injection on (W, π, U, V) with $U \neq \emptyset$ then its extension will be automatically a generic injection. However in the definition above we do not exclude the case when $U = \emptyset$.

Definition 2.7. *If for any point $z \in V$ there exists a neighborhood $V(z)$ such that the restriction $f|_{W|_{V(z)}}$ extends after a reparametrization onto a complete Hartogs figure $(\tilde{W}|_{V(z)}, \tilde{\pi}, V(z))$ over $V(z)$ then we say that f locally extends after a reparametrization.*

Let us be very precise at this point: by saying that the “restriction $f|_{W|_{V(z)}}$ extends” we mean here that one is taking the restriction of f onto the Hartogs subfigure $(W|_{V(z)}, \pi, V(z) \cap U, V(z))$ and this restriction extends in the sense of Definition 2.6. I.e., a reparametrization is supposed to be made near $\partial_0 W|_{V(z)}$ only.

If a generically injective (!) mapping f extends locally then it extends globally. Namely the following is true:

Lemma 2.1. *Let $V \supset V_1 \cup V_2$ with (may be empty) intersection $V_{12} := V_1 \cap V_2$. Let (W, π, \emptyset, V) be a generalized Hartogs figure over V and $f : W \rightarrow X$ be a generically injective holomorphic map into a complex manifold X such that f meromorphically extends onto a complete Hartogs figures $(W|_{V_k}, \pi, V_k)$ for $k = 1, 2$ after a reparametrization. Then f extends after a reparametrization onto a figure $(W, \pi, V_1 \cup V_2, V)$.*

Proof. *Step 1. Extending to $(W|_{V_1 \cup V_2}, \pi, V_1 \cup V_2)$.* Denote by $h_k : (\partial_0 \tilde{W}_k, \tilde{\pi}_k, \emptyset, V_k) \rightarrow (\partial_0 W|_{V_k}, \pi, \emptyset, V_k)$ the corresponding foliated biholomorphisms. The fact that f is a generic injection imply that $h_2^{-1} \circ h_1 = (f \circ h_2)^{-1} \circ (f \circ h_1) : \partial_0 \tilde{W}_{1,z} \rightarrow \partial_0 \tilde{W}_{2,z}$ extends for every $z \in V_{12}$ onto a corresponding disk and therefore extends to a foliated biholomorphism between complete figures $(\tilde{W}_1|_{V_{12}}, \tilde{\pi}_1, V_{12})$ and $(\tilde{W}_2|_{V_{12}}, \tilde{\pi}_2, V_{12})$. Therefore complete figures $(\tilde{W}_1, \tilde{\pi}_1, V_1)$ and $(\tilde{W}_2, \tilde{\pi}_2, V_2)$ glue together to a complete figure $(\tilde{W}_3, \tilde{\pi}, V_1 \cup V_2)$ and reparametrization maps h_k glue to a reparametrization map $h : (\partial_0 \tilde{W}_3, \tilde{\pi}, \emptyset, V) \rightarrow (\partial_0 W|_{V_1 \cup V_2}, \pi, \emptyset, V)$. Mapping f extends, after being reparametrized by h onto the complete figure $(W|_{V_1 \cup V_2}, \pi, V_1 \cup V_2)$.

Step 2. Extending to $(W, \pi, V_1 \cup V_2, V)$. Reparametrization $h : \partial_0 \tilde{W}_3 \rightarrow \partial_0 W|_{V_1 \cup V_2}$ constructed in the first Step aloud us to glue figures $(\tilde{W}_3, \tilde{\pi}, V_1 \cup V_2)$ and (W, π, \emptyset, V) together to a figure $(W, \pi, V_1 \cup V_2, V)$. In the case $V_1 \cap V_2 \neq \emptyset$ we are done.

When $V_1 \cap V_2 = \emptyset$ the first step is not needed and one should fulfill Step 2 twice.

□

From this lemma we obtain the following

Corollary 2.1. *Let $f : W \rightarrow X$ be a generically injective holomorphic immersion of a generalized Hartogs figure (W, π, U, V) into a complex manifold X . Then there exists a maximal open $U \subset \hat{U} \subset V$ such that f meromorphically extends onto the complete figure (W, π, \hat{U}) after a reparametrization .*

2.3. Key lemma. The following statement is the key lemma for this section. It replaces the Lemma 2.3 from [Iv3] for our “parametric” case. In fact it states even more general result that we need in this paper, see Remark 2.4. But we are including for the future references. We suppose that our complex manifold X is equipped with some Riemannian metric. The condition (i) in the following lemma, where this metric is used, doesn’t depend, in fact, on a particular choice of a metric.

Lemma 2.2. *Let $f : W \rightarrow X$ be a generically injective holomorphic map of a trivial Hartogs figure (W, π, \emptyset, V) into a complex manifold X . Suppose that $\dim V = 1$ and that for some sequence $z_n \rightarrow z_0 \in V$ restrictions $f|_{W_{z_n}} : W_{z_n} \rightarrow X$ holomorphically extend to a generic injections onto a disks \tilde{W}_{z_n} after a reparametrization. Suppose additionally that:*

- i) $\text{area} \left(\tilde{f}|_{\tilde{W}_{z_n}}(\tilde{W}_{z_n}) \right)$ is uniformly bounded;
- ii) $\tilde{f}|_{\tilde{W}_{z_n}}(\tilde{W}_{z_n})$ stay in some compact of X .

Then there exists a neighborhood $D \ni z_0$ such that f extends meromorphically onto a figure (\tilde{W}, π, D, V) after a reparametrization. Moreover, the extension \tilde{f} will be a generically injective meromorphic map.

Proof. Writing $\tilde{f}|_{\tilde{W}_{z_n}}$ in the statement of this lemma we mean that for every n a reparametrization map $h_{z_n} : \partial\tilde{W}_{z_n} \rightarrow \partial W_{z_n}$ is given such that $\tilde{f}|_{\tilde{W}_{z_n}} := f|_{W_{z_n}} \circ h_{z_n}$ extends as a (not necessarily generically injective) holomorphic map to the disk \tilde{W}_{z_n} . The proof will use in a crucial way the description of convergence of analytic disks obtained in [IS2] and the structures of Banach neighborhoods of stable curves obtained in [IS1].

Set $f_{z_n} = f|_{\tilde{W}_{z_n}}$ and consider them as complex disks over X , parameterized by a fixed disk Σ (see §3 from [IS1] or §2 from [IS2] for exact definitions). Applying Theorem 1 from [IS2] we can find a subsequence from $\{f_{z_n}\}$ that converge in the sense of Definition 2.5 from [IS2] to a stable curve f_0 over X , parameterized again by a disk. Be careful, this $f_0(\Sigma)$ may have compact components.

By Theorem 3.4 from [IS1] the space of disks over X which are close to f_0 is a Banach analytic set of finite codimension. Denote it by \mathcal{C} . By the Theorem of Ramis, see [Ra], \mathcal{C} is the union of finitely many irreducible components \mathcal{C}_j and each \mathcal{C}_j is a finite ramified covering over a Banach ball. Take a component which contains infinitely many of f_{z_n} -s. In order not to complicate our notations we suppose that \mathcal{C} is irreducible itself and contains all f_{z_n} .

For the sequel it is important to understand how \mathcal{C} was constructed in [IS1]:

1) The parameterizing disk Σ is covered by finite number of disks, annuli and pants Σ_α (the boundary annulus is one of them, denote it as Σ_{α_0}). This covering has that property that each intersecting pair $\Sigma_\alpha, \Sigma_\beta$ intersect by an annulus denoted as $\Sigma_{\alpha,\beta}$.

2) For each α a Banach manifold H_α of holomorphic maps from Σ_α to X is considered. The same type manifolds $H_{\alpha,\beta}$ of holomorphic maps $\Sigma_{\alpha,\beta} \rightarrow X$ for intersecting Σ_α and Σ_β are considered.

3) For every pair of intersecting Σ_α and Σ_β a transition map $\psi_{\alpha,\beta} : H_\alpha \times H_\beta \rightarrow H_{\alpha,\beta}$ is defined.

Now \mathcal{C} comes out as the zero set of $\Psi = \{\psi_{\alpha,\beta}\} : \bigcup_\alpha H_\alpha \rightarrow \bigcup_\alpha H_{\alpha,\beta}$. By construction \mathcal{C} restricts as a Banach analytic subset to each of H_α .

All what is left to do is to replace H_{α_0} (the manifold of maps from the annulus adjacent to the boundary) by a 1-dimensional manifold $\mathcal{W} := \{f|_{W_z} : z \text{ in a neighborhood of } z_0\}$ (W_z is an annulus!). The obtained Banach analytic set, we still denote it as \mathcal{C} , is of finite dimension (the proof goes along the same lines as the proof of Lemma 1.1 from [Iv3]). In fact it is clearly of dimension not more than one. But since it contains the sequence $\{f|_{\tilde{W}_{z_n}}\}$ its dimension is actually one. Therefore \mathcal{C} is a usual analytic set by Barlet-Mazet theorem, [M], i.e., is a complex curve in our case. Restriction $\mathcal{C} \rightarrow \mathcal{W}$ is an analytic map and it is *proper* (!), because a nondegenerate analytic maps between complex curves are always proper. Therefore its image is the whole \mathcal{W} . We get an extension \tilde{f}_z for all z close to z_0 as a family by a tautological map $\tilde{f} : \tilde{\mathcal{W}} \rightarrow X$. Here $\tilde{\mathcal{W}}$ is a tautological family of disks over \mathcal{W} . □

Remark 2.4. (a) An analogous statement can be proved also in the case $\dim V \geq 2$, but then one should require the boundedness of rational cycle geometry of X as in [Iv3] (only cycles tangent to \mathcal{L} are relevant). We shall do this later, see Lemma 2.6.

(b) One can seriously simplify the proof of this lemma if one imposes *ad hoc* the condition that (some subsequence of) the sequence $\{\tilde{f}|_{\tilde{W}_{z_n}}(\tilde{W}_{z_n})\}$ converges to f_0 without bubbles. The proof is then almost immediate, since then only one Banach manifold H_0 appears (no Banach analytic sets), that of deformations of f_0 and it has dimension at least one because it contains a sequence.

2.4. Two-dimensional case. Recall that a complex manifold X is called disk-convex if for any compact $K \subset X$ there exists a compact \hat{K} in X such that for any holomorphic map $\varphi : \bar{\Delta} \rightarrow X$ such that $\varphi(\partial\Delta) \subset K$ one has $\varphi(\bar{\Delta}) \subset \hat{K}$. Let's adapt this notion to the foliation theory:

Definition 2.8. A complex foliated manifold (X, \mathcal{L}) is called disk-convex if for any compact $K \subset X$ there exists a compact $\hat{K}_{\mathcal{L}}$ in X such that for any holomorphic map $\varphi : \bar{\Delta} \rightarrow X$ tangent to \mathcal{L} and such that $\varphi(\partial\Delta) \subset K$ one has $\varphi(\bar{\Delta}) \subset \hat{K}_{\mathcal{L}}$.

A holomorphic mapping $\varphi : \bar{\Delta} \rightarrow X$ is called tangent to \mathcal{L} if it takes (almost all) its values in some leaf of \mathcal{L} . Note that for disk-convex (X, \mathcal{L}) and foliated mappings $f : (W, \pi, \emptyset, V) \rightarrow (X, \mathcal{L})$ the condition (ii) in Lemma 2.2 is satisfied automatically.

Let ω be a (1,1)-form on X .

Definition 2.9. We call ω plurinegative (dd^c -negative) if $dd^c\omega \leq 0$. We call ω pluriclosed (dd^c -closed) if $dd^c\omega = 0$.

Denote by $\mathcal{E}_{\mathbb{R}}^{p,p}$ the sheaf of smooth real (p,p) -forms and $\mathcal{E}^{p,g}$ the sheaf of smooth complex valued (p,q) -forms on X . Likewise by $\mathcal{E}_{p,q}$ we denote the dual to $\mathcal{E}^{p,g}$ space of currents of bidimension (p,q) and by $\mathcal{E}_{p,p}^{\mathbb{R}}$ the space of real currents of bidimension (p,p) .

Fix some strictly positive (1,1)-form Ω on X . Given a holomorphic foliation by curves \mathcal{L} on X define the following convex compact $K_{1,1}(\mathcal{L}) \subset \mathcal{E}_{1,1}^{\mathbb{R}}(X)$. For every point $z \in X^0$ take a (1,1)-vector $\frac{i}{2}v \wedge \bar{v}$ tangent to \mathcal{L}_z^0 such that $\langle \frac{i}{2}v \wedge \bar{v}, \Omega \rangle = 1$.

For $z \in \mathcal{L}^{\text{sing}}$ take any sequence $z_n \rightarrow z$, $z_n \in X^0$ and any $\frac{i}{2}v_n \wedge \bar{v}_n$ tangent to $\mathcal{L}_{z_n}^0$ such that $\langle \frac{i}{2}v_n \wedge \bar{v}_n, \Omega \rangle = 1$ for all n . Subtract a converging subsequence from $\frac{i}{2}v_n \wedge \bar{v}_n$ and denote by $\frac{i}{2}v_0 \wedge \bar{v}_0$ its limit. In this way we obtain all positive bidimension $(1, 1)$ δ -currents tangent to \mathcal{L} . $K_{1,1}(\mathcal{L})$ is the closure of the convex hull of these δ -currents.

Definition 2.10. A $(1, 1)$ -form ω we call a taming form for \mathcal{L} if $\langle T, \omega \rangle > 0$ for every $T \in K_{1,1}(\mathcal{L})$.

Lemma 2.3. Let (X, \mathcal{L}) be a disk-convex foliated complex manifold which admits a dd^c -negative taming form ω and let $f : (W, \pi, \emptyset, V) \rightarrow (X, \mathcal{L})$ be a generically injective foliated holomorphic mapping. Suppose that $\dim V = 1$ and that for some sequence $z^n \rightarrow z^0 \in V$ restrictions $f_n := f|_{W_{z^n}}$ extend onto disks \tilde{W}_{z^n} after a reparametrization as a generic injections $\tilde{f}|_{\tilde{W}_{z^n}}$ and that conditions (i) , (ii) of Lemma 2.2 are satisfied. Then mapping f extends after a reparametrization as a generically injective meromorphic map onto a complete Hartogs figure (W, π, V) over V minus a closed "essential singularity" set S of the form $S = \bigcup_{z \in S_1} S_z$, where S_1 is a closed complete polar subset of V and S_z for every $z \in S_1$ is a compact in the disk W_z .

Remark 2.5. Condition about a common point is always satisfied if f was defined and immersive on (W, π, U, V) with not empty U .

Proof. Denote by \hat{U} the maximal open subset of V such that f meromorphically extends onto a complete Hartogs figure (W, π, \hat{U}, V) after a reparametrization. By Lemma 2.2 we know that $\hat{U} \neq \emptyset$. We are going to prove that $\partial \hat{U} \cap V$ is a closed complete polar subset of V .

For points $z \in \hat{U}$ the following area function is well defined:

$$a(z) = \text{area}(f|_{W_z}(W_z)), \quad (2.2)$$

and it doesn't depend on a reparametrization. We dropped tildes everywhere. Here $f|_{W_z}$ is the restriction of the meromorphic mapping f to the disk W_z , which therefore a holomorphic mapping from this disk to X . If we denote by T the current $f^*\omega$ then (2.2) writes as

$$a(z) = \int_{W_z} T. \quad (2.3)$$

Step1. Laplacian of the area function. After shrinking W , if necessary, we can suppose that f is defined and smooth up to $\partial_0 W$. Take a disk $D \subset \hat{U}$ such that f extends after a reparametrization to a complete figure $(W|_D, \pi, D)$ over D and such that in $W|_D$ one can introduce global coordinates $z_1 = z$ and z_2 - a vertical coordinate. This clearly can be done if D is taken small enough. In these coordinates we can write $T = \frac{i}{2}t^{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$ and therefore (2.3) becomes

$$a(z_1) = \frac{i}{2} \int_{W_{z_1}} t^{2\bar{2}}(z_1, z_2) dz_2 \wedge d\bar{z}_2. \quad (2.4)$$

Remark that T is plurinegative as well as ω and this condition writes as

$$\frac{\partial^2 t^{1\bar{1}}}{\partial z_2 \partial \bar{z}_2} + \frac{\partial^2 t^{2\bar{2}}}{\partial z_1 \partial \bar{z}_1} - \frac{\partial^2 t^{1\bar{2}}}{\partial z_2 \partial \bar{z}_1} - \frac{\partial^2 t^{2\bar{1}}}{\partial z_1 \partial \bar{z}_2} \leq 0. \quad (2.5)$$

Now we can rewrite the Laplacian of a in D as follows: $\Delta a(z_1) =$

$$\begin{aligned} &= 2i \int_{W_{z_1}} \frac{\partial^2 t^{2\bar{2}}}{\partial z_1 \partial \bar{z}_1} dz_2 \wedge d\bar{z}_2 \leq 2i \int_{W_{z_1}} \left(-\frac{\partial^2 t^{1\bar{1}}}{\partial z_2 \partial \bar{z}_2} + \frac{\partial^2 t^{1\bar{2}}}{\partial z_2 \partial \bar{z}_1} + \frac{\partial^2 t^{2\bar{1}}}{\partial z_1 \partial \bar{z}_2} \right) dz_2 \wedge d\bar{z}_2 = \\ &= 2i \int_{\partial W_{z_1}} \frac{\partial t^{1\bar{1}}}{\partial z_2} dz_2 + 2i \int_{\partial W_{z_1}} \frac{\partial t^{1\bar{2}}}{\partial \bar{z}_1} d\bar{z}_2 - 2i \int_{\partial W_{z_1}} \frac{\partial t^{2\bar{1}}}{\partial z_1} dz_2 =: \psi(z_1). \end{aligned} \quad (2.6)$$

Remark 2.6. One might think that ψ , defined by the right hand side of (2.6) is likely to be a smooth function on the whole of V . But it is not (!), it simply doesn't exist on V because to define it one would need global coordinates z_1, z_2 in the region where one conducts the computations. An example in [CI] explained on the Figure 4 precisely means that there could be no such global coordinates over the whole of V (or even over a given relatively compact part of it). Therefore the arguments from [Lv3] cannot be directly applied here. However, we shall prove now that the integral of ψ (but not ψ itself!) is independent of a reparametrization.

Step 2. Integral of ψ is independent of a reparametrization .

By reparametrizations we mean here the reparametrizations in a neighborhood of $\partial_0 W$ over the whole of V , where a and Δa are not defined. For this remark that reparametrizations are coordinate changes of the form

$$\tilde{z}_1 = z_1, \quad \tilde{z}_2 = h(z_1, z_2). \quad (2.7)$$

Write

$$\begin{aligned} d^c T &= i(\partial - \bar{\partial})T = i\left(\frac{\partial t^{1\bar{1}}}{\partial z_2} dz_2 \wedge dz_1 \wedge d\bar{z}_1 - \frac{\partial t^{1\bar{1}}}{\partial \bar{z}_2} d\bar{z}_2 \wedge dz_1 \wedge d\bar{z}_1 + \frac{\partial t^{1\bar{2}}}{\partial z_2} dz_2 \wedge dz_1 \wedge d\bar{z}_2 - \right. \\ &\quad \left. - \frac{\partial t^{1\bar{2}}}{\partial \bar{z}_1} d\bar{z}_1 \wedge dz_1 \wedge d\bar{z}_2 + \frac{\partial t^{2\bar{1}}}{\partial z_1} dz_1 \wedge dz_2 \wedge d\bar{z}_1 - \frac{\partial t^{2\bar{1}}}{\partial \bar{z}_2} d\bar{z}_2 \wedge dz_2 \wedge d\bar{z}_1 + \right. \\ &\quad \left. + \frac{\partial t^{2\bar{2}}}{\partial z_1} dz_1 \wedge dz_2 \wedge d\bar{z}_2 - \frac{\partial t^{2\bar{2}}}{\partial \bar{z}_1} d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2\right) \end{aligned} \quad (2.8)$$

Take a relatively compact disk D in V such that the trivial Hartogs figure $(W|_D, \pi, \emptyset, D)$ possesses global coordinates z_1, z_2 and integrate (2.8) over the distinguished part $\partial_0 W|_D$ of the boundary of the cylinder $W|_D$. When integrating over $\partial_0 W_D$ one takes into account that terms containing $dz_2 \wedge d\bar{z}_2$ vanish. Therefore

$$\begin{aligned} \int_{\partial_0 W_D} d^c T &= i \int_{\partial_0 W_D} \left(\frac{\partial t^{1\bar{1}}}{\partial z_2} dz_2 \wedge dz_1 \wedge d\bar{z}_1 - \frac{\partial t^{1\bar{1}}}{\partial \bar{z}_2} d\bar{z}_2 \wedge dz_1 \wedge d\bar{z}_1 - \right. \\ &\quad \left. - \frac{\partial t^{1\bar{2}}}{\partial \bar{z}_1} d\bar{z}_1 \wedge dz_1 \wedge d\bar{z}_2 + \frac{\partial t^{2\bar{1}}}{\partial z_1} dz_1 \wedge dz_2 \wedge d\bar{z}_1 \right) = \\ &= \int_D \psi(z_1) dz_1 \wedge d\bar{z}_1 - i \int_{\partial_0 W_D} \frac{\partial t^{1\bar{1}}}{\partial \bar{z}_2} d\bar{z}_2 \wedge dz_1 \wedge d\bar{z}_1. \end{aligned} \quad (2.9)$$

Here ψ is just the expression (2.6) and remark that it is obviously defined in coordinates z_1, z_2 which are valid only near $\partial_0 W|_D$. Note that the function $\varphi(z_1) := i \int_{\partial_0 W_{z_1}} \frac{\partial t^{1\bar{1}}}{\partial \bar{z}_2} d\bar{z}_2$

is invariant with respect to coordinate changes (2.7). Therefore $i \int_D \psi(z_1) dz_1 \wedge d\bar{z}_1 = \int_{\partial_0 W_D} d^c T - \int_D \varphi(z_1) dz_1 \wedge d\bar{z}_1$ is invariantly defined and admits the following bound:

$$i \int_D \psi(z_1) dz_1 \wedge d\bar{z}_1 \leq C \text{area}(D), \quad (2.10)$$

where constant C does not depend on D if it is contained in some fixed relatively compact subdomain $V_1 \Subset V$. Now the bound (2.10) implies the bound of the corresponding integral of the Laplacian over the part of $D \cap \hat{U}$ where the last is defined, *i.e.*, we get

$$i \int_{D \cap \hat{U}} \Delta a(z_1) dz_1 \wedge d\bar{z}_1 \leq C \text{area}(D \cap \hat{U}). \quad (2.11)$$

Step 3. Polarity of $\partial \hat{U}$.

