WODZICKI'S NONCOMMUTATIVE RESIDUE AND TRACES FOR OPERATOR ALGEBRAS ON MANIFOLDS WITH CONICAL SINGULARITIES

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Introduction

In 1984 M. Wodzicki found a trace on the algebra $\Psi_{cl}(M)$ of all classical pseudodifferential operators on a closed compact manifold M; he called it the *noncommutative residue.* This trace vanishes on the ideal $\Psi^{-\infty}(M)$ of smoothing operators; it even is the unique trace (up to constant multiples) on $\Psi_{cl}(M)/\Psi^{-\infty}(M)$, provided M is connected and $\dim M > 1$.

Although it first seems a rather exotic object, this trace has found a wide range of applications both in mathematics and in mathematical physics. In appreciation of Wodzicki's accomplishment the name *Wodzicki residue* has become generally accepted.

Also various extensions and analogs of the noncommutative residue have been established, e.g. for certain algebras of Fourier integral operators (Guillemin [11]), manifolds with boundary (Fedosov, Golse, Leichtnam, and Schrohe [7, 8]), manifolds with conical singularities (Schrohe [26]), or cusp pseudodifferential operators (Melrose and Nistor [21]).

In these four lectures I shall first give a short review of Wodzicki's residue and some of its applications. Next I will explain the idea of B.-W. Schulze's 'cone algebra', a pseudodifferential calculus for manifolds with conical singularities. For every conical singularity we shall obtain a trace on this algebra. These traces vanish on operators supported in the interior and are therefore different from Wodzicki's. On the other hand, there is a natural ideal in the cone algebra having a trace which extends the classical noncommutative residue. All these traces vanish on smoothing operators. They are moreover seen to be the unique continuous traces with this property on a slightly extended version of the cone algebra. In view of the fact that this ASI focuses on microlocal analysis and spectral theory, I shall finally sketch Connes' theorem linking Wodzicki's residue to Dixmier's trace. For one thing this makes the noncommutative residue an important tool for explicit computations in noncommutative geometry, see Connes [3]; it also shows Weyl's law on the asymptotics of the eigenvalues of the Laplacian.

Lecture 1: Wodzicki's Noncommutative Residue for Pseudodifferential Operators

1.1 Definition. Let $\mathcal{A}$ be an algebra over $\mathbb{C}$. A linear map $\tau : \mathcal{A} \to \mathbb{C}$ is called a *trace* if it vanishes on commutators, i.e., if

$$\tau[P, Q] = \tau(PQ - QP) = 0 \text{ for all } P, Q \in \mathcal{A}.$$

Clearly, if τ is a trace, then $\lambda\tau$ is a trace for each λ in $\mathbb{C}$; moreover, the zero map is always a trace. When we speak of a unique trace, we shall mean that it is non-zero and the only one up to multiples.

1.2 Example. On $M_r(\mathbb{C})$, the algebra of $r \times r$ matrices over $\mathbb{C}$, there is a unique trace, namely the standard one, $\mathrm{Tr} : A \mapsto \sum_j^r A_{jj}$. Indeed, let $E_{j,k}$ denote the matrix having a single 1 at position j, k (and zeros else). Then the statement is immediate from the observation that $[E_{j,k}, E_{k,k}] = E_{j,k}$ for $j \neq k$ and $[E_{j,k}, E_{k,j}] = E_{j,j} - E_{k,k}$.

In this lecture we shall be concerned with the following theorem, proven by M. Wodzicki in 1984, as well as with several of its applications.

1.3 Theorem. Let M be closed, compact, connected, $\dim M > 1$. Let $\mathcal{A} = \Psi_{cl}(M)/\Psi^{-\infty}(M)$ be the algebra of all classical pseudodifferential operators on M modulo the ideal of the regularizing elements. Then there is a unique trace on $\mathcal{A}$, the so-called noncommutative residue or Wodzicki residue.

1.4 Applications. (a) As mentioned before, the noncommutative residue plays a crucial role in Connes' noncommutative geometry due to Connes observation that it coincides with Dixmier's trace on pseudodifferential operators of order $-\dim M$, cf. [2].
(b) As Wodzicki observed, it also is closely related to the residues of zeta functions of elliptic pseudodifferential operators that were computed by Seeley [30] as well as to the coefficients in heat kernel expansions.
(c) Wodzicki's trace is the multi-dimensional analog of the residue Manin [19] and Adler [1] had found in 1978/79 in connection with their work on algebraic aspects of Korteweg-de Vries equations in dimension one.
(d) Guillemin [10] had discovered the noncommutative residue independently as an essential ingredient in his 'soft' proof of Weyl's formula on the asymptotic distribution of eigenvalues. Under rather general axiomatic conditions linking 'classical observables', i.e. functions p on a symplectic manifold, with their 'quantum mechanical counterparts', namely self-adjoint operators on a suitable Hilbert space, he showed that the counting function $N_P(\lambda)$ of the eigenvalues of P satisfies the relation $N_P(\lambda) \sim c \operatorname{vol}\{p \leq \lambda\}$ with a constant c independent of p or P.
(e) The noncommutative residue has been used in conformal field theory in order to construct central extensions of the algebra of pseudodifferential symbols on the circle, cf. Khesin and Kravchenko [16].
(f) It has been applied to derive the Einstein-Hilbert action in the theory of gravitation (Kalau and Walze [13], Kastler [15]).

We shall now go more into the details. We first recall a few facts about pseudodifferential operators:

1.5 Classical pseudodifferential operators on manifolds. Let $m \in \mathbb{Z}$ and let a be a symbol in Hörmander's class $S^m = S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. It defines the linear operator $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ by

$$Au(x) = \int e^{ix\xi} a(x, \xi) \hat{u}(\xi) đ\xi, \quad u \in \mathcal{S}(\mathbb{R}^n).$$

We say that A is a pseudodifferential operator of order m on $\mathbb{R}^n$ and refer to a as its symbol; it is uniquely determined by A, see [18]. We call a classical if it has an asymptotic expansion $a \sim \sum_{j=0}^{\infty} a_{m-j}$ with $a_j \in S^j$ homogeneous of degree j in ξ for large $|\xi|$, i.e., $a_j(x, \lambda\xi) = \lambda^j a_j(x,\xi)$ for $\lambda \geq 1$ and $|\xi| \geq R$. The $\sim$ indicates that upon subtracting the first N summands from a we obtain an element in S^{m-N}.

In the following we let M be a compact manifold of dimension n, E a vector bundle over M.

We say that a linear operator $A : C^\infty(M,E) \to C^\infty(M,E)$ is classical pseudodifferential operator and write $A \in \Psi_{cl}(M)$ if, in each coordinate neighborhood, the action of A is given by a pseudodifferential operator with a classical symbol, modulo an operator with smooth integral kernel, a so-called *smoothing operator.* We denote those by $\Psi^{-\infty}(M)$. Note that an operator will be smoothing whenever its symbol is in $S^{-\infty} = \bigcap_m S^m$ and that $\Psi^{-\infty}(M)$ is an ideal in $\Psi_{cl}(M)$. In the following we let $\mathcal{A} = \Psi_{cl}/\Psi^{-\infty}$.

Any smooth change of the symbol a_j on $\{|\xi| \leq R\}$ modifies a_j by an element in $S^{-\infty}$. Over each coordinate neighborhood U, the equivalence class of a pseudodifferential operator of order m in $\mathcal{A}$ can be therefore be identified with a formal sum of homogeneous functions (taking values in square matrices), $\sum_{j=0}^{\infty} a_{m-j}(x,\xi)$, with $a_j(x,\xi) \in C^\infty(U \times (\mathbb{R}^n \setminus \{0\}))$ homogeneous in ξ of degree j. There are well-known rules for the behavior of $\sum a_j$ under changes of coordinates.