Set $A(z_1) = a(z_1) - 4C|z_1|^2$. Then A is bounded from below in $\hat{U} \cap V_1$ and

$$\int_{\Delta} \Delta A(z_1) dz_1 \wedge d\bar{z}_1 \leq 0 \quad (2.12)$$

for every sufficiently small disk Δ in $\hat{U} \cap V_1$. From the Vitali theorem, see [F] p.151, we get that the measure ΔA is non-positive and therefore that A is superharmonic.

All what is left to do is to remark that $a(z_1)$ and therefore $A(z_1)$ tend to $+\infty$ when $z_1 \rightarrow \partial \hat{U} \cap V_1$ by Lemma 2.2. Therefore, as in the proof of Lemma 2.4 from [Iv3], one concludes that $\partial \hat{U} \cap V_1$ is complete polar in V_1 . Since V_1 was arbitrary $\partial \hat{U} \cap V$ is complete polar in V .

Denote by $(\tilde{W}, \tilde{\pi}, \tilde{U})$ the figure and by \tilde{f} the extensions obtained (this time we distinguish f from its extension after a reparametrization \tilde{f}). Note that for $z_1 \in \hat{U}$ the fiber W_z is a disk, but for $z_1 \in \partial \hat{U} \cap V$ it is an annulus. By the standard "rationalization trick", see Lemma 2.1 in [Br2] for details, W can be realized, preserving the fibration, as a fibred manifold over V with all fibers being disks. Here one essentially uses the polarity of $V \setminus \hat{U}$. \square

Remark 2.7. One can modify the proof of Lemma 2.3 along the lines of [Iv2] and then the Remark 2.4 would be sufficient. We shall not do that.

2.5. Condition $\int d^c T = 0$ and foliated shells. In this subsection we suppose that f is already extended onto $W \setminus S$, see Lemma 2.3. The polar set $\partial \hat{U} \cap V$ we denote as S_1 . Therefore the "essential singularity" set S of the extended map is actually $S = \bigcup_{z \in S_1} S_z$. Now we shall see how this leads to a foliated shell. We suppose therefore that S is not empty. Take a point $s_0 \in S_1$.

Restrict V to a small neighborhood of s_0 such that $\partial V \cap S_1 = \emptyset$. Shrinking W we can suppose that $(W, \pi, V) = (\Delta^2, \pi, \Delta)$ where $\pi : (z_1, z_2) \rightarrow z_1$ is the vertical projection. We can also suppose that $s_0 = 0$.

In (2.9) we had computed $\int_{\partial_0 W|_V} d^c T$. Let us do that over an another piece of the boundary. Take a relatively compact disk $D \Subset \Delta$ such that $\partial D \cap S_1 = \emptyset$.

Lemma 2.4. *One has*

$$\int_{W|_{\partial D}} d^c T = i \int_{\partial_0 W|_{\partial D}} t^{2\bar{1}} d\bar{z}_1 \wedge dz_2 - i \int_{\partial_0 W|_{\partial D}} t^{1\bar{2}} dz_1 \wedge d\bar{z}_2 - i \int_{\partial D} d_{z_1}^c a(z_1). \quad (2.13)$$

Proof. Again use (2.8), but this time take into account that integrals of all terms which contain $dz_1 \wedge d\bar{z}_1$ vanish:

$$\begin{aligned}
\int_{W|_{\partial D}} d^c T &= i \int_{W|_{\partial D}} \frac{\partial t^{1\bar{2}}}{\partial z_2} dz_2 \wedge dz_1 \wedge d\bar{z}_2 + i \int_{W|_{\partial D}} \frac{\partial t^{2\bar{1}}}{\partial \bar{z}_2} d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 + \\
&+ i \int_{W|_{\partial D}} \frac{\partial t^{2\bar{2}}}{\partial z_1} dz_1 \wedge dz_2 \wedge d\bar{z}_2 - i \int_{W|_{\partial D}} \frac{\partial t^{2\bar{2}}}{\partial \bar{z}_1} d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 = -i \int_{W|_{\partial D}} d_{z_2} \left(t^{1\bar{2}} dz_1 \wedge d\bar{z}_2 \right) + \\
&+ i \int_{W|_{\partial D}} d_{z_2} \left(t^{2\bar{1}} d\bar{z}_1 \wedge dz_2 \right) + \int_{\partial D} (\partial_{z_1} - \bar{\partial}_{z_1}) i \int_{W_{z_1}} t^{2\bar{2}} dz_2 \wedge d\bar{z}_2 = \\
&= i \int_{\partial_0 W|_{\partial D}} t^{2\bar{1}} d\bar{z}_1 \wedge dz_2 - i \int_{\partial_0 W|_{\partial D}} t^{1\bar{2}} dz_1 \wedge d\bar{z}_2 - i \int_{\partial D} d_{z_1}^c \int_{W_{z_1}} t^{2\bar{2}} dz_2 \wedge d\bar{z}_2. \tag{2.14}
\end{aligned}$$

□

Lemma 2.5. *Suppose that for a relatively compact disk $D \Subset \Delta$ such that $\partial D \cap S_1 = \emptyset$ one has*

$$\int_{\partial W|_D} d^c T = 0. \tag{2.15}$$

Then f meromorphically extends onto $W|_D$.

Proof. Note that if the taming form ω is pluriclosed inequalities (2.5) and (2.6) become an equalities. Therefore $\Delta a(z_1)$ smoothly extends onto Δ_{z_1} (and a is positive!). Subtracting from a an appropriate smooth function b in Δ_{z_1} we get that $a - b$ is harmonic on $\Delta_{z_1} \setminus S_1$ with S_1 complete polar and is locally bounded from below in Δ_{z_1} . Therefore $a - b$ extends to a superharmonic function on Δ_{z_1} . In particular $\Delta(a - b)$ is a negative measure supported on S_1 .

From (2.4) for every disk $D \Subset \Delta_{z_1}$ as above we get:

$$\int_{\partial D} d_{z_1}^c a(z_1) = \int_{\partial_0 W|_{\partial D}} t^{2\bar{1}} d\bar{z}_1 \wedge dz_2 - \int_{\partial_0 W|_{\partial D}} t^{1\bar{2}} dz_1 \wedge d\bar{z}_2 + i \int_{\partial_0 W_D} d^c T. \tag{2.16}$$

The right hand side of (2.16) is the integral of smooth forms in $\partial_0 W_D$ and therefore from Stokes formula it follows that for every compact subdisk $D_r \Subset \Delta_{z_1}$ as above one has

$$\left| -i \int_{D_r} \Delta a(z_1) dz_1 \wedge d\bar{z}_1 \right| = \left| - \int_{D_r} dd_{z_1}^c a(z_1) \right| \leq Cr^2, \tag{2.17}$$

where r is the radius of D_r . Δb trivially possesses such an estimate and therefore

$$-i \int_{D_r} \Delta(a - b) dz_1 \wedge d\bar{z}_1 \leq Cr^2. \tag{2.18}$$

Therefore $-\Delta a \in L_{loc}^\infty \subset L_{loc}^p$ for all $p < \infty$. By the standard elliptic regularity $a \in L_{loc}^{2,p}$ for all $p < \infty$. By Sobolev imbedding theorem $a \in \mathcal{C}^{1,\alpha}$ with $\alpha = 1 - \frac{2}{p}$. In particular, the area function a is locally bounded in D .

Lemma 2.2 implies now the extendibility of f onto $W|_D$.

□

We conclude with the following

Corollary 2.2. *If in the conditions of Lemma 2.3 f is additionally supposed to be an immersion and the singularity set S of the extended mapping is non-empty then (X, \mathcal{L}) contains a foliated shell.*

Really, the extended mapping $\tilde{f} : \tilde{W} \setminus S \rightarrow X$ might fell to be an immersion in this case only on a discrete subset of $\tilde{W} \setminus (S \cup I_{\tilde{f}})$. We add this subset to S together with indeterminacy set $I_{\tilde{f}}$ of \tilde{f} to get \tilde{S} . The projection $\tilde{S}_1 = \pi(\tilde{S})$ will stay complete polar. Let $0 \in S$. Take a small disk Δ_r around 0 in such a way that $\partial\Delta_r \subset V \setminus \tilde{S}_1$ and such that $\tilde{W}|_{\Delta_r}$ is a bidisk (after a slight shrinking of its outer boundary $\partial_0\tilde{W}|_{\Delta_r}$), i.e., $\tilde{W}|_{\Delta_r} = \Delta_r \times \Delta$ as foliated manifolds. In $\tilde{W}|_{\Delta_r}$ take a bidisk $P = \Delta_r \times \Delta_{1-\varepsilon}$ for $\varepsilon > 0$ small enough to insure the immersivity of \tilde{f} near the boundary B of P . Lemma 2.5 says now that $\int_{\tilde{f}(B)} \omega \neq 0$, i.e., we got a foliated shell.

Remark that we also proved the Proposition 1 from the Introduction:

Corollary 2.3. *A generic holomorphic injection $h : (B^\varepsilon, \mathcal{L}^\pi) \rightarrow (X^0, \mathcal{L}^0, \omega)$ into a disk-convex pluritamed foliated manifold defines a foliated shell if and only if it is an immersion and*

$$\int_{h(B)} d^c \omega \neq 0. \quad (2.19)$$

2.6. Nonparametric extension in all dimensions. Let $n \geq 1$ be the dimension of the base V .

Theorem 2.1. *Let (X, \mathcal{L}) be a disk-convex foliated manifold admitting a plurinegative taming form ω and let $f : (W, \pi, U, V) \rightarrow (X, \mathcal{L})$ be a generically injective foliated holomorphic map from a non-trivial Hartogs figure (i.e., $U \neq \emptyset$) into X . Then f extends after a reparametrization to a foliated meromorphic map $\tilde{f} : (\tilde{W}, \tilde{\pi}, V) \setminus S \rightarrow X$ of a complete Hartogs figure minus a closed subset S of the form $S = \cup_{z \in S_1} S_z$. Here:*

(a₁) S_1 is a complete $(n-1)$ -polar subset of V of Hausdorff dimension $2n-2$.

(a₂) S_z is a compact in the disk of \tilde{W}_z for every $z \in S_1$.

(a₃) If $\dim W = \dim X$ and f was an immersion then the extended map \tilde{f} is a meromorphic immersion.

$(n-1)$ -polarity of S_1 means that every point $0 \in S_1$ admits a neighborhood $\Delta^n = \Delta^{n-1} \times \Delta$ with coordinates (λ, z_1) such that for every λ the disk $\Delta_\lambda := \{\lambda\} \times \Delta$ intersects S_1 by a complete polar compact set. Hausdorff zero-dimensionality of $\Delta_\lambda \cap S_1$ follows. For the purposes of this paper we will need the following, more precise version for pluriclosed taming forms. Tildes over the extended objects will be omitted.

Theorem 2.2. *Under the conditions of Theorem 2.1 suppose additionally that ω is pluriclosed. If the singular set S is non-empty then it has the following structure:*

(b₁) For every point $s^0 \in S$ one can find a coordinate n -disk $\Delta^n \ni 0 = \pi(s^0) \in S_1$ and a neighborhood P of S_0 in W biholomorphic to $\Delta^n \times \Delta$ and such that $\pi|_P \rightarrow \Delta^n$ is the vertical projection $\Delta^n \times \Delta \rightarrow \Delta^n$.

(b₂) In its turn Δ^n can be presented as $\Delta^n = \Delta^{n-1} \times \Delta$ with coordinates (λ, z_1) in such a way that the restriction to $S \cap \Delta^{n+1}$ of an another vertical projection $\pi_2 : \Delta^{n+1} \rightarrow \Delta^{n-1}$ such that $(\lambda, z_1, z_2) \rightarrow \lambda$ is proper and surjective (z_2 will denote the coordinate along the fiber of π). In another words for every $\lambda \in \Delta^{n-1}$ the intersection $S_\lambda := \Delta_\lambda^2 \cap S$ is nonempty.

(b₃) For every $\lambda \in \Delta^{n-2}$ let $B_\lambda = \partial\Delta_\lambda^2$. Then $f(\partial B_\lambda)$ is not homologous to zero in X , i.e., it is a foliated shell in (X, \mathcal{L}) provided that f was in addition an immersion.

Here by $\Delta_{\lambda^n, z_1^n}$ we mean the vertical leaf $\{(\lambda^n, z_1^n)\} \times \Delta$ of W . In all applications S will be supposed to be minimal closed such that f extends onto $W \setminus S$. The proof is not a direct generalization of two-dimensional case. First of all we need to introduce one object relevant to a complex foliated manifold (X, \mathcal{L}) . Denote by $\mathcal{R}_\mathcal{L}$ the analytic space of rational cycles on X tangent to \mathcal{L} . Recall that a rational cycle is a finite linear combination of rational curves with integer coefficients: $C = \sum_j n_j C_j$. Here each C_j is a rational curve in X . We fix a Hermitian metric on X and denote by ω its associated (1,1)-form. The area of C is defined as

$$v_\omega(C) = \sum_j n_j \int_{C_j} \omega. \quad (2.20)$$

Definition 2.11. Let us say that (X, \mathcal{L}) has unbounded rational cycle geometry if there exists a path $\gamma : [0, 1[\rightarrow \mathcal{R}_\mathcal{L}$ such that

- 1) $C_{\gamma(t)}$ stays in some compact K of X for all $t \in [0, 1[$;
- 2) $v_\omega(C_{\gamma(t)}) \rightarrow +\infty$ when $t \nearrow 1$.

Here $C_{\gamma(t)}$ is the rational cycle in X corresponding to the point $\gamma(t) \in \mathcal{R}_\mathcal{L}$. This notion doesn't depend on the particular choice of ω and presents a pure a complex-geometric property of (X, \mathcal{L}) .

In Lemma 5.1 we shall prove that if (X, \mathcal{L}) admits a plurinegative taming form then the rational cycle geometry of (X, \mathcal{L}) is bounded. Recall finally, that a subset A of a complex manifold V is said to be *thick* at the point $z^0 \in V$ if for any neighborhood U of z^0 $A \cap U$ is not contained in a proper analytic subset of U . Now we can state the needed lemma:

Lemma 2.6. Let $f : W \rightarrow X$ be a generically injective foliated holomorphic mapping of a trivial Hartogs figure (W, π, \emptyset, V) into a complex foliated manifold (X, \mathcal{L}) . Suppose that $\dim V \geq 2$ and that for all z in some subset $A \subset V$ thick at z^0 all restrictions $f|_{W_z} : W_z \rightarrow X$ holomorphically extend onto disks \tilde{W}_z after a reparametrization. Suppose additionally that:

- i) $\text{area}(\tilde{f}|_{\tilde{W}_z}(\tilde{W}_z))$ is uniformly bounded for $z \in A$;
- ii) $\tilde{f}|_{\tilde{W}_z}(\tilde{W}_z)$ stay in some compact K of X ;
- iii) (X, \mathcal{L}) has bounded rational cycle geometry.

Then there exists a neighborhood $D \ni z^0$ such that f extends meromorphically onto a complete Hartogs figure (\tilde{W}, π, D) after a reparametrization.

Proof. We keep the notations used in the proof of Lemma 2.2. Only for the annulus Σ_{α_0} adjacent to the boundary of the disk Σ the manifold $\mathcal{W} := \{f|_{\tilde{W}_z} : z \text{ in a neighborhood of } z^0\}$ now has dimension $n = \dim V \geq 2$.

Let $\nu > 0$ be the minimum of areas of rational curves tangent to \mathcal{L} and contained in the compact K . We divide A into a finite union of increasing closed subsets: $A_1 \subset \dots \subset A_k \subset \dots \subset A_K = A$ where $A_k = \{z \in A : \text{area}_\omega(\tilde{f}|_{\tilde{W}_z}) \leq k \frac{\nu}{2}\}$. For some k the set $A_k \setminus A_{k-1}$ is thick at origin. In the sequel we take it as A . As a result every converging sequence $\{\tilde{f}|_{\tilde{W}_{z_n}} : z_n \in A, z_n \rightarrow z^0\}$ has the same limit. Really two different limits should differ by a rational cycle. Therefore their areas should differ at least by ν . Contradiction.

The Banach analytic set \mathcal{C} obtained literally as in the proof of Lemma 2.2 is again finite dimensional. But the problem is that the restriction map $r : \mathcal{C} \rightarrow \mathcal{W}$ might be not proper. That mean that for some z close to z^0 the preimage $r^{-1}(\tilde{f}|_{\tilde{W}_z})$ is not compact. But a cycle in this preimage is different from $\tilde{f}|_{\tilde{W}_z}$ itself by a rational cycle tangent to \mathcal{L} . Therefore we got a contradiction with the boundedness of rational cycle geometry condition (iii) of this Lemma. So r is proper and by Remmert proper mapping theorem $r(\mathcal{C})$ is an analytic set in \mathcal{W} . Since it contains a thick subset it is the whole \mathcal{W} . We again get extension of all f_z for z close to z^0 as a family by a tautological map $\tilde{f} : \tilde{\mathcal{W}} \rightarrow X$.

□

Proof of Theorems 2.1 and 2.2. Let again \hat{U} be the maximal open subset of V such that f extends onto (W, π, \hat{U}) after a reparametrization.

Now we can proceed exactly as in the proof of Steps 1 and 2 on the page s 817-818 of [Iv3]. Our present situation is even somewhat simpler because f is holomorphic on the Hartogs figure. Reparametrizations do not cause any additional difficulties. In this way we get we get that $S_1 := \partial\hat{U} \cap V$ is of Hausdorff dimension $2n - 2$. By the “rationalization trick” we extend our map f onto a complete Hartogs figure (W, π, V) over V minus the closed set of the form $S = \bigcup_{(\lambda, z_1) \in S_1} S_{\lambda, z_1}$ with S_{λ, z_1} being compact subsets of the disks Δ_{λ, z_1} . Due to our localization Lemma 2.1 we need to work in a neighborhood of a point $(\lambda, z_1) \in S_1$ only. Lemma 2.6 shows that for every natural N the set $\{z \in V : f|_{W_z} \text{ extends onto a disk } \tilde{W}_z \text{ and } \text{area}(\tilde{f}|_{\tilde{W}_z}) \leq N\}$ is thin in a neighborhood of S_1 . The rest is obvious and we get the Theorem 2.1. In particular, one gets that for any two-dimensional submanifold $U \subset V$ the domain $U \setminus (S_1 \cap U)$ is the maximal domain over which the restricted map $f|_{W|_U}$ extends after a reparametrization.

Starting from this point further reparametrizations are needed. Therefore the proof of $(a_1), (a_2), (b_1)$, and (b_2) is done.

(a_3) is clear, because \tilde{f} could fail to be immersion only along a divisor, which should then intersect $\tilde{W}|_U$. But this is not the case.

(b_3) is exactly the Corollary 2.2 from the preceding subsection.

□

Item (b_3) easily implies the following:

Corollary 2.4. *In the conditions (and notations) of Theorem 2.2 denote by S_λ^0 the minimal closed subset of Δ_λ^2 such that the restriction $f_\lambda := f|_{\Delta_\lambda^2}$ extends onto $\Delta_\lambda^2 \setminus S_\lambda^0$. Then $S_\lambda^0 = S_\lambda$.*

Let’s repeat once more that S in Theorem 2.2 is always understood as being the minimal closed subset that f meromorphically extends to its complement.

Suppose now that a polydisk $P = \Delta^n \times \Delta_{1+\varepsilon}$ is fixed, a closed subset $S \subset P$ of the form $S = \bigcup_{z \in S_1} S_z$ in P is given, where S_z is a compact subset of the leaf $P_z := \{z\} \times \Delta_{1+\varepsilon}$ for every $z \in S_1$. Suppose that $S_1 \ni 0$ and that 0 is an accumulation point for $\Delta^n \setminus S_1$. Finally, let a meromorphic foliated generic injection $f : (P \setminus S, \pi) \rightarrow (X, \mathcal{L})$ into a disk-convex foliated manifold be given. We shall make use of the following

Theorem 2.3. *For a fixed constant $C > 0$ let A_C denote a set of (λ, z_1) in a neighborhood of $0 \in \Delta^n$ such that $(\lambda, z_1) \notin S_1$ and that*

$$\int_{\Delta_{\lambda, z_1^n}} f^* \omega \leq C. \quad (2.21)$$

If (X, \mathcal{L}) has bounded rational cycle geometry (ex. admits a plurinegative taming form) then A_C is contained in a germ of a proper analytic subset of Δ^n at 0.

The proof follows immediately from Lemmas 2.6 and 5.1.

3. VANISHING CYCLES, COVERING CYLINDERS AND FOLIATED SHELLS

3.1. Vanishing ends and holonomy covering cylinders. Let (X, \mathcal{L}) be a foliated manifold. Saying that (X, \mathcal{L}) is a foliated manifold we shall always mean that \mathcal{L} is a holomorphic foliation by curves on a complex manifold X , if the opposite is not specified. Let $\mathcal{L}^{\text{sing}}$ denote the singular set of \mathcal{L} . This is an analytic subset of X of codimension at least two. Set $X^0 = X \setminus \mathcal{L}^{\text{sing}}$ and $\mathcal{L}^0 = \mathcal{L}|_{X^0}$. Take a point $z^0 \in X^0$ and denote by $\mathcal{L}_{z^0}^0$ the leaf of \mathcal{L}^0 passing through z^0 . Recall that a parabolic end of $\mathcal{L}_{z^0}^0$ is a closed subset $E \subset \mathcal{L}_{z^0}^0$ which is biholomorphic to the closed punctured disk $\bar{\Delta}^* = \{\zeta \in \mathbb{C} : 0 < |\zeta| \leq 1\}$. By ∂E we shall denote the biholomorphic image of the circle $\{|\zeta| = 1\}$ - the outer boundary of the end E . Foliation \mathcal{L} may have a nontrivial holonomy along ∂E , which can be finite or infinite.

Consider the case when holonomy is finite. Recall what does that mean. Take a transversal D through z^0 . Transversals will be always taken small enough, in particular, we shall always suppose that $D \subset X^0$ and that D is transversal to \mathcal{L}^0 “up to its boundary ∂D ”. Take a path γ_{z^0} on $\mathcal{L}_{z^0}^0$ which goes from z^0 to some point $q \in \partial E$, then goes one time around ∂E and goes back to z^0 . If one takes a point $z \in D$ close to z^0 and travels on \mathcal{L}_z^0 along the path γ_z close to γ_{z^0} then one certainly hits D in a neighborhood of z^0 by a point $g(z)$. This defines a local biholomorphism $g : (D, z) \rightarrow (D, z)$ which generates a subgroup $\langle g \rangle$ of the group $\text{Bihol}(D, z^0)$ of local biholomorphisms of D fixing z^0 . We suppose that $\langle g \rangle$ is finite, i.e., $g^d = \text{Id}$ for some $d \geq 1$ and this d is always taken to be the minimal satisfying this property. This d is called the order of the holonomy of \mathcal{L} along ∂E .

Lemma 3.1. *There exists a foliated holomorphic immersion $f : \Delta^n \times A_{1-\varepsilon, 1+\varepsilon} \rightarrow \mathcal{L}_D^0$ (for sufficiently small $\varepsilon > 0$) such that:*

i) $f(\{0\} \times A_{1-\varepsilon, 1+\varepsilon}) \subset \mathcal{L}_{z^0}^0$ and the restriction $f|_{\{0\} \times A_{1-\varepsilon, 1+\varepsilon}} : \{0\} \times A_{1-\varepsilon, 1+\varepsilon} \rightarrow \mathcal{L}_{z^0}^0$ is a regular covering of order d (i.e., covers d -times some imbedded annulus in $\mathcal{L}_{z^0}^0$ and $f(\{0\} \times \partial \Delta) = d \cdot \partial E$).

ii) For all $z \in \Delta^n$ outside of a proper analytic subset $A \subset \Delta^n$ the restriction $f|_{\{z\} \times A_{1-\varepsilon, 1+\varepsilon}} : \{z\} \times A_{1-\varepsilon, 1+\varepsilon} \rightarrow \mathcal{L}_z$ is an imbedding.

Proof. Take an annulus A_0 on $\mathcal{L}_{z^0}^0$ around ∂E . Let $g \in \text{Bihol}(z^0, D)$ generates the holonomy of \mathcal{L} along ∂E as above. Denote by A the germ of a proper analytic subset of D at z^0 which consists from those $z \in D$ that the orbit of the corresponding holonomy has cardinality l strictly less than d . When one travels from $z \in D$ to z in the leaf \mathcal{L}_z^0 along a curve close to $d \cdot \partial E$ one in a standard way cuts an imbedded annulus A_z on \mathcal{L}_z^0 . For z in the exceptional set A one sweeps A_z l times for z outside from A only once. The union $W = \bigcup_{z \in D} A_z$ has a natural structure of a complex manifold and possesses a natural foliated holomorphic immersion into itself $f : D \times A_{1-\varepsilon, 1+\varepsilon} \rightarrow W$ coming from the

construction, which is a generic injection. f sends each annulus $\{z\} \times A_{1-\varepsilon, 1+\varepsilon}$ onto the corresponding A_z with corresponding multiplicity. For that one might need to shrink D and annuli A_z for $z \in D$. Now we can suppose that D is biholomorphic to Δ^n . The rest is obvious. \square

As we remarked in the proof our f is a *generic injection* of the trivial Hartogs figure $\Delta^n \times A_{1-\varepsilon, 1+\varepsilon}$ over a polydisk in the sense of Definition 2.4 and results of the previous section are applicable to such f .

Definition 3.1. A parabolic end E is called a vanishing end of order d if:

- i) the holonomy of \mathcal{L} along ∂E is finite of order $d \geq 1$;
- ii) the generic injection $f : \Delta^n \times A_{1-\varepsilon, 1+\varepsilon} \rightarrow \mathcal{L}_D^0$, constructed above, extends as a foliated meromorphic immersion $\tilde{f} : \tilde{W} \rightarrow X$ from a complete Hartogs figure $(\tilde{W}, \pi, \Delta^n)$ over Δ^n to X after a reparametrization.
- iii) The intersection of $\tilde{W}_0 := \pi^{-1}(0)$ with the set of points of indeterminacy $I_{\tilde{f}}$ of \tilde{f} consists of a single point $a \in \tilde{W}_0$.

The point $q = \tilde{f}|_{\tilde{W}_0}(a)$ will be called the *endpoint* of the vanishing end E (or of the leaf \mathcal{L}_z^0). Following Brunella, see [Br2], we add all vanishing endpoints to the leaf \mathcal{L}_z^0 and call the curve obtained a *completed leaf* through z . Completed leaf will be denoted as \mathcal{L}_z .

Remark 3.1. Let us give two very simple examples explaining this notion.

1. Consider the radial foliation in \mathbb{C}^2 , i.e., $\mathcal{L}_c = \{z_2/z_1 = c\}$ for $c \in \mathbb{C}\mathbb{P}^1$. The origin of \mathbb{C}^2 is a parabolic end for every leaf \mathcal{L}_c^0 . But it is never a vanishing end! Really, one cannot construct a foliated meromorphic immersion as in Definition 3.1 in this case. Any \tilde{f} will contract some complex curve to a point.
2. Let \mathcal{L} be the vertical foliation in \mathbb{C}^2 , i.e., $\mathcal{L}_c = \{z_1 = c\}$ for $c \in \mathbb{C}$. Blow-up the origin $\pi : \hat{\mathbb{C}}^2 \rightarrow \mathbb{C}^2$ and lift our foliation to $\hat{\mathbb{C}}^2$. The leaf \mathcal{L}_0^0 has now a parabolic end at its point of intersection with the exceptional divisor and this end is a vanishing end. The role of $f = \tilde{f}$ plays π^{-1} .
3. Let D be a transversal through z and let E be a vanishing end of \mathcal{L}_z^0 . Remark that for points z' close to z on D only those ones which belong to some proper analytic subset could be such that $\mathcal{L}_{z'}^0$ has a vanishing end E' with $\partial E'$ close to E . Really, such z' should lie in the projection $\tilde{\pi} : \tilde{W} \rightarrow D$ of a point of indeterminacy of \tilde{f} . Therefore a generic leaf of \mathcal{L} has no vanishing ends at all.
4. If the holonomy along ∂E is infinite then E is never a vanishing end by definition.

For each $z \in D$ take a holonomy cover $\hat{\mathcal{L}}_z^0$ of the leaf \mathcal{L}_z^0 . Recall that a holonomy cover of \mathcal{L}_z^0 is a cover with respect to the holonomy subgroup $\mathbf{Hol}(z, \mathcal{L}_z^0)$ of the fundamental group $\pi(z, \mathcal{L}_z^0)$. That means that in the construction of $\hat{\mathcal{L}}_z^0$ two paths γ_1, γ_2 from z to some $w \in \mathcal{L}_z^0$ define the same point of $\hat{\mathcal{L}}_z^0$ if and only if $\gamma_1 \circ \gamma_2^{-1} \in \mathbf{Hol}(z, \mathcal{L}_z^0)$, i.e., if the holonomy along $\gamma_1 \circ \gamma_2^{-1}$ is trivial.

Set

$$\hat{\mathcal{L}}_D^0 = \bigcup_{z \in D} \hat{\mathcal{L}}_z^0. \quad (3.1)$$

This set (introduced by Suzuki in [Sz] under the name of “tube normaux”) has the natural structure of a complex manifold together with the natural projection $\pi : \hat{\mathcal{L}}_D^0 \rightarrow D$ which

sends $\hat{\mathcal{L}}_z^0$ to z . It admits also the natural locally biholomorphic foliated map $p : \hat{\mathcal{L}}_D^0 \rightarrow \mathcal{L}_D^0 \subset X^0$ which sends $\hat{\mathcal{L}}_z^0$ to \mathcal{L}_z^0 with $p|_{\hat{\mathcal{L}}_z^0} : \hat{\mathcal{L}}_z^0 \rightarrow \mathcal{L}_z^0$ being the canonical holonomy covering map. Call $\hat{\mathcal{L}}_D^0$ the *holonomy covering cylinder* of \mathcal{L} over D .

Vanishing ends of $\hat{\mathcal{L}}_z^0$ are defined similarly to that of \mathcal{L}_z^0 . Let E be a parabolic end of $\hat{\mathcal{L}}_{z^0}^0$. Take $f : \Delta^n \times A_{1-\varepsilon, 1+\varepsilon} \rightarrow \hat{\mathcal{L}}_D^0$ such that:

- i) $f : \Delta^n \times A_{1-\varepsilon, 1+\varepsilon} \rightarrow \hat{\mathcal{L}}_D^0$ is an imbedding;
- ii) $f(\{0\} \times \partial\Delta) = \partial E$ (note that $d = 1$ in this case).

The only difference that now f takes values in $\hat{\mathcal{L}}_D^0$ and f is an imbedding. The last is because the holonomy of the foliation $\hat{\mathcal{L}}^0$ on $\hat{\mathcal{L}}_D^0$ is trivial.

Definition 3.2. *E is called a vanishing end of $\hat{\mathcal{L}}_{z^0}^0$ if $h = p \circ f$ extends to a meromorphic foliated immersion $\tilde{h} : \tilde{W} \rightarrow X$ after a reparametrization (not f itself as in Definition 3.1) and \tilde{W}_0 intersects the indeterminacy set $I_{\tilde{h}}$ of \tilde{h} by exactly one point.*

The union of $\hat{\mathcal{L}}_z^0$ with all its vanishing endpoints equipped with an obvious complex structure will be denoted as $\hat{\mathcal{L}}_z$. We shall call it also a *completed holonomy covering leaf* of the leaf \mathcal{L}_z^0 . Set $\hat{\mathcal{L}}_D := \bigcup_{z \in D} \hat{\mathcal{L}}_z$ and call it the *completed holonomy covering cylinder* over D . Now let us bring together the principal properties of $\hat{\mathcal{L}}_D$, which will be repeatedly used along this paper.

Lemma 3.2. i) *The completed holonomy covering cylinder possesses the natural structure of a foliated complex manifold with foliation given by the natural projection $\pi : \hat{\mathcal{L}}_D \rightarrow D$ defined as above by $\pi(\hat{\mathcal{L}}_z) = z$.*

ii) *The natural holomorphic immersion $p : \hat{\mathcal{L}}_D^0 \rightarrow \mathcal{L}_D^0$ extends to a meromorphic foliated immersion $p : \hat{\mathcal{L}}_D \rightarrow X$ and its restrictions $p|_{\hat{\mathcal{L}}_z} : \hat{\mathcal{L}}_z \rightarrow \mathcal{L}_z$ are ramified at vanishing ends.*

Proof. (i) Cylinder $\hat{\mathcal{L}}_D^0$ has a natural complex structure. Therefore we need to add vanishing ends to some leaves and extend this structure to a neighborhood of each added end. Take a vanishing endpoint $a \in \hat{\mathcal{L}}_{z^0}^0$. Let $f : \Delta^n \times A_{1-\varepsilon, 1+\varepsilon} \rightarrow \hat{\mathcal{L}}_D^0$ be an imbedding from the Definition 3.2 with $h = p \circ f$ already extended to a meromorphic foliated immersion of $(\Delta^n \times \Delta_{1+\varepsilon}, \mathcal{L}^\pi)$ into (X, \mathcal{L}) . Let I_h be the indeterminacy set of h . For $z \notin A := \pi(I_h)$ the restriction $h|_{\{z\} \times \Delta_{1+\varepsilon}} : \{z\} \times \Delta_{1+\varepsilon} \rightarrow X$ is an imbedding and therefore so is also the $f|_{\{z\} \times \Delta_{1+\varepsilon}} : \{z\} \times \Delta_{1+\varepsilon} \rightarrow \hat{\mathcal{L}}_D^0$. This implies that f is an imbedding on $\Delta^n \setminus (A \times \Delta)$. This immediately implies that f is an imbedding on $\Delta^n \setminus I_h$. Therefore we can complete $\hat{\mathcal{L}}_D^0$ by I_h over the image $(\pi \circ f)(\Delta^n) \subset D$. This defines the structure of a complex manifold on $\hat{\mathcal{L}}_D$. The rest is obvious.

(ii) This item follows readily from the construction above. □

Remark 3.2. Let us make a remark which will be important for the future (but is irrelevant for this Section). The covering $p_{z^0} : \hat{\mathcal{L}}_{z^0} \rightarrow \mathcal{L}_{z^0}$ is an orbifold covering. That means that its ramification index at point a depends only on $b := p_{z^0}(a)$. This is also an unbounded covering in the sense that for every a there exists a disk-neighborhood $V \ni b$ such that $p_{z^0}^{-1}(V)$ is a disjoint union of disks W_j with centers a_j - preimages of b , such that every restriction $p_{z^0}|_{W_j} : W_j \rightarrow V$ is a proper covering ramified over b .

3.2. Vanishing cycles. Let now $\gamma_0 : [0, 1] \rightarrow \hat{\mathcal{L}}_{z^0}^0$ be a loop in $\hat{\mathcal{L}}_{z^0}^0$ which is not homotopic to zero in $\hat{\mathcal{L}}_{z^0}^0$.

Definition 3.3. We say that $\hat{\gamma}_0$ is a vanishing cycle if for some sequence $z_n \rightarrow z^0$ there exist loops $\hat{\gamma}_n$ in $\hat{\mathcal{L}}_{z_n}$ uniformly converging to $\hat{\gamma}_0$ which are homotopic to zero in the corresponding leaves $\hat{\mathcal{L}}_{z_n}$.

(a) We say that $\hat{\gamma}_0$ is an algebraic vanishing cycle if γ_0 is not homotopic to zero in $\hat{\mathcal{L}}_{z^0}^0$ but is homotopic to zero in the completed leaf $\hat{\mathcal{L}}_{z^0}$.

(b) If $\hat{\gamma}_0$ is not homotopic to zero also in the completed leaf $\hat{\mathcal{L}}_{z^0}$ we call it an essential vanishing cycle.

There is an analogy (rather deep in fact) between algebraic/resp. essential vanishing cycles and poles/resp. essential singularities of meromorphic functions. Really, pole of a meromorphic function f becomes a regular point if one completes \mathbb{C} to \mathbb{CP}^1 and considers f as a holomorphic mapping into the latter manifold. However, an essential singular point stays to be a singularity of f after this operation. The same with cycles. For the moment let us say that:

- If $\mathcal{L}^{\text{sing}} = \emptyset$, i.e., if \mathcal{L} has no singularities, then every vanishing cycle is an essential vanishing cycle.
- Algebraic vanishing cycles in the leaf $\hat{\mathcal{L}}_z^0$ can be removed (i.e., one can make these cycles homotopic to zero) by adding to $\hat{\mathcal{L}}_z^0$ vanishing ends.
- It is known also (it follows from [Br2]) that if X is Kähler, then all vanishing cycles (of any \mathcal{L}) are algebraic.

In this paper we shall concentrate our attention on essential vanishing cycles only. Our goal in this subsection is to show that if $\hat{\mathcal{L}}_z$ contains an essential vanishing cycle then it contains an imbedded essential vanishing cycle.

We shall need the following (probably well known) statement. Since an exact reference seems to be missing in the literature we shall sketch a proof here. Take an immersed loop γ in a Riemann surface R which has only transversal self-intersections. Denote by N the closure of a sufficiently small tubular neighborhood of γ . Add to N all disks bounded by circles - components of ∂N , and denote the obtained compact as \bar{N} .

Lemma 3.3. Imbedding $\bar{N} \subset R$ induces the natural injection $\pi_1(\bar{N}) \rightarrow \pi_1(R)$.

Proof. Suppose that there exists a loop β in \bar{N} not homotopic to zero in \bar{N} which is homotopic to zero in R . Then the homotopy of β to zero is supported in a compact part of R and therefore we can suppose that R has finite topology, i.e., finite number of handles and boundary circles. In the sequel the trivial case when \bar{N} or R is a disk or an annulus will be omitted. Now we perform the following manipulations which obviously do not change the homotopy type of \bar{N} .

(a) Every connected component of $R' := R \setminus \bar{N}$ which is an annulus adjacent to ∂R we add to \bar{N} .

(b) If some component C of R' is an annulus with both boundary circles belonging to $\partial \bar{N}$ then we cut C on two annuli C_1 and C_2 . Each of them we add to \bar{N} and think about \bar{N} as having ∂C_1 and $\partial C_2 = -\partial C_1$ as two boundary components.

Denote by g the Riemannian metric on R of curvature -1 having boundary circles as geodesics. Every loop γ in R is now homotopic to a unique geodesic $\tilde{\gamma}$ in metric g which

is either not intersecting ∂R or is a component of ∂R , see for example [Bu] Theorem 1.6.6. We deform all boundary circles of \bar{N} one by one to geodesics. If in the process of deformation a curve is touching β we move β appropriately enlarging (or contracting) \bar{N} in a way to keep β inside.

We end up with having all boundary circles of \bar{N} geodesics in g . Now we do the same with β getting from it a geodesic $\tilde{\beta}$ in \bar{N} . Note that it stays in \bar{N} and do not intersect also $\partial C_2 = -\partial C_1$ from (b) (or coinciding with one of them). But this $\tilde{\beta}$ stays to be geodesic in g on the whole of R and therefore is not homotopic to zero. Contradiction. \square

Now we are going to reduce the question of existence of essential vanishing cycles in $\hat{\mathcal{L}}_D$ to the existence of *imbedded* essential vanishing cycles in $\hat{\mathcal{L}}_D$. Namely, we shall prove that the following is true:

Lemma 3.4. *If there exists an essential vanishing cycle in $\hat{\mathcal{L}}_{z^0}$ then there exists an imbedded essential vanishing cycle in $\hat{\mathcal{L}}_{z^0}$.*

Proof. Let $\hat{\gamma}_0 : [0, 1] \rightarrow \hat{\mathcal{L}}_{z^0}$ be our essential vanishing cycle. After perturbing it, if necessary, we can suppose that $\hat{\gamma}_0$ is an immersion with only transversal self-intersections. For every point $\hat{\gamma}_0(t)$ take an $(n-1)$ -disk $Q_{\hat{\gamma}_0(t)}$ in $\hat{\mathcal{L}}_D$ transversal to the leaf $\hat{\mathcal{L}}_{z^0}$ and cutting it by the point $\hat{\gamma}_0(t)$. Make these disks depend smoothly on $\hat{\gamma}_0(t)$ in such a way that for $\hat{\gamma}_0(t_1) \neq \hat{\gamma}_0(t_2)$ the corresponding $(n-1)$ -disks do not intersect. Let's stress explicitly that $Q_{\hat{\gamma}_0(t)}$ depends only on the image point $\hat{\gamma}_0(t)$ on the curve and not on t . We have therefore a natural projection $\Pi : \bigcup_{\hat{\gamma}_0(t)} Q_{\hat{\gamma}_0(t)} \rightarrow \hat{\gamma}_0$. Extend these data over a closure of a small tubular neighborhood N_0 of $\hat{\gamma}_0$. I.e., set $Q := \bigcup_{\tau \in N_0} Q_\tau$ and now Π maps this Q onto N_0 .

For every z in our transversal D , which is close to z^0 each Q_τ cuts the leaf \mathcal{L}_z^0 exactly by one point and when τ runs over N_0 our disks Q_τ cuts a closure of a tubular neighborhood N_z of some closed curve γ_z which covers $\hat{\gamma}_0$ under the projection $\Pi|_{\gamma_z} : \gamma_z \rightarrow \hat{\gamma}_0$. Remark also that $\Pi|_{N_z} : N_z \rightarrow N_0$ is bijective. Denote by \bar{N}_0 the union of N_0 with all disks bounded by circles components of ∂N_0 . Denote likewise by \bar{N}_z the union of N_z with all disks bounded by circles components of ∂N_z .

Take some component $\hat{\gamma}'_0$ of ∂N_0 bounding a disk in $\hat{\mathcal{L}}_{z^0}$. Then the corresponding component $\hat{\gamma}'_{z_n}$ of ∂N_{z_n} bounds a disk in $\hat{\mathcal{L}}_{z_n}$, say D'_0 and then $\Pi|_{N_{z_n}} : N_{z_n} \rightarrow N_0$ extend to a homeomorphism $\Pi|_{N_{z_n} \cup D'_{z_n}} : N_{z_n} \cup D'_{z_n} \rightarrow N_0 \cup D'_0$.

If $\hat{\gamma}'_0$ doesn't bound a disk in $\hat{\mathcal{L}}_{z^0}$ but $\hat{\gamma}'_{z_n}$ do bounds a disk in $\hat{\mathcal{L}}_z$ we get an imbedded essential vanishing cycle in $\hat{\mathcal{L}}_{z^0}$.

So, unless an imbedded vanishing cycle was found in $\hat{\mathcal{L}}_{z^0}$ we end up with extending Π to a homeomorphism $\tilde{\Pi} : \bar{N}_{z_n} \rightarrow \bar{N}_0$.

Since γ_{z_n} is homotopic to zero in $\hat{\mathcal{L}}_{z_n}$ it will be homotopic to zero in \bar{N}_{z_n} by Lemma 3.1. Therefore $\hat{\gamma}_0$ should be homotopic to zero in \bar{N}_0 and therefore in $\hat{\mathcal{L}}_{z^0}$. Contradiction. Therefore the only possibility left is that some component $\hat{\gamma}'_0$ of ∂N_0 doesn't bound a disk while $\hat{\gamma}'_{z_n}$ do bound disk, i.e., $\hat{\gamma}'_0$ is an imbedded essential vanishing cycle in $\hat{\mathcal{L}}_{z^0}$. \square

Remark 3.3. Remark that if $\hat{\gamma}_0$ is an imbedded ess. vanishing cycle in $\hat{\mathcal{L}}_{z^0}$ then a sequence $z_n \rightarrow z$ such that there exists γ_{z_n} bounding a disk in $\hat{\mathcal{L}}_{z_n}$ and γ_{z_n} uniformly converging to γ_0 when $z_n \rightarrow z^0$ can be taken generic.