1.6 Definition and Lemma. On $\mathbb{R}^n$, $n \geq 2$, define the $(n-1)$-form

$$\sigma(\xi) = \sum_{j=1}^{n} (-1)^{j+1} \xi_j d\xi_1 \wedge \ldots \wedge \widehat{d\xi_j} \wedge \ldots \wedge d\xi_n.$$

The hat indicates that this differential is omitted. Let p be a smooth function on $\mathbb{R}^n \setminus \{0\}$ which is homogeneous of degree $-n$. Euler's identity $\sum \xi_j \partial_{\xi_j} p = -np$ implies that the form $p\sigma$ is closed:

$$d(p\sigma) = (dp) \wedge \sigma + p d\sigma = -npd\xi_1 \wedge \ldots d\xi_n + pnd\xi_1 \wedge \ldots d\xi_n = 0.$$

The restriction of σ to the unit sphere S^{n-1} is the surface measure.

We can now define the Wodzicki residue $\operatorname{res} A$ of an operator A:

1.7 Theorem. *Let $A \in \Psi_{cl}(M)$, $x \in M$. Suppose that in a neighborhood U of x, the symbol A has the asymptotic expansion $\sum a_j$ with a_j homogeneous of degree j for $|\xi| \geq 1$. Denote by* Tr *the trace on $\mathcal{L}(E)$ and define*

$$\operatorname{res}_x A = \left(\int_{S^{n-1}} \operatorname{Tr} a_{-n}(x,\xi)\sigma(\xi) \right) dx_1 \wedge \ldots \wedge dx_n.$$

This is a density on M. It therefore makes sense to set

$$\operatorname{res} A = \int_M \operatorname{res}_x A. \tag{1}$$

Then res *only depends on the equivalence class of A in $\mathcal{A}$. It is a trace:* $\operatorname{res}[A,B] = 0$ *for all $A, B \in \mathcal{A}$. If M is connected, then any other trace on $\mathcal{A}$ is a multiple of* res.

Note. The local density $a_{-n}(x,\xi)\sigma(\xi) \wedge dx_1 \wedge \cdots \wedge dx_n$ can be patched to a global density Ω_A with $\operatorname{res} A = \int_{S^*M} \Omega_A$: Denoting by ω the canonical symplectic form on T^*M and by ρ the radial vector field one has

$$a_{-n}\sigma \wedge dx_1 \wedge \cdots \wedge dx_n = (-1)^{n(n-1)/2}\,\frac{1}{n!}(a\,\rho \rfloor \omega^n)_0,$$

where $(\ldots)_0$ is the homogeneous component of degree 0 in an asymptotic expansion of $a\,\rho \rfloor \omega^n$ into homogeneous forms ($\rfloor$ stands for the contraction of forms with vector fields).

The proof relies on the following simple lemma. For a proof see e.g. [8].

1.8 Lemma. (a) *Let the function p be a derivative of a smooth homogeneous function q of degree $-(n-1)$ on $\mathbb{R}^n \setminus \{0\}$, say $p = \partial_{\xi_k} q$. Then $\int_{S^{n-1}} p\sigma = 0$.*
(b) *Let p be a homogeneous function on $\mathbb{R}^n \setminus \{0\}$. Each of the following conditions is sufficient for p to be a sum of derivatives:*

(i) $\deg p \neq -n$.

(ii) $\deg p = -n$ *and* $\int_S p\sigma = 0$.