3.3. Universal covering cylinder. Further, for $z \in D$ denote by $\tilde{\mathcal{L}}_z$ the universal cover of the completed holonomy leaf $\hat{\mathcal{L}}_z$. I.e., we take the orbifold universal covering of \mathcal{L}_z , see Remark 3.2. On the union

$$\tilde{\mathcal{L}}_D = \bigcup_{z \in D} \tilde{\mathcal{L}}_z \quad (3.2)$$

one defines a natural topology in the following way. An element of $\tilde{\mathcal{L}}_D$ is a path γ in some leaf $\hat{\mathcal{L}}_z$ starting from z and ending at some point $w \in \hat{\mathcal{L}}_z$. γ and γ' define the same point if their ends coincide and they are homotopic (inside $\hat{\mathcal{L}}_z$) with ends fixed. A neighborhood of $\gamma \subset \hat{\mathcal{L}}_z$ in $\tilde{\mathcal{L}}_D$ is the set of pathes γ' -s in the leaves $\hat{\mathcal{L}}_{z'}$ with z' close to z which are themselves close to γ . γ' “close” to γ is understand here as closed in the topology of uniform convergency in the space $\mathcal{C}([0,1], X)$ of continuous mappings from $[0,1]$ to X .

Definition 3.4. $\tilde{\mathcal{L}}_D$ with the topology just described is called the universal covering cylinder of \mathcal{L} over D .

The natural projection $\pi : \hat{\mathcal{L}}_D \rightarrow D$ lifts to $\pi : \tilde{\mathcal{L}}_D \rightarrow D$ (and will be denoted with the same letter). There is a distinguished section $\sigma : D \rightarrow \tilde{\mathcal{L}}_D$ sending z to z . The mapping $p : \hat{\mathcal{L}}_D \rightarrow X$ lifts to $\tilde{\mathcal{L}}_D$ and stays to be a *meromorphic foliated immersion* $\tilde{p} : \tilde{\mathcal{L}}_D \rightarrow X$ in the sense that it is a foliated immersion outside of its indeterminacy set.

Due to the eventual presence of essential vanishing cycles the natural topology on the covering cylinder might be not Hausdorff. Let us explain this in more details. Non-separability of the natural topology on $\tilde{\mathcal{L}}_D$ means that:

- there exist $z \in D$ and $w \in \hat{\mathcal{L}}_z$ and two pathes γ_1, γ_2 from z to w such that $\gamma_1 \circ \gamma_2^{-1}$ is not homotopic to zero in $\hat{\mathcal{L}}_z$;
- there exist some sequence $z_n \rightarrow z$ in D , some sequence $w_n \in \hat{\mathcal{L}}_{z_n}$ converging to w , some sequences of pathes γ_1^n and γ_2^n from z_n to w_n each converging uniformly to γ_1 and γ_2 such that $\gamma_1^n \circ (\gamma_2^n)^{-1}$ are homotopic to zero in \mathcal{L}_{z_n} .

And that exactly means that $\gamma_1 \circ \gamma_2^{-1}$ is an essential vanishing cycle.

Vice versa, if $\gamma : [0,1] \rightarrow \hat{\mathcal{L}}_z$ is an essential vanishing cycle starting and ending at z , then γ and the trivial path $\beta \equiv z$ represent two non-separable points in $\tilde{\mathcal{L}}_D$.

3.4. Proof of Theorem 1. In the presence of a pluriclosed taming form the problem of the separability of the topology of $\tilde{\mathcal{L}}_D$ can be resolved by Theorem 1 from the Introduction. Now we shall state and prove somewhat more general and precise statement which contains the aforementioned result from the Introduction. To make the statement more precise let's turn to the Definition 1.2 from the Introduction. By the Theorem 2.2 we know that mapping $h : (B^\varepsilon, \mathcal{L}^\pi) \rightarrow (X, \mathcal{L})$, which defines a foliated shell in a pluritamed foliated manifold extends onto $P^\varepsilon \setminus \bigcup_{z_1 \in S_1} S_{z_1}$, where S_1 is a complete polar compact in Δ . Let us say that this shell is centered at z_1^0 if $z_1^0 \in S_1$. One more remark: for a transversal $D \subset X^0$ and an imbedded disk $\Delta \subset D$ the restriction \mathcal{L}_Δ^0 is well defined and is even a locally closed submanifold in X^0 (we don't need to make the latter more precise).

Theorem 3.1. *Let (X, \mathcal{L}) be a disk-convex foliated manifold which admits a dd^c -closed taming form and let $z^0 \in X^0$ be a point. Then the following statements are equivalent:*

- i) *The leaf $\hat{\mathcal{L}}_{z^0}$ contains an essential vanishing cycle.*
- ii) *For every transversal $D \ni z^0$ there exists an imbedded disk $z^0 \in \Delta \subset D$ such that \mathcal{L}_Δ^0 contains a spherical shell centered at z^0 .*

Remark 3.4. (a) Let us explain that the item (i) of this Theorem is equivalent to the item (i) of Theorem 1 from the Introduction in the case when \mathcal{L} is smooth (i.e., without singularities). In that case vanishing ends do not exist and, in particular, $p_{z^0} : \hat{\mathcal{L}}_{z^0} \rightarrow \mathcal{L}_{z^0}$ is an unramified covering. Let $\gamma_0 \subset \mathcal{L}_{z^0}$ be a vanishing cycle and $\gamma_n \subset \mathcal{L}_{z_n}$ be cycles homotopic to zero converging to γ_0 . All γ_n lift to cycles $\hat{\gamma}_n \subset \hat{\mathcal{L}}_{z_n}$ converging to the lift $\hat{\gamma}_0 \subset \hat{\mathcal{L}}_{z^0}$ of γ_0 . All $\hat{\gamma}_n$ are homotopic to zero. But $\hat{\gamma}_0$ cannot be homotopic to zero. Therefore we get a vanishing cycle $\hat{\gamma}_0$ in $\hat{\mathcal{L}}_{z^0}$. Vice versa, let $\hat{\gamma}_0$ and $\hat{\gamma}_n$ be as above in the holonomy covering leaves. Then γ_n project to cycles homotopic to zero in corresponding leaves. But $\hat{\gamma}_0$ project to some γ_0 which cannot be homotopic to zero because in the latter case its lift $\hat{\gamma}_0$ (as lift of any curve homotopic to zero) should be homotopic to zero itself. Contradiction. Therefore γ_0 is a vanishing cycle in \mathcal{L}_{z^0} .

(b) The item (ii) specifies that the ‘‘support’’ $\Sigma = h(B)$ of the foliated shell is in \mathcal{L}_D^0 (but it is not homologous to zero in the whole of X !) Remark also that the existence of an essential vanishing cycle in $\hat{\mathcal{L}}_{z^0}$ is unrelated to the choice of a transversal $D \ni z^0$. Therefore if for some transversal $D \ni z^0$ there is a shell in \mathcal{L}_D then it persists in all others.

Proof. (i) \Rightarrow (ii) By Lemma 3.4 we can suppose that our vanishing cycle $\hat{\gamma}_0$ is imbedded into $\hat{\mathcal{L}}_{z^0} \subset \hat{\mathcal{L}}_D$. Deforming it, if necessary, we suppose that $\hat{\gamma}_0$ is contained in $\hat{\mathcal{L}}_{z^0}^0$. Therefore we can suppose that for an imbedded loop $\hat{\gamma}_0 \subset \hat{\mathcal{L}}_{z^0}$ started at z^0 the following holds:

- $\hat{\gamma}_0$ does not bounds a disk in $\hat{\mathcal{L}}_{z^0}$;
- but for a generic sequence $z_n \rightarrow z^0$ and a sequence of imbedded loops $\hat{\gamma}_n \subset \hat{\mathcal{L}}_{z_n}$ uniformly converging to $\hat{\gamma}_0$ every $\hat{\gamma}_n$ bounds a disk Δ_n in $\hat{\mathcal{L}}_{z_n}$.

Take a neighborhood U of some z_N such that for every $z \in U$ there is an imbedded loop $\hat{\gamma}_z$ close to $\hat{\gamma}_N$ bounding a disk Δ_z in $\hat{\mathcal{L}}_z$. We can suppose that $\hat{\gamma}_z$ smoothly depend on $z \in U$. Take some open cell $V \subset D$ containing U and z^0 and extend our family $\Gamma := \{\hat{\gamma}_z\}$ smoothly over $z \in V$ (after shrinking it over U , if necessary) in such a way that $\hat{\gamma}_{z^0}$ coincides with $\hat{\gamma}_0$. Perturbing the family Γ , if necessary, we can suppose that some neighborhood W of $\Gamma \cup \Delta_{z_N}$ in $\hat{\mathcal{L}}_D$ forms a generalized Hartogs figure (W, π, U, V) . Projection $\pi : W \rightarrow V$ here is the restriction to W of the natural projection $\pi : \hat{\mathcal{L}}_D \rightarrow D$.

Mapping $p : \hat{\mathcal{L}}_D \rightarrow X$ restricted to W will be likewise denoted as $p : W \rightarrow X^0 \subset X$ and it is a holomorphic foliated immersion, because the construction can be obviously fulfilled in such a way that $W \subset \hat{\mathcal{L}}_D^0$. Note also that p is a generic injection because for generic z_N our $p|_{\gamma_N}$ is an imbedding. But $p|_{\hat{\gamma}_0}$ might be only an immersion in general. By Theorems 2.1 and 2.2 p extends after a reparametrization onto $\tilde{W} \setminus S$, where \tilde{W} is a complete Hartogs figure over V and S is of the form $S = \bigcup_{z \in S_1} S_z$ with S_1 being a complete $(n-1)$ -polar subset of V and every S_z is a compact subdisk of the corresponding disk \tilde{W}_z . This extension \tilde{p} is a foliated meromorphic immersion, i.e., is an immersion outside of its indeterminacy set $I_{\tilde{p}}$ and takes values in X (not more in X^0). The family, which

corresponds in \tilde{W} to our family Γ will be denoted still by Γ and no new notation for the loops $\hat{\gamma}_z$ will be introduced.

Observe that $z^0 \in S_1$. Otherwise take an n -disk Δ^n around z^0 in D such that $\Delta^n \cap S_1 = \emptyset$ and such that:

- $\pi^{-1}(\Delta^n)$ is biholomorphic to $\Delta^n \times \Delta_{1+\varepsilon}$ with π being the vertical projection $\Delta^n \times \Delta \rightarrow \Delta^n$ (one might need to shrink Δ^n and \tilde{W} to achieve this).
- For $z \in \Delta^n$ circles $\hat{\gamma}_z = \partial\Delta_z$ belong to our family Γ (for this one might need to perturb Γ).

Our \tilde{p} now is meromorphically extended to $\Delta^n \times \Delta_{1+\varepsilon}$. But that means (by the very definition of vanishing ends) that $p^{-1} \circ \tilde{p}$ lifts to a holomorphic map $\tilde{f} : \Delta^n \times \Delta_{1+\varepsilon} \rightarrow \hat{\mathcal{L}}_D$. Therefore $\tilde{f}|_{\{z^0\} \times \Delta}$ realizes the homotopy of $\hat{\gamma}_0 = \tilde{f}(\{z^0\} \times \partial\Delta)$ to zero. Contradiction.

Denote by A the proper analytic subset in a neighborhood of z^0 on D which consists from points z such that $\tilde{p}|_{\partial_0 \tilde{W}_z}$ is not a generic injection. Again we locally represent \tilde{W} as a product $\tilde{W} = \Delta^n \times \Delta_{1+\varepsilon}$ with π being the vertical projection $\pi : \Delta^n \times \Delta_{1+\varepsilon} \rightarrow \Delta^n$ and with z^0 being the origin in these coordinates. Decompose $\Delta^n = \Delta^{n-1} \times \Delta$ in such a way that $(\{0\} \times \Delta) \cap A = \{ \text{a finite set} \}$. Then by Theorem 2.2 for $\lambda \in \Delta^{n-1}$ close to 0 (if $n \geq 2$), or equal to 0 (if $n = 1$) we have $\tilde{p}(\partial\Delta_\lambda^2) \not\sim 0$ (not homologous to zero in X). Moreover, for every $z = (\lambda, z_1)$ in Δ^n of the form $(0, z_1)$, $z_1 \neq 0$ we have that $\tilde{p}|_{\{z\} \times \partial\Delta}$ is an imbedding.

Therefore we get a foliated shell in \mathcal{L}_D^0 (or a family of foliated shells $\tilde{p}(\partial\Delta_\lambda^2)$ if $n \geq 2$).

(ii) \Rightarrow (i) Suppose now that \mathcal{L}_D^0 contains a foliated shell $h : (B^\varepsilon, \mathcal{L}^\pi) \rightarrow (X^0, \mathcal{L}^0)$. By the Theorem 2.2 this can be extended to a foliated meromorphic map (in particular this extension stays to be a generic injection) $h : (P^\varepsilon \setminus S, \mathcal{L}^\pi) \rightarrow (X, \mathcal{L})$, where P^ε is the ε -neighborhood of the polydisk P and $S = S_1 \times \bar{\Delta}_{1-\varepsilon}$ for some non-empty (!) complete polar compact $S_1 \subset \Delta_{1-\varepsilon}$. Note that we don't need to make any reparametrizations here. Without loss of generality we suppose that $S_1 \ni \{0\}$. Denote by z^0 the image under h of the point $q = (0, 1) \in P^\varepsilon$ - the future reference point for the leaf \mathcal{L}_{z^0} which contains $h(\{0\} \times \partial\Delta)$. Since $h|_{\{z_1\} \times \partial\Delta}$ is not an imbedding only for finite set of z_1 -s we can shrink Δ and suppose that for all $z_1 \neq 0$ the restriction $h|_{\{z_1\} \times \partial\Delta}$ is an imbedding. In fact it will be an imbedding on some annulus $A_{1-\varepsilon, 1+\varepsilon}$ for some $\varepsilon > 0$ - the same for all $z_1 \neq 0$ - and therefore also it will be an imbedding on the disk $\Delta_{1+\varepsilon}$ provided $z_1 \notin S_1$.

Now we can lift $h|_{\Delta_\varepsilon \times A_{1-\varepsilon, 1+\varepsilon}}$ for some $\varepsilon > 0$, small enough, to an imbedding $f := p^{-1} \circ h : \Delta_\varepsilon \times A_{1-\varepsilon, 1+\varepsilon} \rightarrow \hat{\mathcal{L}}_D^0$. This should be explained in more details. Consider $p^{-1} \circ h$ along $\partial\Delta_0$. It cannot be multivalued because for $z_1 \sim 0, z_1 \notin S_1$ this map is defined and singlevalued on the disk Δ_{z_1} . Moreover $(p^{-1} \circ h)|_{\partial\Delta_0}$ is also univalent. And this again follows from the same property of $(p^{-1} \circ h)_{\Delta_{z_1}}$ and Rouché's theorem. The rest is clear.

If $n \geq 2$ we can extend this lifting to a holomorphic foliated imbedding $f : \Delta_\varepsilon^n \times A_{1-\varepsilon, 1+\varepsilon} \rightarrow \hat{\mathcal{L}}_D^0$ (taking a smaller $\varepsilon > 0$ if necessary). This follows from the fact that $\Delta_\varepsilon \times A_{1-\varepsilon, 1+\varepsilon}$ is Stein, so $f(\Delta_\varepsilon \times A_{1-\varepsilon, 1+\varepsilon})$ has a Stein neighborhood (after shrinking ε , see [Si]) and from the absence of holonomy on $\hat{\mathcal{L}}_D$. At this moment we fix an *imbedded* (!) transversal $h(\Delta_\varepsilon^n \times \{q\})$ and name it as D . From that moment our mappings f (respectively h) and their future reparametrizations are mappings over Δ_ε^n (in the regions where this makes sense), i.e., fibers Δ_z are mapped into fibers $\hat{\mathcal{L}}_z$ (or \mathcal{L}_z respectively).

We know already that for $z = (0, \dots, 0, z_1) \in \Delta_\varepsilon^{n-1} \times \Delta_\varepsilon, z_1 \notin S_1$ $h|_{\{z\} \times \partial\Delta}$ extends to an imbedding of $\{z\} \times \Delta_{1+\varepsilon}$ to \mathcal{L}_D^0 . Therefore it extends after a reparametrization onto $\{z\} \times \Delta_{1+\varepsilon}$ for z -s in an open non-empty subset of Δ_ε^n (a neighborhood of any such $(0, \dots, 0, z_1)$). The same is true therefore for $f = p^{-1} \circ h$. Theorem 2.2 gives us an extension \tilde{h} of h after a reparametrization onto $\tilde{W} \setminus \tilde{S}$ and this extension is a foliated meromorphic immersion which is generically injective. Therefore the same is true for f , i.e., f extends after a reparametrization to $\tilde{f} : \tilde{W} \setminus \tilde{S} \rightarrow \hat{\mathcal{L}}_D$. Remark that \tilde{S} is not empty and up to introducing new coordinates (locally near the fiber \tilde{W}_0) we can suppose that $\tilde{W} = \Delta^n \times \Delta_{1+\varepsilon}$, $\tilde{S} = \tilde{S}_1 \times \Delta_{1-\varepsilon}$ where \tilde{S}_1 as in Theorem 2.2 and $\tilde{S}_1 \ni 0$. The following diagram could be useful here:

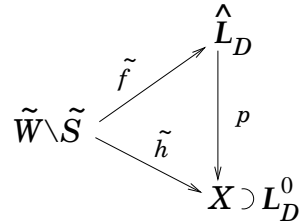


FIGURE 5. Diagram relating \tilde{f} , p and \tilde{h} : \tilde{f} is a foliated imbedding (i.e., it is holomorphic), while both \tilde{h} and p are, in general, meromorphic immersions.

We claim that $\hat{\gamma}_0 := \tilde{f}|_{\Delta_0}(\partial\Delta)$ is a vanishing cycle in $\hat{\mathcal{L}}_{z^0}$. Since for all $z_1 \notin S_1$ the restriction $\tilde{f}|_{\{(0, z_1)\} \times \partial\Delta_{1+\varepsilon}}$ is an imbedding, we get that $\tilde{f}|_{\{(0, z_1)\} \times \Delta_{1+\varepsilon}}$ is an imbedding to and therefore $\hat{\gamma}_{z_1} := \tilde{f}|_{\{(0, z_1)\} \times \partial\Delta}$ is homotopic to zero in the corresponding leaf. All is left to prove is that $\hat{\gamma}_0$ doesn't bound a disk in $\hat{\mathcal{L}}_{z^0}$. But would $\hat{\gamma}_0$ bound a disk Δ^0 in $\hat{\mathcal{L}}_{z^0}$ our foliation on $\hat{\mathcal{L}}_D$ in a neighborhood of Δ^0 would be biholomorphic to the product $\Delta^n \times \Delta$ with $\Delta^0 := \{0\} \times \Delta$ and $\Delta^z := \{z\} \times \Delta$ being the leaves of $\hat{\mathcal{L}}_D$. Now, for all $(\lambda, z_1) \in \Delta_\varepsilon^n$ (with $\varepsilon > 0$ again to be taken small enough) \tilde{f} sends $\partial\Delta_{\lambda, z_1}$ to some imbedded loop in some $\Delta^{\varphi(\lambda, z_1)}$, where $\varphi : \Delta_\varepsilon^n \rightarrow \Delta^n$ is some holomorphic map sending 0 to 0.

Now observe that $\text{area}(p(\Delta_{\varphi(\lambda, z_1)}))$ stays bounded when $\varphi(\lambda, z_1)z \rightarrow 0$. All is left is to remark that $p(\Delta_{\varphi(\lambda, z_1)}) = \tilde{h}(\Delta_{(\lambda, z_1)})$ for $(\lambda, z_1) \notin \tilde{S}_1$. Therefore this implies that $\text{area}(\tilde{h}(\Delta_{(\lambda, z_1)}))$ stays bounded as $(\lambda, z_1) \rightarrow 0$ and $(\lambda, z_1) \notin \tilde{S}_1$. But this contradicts to (2.21) and to the fact that $0 \in \tilde{S}_1$ is an essential singular point of \tilde{h} .

□

In the process of proof of Theorem 1 we saw that vanishing cycles appear exactly in the fibers $\hat{\mathcal{L}}_z$ for z belonging to the closed $(n-1)$ -polar set S_1 of Hausdorff dimension $2n-2$. Therefore we obtain the following:

Corollary 3.1. *Let \mathcal{L} be a holomorphic foliation by curves on a disk-convex $(n+1)$ -dimensional complex manifold X which admits a pluriclosed taming form and let D be a transversal. Then the subset $S_1 \subset D$ of points s such that the completed holonomy leaf $\hat{\mathcal{L}}_s$ contains a vanishing cycle is complete $(n-1)$ -polar of Hausdorff dimension $2n-2$.*

We also have the following

Corollary 3.2. *If a disk-convex pluritamed foliated manifold (X, \mathcal{L}, ω) contains a foliated shell that there exists a nontrivial closed, positive $(1,1)$ -current T tangent to \mathcal{L} .*

From Theorem 2.2 we see that (2.21) holds in our case. $h|_{\partial\Delta_{q^n}}$ converges but $\text{area}(h(\Delta_{q^n}))$ diverges to infinity. Therefore by standard (and obvious) reasoning currents

$$T_n = \frac{[h(\Delta_{q^n})]}{\text{area}(h(\Delta_{q^n}))} \quad (3.3)$$

converge to a closed, positive current of mass one tangent to \mathcal{L} . □

3.5. Imbedded vanishing cycles and proof of Theorem 2. First of all let us make precise what we mean by an imbedded essential vanishing cycle. Let $\gamma_0 \subset \mathcal{L}_{z^0}^0$ be an imbedded loop and let $d \geq 1$ be the order of the holonomy of \mathcal{L} along γ_0 . Denote by $\hat{\gamma}_0 \subset \hat{\mathcal{L}}_{z^0}^0$ the lift of γ_0 . Then $p|_{\hat{\gamma}_0} : \hat{\gamma}_0 \rightarrow \gamma_0$ is a regular covering of order d .

Definition 3.5. *An imbedded essential vanishing cycle in \mathcal{L}_{z^0} is a loop $\gamma_0 \subset \mathcal{L}_{z^0}$ for which the following items are satisfied:*

- γ_0 is imbedded in $\mathcal{L}_{z^0}^0$, it admits a lift $\hat{\gamma}_0$ which is imbedded in $\hat{\mathcal{L}}_{z^0}^0$ and regularly covers γ_0 with degree d .
- $\hat{\gamma}_0$ doesn't bound a disk on $\hat{\mathcal{L}}_{z^0}^0$.
- For some (and therefore for a generic) sequence $\{z_n\} \subset D$ converging to z^0 there are imbedded loops $\hat{\gamma}_n$ in $\hat{\mathcal{L}}_{z_n}^0$ uniformly converging to $\hat{\gamma}_0$, each bounding a disk D_{z_n} in $\hat{\mathcal{L}}_{z_n}^0$.

Remark 3.5. Condition γ_0 to be in $\mathcal{L}_{z^0}^0$ and not just in \mathcal{L}_{z^0} is not innocent at all. One may not be able to perturb an imbedded $\gamma_0 \subset \mathcal{L}_{z^0}$ (which admits a lift) in the way that this perturbation still admits a lift lifts to $\hat{\mathcal{L}}_{z^0}^0$. And this will be needed in the proof (and it is actually an important issue).

Now we state the precise version of the Theorem 2 from the Introduction.

Theorem 3.2. *Let (X, \mathcal{L}) be a disk-convex foliated manifold which admits a dd^c -closed taming form and let $D \subset X^0$ be a transversal. Then the following statements are equivalent:*

- i) *Some leaf $\mathcal{L}_{z^0} \subset \mathcal{L}_D$ contains an imbedded essential vanishing cycle.*
- ii) *\mathcal{L}_D contains an imbedded foliated cyclic shell.*

Proof. (i) \Rightarrow (ii) For a given transversal $D \ni z^0$ we need to produce from an *imbedded essential vanishing cycle* in $\mathcal{L}_{z^0} \subset \mathcal{L}_D$ an *imbedded foliated cyclic shell* in \mathcal{L}_D^0 .

Take open cells $U \ni z_n, V \ni z^0, U \subset V \subset D$ such that for an appropriate Hartogs figure $(W, \pi, U, V) \subset \hat{\mathcal{L}}_V^0$ mapping $p : \hat{\mathcal{L}}_V^0 \rightarrow X$ restricted to W is a foliated holomorphic immersion, which extends (after a reparametrization) to a foliated meromorphic immersion $p : W \setminus S \rightarrow X$ as in Theorem 2.2 (we drop tildes for the simplicity of our notations).

Note that d is the maximal cardinality of the holonomy along loops $\gamma_z := p(\partial\Delta_z) \subset \mathcal{L}_z$ close to $\gamma_{z^0} = \gamma_0$ for z in a neighborhood of z^0 . Find a coordinate system $\Delta^{n-1} \times \Delta^2$ in a neighborhood of W_{z^0} in W as in Theorem 2.2, actually we shrink W to have $W = \Delta^{n-1} \times \Delta^2$ in the sequel. We keep noting coordinates in $\Delta^{n-1} \times \Delta^2$ as (λ, z_1, z_2) . Note that (λ, z_1) are coordinates in a neighborhood of z^0 on D . Actually we shrink D and

suppose that $D = \Delta^n$ with coordinates λ, z_1 . Coordinates are chosen in such a way that z^0 correspond to $(\lambda = 0, z_1 = 0)$.

Due to Theorem 2.2 the restriction to S of the natural projection $\pi_2 : \Delta^{n-1} \times \Delta^2 \rightarrow \Delta^{n-1}$ is proper and surjective. Of course, one should remark here that $S \neq \emptyset$ and, moreover, $\pi(S) = S_1 \ni z^0$, because otherwise γ_{z^0} would not be an essential vanishing cycle! By our assumption the restriction $p|_{W_{z^0}} : W_{z^0} \setminus S_{z^0} \rightarrow \mathcal{L}_{z^0} \subset X^0$ is a regular covering of order $d \geq 1$ between an appropriate annuli in the source and target curves. For $1 \leq l \leq d$ denote by A_l the analytic set in Δ^n which consists from points q such that the cardinality of the holonomy along γ_q is at least l . Remark that $z^0 = 0 \in A_d$, $A_1 = D$ and we set by definition $A_{d+1} = \emptyset$. Take a minimal l such that $S_1 \cap (A_l \setminus A_{l+1}) \neq \emptyset$. Call it l_0 . Here, as usual, $S_1 = \pi(S)$ is the image of the singularity set S under the natural projection $\pi : \Delta^{n+1} \rightarrow \Delta^n$ (see Theorem 2.2).

Case 1. $l_0 = 1$.

Take a point $s_1 \in S_1 \cap (A_1 \setminus A_2)$ and shrink our transversal D once more to a polydisk $D = \Delta^n$ - a neighborhood of s_1 . We can suppose that $s_1 = 0$ in these coordinates. If this neighborhood was taken small enough our foliation has no holonomy along γ_z for $z \in D$. Therefore $p : \partial_0 W|_D \rightarrow X^0$ is an imbedding. Consider the disk $\Delta_0 := \{0\} \times \Delta \subset \Delta^{n-1} \times \Delta$ and consider the restriction $W|_{\Delta_0} = \Delta_0 \times \Delta$ and the restriction of p to $W|_{\Delta_0}$. Recall that $\Delta_0 \cap S_1$ is a complete polar compact subset of Δ_0 and, in particular, its Hausdorff dimension is zero.

Lemma 3.5. *There exists a finite union of imbedded loops $\beta \subset \Delta_0$ which bound a relatively compact domain $G \subset \Delta_0$ such that:*

- a) $G \cap S_1 \neq \emptyset$ and $\partial G \cap S_1 = \emptyset$.
- b) $p|_{\bigcup_{z \in \beta} W_z}$ is injective.
- c) Moreover, $p\left(\bigcup_{z \in \beta} W_z \cup \partial_0 W|_G\right)$ is an imbedding.

Proof. As in Section 2 consider the area function $a(z_1) = \int_{W_{z_1}} p^* \omega$ for $z_1 \in \Delta_0 \setminus S_1$. Function a is positive, smooth and tends to infinity when $z_1 \rightarrow S_1$, see Corollary 2.4 (by S_1 here we understand now $S_1 \cap \Delta_0$ - but we not introduce any new notations). By Sard's lemma for a generic positive c the level set $\beta_c = \{z_1 : a(z_1) = c\}$ is a union of smooth curves in Δ_0 . In the sequel c will be always taken bigger then $\inf\{a(z_1) : z_1 \in \partial\Delta_0\}$, i.e., our curves will be all closed and situated away from $\partial\Delta_0$.

Claim 1. β_c has finite number of irreducible components. Suppose not and denote by β_c^i the sequence of irreducible components of β_c . Let q be an accumulation point of β_c^i . q belongs to S_1 , because $\bigcup_i \beta_c^i$ is a smooth manifold. But this contradicts Lemma 2.2. Really, $\bigcup_i \beta_c^i$ is thick at q and therefore p should extend to a neighborhood of W_q . This contradicts to the fact that W_q is an essential singular point of p .

Remark that we are working here with $p|_{W_{\Delta_0}}$ and use the fact that W_q contains a singular point of this restriction. This follows from the homological characterization (b_3) of essential singularities of p in Theorem 2.2.

Claim 2. p is injective on $W|_{\beta_c}$. First of all p is injective on each W_{z_1} , $z \in \Delta \setminus S_1$ because it is injective on ∂W_{z_1} . Suppose that for some $z_1, z_2 \in \beta_c$, $z_1 \neq z_2$ one has $p(W_{z_1}) \cap p(W_{z_2}) \neq \emptyset$. Since $p(\partial W_{z_1}) \cap p(\partial W_{z_2}) = \emptyset$ we have that $p(W_{z_1}) \subset p(W_{z_2})$ (or vice versa). But this contradicts to the fact that $\text{area}(p(W_{z_1})) = \text{area}(p(W_{z_2})) = c$.

For every i denote by D^i the compact component of $\Delta_0 \setminus \beta_c^i$. Fix some point $s_1 \in S_1$. Take one of D^i -s, namely such that $D^i \ni s_1$. Denote it as D^1 and its boundary curve as β^1 . If p is not injective on $\partial_0 W|_{D^1} \cup W|_{\beta^1}$ then there exists $z_1 \in D^1$ such that $p(W_{z_1}) \cap p(W_{z_2}) \neq \emptyset$ for some $z_2 \in \beta^1$. Since $p(\partial W_{z_1}) \cap p(\partial W_{z_2}) = \emptyset$ we have two possibilities. First: $p(W_{z_1}) \supset p(W_{z_2})$ but this simply doesn't imply that p is not injective on $\partial_0 W|_{D^1} \cup W|_{\beta^1}$. Therefore we are left with the second one: $p(W_{z_1}) \subset p(W_{z_2})$.

Claim 3. If $p(W_{z_1}) \subset p(W_{z_2})$ then there exists $\beta_c^j \in D^1$. This is obvious, take a path from z_1 to S_1 inside D^1 . Then it will contain a point z with $a(z) = c$.

If this β_c^j surrounds our point s_1 call it β^2 and the compact component of $\Delta_0 \setminus \beta^2$ call D^2 . If this is not the case call $\beta^1 \cup \beta_c^j$ as β^2 and the region bounded by them as D^2 . Note that in both cases D^2 contains s_1 .

The process $D^1 \supset D^2 \supset \dots$ is finite because the number of β_c^i -s is finite. Therefore after a finite number of steps we will get $D^N =: G$ and $\beta^N =: \beta = \partial G$ such that p injective on $\partial_0 W|_G \cup W|_\beta$ and G has the required properties. \square

Since (taking initially Δ_0 small enough) we can suppose that $W|_\Delta$ is biholomorphic to $\Delta \times \Delta$ we get a pseudoconvex domain $G \times \Delta \subset W|_\Delta$ such that p has an essential singularity inside of this domain. By Theorem 2.2 this means that $p(\partial(G \times \Delta))$ is not homologous to zero in X . Set $B = \partial(G \times \Delta)$, then $p(G)$ is an imbedded foliated shell in (X, \mathcal{L}) .

Remark 3.6. (a) Note that cyclic quotients didn't appear at this case, but the topology of the shell became complicated.

(b) Note also that G is found such that it contains an ad hoc taken point $s_1 \in S_1 \cap (A_1 \setminus A_2)$, i.e., the constructed shell is centered at this s_1 . This will be used in the sequel.

Case 2. $l_0 > 1$.

Set $A = \bigcup_{l \geq 2} A_l$. This is a proper analytic subset of D . Changing the slope of z_1 -coordinate and shrinking a neighborhood of z^0 , if necessary, we can suppose that the projection $\pi_1|_A : A \rightarrow \Delta^{n-1}$ is proper. Here $\pi_1 : \Delta^n \rightarrow \Delta^{n-1}$ is the natural projection $(\lambda, z_1) \rightarrow \lambda$. Take an irreducible component A' of $A_{l_0} \setminus A_{l_0+1}$ which contains a point $s_1 \in S_1$. Shrinking D , if necessary, we can suppose that $A_{l_0+1} \cap D = \emptyset$ and D intersect A_{l_0} only by A' .

Claim 4. A' has pure dimension $n - 1$ and is entirely contained in S_1 .

Choose coordinates (λ, z_1) in a neighborhood of $s_1 = (0, 0)$ in $D = \Delta^{n-1} \times \Delta$ in such a way that $\pi_1|_{A'}$ is proper. If $\dim A' < n - 1$ then $\dim \pi_1(A') < n - 1$. But we know that for every $\lambda \in \Delta^{n-1} \setminus \pi_1(A')$ there exists at least one z_1 such that $(\lambda, z_1) \in S_1$. Remark also that the holonomy along γ_{λ, z_1} for such $z = (\lambda, z_1)$ is less than l_0 . Contradiction to the definition of l_0 .

Therefore $\dim A' = n - 1$ and $\pi_1|_{A'} : A' \rightarrow \Delta^{n-1}$ is a (ramified) covering. Note that $S_1 \subset A'$ by the definition of l_0 . If there exists a point $q \in A' \setminus S_1$ then from homological characterization (b_3) in Theorem 2.2 it follows that no point of A' belongs to S_1 . Contradiction.

From now on we can suppose that $A' = S_1$ is smooth and given by the equation $z_1 = z_2 = 0$. Let $g : D \rightarrow D$ be a local biholomorphism generating the holonomy along γ_0 . Remark that $g|_{A'} \equiv \text{Id}$ and $g^{l_0} \equiv \text{Id}$.

Claim 5. In an appropriate coordinates with center at s_1 the automorphism g has the form $g(\lambda, z_1) = (\lambda, e^{\frac{2\pi il}{l_0}} z_1)$ for some $l \in \{1, \dots, l_0\}$ relatively prime with l_0 .

This is a nearly standard fact which easily follows from the famous Bochner's linearization theorem, see [Bo]. Really, due to [Bo] one can find coordinates in which g is linear and still preserving $A' = \{z_1 = 0\}$. Therefore g has the form $g(\lambda, z_1) = (\lambda, e^{\frac{2\pi il}{l_0}} z_1)$ for some $l \in \{1, \dots, l_0 - 1\}$ relatively prime with l_0 .

Factorize $\Delta^{n-1} \times \Delta \times \Delta$ by the action $(\lambda, z_1, z_2) \rightarrow (\lambda, e^{\frac{2\pi il}{l_0}} z_1, e^{\frac{2\pi i}{l_0}} z_2)$ to get $\Delta^{n-1} \times \mathcal{X}^{l, l_0}$, where \mathcal{X}^{l, l_0} is a surface with cyclic quotient singularity. We get a holomorphic foliated immersion $p: \Delta^{n-1} \times \mathcal{X}^{l, l_0} \setminus S^{l, l_0} \rightarrow X$, where S^{l, l_0} - image of S under the factorization.

Remark that $p|_{\partial(\{0\} \times \mathcal{X}^{l, l_0})}$ is now an imbedding. Therefore we can repeat arguments of Lemma 3.5 and prove that $p|_{\partial(\{0\} \times \mathcal{X}^{l, l_0})}$ is injective in an neighborhood of the boundary B of the domain $W_{l, l_0} = \bigcup_{z \in G} W_z$ for some $G \Subset \Delta$. Would $p(B)$ be homologous to zero in X then by (b_3) of the Theorem 2.2 would imply the extensibility of p onto our domain W_{l, l_0} and this is not the case. I.e. we got an *imbedded* foliated cyclic shell.

Remark 3.7. We silently used here a version of Theorem 2.2 in the spaces with cyclic singularities. One can either prove such version directly, or "lift" the problem to the covering of \mathcal{X} (which is a bicylinder), apply extension there and push the extended map down. This is possible, because the extended map will be also invariant under the action of the cyclic group by the uniqueness theorem for holomorphic functions.

$(ii) \Rightarrow (i)$ Let $h: (B^\varepsilon, \mathcal{L}^\pi) \rightarrow (X, \mathcal{L})$ be an imbedded foliated cyclic shell. We can proceed literally as in the proof of $(ii) \Rightarrow (i)$ of Theorem 3.1. All we need to do is to see that the cycle $h|_{\{0\} \times \partial\Delta}: \{0\} \times \partial\Delta \rightarrow \mathcal{L}_{z^0}^0$ - proved to be a vanishing one - was imbedded from the every beginning. Further details will be omitted. Theorem is proved. □

Remark 3.8. Notice, that in the Theorem 3.2 the place for the shell is less precise then in Theorem 3.1. But let us still make a precision here. Let $D = \Delta^{n-1} \times \Delta$ in a neighborhood of $z^0 = (0, 0)$ as above. Then we proved, in fact, that we can find $\lambda \in \Delta^{n-1}$ arbitrarily closed to 0 such that $\mathcal{L}_{\Delta_\lambda}^0$ will contain an imbedded foliated shell centered at given $s_1 \in \Delta_\lambda$. If $n = 1$ then this λ is 0. (remember of able to center our shell in a generic point s_1 on S_1 near z^0).

4. PLURICLOSED METRIC FORMS AND FOLIATED SPHERICAL SHELLS

4.1. Pluriclosed metric forms and foliated spherical shells. Up to now our immersed shells were boundaries of the bicylinder (or pseudoconvex hypersurfaces close to it). One might ask if the CR -geometry is relevant here? The test question would be: can one take as shells the images of the standard spheres (with the standard vertical foliation) and not such a Levi-flat objects as boundaries of bicylinders? Every book in complex analysis explains in its first chapter how different these objects are. In the context of this paper this issue goes together in one line with reducing of the size of the essential singularity set S that is "virtually present" in the heart of all our proofs. And this task is of capital importance. It appears to be crucial for getting from vanishing cycles the *imbedded* ones.

At present we are able to reduce the size of S (equivalently to pass to spheres as shells) only in the case when our dd^c -closed taming form is actually a metric form on X , i.e.,

ω should be not just a dd^c -closed form positive in the directions tangent to \mathcal{L} but in all directions in TX .

Let $B = \mathbb{S}^3 = \{z = (z_1, z_2) \in \mathbb{C}^2 : \|z\| = 1\}$ denote the unit sphere in \mathbb{C}^2 , $P = \{z \in \mathbb{C}^2 : \|z\| < 1\}$ - the unit ball. For some $0 < \varepsilon < 1$ set $B^\varepsilon = \{z \in \mathbb{C}^2 : 1 - \varepsilon < \|z\| < 1 + \varepsilon\}$ - a shell around \mathbb{S}^3 . Denote by $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$ the canonical projection $\pi(z) = z_1$ onto the first coordinate of \mathbb{C}^2 . Note that B^ε is foliated by π over the disk $\Delta_{1+\varepsilon}$ of radius $1 + \varepsilon$ (Δ_r denotes in the sequel the disk of radius $r > 0$). Denote this foliation as \mathcal{L}^π . Its leaves $\mathcal{L}_{z_1} := \pi^{-1}(z_1)$ are disks if $1 - \varepsilon < |z_1| < 1 + \varepsilon$ and are annuli if $|z_1| < 1 - \varepsilon$.

Definition 4.1. The pair $(B^\varepsilon, \mathcal{L}^\pi)$ we shall call the *standard foliated spherical shell*.

Let $h : (B^\varepsilon, \mathcal{L}^\pi) \rightarrow (X^0, \mathcal{L}^0)$ be some generically injective foliated holomorphic immersion of the standard foliated spherical shell into (X^0, \mathcal{L}^0) . Denote by Σ the image of the unit sphere \mathbb{S}^3 under h .

Definition 4.2. $h(B^\varepsilon)$ is called a *foliated spherical shell* in (X, \mathcal{L}) if Σ is not homologous to zero in X .

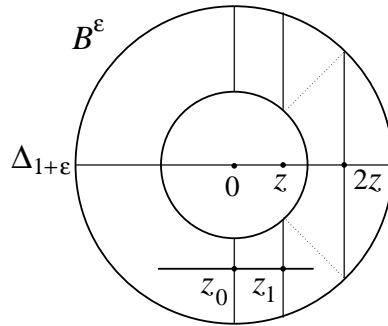


FIGURE 6. The "vertical foliation" on the Hopf surface is again the simplest example. The leaf \mathcal{L}_0^0 is a torus, $\mathcal{L}_z^0 = \mathcal{L}_{2z}^0$ is a disk for $z \neq 0$. The cycle $\gamma = \{(0, z) : |z| = 1\}$ is a vanishing cycle. Image of the ε - neighborhood of \mathbb{S}^3 together with the "vertical foliation" under the natural projection $\mathbb{C}^2 \setminus \{0\} \rightarrow H$ is a foliated spherical shell in (H^2, \mathcal{L}) .

Remark 4.1. Let us recall the Main Theorem from [Iv3], where we worked with pluriclosed *metric* forms. We proved there that the singularity set S from the Theorem 2.2 is "small" in the sense that for every $\lambda \in \Delta^{n-1}$ (see notations in Theorem 2.2) the set $S_\lambda := S \cap \Delta_\lambda^2$ is a complete pluripolar compact of Δ_λ^2 of Hausdorff dimension zero.

Proposition 4.1. *Let (X, \mathcal{L}, ω) be a disk-convex pluritamed holomorphic foliation by curves. Suppose that ω is actually a metric form and that a foliated manifold (X, \mathcal{L}) contains a foliated shell $h : (B^\varepsilon, \mathcal{L}^\pi) \rightarrow (X^0, \mathcal{L}^0)$. Then:*

- i) h extends to a foliated meromorphic immersion of $P^\varepsilon \setminus S$ where S is a complete pluripolar compact subset of P of Hausdorff dimension zero.
- ii) (X, \mathcal{L}) contains a foliated spherical shell.

Proof. The proof is immediate because as a shell we can take a standard 3-sphere $\mathbb{S}_r^3(s)$ around any point s of S . A radius r should be chosen in such a way that $\mathbb{S}_r^3(s) \cap S = \emptyset$. And this is possible due to the null-dimensionality of S .

□

Remark 4.2. (a) In this proof we didn't use the condition on h to be a foliated map, because any holomorphic mapping $h : B^\varepsilon \rightarrow X$, where X admits a pluriclosed metric form, extends to a meromorphic map from $P^\varepsilon \setminus S$ to X with S being a complete polar compact subset of P of Hausdorff dimension zero.

(b) The conclusion of this Proposition remains valid if ω is supposed to be a pluriclosed taming form for \mathcal{L} and there exists some other plurinegative metric form ω_1 on X (irrelevant to \mathcal{L}). Really, all we need is to reduce the size of the essential singularity set S in the Theorem 2.2 and this can be done with the help of ω_1 .