(iii) $p = \xi^\alpha \partial^\beta q$, *where q is a homogeneous function and $|\beta| > |\alpha|$.*

Proof of Theorem 1.7. Under a change of variables χ the symbol a transforms to a symbol b with

$$b(y, {}^t\chi'(y)\xi) \sim \sum_{|\alpha|\geq 0} \partial_\xi^\alpha a(\chi(y),\xi)\varphi_\alpha(y,\xi), \tag{1}$$

where the $\varphi_\alpha(y,\xi)$ are polynomials in ξ of degree $\leq |\alpha|/2$ and $\varphi_0 = 1$ (see Hörmander [12, (18.1.30)]). Changing the variable in the integral, and applying first (1), then Lemma 1.8(b.iii) we get

$$\begin{aligned} \int_S b_{-n}(y,\eta)\sigma(\eta) &= |\det \chi'(y)| \int_S b_{-n}(y, {}^t\chi'(y)\xi)\sigma(\xi) \\ &= |\det \chi'(y)| \sum_{|\alpha|\geq 0} \int_S (\partial_\xi^\alpha a(\chi(y),\xi)\varphi_\alpha(y,\xi))_{-n}\sigma(\xi) \\ &= |\det \chi'(y)| \int_S a_{-n}(\chi(y),\xi)\sigma(\xi). \end{aligned} \tag{2}$$

Hence res transforms like a density.

For the proof of the trace property we may employ the linearity of res to confine ourselves to the case of two operators A, B, with symbols a and b supported in the same chart U. Also we may assume that we are in the scalar case, since everything commutes under Tr. The symbol of $[A,B]$ is given by

$$\sum_{|\alpha|\geq 0} \frac{(-i)^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha a\, \partial_x^\alpha b - \partial_\xi^\alpha b\, \partial_x^\alpha a). \tag{3}$$

We may rewrite this expression as $\sum_{j=1}^n \partial_{\xi_j} A_j + \partial_{x_j} B_j$, where A_j and B_j are bilinear expressions in a and b and their derivatives; they vanish for $x \notin U$. Thus, the integrals over S of $(\partial_{\xi_j} A_j)_{-n}\sigma$ are zero by Lemma 1.8(a). The same holds for the integrals of $(\partial_{x_j} B_j)_{-n}$ over U, since all B_j have compact x-support in U.

To prove uniqueness, suppose τ is another trace on $\mathcal{A}$, and consider an operator A of order m with symbol $a \sim \sum a_j$ supported in U. Let $\widehat{x}_j$ and $\widehat{\xi}_j$ denote any symbols with x-supports in U coinciding with x_j and ξ_j on the support of a. The symbols of the commutators $[A, \operatorname{op} \widehat{x}_j]$ and $[A, \operatorname{op} \widehat{\xi}_j]$ then are $-D_{\xi_j} a$ and $D_{x_j} a$, respectively. Since the trace τ vanishes on commutators, it vanishes on all symbols that are derivatives with respect to either x or ξ.

Define $\overline{a}(x) = \frac{1}{\operatorname{vol} S} \int_S a_{-n}(x,\xi)\sigma_\xi$. Applying Lemma 1.8(b) to a_j for all $j \neq -n$, there exist n functions $b_{kj}(x,\xi)$, $k = 1, \ldots, n$, homogeneous of degree $j+1$ in ξ such that $a_j = \sum_{k=1}^n \partial_{\xi_k} b_{kj}$. Let $b_k(x,\xi) \sim \sum_{j \le m, j \neq -n} b_{kj}$. Then

$$a(x,\xi) - \overline{a}(x)|\xi|^{-n} = \sum_{k=1}^{n} \partial_{\xi_k} b_k(x,\xi) + (a_{-n}(x,\xi) - \overline{a}(x)|\xi|^{-n}).$$