4.2. Imbedded vanishing cycles. It occurs that the reduction of the size of S already made is still not enough. We need S to be almost an analytic set. Let us formalize this by giving the following:

Definition 4.3. *A foliated manifold (X, \mathcal{L}) of dimension $n+1 \geq 2$ is called almost Hartogs if the following is satisfied. Given any transversal $D \subset X$. Then:*

i) *Every foliated holomorphic immersion $h : (W, \pi, U, D) \rightarrow (X, \mathcal{L})$ of a non-trivial generalized Hartogs figure of dimension $\dim X$ extends to a foliated meromorphic immersion of $(\tilde{W} \setminus S, \pi, D)$ into (X, \mathcal{L}) after a reparametrization.*

ii) *The essential singularity set S is a closed subset of \tilde{W} , which is at most countable union of analytic subsets of \tilde{W} of pure codimension two.*

As usual “meromorphic foliated immersion” means here that the extended h is a foliated immersion outside of its indeterminacy set. However, one should remark that the only point here is to extend h : if a meromorphic extension of h onto $\Delta_{1+\varepsilon}^{n+1} \setminus A$ is possible then it will be automatically a foliated immersion outside of its indeterminacy set. If A happens to be empty for every such mapping into (X, \mathcal{L}) then the latter is called simply “Hartogs”.

Our goal in this subsection is to reduce the problem of finding imbedded vanishing cycles in a shelled foliations to the proof of the almost Hartogs extension property of (X, \mathcal{L}) . This can be done in many interesting cases, see [Iv1, Iv2, Iv3, Iv4]. In particular, the Theorem 3.3 from [Iv3] can be restated in the following form:

Proposition 4.2. *Suppose that a foliated manifold (X, \mathcal{L}) admits a pluriclosed taming form ω , such that ω is actually a metric form and $d^c\omega \in H^3(X, \mathbb{Z})$. Then (X, \mathcal{L}) is almost Hartogs.*

One more example is a result from [Iv4] (it doesn't requires any metric form on the total space X):

Proposition 4.3. *Suppose that the manifold X is an elliptic fibration (with possibly singular fibers) over a disk-convex Kähler manifold Y . Then every holomorphic foliation by curves on X is almost Hartogs.*

Really, let $f : (W, \pi, U, V) \rightarrow X$ be a holomorphic map. If $pX \rightarrow Y$ is the holomorphic mapping defining the elliptic fibration then the composition $p \circ f$ extends onto W after a reparametrization by [Iv2] and [Br2]. Following the arguments in [Iv4] one gets an extension of f onto $W \setminus S$ where S is the indeterminacy set of $p \circ f$ (reparametrizations do not cause any problems here).

Now we are prepared to state the main result of this Section:

Theorem 4.1. *Let (X, \mathcal{L}) be a disk-convex foliated manifold and let $f : (H_\varepsilon, \mathcal{L}^\pi) \rightarrow \hat{\mathcal{L}}_D$ be a foliated holomorphic imbedding of the standard Hartogs figure into the holonomy covering cylinder $\hat{\mathcal{L}}_D$ for some transversal $D \subset X^0$. Suppose that:*

1) $h := p \circ f$ extends as a foliated meromorphic immersion to a complement of a closed subset $S \subset \Delta_{1+\varepsilon}^{n+1}$ which is at most countable union of analytic subsets of $\Delta_{1+\varepsilon}^{n+1}$ of pure codimension two and such that each point of S is an essential singular point for h ;

2) For some $z^0 \in \Delta_{1+\varepsilon}^n$ and the disk $\Delta_{z^0} := \{z^0\} \times \Delta_{1+\varepsilon}$ the cycle $\hat{\gamma}_0 := f|_{\Delta_{z^0}} : \partial\Delta \rightarrow \hat{\mathcal{L}}_{z^0}$ is an imbedded essential vanishing cycle in the holonomy covering leaf $\hat{\mathcal{L}}_{z^0} \supset f|_{\Delta_{z^0}}(\partial\Delta)$.

Then the leaf \mathcal{L}_{z^0} itself contains an imbedded essential vanishing cycle $\gamma_0 \subset \mathcal{L}_{z^0}^0$.

Proof. Note that from (1) we get that f itself extends as a foliated imbedding of $\Delta_{1+\varepsilon}^{n+1} \setminus S$ into $\hat{\mathcal{L}}_D$. The condition that every point $s \in S$ is an essential singularity of h (and therefore also of f) means that there exists no neighborhood $V \ni s$ such that h (and f) meromorphically (holomorphically) extends to V . Note also that Δ_{z^0} intersects S , otherwise $f|_{\Delta_{z^0}}(\partial\Delta)$ cannot be a vanishing cycle.

We shall work locally around point $z^0 \in \Delta_{1+\varepsilon}^n$ and therefore we shall take coordinates in which this point is the origin 0. $h_0 := h|_{\Delta_0} : \Delta_0 \setminus S_0 \rightarrow \mathcal{L}_{z^0}$ is a holomorphic mapping of a pluri-punctured disk $\Delta_0 \setminus S_0$ to the Riemann surface \mathcal{L}_{z^0} which factors as $h_0 = p \circ f_0$ through the holomorphic imbedding $f_0 : \Delta_0 \setminus S_0 \rightarrow \hat{\mathcal{L}}_{z^0}$. Here $S_0 := S \cap \Delta_0$ is at most countable compact in Δ_0 . Take some isolated point in S_0 , suppose it is an origin and remark that for a boundary of a small disk around the origin its image by f is an imbedded essential vanishing cycle in $\hat{\mathcal{L}}_{z^0}$. Therefore we shrink our polydisk to Δ^{n+1} to be as small as necessary to have that 0 is the only intersection point of Δ_0 with S , i.e., of $\{0\} = S_0 = S \cap \Delta_0$.

Remark that $f_0 : \Delta_0 \setminus \{0\} \rightarrow \hat{\mathcal{L}}_{z^0}$ extends to a holomorphic imbedding of the disk Δ_0 into a Riemann surface \hat{R} which is obtained from $\hat{\mathcal{L}}_{z^0}$ by adding to it a point, i.e., $\hat{R} \setminus \hat{\mathcal{L}}_{z^0} = \{a_0\}$ and a_0 is the image of 0 under the extended map (which we still denote as $f_0 : \Delta_0 \rightarrow \hat{R}$). This follows easily from the fact that $f_0 : \Delta_0 \setminus \{0\} \rightarrow \hat{\mathcal{L}}_{z^0}$ is an imbedding.

Now we need to study the behavior of h_0 . Here we have several cases.

Case 1. \mathcal{L}_{z^0} is hyperbolic.

Then h_0 extends to zero as a mapping from Δ_0 to $R := \mathcal{L}_{z^0} \cup \{b_0\}$ with $b_0 := h_0(0)$. But this is a much less trivial fact (because h_0 is not an imbedding) and is due to Ohtsuka, see [Oh1, Oh2] and for a much simpler proofs see [Re, Ro]. Now p restricted to $\hat{\mathcal{L}}_{z^0}$ extends to a holomorphic map $p_0 : \hat{R} \rightarrow R$ which is a ramified covering between neighborhoods of a_0 and b_0 . And this results to an imbedded essential vanishing cycle in \mathcal{L}_{z^0} .

In all other cases \mathcal{L}_{z^0} will be parabolic, i.e., torus, sphere, plane or punctured plane. If h_0 extends to zero as a mapping from Δ_0 to $R = \{b_0\} \cup \mathcal{L}_{z^0}$ (or, even to \mathcal{L}_{z^0} itself) then everything goes like in the Case 1. Therefore below we shall be concerned with h_0 not extending to the origin in the sense described in Case 1, i.e., with p_0 having an essential singularity at “added” point a_0 .

For all notions and facts about orbifold Riemann surfaces that we are going to use below we refer to the beautiful book of Milnor [Mil] and references there. For the rudiments about Fuchsian groups see [Be]. To start with recall the Remark 3.2, i.e., that our mapping $p_0 : \hat{\mathcal{L}}_{z^0} \rightarrow \mathcal{L}_{z^0}$ is an orbifold covering map (this follows from the very definition

of a vanishing end). Therefore we are well in the framework of [Mi1], Appendix E. Recall that for a Riemann surface orbifold (R, ν) the Euler characteristic is defined as

$$\chi(R, \nu) = \chi(R) + \sum_j \left(\frac{1}{\nu(z_j)} - 1 \right), \quad (4.1)$$

where $\chi(R)$ is the Euler characteristic of the underlying Riemann surface R and $\nu(z_j)$ is the value of ramification function ν at ramification point z_j .

Case 2. $\hat{\mathcal{L}}_{z^0}$ is hyperbolic and \mathcal{L}_{z^0} is parabolic.

In this case $\chi(R, \nu) < 0$ and $\chi(R) \geq 0$. Therefore we necessarily have at least one ramification point. Since $\hat{\mathcal{L}}_{z^0}$ is hyperbolic its universal cover $\tilde{\mathcal{L}}_{z^0}$ is the unit disk Δ .

Lemma 4.1. *Let a branched covering map $p_0 : H \rightarrow P$ of a hyperbolic Riemann surface H to a parabolic one P be given. Suppose that H has a puncture, i.e., that H is contained in some Riemann surface \hat{H} such that $\hat{H} \setminus H = \{a_0\}$ - a point in \hat{H} . Then the covering map p_0 cannot have an essential singularity at a_0 .*

Proof. Suppose that the opposite occurred. The Fuchsian group of deck transformations G_H giving $\Delta/G_H = H$ contains at least one translation T , coming from the puncture and this T is primitive. Let G_T be the cyclic group generated by T . Then Δ/G_T is the punctured disk Δ^* if T is chosen (and one can always arrange this) to be the translation by 1 (in the upper half-plane realization of Δ). Identification of Δ/G_T with Δ^* is given by $e^{2\pi i}$. In that case our original puncture can be considered as an imbedded subdisk of radius $e^{-\pi}$ in Δ^* , see Lemma 1 in [Ro]. We fix the standard inclusion $\Delta_{e^{-\pi}}^* \subset \Delta^*$ and an unramified covering $p_2 : \Delta/G_T \rightarrow \Delta/G_H = H$ induced by the inclusion $G_T \subset G_H$. This gives a mapping $i : \Delta_{e^{-\pi}}^* \rightarrow H$ which is an imbedding by Royden's Lemma, see the left half of the Figure 7.

Let G_P be the Fuchsian group of deck transformations of the cover $\Delta \rightarrow P$, i.e., $\Delta/G_P = P$. Let us explain that $G_P \supset G_H$. Really, elements of G_H , if one view them as deck transformations of the cover $\Delta \rightarrow H$, are generated by loops in H (all starting from some fixed reference point). But every such loop projects to a loop in P by our map $p_0 : H \rightarrow P$. And after that the obtained loop in P generates a deck transformation of the cover $\Delta \rightarrow P$, i.e., an element of G_P . The constructed homomorphism $G_H \rightarrow G_P$ is a monomorphism and we are done. I.e., our map p_0 fits into our diagram:

$$p_0 : H = \Delta/G_H \rightarrow \Delta/G_P = P, \quad (4.2)$$

see the lower row of our Figure 7.

Case 1. Suppose that for some $0 < r < e^{-\pi}$ the restricted mapping $p_0|_{\Delta_r^} : \Delta_r^* \rightarrow P$ maps Δ_r^* away from the ramification points of the covering $p_0 : H \rightarrow P$.*

In that case there can be only one ramification point and $P = \mathbb{C}$ or $P = \mathbb{CP}^1$ with one or two ramification points (Big Picard Theorem). \mathbb{CP}^1 with one point cannot occur because it doesn't admit ramified coverings (i.e., is a "tear drop" and for the same reason if it has two ramification the indices of them should be equal). In all these cases $P \setminus \{\text{points}\}$ stays parabolic and $H \setminus \{\text{preimages of this points}\}$ then should be also parabolic. But then H is parabolic to and this is not our case.

Therefore the mapping $p_0|_{\Delta_{e^{-\pi}}^*} : \Delta_{e^{-\pi}}^* \rightarrow P$ doesn't omit some ramification point when approaching to the puncture. I.e., p_0 takes this value, denote it as $e \in P$, infinitely many times. In that case the group G_P contains a sequence of elliptic elements E_n with centers

$$\begin{array}{ccc}
\Delta/G_T = \Delta^* \xrightarrow{q} & \Delta/G_F & \\
p_2 \downarrow & \Delta_{e^{-\pi}}^* \supset \Delta_r^* & \downarrow p_1 \\
\Delta/G_H = H \xrightarrow{p_0} & P = \Delta/G_P &
\end{array}$$

FIGURE 7

e_n converging to the puncture a_0 and all $E_n, n \geq 2$ are conjugate to E_1 . We add to this sequence also all other elliptic elements conjugated to E_1 in G_P . All E_n have the same ramification index and their centers e_n are mapped by the covering $p_0 : H \rightarrow P$ to the same point e . We should distinguish two more cases.

Case 2. Group G_P contains elliptic elements $\{F_1, F_2, \dots\}$ other than $\{E_1, \dots, E_n, \dots\}$.

Add to this list $F := \{F_1, F_2, \dots\}$ all elliptic elements of G_P other than in the list E . Let $f := \{f_1, f_2, \dots\}$ be the images of centers of F_1, F_2, \dots . Let G_F be the subgroup of G_P normally generated by T and F . Remark that $p_0|_{\Delta_{e^{-\pi}}^*} : \Delta_{e^{-\pi}}^* \rightarrow P$ lifts to some $q : \Delta_{e^{-\pi}}^* \rightarrow \Delta/G_F$ (this is because $G_F \supset G_T$), i.e., is a composition of q with the projection $p_1 : \Delta/G_F \rightarrow P$ (this is because $G_P \supset G_F$), see our picture.

If the set of centers f consists of not more than two points we add the list F to E and we remark that orders of elements from E stay bounded and now E contains all elliptic elements of G_P . This situation will be considered in the Case 3 below. Therefore suppose that f contains at least three points. But then:

- Δ/G_F is hyperbolic.
- Δ_r^* properly covers some puncture $q(\Delta_r^*) = \Delta_\rho^* \subset \Delta/G_F$ (by Ohtsuka's theorem).
- p_0 factors as q composed with projection $p_1 : \Delta/G_F \rightarrow P$, in particular, p_0 has an essential singularity at puncture of Δ_r^* if and only if p_1 has it at the puncture of Δ_ρ^* .

At the same time remark that the group G_P/G_F of the deck transformations of the cover $p_1 : \Delta/G_F \rightarrow P$ has as elliptic elements only the images of elements of E under the factorization $G_P \rightarrow G_P/G_F$. And their orders are uniformly bounded. Therefore we land to the following:

Case 3. Group G_P contains no other elliptic elements than $\{E_1, \dots, E_n, \dots\}$ and their orders are uniformly bounded.

Since the orders of all elliptic elements of G_P are bounded we can apply the remarkable result of Purzitsky, see [Pr]: G_P contains a torsion free subgroup G_1 of finite index. The strength of this result lies in the fact then one doesn't need G_P to be finite generated (and this is precisely our case!). Therefore $\Delta \rightarrow \Delta/G_1$ is an unbranched covering and therefore Δ/G_1 is hyperbolic. So the theorem of Ohtsuka applies again: Δ_r^* projects properly to Δ/G_1 for some $r > 0$ onto some puncture Δ_ρ^* . But Δ/G_1 is a finite branched covering of $\Delta/G_P = P$. Therefore under the resulting covering Δ_r^* is mapped onto a neighborhood of a puncture in Δ/G_P . But this is impossible because the mapping $\Delta_r^* \rightarrow \Delta/G_1$ has a sequence of critical points converging to the puncture, i.e., it should be constant. Contradiction.

□

Applying this Lemma to the our Case we get that $p_0 : \hat{\mathcal{L}}_{z^0} \rightarrow \mathcal{L}_{z^0}$ cannot have an essential singularity at the puncture a_0 and therefore we conclude as in the Case 1.

Case 3. Both $\hat{\mathcal{L}}_{z^0}$ and \mathcal{L}_{z^0} are parabolic.

These few cases can be listed with the help of [Mi1, Mi2]. Note that in all these cases we have both $\chi(\mathcal{L}_{z^0}) \geq 0$ and $\chi(\hat{\mathcal{L}}_{z^0}) \geq 0$.

Subcase 3a. \mathcal{L}_{z^0} is non-compact, i.e., is \mathbb{C} or \mathbb{C}^ .*

Formula (4.1) tells us that \mathcal{L}_{z^0} can be either \mathbb{C} with one ramification point, or \mathbb{C} with two of index two, or \mathbb{C}^* with no ramifications. All these cases are trivial, i.e., we always get an imbedded vanishing cycle.

Subcase 3b. \mathcal{L}_{z^0} is a torus.

In that case formula (4.1) tells us that $p : \hat{\mathcal{L}}_{z^0} \rightarrow \mathcal{L}_{z^0}$ is an unramified covering. Now every loop in \mathbb{T}^2 is homotopic to a multiply covered imbedded one and therefore $h_0(\partial\Delta_0)$ is homotopic to a multiply covered imbedded loop γ_0 and this homotopy lifts again to $\hat{\mathcal{L}}_{z^0}$. This again produces an imbedded essential vanishing cycle.

In the last two cases \mathcal{L}_{z^0} is a sphere. Then the formula (4.1) tells that $p : \hat{\mathcal{L}}_{z^0} \rightarrow \mathcal{L}_{z^0}$ is a ramified covering with either three or four ramification points $\{z_j\}$ with multiplicity function ν satisfying

$$\sum_j \left(1 - \frac{1}{\nu(z_j)}\right) = 2. \quad (4.3)$$

There are only four integer solutions of (4.3), see [Mi1] Remark E.6 and [Mi2] Corollary 4.5 for more details. Here we only list them together with the needed facts.

- The (orbifold) universal covering $\tilde{\mathcal{L}}_{z^0}$ if \mathcal{L}_{z^0} (i.e., the usual universal covering of $\hat{\mathcal{L}}_{z^0}$) is \mathbb{C} in all these cases and the group of deck transformations of the covering $\tilde{p}_{z^0} : \tilde{\mathcal{L}}_{z^0} \rightarrow \mathcal{L}_{z^0}$ is the extension of \mathbb{Z}^2 by a finite group \mathbb{Z}_n of n -roots of 1 for $n = 2, 3, 4, 6$ (i.e., one has four options). In another words the group in question is $\mathbb{Z}^2 \rtimes \mathbb{Z}_n$ - the semidirect product of \mathbb{Z}^2 with \mathbb{Z}_n .
- \mathbb{Z}^2 acts on \mathbb{C} by translations along some lattice Λ and \mathbb{Z}_n by rotations onto the angle $e^{\frac{2\pi i}{n}}$.
- In the case (2, 2, 2, 2) the lattice Λ is generated by 1 and τ where τ is an arbitrary complex number which belongs to the Siegel region $\mathcal{S} := \{\tau : |\tau| \geq 1, \operatorname{Re}(\tau) \leq 1/2, \operatorname{Im}(\tau) > 0 \text{ and } \operatorname{Re}(\tau) \geq 0 \text{ if } |\tau| = 1 \text{ or } |\operatorname{Re}(\tau)| = 1/2\}$ - the fundamental domain of $SL(2, \mathbb{Z})$. The finite group is $\mathbb{Z}_2 = \{\pm 1\}$ in this case.
- In all other cases the lattice is rigid, i.e., unique, and is determined by the condition to be invariant under the rotations from \mathbb{Z}_n for $n = 3, 4, 6$.

With this information at hand one should distinguish here two cases.

Subcase 3c. \mathcal{L}_{z^0} is a sphere and the ramification function is one of (2, 4, 4), (2, 3, 6), (3, 3, 3).

Recall the following well known fact:

The group $G_n := \mathbb{Z}^2 \rtimes \mathbb{Z}_n$ for $n = 3, 4, 6$ has no nontrivial normal subgroups of infinite index.

Really, let $N \triangleleft G_n$ be a nontrivial normal subgroup. We see it as acting on \mathbb{C} as described. Suppose N contains a rotation ρ . Take any translation $t \in G_n$. Then the commutator $t_1 := [\rho, t]$ is (obviously) a translation and it belongs to N because $[\rho, t] = \rho(t\rho^{-1}t^{-1})$ and

N is normal. But $t_2 := \rho t \rho^{-1}$ is a translation transversal to t_1 (if $n \neq 2$) and therefore $N \supset \mathbb{Z} \cdot t_1 \times \mathbb{Z} \cdot t_2$ and we are done.

Remark now that the group N of deck transformations of the covering $\tilde{\mathcal{L}}_{z^0} \rightarrow \hat{\mathcal{L}}_{z^0}$ should be a normal subgroup of the group G_n of the deck transformations of the covering $\tilde{\mathcal{L}}_{z^0} \rightarrow \mathcal{L}_{z^0}$. By the fact, just mentioned, N is either trivial or of finite index in G_n . In both cases there cannot be any vanishing cycles in $\hat{\mathcal{L}}_{z^0}$.

Subcase 3d. \mathcal{L}_{z^0} is a sphere and the ramification function is $(2, 2, 2, 2)$.

This case is not rigid in the sense that there is one conformal parameter, namely the cross-ratio of four (ramification) points on \mathbb{CP}^1 . But this doesn't matter. Again, if $N \triangleleft G_2$ is a nontrivial normal subgroup then it contains a translation t_1 as it was explained above. So $N \supset \mathbb{Z} \cdot t_1$. If N contains also a rotation then it contains also another translation $t_2 = [\rho, t]$ transversal to t_1 if t was taken transversal to t_1 , the proof goes exactly as above.

Therefore we are left with the case $N = \mathbb{Z} \cdot t$, i.e., the group $N \triangleleft G_2$ of the deck transformations of the cover $\tilde{\mathcal{L}}_{z^0} \rightarrow \hat{\mathcal{L}}_{z^0}$ can be only $\mathbb{Z} \cdot t$ in this case (other cases are trivial). Take $t = k$ for simplicity (after and appropriate choice of a basis for Λ , i.e., $1, \tau$ as above). Then $\hat{\mathcal{L}}_{z^0}$ is a cylinder $\mathbb{C}/\mathbb{Z} \cdot t$, i.e., $\hat{\mathcal{L}}_{z^0} = \mathbb{C}/k\mathbb{Z} = [0, k] + \mathbb{R}\tau$ with left and right boundary lines identified by $z \rightarrow z + k$. Every imbedded loop $\hat{\gamma}_0$ in this cylinder is homotopic to the interval $[0, k]$. Covering $\hat{\mathcal{L}}_{z^0} \rightarrow \mathcal{L}_{z^0}$ is a composition of a unramified k -sheeted covering $p_1 : \mathbb{C}/\mathbb{Z} \cdot k \rightarrow \mathbb{C}/\mathbb{Z}$ and a ramified one $p_2 : \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}/\mathbb{Z}^2 \times \mathbb{Z}_2$. Under the first mapping $\hat{\gamma}_0$ maps to a k -times taken imbedded loop $[0, 1]$. This loop is homotopic to $(k\text{-times taken}) [0, 1] + \frac{i}{4}$ and the last lies entirely in the fundamental domain of G_2 (only the ends are identified). Therefore it projects to an imbedded loop γ_0 in the factor \mathcal{L}_{z^0} . As a result we got an imbedded vanishing cycle. Figure 5 from [Mi2] might be helpful for better understanding the last few lines above.

Proposition is proved. □

Theorem 3 from the Introduction follows now immediately from this Corollary and the Theorem 4.1. More precisely, we obtain the following result.

Corollary 4.1. *Let (X, \mathcal{L}, ω) be a disk-convex, singular holomorphic foliation by curves such that the pluritaming form ω is:*

- i) ω is actually a metric form;
- ii) $d^c \omega \in H^3(X, \mathbb{Z})$.

If some leaf \mathcal{L}_{z^0} of \mathcal{L} contains an essential vanishing cycle then it contains also an imbedded essential vanishing cycle.

Remark 4.3. The same is true for disk-convex foliated manifolds (X, \mathcal{L}) provided X is a total space of an elliptic fibration (with possibly singular fibers) over a disk-convex Kähler manifold.

4.3. Imbedded shells in dimension two. It would be instructive to understand something to the very end. Also it is a time to get more examples. That's why let us look closely to foliations on compact complex surfaces. X in this subsection will denote a compact complex surface, i.e., a complex manifold of dimension two. \mathcal{L} will be a singular holomorphic foliation by curves on X . We will work only with $(X, \mathcal{L}) \in \mathcal{S}$ in this subsection.

As we know on a compact complex surface there always exists a dd^c -closed metric form. This was for the first time observed by Gauduchon in [Ga]. Moreover all compact complex

surfaces are almost Hartogs, this is explained in [Iv1, Iv4]. Really, the Kähler ones are simply Hartogs, elliptic ones are served by Proposition 4.3 and that of class VII by the Proposition 4.2. Therefore results of this paper are applicable to compact complex surface in the full scale.

The following beautiful and extremely powerful idea (I call it a “pseudoconvex surgery”) is due to Kato, we shall step by step use his results from [K1, K2] adapting them to our “foliated” case.

Pseudoconvex surgery. Let $h : (P^\varepsilon \setminus \{0\}, \mathcal{L}^\pi) \rightarrow (X, \mathcal{L})$ be an imbedded foliated shell. We keep the notations of the Introduction and of the proof of Theorem 3.2. Recall that $P = G \times \Delta$ for a domain $G \ni 0$. In an ε -neighborhood B^ε of the boundary $B = \partial P$ the mapping h is a foliated imbedding (but it is only immersion on the whole of $P \setminus \{0\}$). The origin $\{0\}$ is the only essential singular point of h . $\gamma_0 := h(\partial\Delta_0)$ is an essential vanishing cycle. Set $\Sigma := h(B)$. Denote by B_\pm^ε one sided neighborhoods of B . Set $\Sigma_\pm^\varepsilon = h(B_\pm^\varepsilon)$ as on the Figure 8. Cut X along Σ to get a connected open set $E := X \setminus \Sigma$.

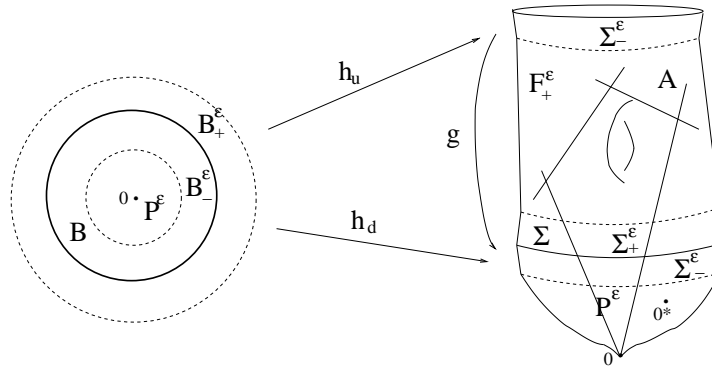


FIGURE 8. Pseudoconvex surgery.

Construct a pseudoconvex manifold F_+^ε by gluing to E the domain P^ε by the biholomorphism $h_d : B_+^\varepsilon \rightarrow \Sigma_+^\varepsilon$ - a copy of h . Note that F_+^ε inherits the foliation \mathcal{L} . Moreover, F_+^ε contains two copies of Σ_-^ε , one near its boundary - second in the interior (see our Figure). There is a natural map g between these two copies of Σ_-^ε , we refer to [K1], §1 for the construction of g . For us it will be important that g is a foliated biholomorphism in its domain of definition. Really, g comes in [K1] and [K2] as a part of a deck transformation \tilde{g} of a certain unramified covering \tilde{X} . The latter inherits a foliation $\tilde{\mathcal{L}}$ which, of course, must be preserved by \tilde{g} and therefore is preserved by g . In fact, one can see g in our Figure: in coordinates on both copies of Σ_-^ε in F_+^ε given by h_u and h_d the mapping g is the identity. h_u is an “upper” copy of h . Anyway, by the Hartogs extension theorem for holomorphic functions g extends onto the whole F_+^ε as a foliated holomorphic map $g : (F_+^\varepsilon, \mathcal{L}) \rightarrow (F_+^\varepsilon, \mathcal{L})$.

F_+^ε also contains a point - the “origin” - it comes from the origin 0 in P^ε when attaching it to E . We keep noting this point as 0 .

Claim 1. (M. Kato, Lemma 1 in [K1], Lemma 2 in [K2].) *There exists a point $0^* \in F_+^\varepsilon$ such that*

$$\bigcap_{n \geq 1} g^n(F_+^\varepsilon) = \{0^*\}. \tag{4.4}$$

Let A be the maximal compact subvariety of F_+^ε . Note that A is contracted by g to points. The set A on our Figure 8 is drawn as a chain of four segments. We need to distinguish two cases.

Case 1. $0^* \notin A$.

0^* may coincide with 0 or not, we treat both cases simultaneously. Remark that due to the fact that g is foliated and $0^* \notin A$ it is a biholomorphism in a neighborhood of 0^* . Take a cyclic quotient $\mathbb{B}_{l,d}$ of the standard ball $\mathbb{B} = \{z \in \mathbb{C}^2 : \|z\| < 1\}$ centered at 0^* and contained in some $g^{n_0}(F_+^\varepsilon)$ if $0^* = 0$ and our shell was (l,d) -cyclic shell. If $0^* \neq 0$ or $d = 1$ then it is just the ball.

Now as in [K2], Lemma 5 one proves that, if $d = 1$, then:

- i) g is a contracting biholomorphism in a neighborhood of 0^* .
- ii) Moreover there exists a strongly plurisubharmonic function φ near 0^* such that for every $c > 0$ small enough $P_c := \{z : \varphi(z) < c\}$ is biholomorphic to \mathbb{B} and g contracts each P_c , i.e., $g(P_c) \Subset P_c$.

The proof is entirely local. Therefore if $d > 1$ one lifts g from $\mathbb{B}_{l,d}$ to \mathbb{B} and has the same properties for the lifted local biholomorphism. In the first case one gets a primary Hopf surface in the second - non-primary. Namely, one masters from the shell between ∂P_c and $g^n(\partial P_c)$ (for appropriate n and $c > 0$) a Hopf surface and proves that our surface X blows down to this one, call it Y and the foliation obtained in Y denote by \mathcal{F} . Bimeromorphic transformations/unramified coverings do not effect essential (!) vanishing cycles. In particular our leaf \mathcal{L}_0 with a vanishing cycle γ_0 descends to the same in Y with foliation \mathcal{F} . Now let us see what happens in Y . Our foliation is vertical in an appropriate coordinates near 0^* . Therefore, after appropriate change in z_1 -coordinate can write the contracting map g (or its lift) in \mathbb{B} in the following form:

$$g(z_1, z_2) = (\alpha_1 z_1, \alpha_2 z_2 + z_1 g_1(z_1, z_2)), \quad (4.5)$$

where $0 < |\alpha_1|, |\alpha_2| < 1$. Now it is obvious that it is the central fiber \mathcal{F}_0 of \mathcal{F} which carries an essential vanishing cycle and this fiber is a torus.

Case 2. $0^* \in A$.

Take a connected component of A containing 0^* and from this moment denote it as A . Let $\lambda : \tilde{F}_+^\varepsilon \rightarrow F_+^\varepsilon$ be the minimal resolution of singularities of F_+^ε and let B be the proper preimage of 0. Remark that in the case of a cyclic quotient singularity all components of B are rational curves, see pp. [BHPV] 107-110. Consider an, a priori meromorphic mapping $\tilde{g} := \lambda^{-1} \circ g \circ \lambda : \tilde{F}_+^\varepsilon \rightarrow \tilde{F}_+^\varepsilon$. Kato proved in [K2] that:

- i) \tilde{g} is holomorphic (and foliated in our case) and there exists n such that $\tilde{g}^n(A \cup B) = \{\text{point}\}$.
- ii) There exists $0^{**} \in B$ such that $\bigcap_{n \geq 1} \tilde{g}^n(\tilde{F}_+^\varepsilon) = \{0^{**}\}$.
- iii) There exists a strongly plurisubharmonic function φ near 0^{**} such that for every c small enough $P_c := \{z : \varphi(z) < c\}$ is biholomorphic to \mathbb{B} and \tilde{g} contracts each P_c , i.e., $\tilde{g}(P_c) \Subset P_c$.
- iv) $\tilde{g} : \tilde{g}^{-1}(P_c \setminus \{0^{**}\}) \rightarrow P_c \setminus \{0^{**}\}$ is a biholomorphism.

See again Lemma 5 in [K2]. Kato then masters from these data (in a clear way) a surface Y with Global Spherical Shell in the terminology of Kato, or a Kato surface (his shell is clearly foliated in our sense) and proves that our X blows down to Y (as well as foliation \mathcal{L} goes down to some \mathcal{F}). On Y one gets a divisor C as factor of $(A \cup B) \setminus \tilde{g}^n(F_+^\varepsilon)$ by \tilde{g}^n for an appropriate n . C is proved to be a chain (or two chains) of rational curves. Again the

foliation in a neighborhood of 0^{**} is vertical in an appropriate coordinates. The image of the leaf \mathcal{L}_0 which supports a vanishing cycle under \tilde{g}^n cannot miss the set $\tilde{g}^n(F_+^\varepsilon) \cap A \cup B$, otherwise the corresponding \mathcal{F}_0 would not contain a vanishing cycle - this was already once explained. Therefore $\mathcal{L}_0 \subset C$. I.e. it is contained in a rational curve and we are done.

Remark 4.4. Using classification of holomorphic foliation of known surfaces, see [DO] and [Br1] one can (eventually) say much more about foliations on surfaces described above, which admit a foliated shell. But we do not enter into the further details. Our aim was to see examples and to understand what can happen with surfaces. I think the answers obtained are complete enough. Remark also that Corollary 3 from the Introduction is proved completely.

It would be instructive to see clearly an example of the Case 2. Let's take the simplest one.

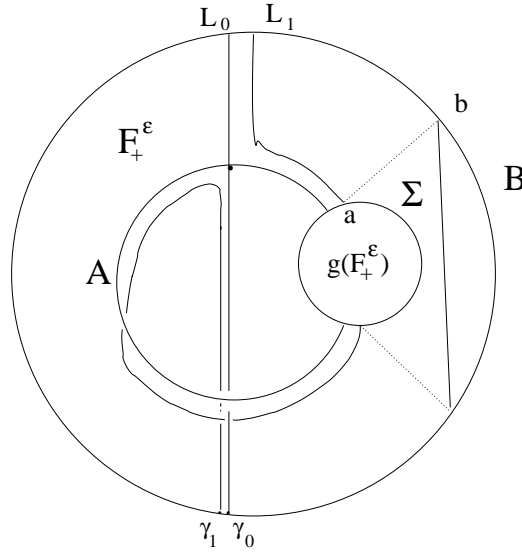


FIGURE 9. Example to the case $0^* \in A$. B is the standard sphere. F_+^ε is the one time blown up unit ball. \tilde{g} is given by $(z_1, z_2) \rightarrow (\frac{1}{2}z_1, \frac{1}{2}z_2)$ and Σ is the image of B under \tilde{g} . The image $\tilde{g}(F_+^\varepsilon)$ of the blown up ball is removed and X is obtained by identifying B with Σ . \mathcal{L}_0 lands to A which is a rational curve with one point of selfintersection. γ_0 is a circle (a point on this Figure) and γ_1 on a nearby leaf \mathcal{L}_1 (complicated curve on the Figure) bounds a disk. To understand this note that circles a and b should be identified.

5. OTHER RESULTS, EXAMPLES AND OPEN QUESTIONS

We still owe the proofs of some statements used in the text of this paper and of some propositions from the Introduction. Moreover it is the time to give more interesting examples (from the point of view of this text) than just foliations on complex surfaces or on Kähler manifolds. Looking on each example in this Section we shall be rather attentive to its Hartogs properties because, as it should be clear from the proofs of this paper, the failure of a foliated manifold (X, \mathcal{L}) to be Hartogs is "almost equivalent" to the presence of essential vanishing cycles/foliated shells in (X, \mathcal{L}) .

5.1. Hartogs foliation on compact non-Hartogs threefold. The following example is due to Nakamura, see [Na]. We only interpret it according to our needs adding a foliation and a pluriclosed taming form to it.

Example 5.1. Take any matrix $A \in SL(2, \mathbb{Z})$ with real eigenvalues $\alpha < 1$ and $1/\alpha$. For example the following one:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}. \quad (5.1)$$

Here $\alpha = 3/2 - \sqrt{5}/2$. Consider the standard integer lattice $\Lambda_0 := \mathbb{Z}^4$ in \mathbb{C}^2 . A preserves Λ_0 and therefore defines a holomorphic automorphism A of the torus $\mathbb{T}_0 := \mathbb{C}^2/\Lambda_0$. Therefore we can construct a compact complex threefold $X_0 := \mathbb{C}^* \times \mathbb{T}_0 / \langle g \rangle$ where $g(z, Z) = (\alpha z, AZ)$. X_0 is a 2-torus bundle over a 1-torus $\mathbb{C}^* / \langle \alpha \rangle$. We fix the coordinate z for \mathbb{C}^* .

Let v be the eigenvector of A with eigenvalue λ and w be that with $1/\alpha$. It will be appropriate for the forthcoming construction to take v, w as the basis in \mathbb{C}^2 , where A acts, and to introduce coordinates $Z = (z_1, z_2)$ in this basis, i.e., now we have: $v = (1, 0)$ and $w = (0, 1)$. In these coordinates A acts as $AZ = (\alpha z_1, 1/\alpha z_2)$. Observe that our lattice Λ_0 is irrational in these coordinates. A foliation on X_0 we construct as follows. Take first the "vertical" foliation $\{z_1 = \text{const}\}$ in \mathbb{C}^2 , factor it by Λ_0 . Due to the irrationality of Λ in the new basis it will have dense leaves. Now we observe that this foliation is obviously invariant under the action of A , which is simply multiplication by $1/\alpha$ on the leaves. Therefore the "vertical" foliation $\mathcal{L}^\pi = \{z = \text{const}, z_1 = \text{const}_1\}$ from $\mathbb{C}^* \times \mathbb{C}^2$ descends to X_0 and we denote it as \mathcal{L}_0 .

Now, following [Na], we shall deform (X_0, \mathcal{L}_0) . In the subspace $\mathbb{C}_{z, z_1}^2 := \mathbb{C}_z \times \mathbb{C}_{z_1}$ of our coordinate space $\mathbb{C}_{z, Z}^3 := \mathbb{C}_z \times \mathbb{C}_{z_1, z_2}^2$ we take a real subspace \mathbb{R}_τ^2 - a deformation of $\{0\}_z \times \mathbb{C}_{z_1}$. Parameter τ here runs in $Gr_{\mathbb{R}}(2, 4)$. This subspace $\mathbb{R}_\tau^2 \subset \mathbb{C}_{z, z_1}^2$ we see as the graph of the uniquely defined \mathbb{R} -linear map $L_\tau : \mathbb{C}_{z_1}^2 \rightarrow \mathbb{C}_z$ and therefore the subspace $\mathbb{R}_\tau^4 := \mathbb{R}_\tau^2 \times \mathbb{C}_{z_2}$ is a graph of $(\text{Id}, L_\tau) : \mathbb{C}_{z_1, z_2}^2 \rightarrow \mathbb{C}_z$. By Λ_τ we denote the image of the lattice Λ_0 under (Id, L_τ) - a deformation of Λ_0 . Denote by \mathbb{T}_τ the torus $\mathbb{R}_\tau^4/\Lambda_\tau$. Remark that A still preserves Λ_τ and therefore $\mathbb{C}_{z, Z}^3 \setminus \{0\}_z \times \mathbb{R}_\tau^4$ factors first by Λ_τ and then by A to a compact complex threefold X_τ which is a real 4-torus bundle over a complex 1-torus $\mathbb{C}^* / \langle \alpha \rangle$. Our "vertical" foliation \mathcal{L}^π descends again to X_τ and we denote the result as \mathcal{L}_τ . The construction of $(X_\tau, \mathcal{L}_\tau)$ is finished.

In the following Proposition \mathcal{V} denotes a sufficiently small neighborhood of $\{0\} \times \mathbb{C}_{z_1}$ in $Gr_{\mathbb{R}}(2, \mathbb{C}_{z, z_1}^2)$.

Proposition 5.1. *The family of foliated 3-folds $\{(X_\tau, \mathcal{L}_\tau) : \tau \in \text{calv}\}$, constructed above, possesses the following properties:*

i) *Manifolds X_τ do not admit a dd^c -closed (even dd^c -negative) metric form for all $\tau \in \mathcal{V} \setminus \mathbb{C}\mathbb{P}^1$ and X_τ is not even almost Hartogs.*

ii) *At the same time foliations \mathcal{L}_τ admit a dd^c -closed taming form for all $\tau \in \mathcal{V}$.*

iii) *Moreover, all $(X_\tau, \mathcal{L}_\tau)$ are Hartogs.*

Proof. (i) The fact that X_τ are not Kähler is explained in [BK], see pp. 82-84. For $\tau \in \mathcal{V} \setminus \mathbb{C}\mathbb{P}^1$ our X_τ has $\mathbb{C}^3 \setminus \mathbb{R}_\tau^4$ as an unramified covering. For this reason it is also not almost Hartogs. Really, the covering map is singular along \mathbb{R}_τ^4 which is much more massive than just a countable union of complex curves. But it is also too massive as a

singularity set for the covering map in the event that X_τ would admit a plurinegative metrik form, see the Main Theorem from [Iv3].

(ii) Take the standard Kähler form $\omega_{st} = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ on \mathbb{C}_Z^2 . Pushforward ω_{st} to \mathbb{R}_τ^4 by (Id, L_τ) and then by a 2 to 1 map A to get $\omega_{\tau au}$ on \mathbb{R}_τ^4 , which is invariant under Λ_τ and A (but is not of the type $(1,1)$). Consider it as a (degenerate) d -closed 2-form on $\mathbb{C}_{z,Z}^3$. This new form will descend to X_τ . Decompose $\omega_\tau = \omega_\tau^{1,1} + \omega_\tau^2$ to a sum of $(1,1)$ -component and components of bedegree $(2,0)$ and $(0,2)$ and observe that $dd^c \omega_\tau^{1,1} = dd^c \omega_\tau^2 = 0$ because dd^c do not mix bedegrees. Therefore $\omega_\tau^{1,1}$ is a needed pluriclosed $(1,1)$ -form on X_τ . If the perturbation τ was small enough it will also be a taming form for \mathcal{L}_τ .

(iii) Let $h : (W, \pi, U, V) \rightarrow (X_\tau, \mathcal{L}_\tau)$ be a foliated generic injection of a three-dimensional generalized Hartogs figure into the foliated manifold $(X_\tau, \mathcal{L}_\tau)$. Without loss of generality assume that $U \subset V$ are bidisks, so W is simply connected. Lift h to a foliated generic injection \tilde{h} of (W, π, U, V) into $(\mathbb{C}^3 \setminus \mathbb{R}_\tau^4, \mathcal{L}^\pi)$. Then it extends after a reparametrization, but as a map with values in $(\mathbb{C}^3, \mathcal{L}^\pi)$. But the fiber of \mathcal{L}^π which touches \mathbb{R}_τ^4 is entirely contained in \mathbb{R}_τ^4 and therefore the extended map never hits \mathbb{R}_τ^4 . After that we can descend the extended map back to $(X_\tau, \mathcal{L}_\tau)$. □

Remark 5.1. Another way to see item (iii) is that, since all leaves of \mathcal{L}_τ are \mathbb{C} it cannot contain a vanishing cycle and therefore foliated shells. Which in its turn is equivalent to the Hartogs property of $(X_\tau, \mathcal{L}_\tau)$.

5.2. Rationality. Recall that by $\mathcal{R}_\mathcal{L}$ we denoted the analytic space of rational cycles on X tangent to \mathcal{L} . Fix a plurinegative taming form ω and consider the area function $v_\omega : \mathcal{R}_\mathcal{L} \rightarrow \mathbb{R}^+$ defined by (2.20). The key observation in the following lemma is that the real function v_ω is plurisuperharmonic on $\mathcal{R}_\mathcal{L}$.