Clearly, $\int_S (a_{-n}(x,\xi) - \overline{a}(x)|\xi|^{-n})\sigma(\xi) = 0$. So Lemma 1.8(b.ii) shows that $a_{-n}(x,\xi) - \overline{a}(x)|\xi|^{-n}$ is a finite sum of derivatives with respect to ξ. Hence $\tau(a) = \tau(\overline{a}(x)|\xi|^{-n})$, so by $T : f \mapsto \tau(f(x)|\xi|^{-n})$ we can define a functional on $C_0^\infty(U)$ which vanishes on derivatives. It is no restriction to assume that U is diffeomorphic to an open ball. Then we easily deduce from Schwartz [29, II.4] that $Tf = c \int f(x)dx$ for a suitable constant c, and we get the assertion for U. A priori, the constant might depend on U, but on the intersection of two coordinate neighborhoods the constants must agree. If M is connected, then all are equal, and the proof is complete. $\triangleleft$

Note that no continuity condition is required for the uniqueness of the noncommutative residue.

1.9 Examples and remarks. (a) Let $A = (I - \Delta)^{-n/2}$. Then $a_{-n}(x,\xi) = |\xi|^{-n}$ and $\operatorname{res} A = \int_M \int_{S^{n-1}} |\xi|^{-n}\sigma(\xi)dx = \operatorname{vol} S^{n-1} \cdot \operatorname{vol} M$. So the volume of M can be found as a noncommutative residue.
(b) If A is a differential operator, then $\operatorname{res} A = 0$.
(c) If the order of A is $< -n$, then $\operatorname{res} A = 0$, so res is not an extension of the usual operator trace. In fact, as we shall see in Lecture 4, Wodzicki's residue coincides with Dixmier's trace on pseudodifferential operators of order $-n$ and therefore *vanishes* on trace class operators.

1.10 Seeley's results on complex powers. We additionally assume A to be invertible of order $m > 0$. In particular, a is elliptic, but we impose a slightly stronger condition: There exists a ray $R_\theta = \{z : z = re^{i\theta}, r \ge 0\}$ in $\mathbb{C}$ with no eigenvalue of of $a_m(x,\xi)$ on R_θ for $\xi \neq 0$. The spectrum of A is discrete. Shifting θ slightly, R_θ will not intersect it. Moreover Seeley [30] showed that:

(i) The norm of $(A-\lambda)^{-1}$ is $O(\lambda^{-1})$, and there exists a family of complex powers $\{A^s : s \in \mathbb{C}\}$, defined by

$$\begin{aligned} A^s &= \frac{i}{2\pi} \int_C \lambda^s (A-\lambda)^{-1} d\lambda, \quad \operatorname{Re} s < 0; \\ A^{s+k} &= A^s A^k, \quad \operatorname{Re} s < 0, k \in \mathbb{N}. \end{aligned}$$

Here C is the path in $\mathbb{C}$ going from infinity along R_θ to a small circle around 0, clockwise about the circle, and back along R_θ.

(ii) A^s is a pseudodifferential operator of order $m \operatorname{Re} s$; $s \mapsto A^s$ is analytic.

(iii) For $\operatorname{Re} s < -n/m$, A^s is an integral operator with a continuous integral kernel $k_s(x,y)$. For each $x \in M$, $s \mapsto k_s(x,x)$ extends to a meromorphic map with at most simple poles in $s_j = \frac{j-n}{m}$, $j = 0, 1, \ldots$. There is no pole in $s = 0$; the residue in s_j is given by an explicit formula. If A is a *differential* operator, then also the residues at the positive integers vanish.

1.11 The noncommutative residue and zeta functions. We use the notation of 1.10. Since the spectrum $\{\lambda_j\}$ of A is discrete and A^s is trace class for $\operatorname{Re} s < -n/m$ we may define the zeta function

$$\zeta_A(s) = \operatorname{trace} A^{-s} = \sum \lambda_j^{-s}, \ \operatorname{Re} s > n/m.$$

This is a holomorphic function. It coincides with $\int_M k_{-s}(x,x)dx$ hence has a meromorphic extension to $\mathbb{C}$ with at most simple poles in the points s_j. Wodzicki used Seeley's explicit formulas to show that

$$\operatorname{Res}_{s=-1}\zeta_A = (2\pi)^n \operatorname{res} A / \operatorname{ord} A; \tag{1}$$

here $\operatorname{ord} A$ is the order of A. More generally,

$$\operatorname{Res}_{s=s_j}\zeta_A = (2\pi)^n \operatorname{res} A^{-s_j} / \operatorname{ord} A. \tag{2}$$

We can use this relation to define res via zeta functions: Let P be an arbitrary pseudodifferential operator. Choose A satisfying the assumptions of 1.10 with $\operatorname{ord} A > \operatorname{ord} P$. Then also $A + uP$, $u \in \mathbb{R}$, will meet the requirements of 1.10, provided $|u|$ is small, and (1) shows that

$$\operatorname{res} P = \frac{d}{du}\operatorname{res}(A+uP)|_{u=0} = (2\pi)^{-n} \operatorname{ord} A \operatorname{Res}_{s=-1} \zeta_{A+uP}.$$

1.12 Heat kernels. Starting from the assumptions in 1.10 we additionally ask that A is a positive operator and that the eigenvalues of the principal symbol matrix a_m lie in the right half-plane. Then one can define

$$e^{-tA} = \int_C e^{-t\lambda}(A-\lambda)^{-1}d\lambda,$$

where C is a suitable contour around the spectrum. The operator e^{-tA} is trace class, and $\operatorname{trace} e^{-tA} = \sum e^{-\lambda_j t}$. The identity

$$\int_0^\infty t^{s-1}e^{-\lambda t}dt = \lambda^{-s}\int_0^\infty (\lambda t)^{s-1}e^{-\lambda t}d(\lambda t) = \lambda^{-s}\Gamma(s)$$

shows that $\Gamma(s)\zeta_A(s) = \int_0^\infty t^{s-1}\operatorname{trace}(e^{-tA})dt$ is the Mellin transform of $\operatorname{trace} e^{-tA}$. It is a well-known property of the Mellin transform that the asymptotic behavior $\sim t^{-s_j}\ln^k t$ near $t = 0$ produces a pole in s_j of order $k+1$ and vice versa. From the above results for the zeta function one immediately deduces the asymptotic expansion near zero:

$$\operatorname{trace} e^{-tA} \sim \sum_{j=0}^{\infty} \alpha_j(A)t^{\frac{j-n}{m}} + \sum_{k=1}^{\infty}\beta_k(A)t^k \ln t$$

Note that there is no term $t^0 \ln t$, since ζ_A is regular in 0 while the Gamma function has a simple pole; for the same reason there are no terms $t^k \ln t$ if A is *differential.*

So we get $\operatorname{res} A = \operatorname{ord} A \cdot \beta_1(A)$. Moreover, we can define the noncommutative residue for a general pseudodifferential operator by choosing an operator A with the above properties and $\operatorname{ord} A > \operatorname{ord} P$, then letting

$$\operatorname{res} P = -\operatorname{ord} A \frac{d}{du} \beta_1(A + uP)|_{u=0}.$$

Classically, A is the Laplace–Beltrami operator Δ associated with a Riemannian metric on M, so that one really deals with the heat equation. It is well-known that the coefficients $a_j(\Delta)$ carry geometric information, see e.g. Gilkey [9].

1.13 Notes and Remarks. The original reference for Wodzicki's residue is [32]; a much more elaborate presentation was given in [33]. Kassel's paper [14] gives a good survey. The proof of Theorem 1.7 here follows [8].

In Theorem 1.7 we asked for simplicity that $n \geq 2$. For $n = 1$ the cosphere bundle has two components. A simpler version of the above arguments then shows that one gets two linearly independent traces when restricting to orientation preserving changes of coordinates otherwise one trace as before.

Lecture 2: The Cone Algebra

In this lecture we shall review the cone calculus for manifolds with conical singularities introduced by B.-W. Schulze. In the next lecture we shall deal with noncommutative residues for these objects.