Lemma 5.1. *Suppose that \mathcal{L} is tamed by a plurinegative form ω . Then every irreducible component of $\mathcal{R}_\mathcal{L}$ is compact and every connected component consists of finitely many irreducible ones. The volume function v_ω is constant on every connected component of $\mathcal{R}_\mathcal{L}$.*

Proof. Let first that \mathcal{K} is an irreducible component of $\mathcal{R}_\mathcal{L}$. Denote by $\mathcal{C}_\mathcal{K}$ the universal family over \mathcal{K} . $\mathcal{C}_\mathcal{K}$ comes with two natural mappings: projection $\pi : \mathcal{C}_\mathcal{K} \rightarrow \mathcal{K}$ and inclusion $p : \mathcal{C}_\mathcal{K} \rightarrow X$. Take an analytic disk $\varphi : \Delta \rightarrow \mathcal{K}$ and the restriction \mathcal{C}_Δ . Then for any test function ψ in Δ we have

$$\begin{aligned} \langle \psi, \Delta(v \circ \varphi) \rangle &= \frac{i}{2} \int_\Delta \Delta \psi \left(\int_{\varphi(\zeta)} p^* \omega \right) d\zeta \wedge d\bar{\zeta} = \frac{i}{2} \int_{\mathcal{C}_\Delta} dd^c(\pi^* \psi) \wedge p^* \omega = \\ &= \frac{i}{2} \int_{\mathcal{C}_\Delta} \pi^* \psi \wedge dd^c(p^* \omega) \leq 0. \end{aligned} \tag{5.2}$$

Therefore v is plurisuperharmonic on \mathcal{K} . Suppose \mathcal{K} is not compact. Take a divergent sequence of points $\{k_n\} \subset \mathcal{C}$. Now two cases could occur:

Case 1. $v(k_n)$ stays bounded (may be on some subsequence).

In that case we can subtract a converging subsequence of rational cycles C_{k_n} . The limit is again a rational cycle C_0 which obviously should belong to our irreducible component \mathcal{K} . Contradiction.

Case 2. $v(k_n) \rightarrow \infty$. So $v(k)$ increases when k goes to infinity in \mathcal{K} , i.e., leaves every compact. But this contradicts to the minimum principle for (pluri)-harmonic functions.

Therefore \mathcal{K} is compact and v is constant on \mathcal{K} . This implies that v is constant on every connected component of $\mathcal{R}_{\mathcal{L}}$. Suppose there exist a sequence \mathcal{K}_n of irreducible components of some connected component \mathcal{N} . Take $k_n \in \mathcal{K}_n$. Then $v(C_{k_n})$ is constant and therefore some subsequence C_{k_n} converges to some rational cycle C_0 which corresponds to a point k_0 in $\mathcal{R}_{\mathcal{L}}$. But in this case $\mathcal{R}_{\mathcal{L}}$ contains a sequence of compact irreducible components having an accumulation point k_0 . This contradicts to the fact that $\mathcal{R}_{\mathcal{L}}$ is a complex space.

Therefore each connected component of $\mathcal{R}_{\mathcal{L}}$ consists from a finite number of compact irreducible ones. □

The following is immediate:

Corollary 5.1. *A foliated manifold (X, \mathcal{L}) which admits a plurinegative taming form has bounded rational cycle geometry.*

Lemma 5.2. *Let $(\tilde{\mathcal{L}}_D, \pi)$ be a covering cylinder of holomorphic foliation by curves \mathcal{L} on a compact complex manifold X which admits dd^c -negative taming form. Suppose that D is biholomorphic to the polydisk and that there exists $z \in D$ such that the fiber $\tilde{\mathcal{L}}_z = \pi^{-1}(z)$ is isomorphic to $\mathbb{C}\mathbb{P}^1$. Then $\pi^{-1}(D) \sim D \times \mathbb{C}\mathbb{P}^1$.*

Proof. The set U of $z \in D$ such that $\tilde{\mathcal{L}}_z \sim \mathbb{C}\mathbb{P}^1$ is clearly open. Each connected component U' of U naturally is included in some irreducible component \mathcal{K} of $\mathcal{R}_{\mathcal{L}}$. Therefore the area function $v_{\omega}(z) = \text{area}_{\omega}(\tilde{\mathcal{L}}_z)$ is constant on U' . But this implies that for any boundary point $z^0 \in \partial U' \cap D$ the fiber $\tilde{\mathcal{L}}_{z^0}$ is again rational. Therefore $U' = D$ and $\tilde{\mathcal{L}}_D = D \times \mathbb{C}\mathbb{P}^1$. □

The statements like in this subsection do not hold true in general. Here is the most striking example.

Example 5.2. *As it is shown in [K3] that there exists a compact complex manifold X of dimension 5 and a smooth holomorphic foliation by curves \mathcal{L} on X such that there exists a non-empty domain $W \subset X$ with $X \setminus \bar{W} \neq \emptyset$ having the following properties:*

- i) *If $z^0 \in W$ then $\mathcal{L}_{z^0} \subset W$ and $\mathcal{L}_{z^0} \cong \mathbb{C}\mathbb{P}^1$.*
- ii) *There exists thin subset S of $X \setminus \bar{W}$ such all compact leaves in $X \setminus \bar{W}$ are contained in S .*

Lemma 5.2 implies now that this (X, \mathcal{L}) doesn't admits a plurinegative taming form.

5.3. Preservation of cycles. Let \mathcal{L} be a foliation by curves on a disk-convex complex manifold X and D be a transversal smooth hypersurface. We shall work on the holonomy covering cylinder $\hat{\mathcal{L}}_D$. If \mathcal{L} is smooth the same works also for \mathcal{L}_D . Take a point $z \in D$ and a loop $\gamma \in \pi_1(\hat{\mathcal{L}}_z)$. Reference point for $\pi_1(\hat{\mathcal{L}}_z)$ will be always z .

Definition 5.1. *The domain of preservation of the homotopy class $[\gamma]$ is a topological space Ω_{γ} defined as follows:*

- 1) *the points of Ω_{γ} are homotopy classes $[\gamma'] \in \pi_1(\hat{\mathcal{L}}_{z'})$ (where z' is any point of D) such that some representative γ' of $[\gamma']$ can be joined by a homotopy γ_t of loops in $\hat{\mathcal{L}}_{z(t)}$ with some representative γ of $[\gamma]$. Here $z(t)$ is a path in D from z' to z .*
- 2) *the topology on Ω_{γ} is defined in a natural way saying that $[\gamma_n]$ converge to $[\gamma]$ if some representatives converge uniformly.*

Let Ω_γ be the domain of preservation of the (homotopy class $[\gamma]$ in fact) of our loop γ . There is a natural projection $p : \Omega_\gamma \rightarrow D$ sending $[\gamma'] \in \pi_1(z')$ to z' .

Proof of Proposition 5. Suppose that for some loop $\gamma \subset \hat{\mathcal{L}}_z$ the space Ω_γ is not Hausdorff. That means that there exists $z^0 \in D$, two loops $\gamma, \beta \subset \hat{\mathcal{L}}_{z^0}$ representing different homotopy classes in $\pi_1(\hat{\mathcal{L}}_{z^0})$ and two sequences of loops $\gamma_n, \beta_n \subset \hat{\mathcal{L}}_{z_n}$, homotopic to each other in $\hat{\mathcal{L}}_{z_n}$, converging to γ and β respectively.

Taking $\alpha_n := \gamma_n \circ \beta_n^{-1}$ we obtain a sequence of loops, homotopic to zero and converging to a loop $\alpha \subset \hat{\mathcal{L}}_{z^0}$ which is not homotopic to zero.

We are exactly in the situation of the proof of the Theorem 3.1 and therefore deduce the existence of a foliated shell in \mathcal{L} .

The local biholomorphicity of the projection p is obvious. □

The phenomena of preservation of cycles to our knowledge it was first studied by Landis-Petrovsky in [LP], see also [Iy1].

5.4. Foliations with compact fibers. We turn to the proof of Proposition 4 from the Introduction.

Proof of Proposition 4. (i) We denote by ω an adapted to \mathcal{L} plurinegative (q, q) -form. Let \mathcal{L}_z be a leaf of \mathcal{L} through the point $z \in X$. If \mathcal{L}_z is compact with finite holonomy we denote by $n(\mathcal{L}_z)$ the cardinality of the holonomy group of \mathcal{L}_z and set

$$v(z) = \text{Vol}(\mathcal{L}_z) = n(\mathcal{L}_z) \int_{\mathcal{L}_z} \omega. \quad (5.3)$$

Denote by Ω the connected component of the set of $z \in X$ such that the leaf \mathcal{L}_z of \mathcal{L} through z is compact and has finite holonomy which contains our compact leaf. By the Reeb local stability theorem Ω is an open set in X .

Case 1. There exists $z^0 \in \partial\Omega$ which is a limit of $z_n \in \Omega$ with $v(z_n)$ uniformly bounded.

For any transversal D to \mathcal{L}_{z^0} the intersection $D \cap \Omega$ is open in D and every \mathcal{L}_{z_n} cuts D by a bounded number of points, say N . This readily follows from the boundedness of volumes of \mathcal{L}_{z_n} . Therefore for every $h \in \text{Hol}(\mathcal{L}_{z^0})$ its order is at most $N!$, i.e., $h^{N!} = \text{Id}$. Therefore the holonomy group $\text{Hol}(\mathcal{L}_{z^0})$ has finite exponent and therefore it is finite itself, see Lemma 2 from [P]. Therefore z^0 is an interior point of Ω . Contradiction.

We are left with the following possibility:

Case 2. $v(z) \rightarrow \infty$ when $z \rightarrow \partial\Omega$.

This case is excluded by Lemma 5.1.

All is left to remark that if $\partial\Omega \neq \emptyset$ we obtain a contradiction with the minimum principle for plurisuperharmonic functions.

Therefore $\Omega = X$ and (i) is proved.

(ii) By the standard observation in foliation theory, see ex. [Go] the set of leaves without holonomy is not thin in X . Therefore we are done by (i). □

Remark 5.2. Lemma 5.1 and Proposition 4 admit an obvious generalizations to higher dimension. Namely, let \mathcal{L} be a smooth holomorphic foliation of dimension $0 < q < n$ on an n -dimensional complex manifold X . We suppose that all leaves of \mathcal{L} are smooth

and compact. If \mathcal{L} admits a plurinegative taming (q, q) -form then items (i) and (ii) of Proposition 4 are valid for \mathcal{L} . The proof is a literal repetition of the proof just made.

5.5. Characterization of pluriexact foliations. In this section X is a compact complex manifold.

Lemma 5.3. *The operator $dd^c : \mathcal{E}_{2,2}^{\mathbb{R}} \rightarrow \mathcal{E}_{1,1}^{\mathbb{R}}$ has closed range.*

Proof. Observe the following resolution of the sheaf $\mathcal{H}_{\mathbb{C}}$ of complex valued pluriharmonic functions on X

$$0 \longrightarrow \mathcal{H}_{\mathbb{C}} \xrightarrow{(-\partial, \text{Id})} \Omega^1 \oplus [\mathcal{H}_{\mathbb{R}} + i\mathcal{E}_{\mathbb{R}}] \xrightarrow{(\text{Id} \oplus \partial)} \mathcal{E}^{1,0} \xrightarrow{(\partial \oplus \bar{\partial})} \mathcal{E}_{\mathbb{R}}^{1,1} \xrightarrow{dd^c} \mathcal{E}_{\mathbb{R}}^{2,2} \xrightarrow{d} \dots \quad (5.4)$$

Here $\mathcal{H}_{\mathbb{R}}$ is the sheaf of real valued pluriharmonic functions, Ω^1 the sheaf of holomorphic 1-forms, $\mathcal{E}_{\mathbb{R}}$ the sheaf of smooth real valued functions. This resolution tells that

$$\text{Ker} \{d : \mathcal{E}_{\mathbb{R}}^{2,2} \rightarrow \mathcal{E}_{\mathbb{R}}^3\} / \text{Im} \{dd^c : \mathcal{E}_{\mathbb{R}}^{1,1} \rightarrow \mathcal{E}_{\mathbb{R}}^{2,2}\} \cong H^4(X, \mathcal{H}_{\mathbb{C}}). \quad (5.5)$$

Therefore $dd^c : \mathcal{E}_{\mathbb{R}}^{1,1} \rightarrow \mathcal{E}_{\mathbb{R}}^{2,2}$ has closed range (in fact of finite codimension). By duality $dd^c : \mathcal{E}_{2,2}^{\mathbb{R}} \rightarrow \mathcal{E}_{1,1}^{\mathbb{R}}$ has also closed range. □

Fix some strictly positive $(1, 1)$ -form Ω on X . Let \mathcal{L} be a holomorphic foliation by curves on X . Denote by $K_{1,1}(\mathcal{L})$ the compact set in $\mathcal{E}_{1,1}^{\mathbb{R}}$ which consists from positive $(1, 1)$ -currents T tangent to \mathcal{L} such that $(\Omega, T) = 1$, i.e., the compact base of currents directed by \mathcal{L} . The following lemma is analogous to Theorem 3.18 from [Go].

Lemma 5.4. *The following two assertions are equivalent:*

- i) (X, \mathcal{L}) admits a pluriclosed taming form.
- ii) There is no dd^c -exact current $T \in K_{1,1}(\mathcal{L})$.

Proof. Let such ω exists. If $dd^c S = T \in K_{1,1}(\mathcal{L})$ for some $S \in \mathcal{E}_{2,2}^{\mathbb{R}}$ then $0 < (\omega, T) = (\omega, dd^c S) = (dd^c \omega, S) = 0$ - a contradiction. Vice versa, if $K_{1,1}(\mathcal{L}) \cap dd^c \mathcal{E}_{2,2}^{\mathbb{R}} = \emptyset$ then, since $dd^c \mathcal{E}_{2,2}^{\mathbb{R}}$ is closed and $K_{1,1}(\mathcal{L})$ is convex, by Hahn-Banach theorem there exists ω such that $\omega|_{K_{1,1}(\mathcal{L})} > 0$ and $\omega|_{dd^c \mathcal{E}_{2,2}^{\mathbb{R}}} = 0$. □

Example 5.3. *Take $X = \mathbb{C}^3 \setminus \{0\} / z \sim 2z$ - the Hopf threefold. Let \mathcal{L} be the vertical foliation $\mathcal{L}_{\mathbb{C}} = \{z_1 = c_1, z_2 = c_2\}$. \mathcal{L} admits a plurinegative taming $(1, 1)$ -form but doesn't admit a pluriclosed one.*

(i) Set $z' = (z_1, z_2)$ and consider the following $(1, 1)$ -form on X :

$$\theta = \frac{i}{2} \frac{(dz', dz')}{\|z'\|^2}. \quad (5.6)$$

θ is a well defined positive bidimension $(2, 2)$ -current on X . One easily checks that $dd^c \theta = -c_4[\mathcal{L}_0]$, where $[\mathcal{L}_0]$ is the current of integration over the central fiber \mathcal{L}_0 of \mathcal{L} and c_4 is the volume of the unit ball in \mathbb{C}^2 . θ is a clear obstruction to the existence of pluriclosed taming form for \mathcal{L} .

(ii) At the same time the $(1, 1)$ -form

$$\omega = \frac{i}{2} \frac{(dz, dz)}{\|z\|^2}, \quad (5.7)$$

where $z = (z_1, z_2, z_3)$, is strictly positive on X (not only on \mathcal{L}) and one easily checks that $dd^c\omega \leq 0$. I.e. ω serves as a plurinegative taming form for any foliation by curves on X .

Example 5.2 of Kato already provided us a foliation without a plurinegative taming form. However it is very inexplicit. Let us give a very simple one.

Example 5.4. (*Foliations on Iwasawa manifold*). Let $H(3)$ be the group of matrices of the form

$$A = \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \quad (5.8)$$

with complex z_1, z_2, z_3 . Denote by $\mathbb{Z}(3)$ the subgroup of $H(3)$ which consists from $z_1, z_2, z_3 \in \mathbb{Z} + i\mathbb{Z}$. The quotient $H(3)/\mathbb{Z}(3)$ is a compact, complex three-dimensional manifold \mathcal{I} - Iwasawa manifold. The holomorphic forms $\omega_1 = dz_1$, $\omega_2 = dz_2$ and $\omega_3 = dz_3 - z_1 dz_2$ are left invariant with respect to the action of $\mathbb{Z}(3)$ and therefore project to holomorphic forms on \mathcal{I} . Define a holomorphic foliation by curves \mathcal{L}_1 on \mathcal{I} by $\omega_1 = \omega_2 = 0$.

Proposition 5.2. *Foliated manifold $(\mathcal{I}, \mathcal{L}_1)$ possesses the following properties:*

- i) *It is Hartogs.*
- ii) *It doesn't admits a plurinegative taming form.*

Proof. (i) Hartogs property is invariant with respect to unramified coverings. Since the universal covering of \mathcal{I} is $H(3) \cong \mathbb{C}^3$ we are done.

(ii) Consider $S := \frac{i}{2}\omega_3 \wedge \bar{\omega}_3$ as a positive $(2,2)$ -current on \mathcal{I} . A simple calculation

$$dd^c S = i\partial\bar{\partial}S = \partial\omega_3 \wedge \bar{\partial}\bar{\omega}_3 = \frac{i^2}{2}\omega_1 \wedge \bar{\omega}_1 \wedge \omega_2 \wedge \bar{\omega}_2 =: T.$$

And T is a positive current directed by \mathcal{L}_1 . A positive current S such that $dd^c S$ is also positive and directed by \mathcal{L} is a clear obstruction to the existence of a plurinegative taming form for \mathcal{L}_1 . □

Remark 5.3. Iwasawa manifold carries also other foliations. For example $\mathcal{L}_2 := \{\omega_1 = \omega_3 = 0\}$. This \mathcal{L}_2 is tamed by the closed form $\frac{i}{2}\omega_2 \wedge \bar{\omega}_2$ and therefore the Proposition 1 from the Introduction applies to \mathcal{L}_2 - it belongs to \mathcal{U} .

5.6. Open questions. In this section we shall formulate some open questions which are important to complete our knowledge about holomorphic foliation by curves on compact complex manifolds and which naturally come out of the discussions in this paper.

Question 1. In the conditions of the Theorem 3.1 let \mathcal{L}_z be the leaf such that $\hat{\mathcal{L}}_z$ contains an essential vanishing cycle. Is it true that its closure $\bar{\mathcal{L}}_z$ is a compact complex curve?

Recall that this is like the proof of Novikov's theorem work, see [Go] for example. If $\dim X = 2$ we proved it in Corollary 3.

Question 2. Prove that a foliation with shells is parabolic.

Question 3. Let $(X, \mathcal{L}) \in \mathcal{S}$ and $\dim X \geq 3$. The (X, \mathcal{L}) should contain a biholomorphic foliated image of a total space of a deformation $(\mathcal{X}, \pi, \Delta^{n-2})$ of foliated compact Hopf or Kato surfaces $\mathcal{X}_\lambda, \lambda \in \Delta^{n-2}$.

Let us give an example showing that the total space of deformation may not sweep the whole of X , *i.e.*, that a foliated shell may "disappear in the limit".

Example 5.5. Let E' be a holomorphic rank two bundle over a Hopf surface $H^2 = \mathbb{C} \setminus \{0\}/z \sim 2z$ which admits a holomorphic section σ vanishing exactly at one point $z_0 \in H^2$ with multiplicity one, see [GH], p.726. Denote by E the bundle dual to E' . Let τ_0 be the zero section of E . The quotient of $E \setminus \tau_0$ by the action $(z, v) \rightarrow (z, \frac{1}{2}v)$ is a compact complex 4-manifold which we denote as X . It is fibered over H^2 and the fiber over $z \in H^2$ we denote as X_z .

$E \setminus \tau_0$ carries a singular holomorphic foliation by curves defined as follows: its leaves in each fiber $E_z \setminus \{0\}, z \neq z_0$ are $\{x \in E_z : \sigma_z(x) = \text{const}\}$. Actually on each $E_z \setminus \{0\}$ it is again our "vertical" foliation. It factors under the chosen action to a foliation \mathcal{L} on X . The singularity set of \mathcal{L} is E_{z_0} . (X, \mathcal{L}) carries an obvious family of foliated shells over $H^2 \setminus \{z_0\}$, and this family extends over z_0 (!) as a family of shells. But \mathcal{L} itself is singular over z_0 and therefore the shell in X_{z_0} is not a foliated one.

Question 4. Is it true that immersed foliated shells could be always made spherical? The same question about imbedded ones. In that case one expects them to be holomorphic foliated images of quotients of the standard sphere in \mathbb{C}^2 with the standard vertical foliation.

Question 5. Let D be a transversal polydisk. Suppose that the skew cylinder $\tilde{\mathcal{L}}_D$ exists (and \mathcal{L} admits a plurinegative adapted form). Prove that $\tilde{\mathcal{L}}_D$ is disk-convex.

This is known for Stein X , [Iy1], in that case $\tilde{\mathcal{L}}_D$ is Stein. It is also known for algebraic X , [Br2].

Question 6. Prove that the set of $z \in D$ such that $\tilde{\mathcal{L}}_z = \mathbb{C}$ is pluripolar in D or is the whole D .

Algebraic case is treated in [Br2].

Question 7. Suppose that the domain Ω_γ of preservation of cycle γ (as in Definition 5.1) exists. Prove that Ω_γ is good in the sense of Landis-Petrovsky, *i.e.*, that for a natural projection $p : \Omega_\gamma \rightarrow D$ the set $\Sigma := D \setminus p(\Omega_\gamma)$ doesn't separates D .

This question for holomorphic foliations by curves on arbitrary compact complex manifolds Il'yashenko calls the generalized Landis-Petrovsky conjecture. The answer is positive for algebraic foliations, but it is wrong for holomorphic ones on Stein manifolds, see [Iy2]. Example of Kato in [K3] leaves little hope for the positive answer in general, but it is not a direct counterexample. In Question 7 We propose to solve the Landis-Petrovsky conjecture when \mathcal{L} admits an adapted pluriclosed (or plurinegative) taming form.

Question 8. Suppose T is a nontrivial, positive, dd^c -exact current directed by \mathcal{L} (as in Lemma 5.4). What are consequences for \mathcal{L} from the existence of such T ? Suppose $T = dd^c R$ where R is also positive. The same question in this case.

Question 9. Is it true that every pluriexact foliation contains a compact cycle?

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