Following the general idea of noncommutative geometry, the information about the underlying space is encoded in a suitable algebra of linear operators. From the analysis of the classical case presented in Section 1, we know that Wodzicki's residue recovers the geometric invariants detected by the heat kernel expansion methods. One might therefore hope that a similar statement holds for the singular case.

In this context the choice of the operator algebra is rather important. Consider for example a manifold M with boundary. One possible operator algebra is, of course, the algebra of classical pseudodifferential operators on the open interior. Yet it is not difficult to see from the proof of Theorem 1.7 that there is *no* trace on this algebra.

On a manifold with boundary, it seems more natural to consider boundary value problems. The canonical analog of the algebra of pseudodifferential operators then is Boutet de Monvel's algebra. As it turns out we then get the desired result [7, 8]:

2.1 Theorem. *There is a trace on the algebra $\mathcal{B}_{cl}(M)$ of classical elements in Boutet de Monvel's calculus on M. It extends Wodzicki's residue, vanishes on the ideal $\mathcal{B}^{-\infty}(M)$ of smoothing elements, and is the unique trace on the quotient algebra $\mathcal{B}_{cl}(M)/\mathcal{B}^{-\infty}(M)$, provided M is connected and* $\dim M > 1$.

We now introduce the basic elements of Schulze's cone calculus.

2.2 Manifolds with conical singularities. A manifold with conical singularities, B, is a second countable Hausdorff space which is, outside a finite number of points $v \in B$, a smooth manifold.

In a neighborhood of each of the so-called *singularities* or *singular points* v, the manifold is diffeomorphic to a cone $X \times [0, \infty)/X \times \{0\}$, whose cross-section, X, is a closed compact manifold.

In the following we shall confine ourselves to the case of one singularity v. We blow up at v and obtain a manifold with boundary; a neighborhood of the boundary can be identified with the collar $X \times [0,1)$. We denote the resulting object by $\mathbb{B}$, while $X^\wedge$ is the cylinder $X^\wedge = X \times \mathbb{R}_+$.

2.3 Idea of the calculus. Apart from technical complications the basic concept is the following :

- On the smooth part of B use the pseudodifferential calculus in its standard form.
- Near the singularities use Mellin calculus on $X \times \mathbb{R}_+$ working with smooth families of meromorphic Mellin symbols taking values in the algebra of pseudodifferential operators on X.

From now on we shall only deal with *classical* pseudodifferential operators. In order to keep the notation short we shall no longer write the subscript *cl.*

2.4 Mellin transform. For $u \in C_0^\infty(\mathbb{R}_+)$ we define the *Mellin transform* Mu by

$$(Mu)(z) = \int_0^\infty t^{z-1}u(t)dt, \quad z \in \mathbb{C}.$$

This furnishes an entire function which is rapidly decreasing along each line $\Gamma_\beta = \{z \in \mathbb{C} : \operatorname{Re} z = \beta\}$. Plancherel's theorem for the Fourier transform shows that M extends to an isomorphism $L^2(\mathbb{R}_+) \to L^2(\Gamma_{1/2})$. The identity $(Mu)|_{\Gamma_{1/2-\gamma}}(z) = M_{t\to z}(t^{-\gamma}u)(z+\gamma)$ motivates the following definition of the *weighted* Mellin transform:

$$M_\gamma u(z) = M(t^{-\gamma}u)(z+\gamma).$$

The inverse of M_γ is given by

$$(M_\gamma^{-1}h)(t) = \frac{1}{2\pi i}\int_{\Gamma_{1/2-\gamma}} t^{-z}h(z)dz.$$

For $v = -t\partial_t u$ one has $Mv(z) = zMu(z)$, in particular $-t\partial_t u = M^{-1}zMu$.

2.5 Cut-off functions. Whenever we speak of a *cut-off function* or use the notation $\omega, \tilde{\omega}, \omega_1, \omega_2, \ldots$ without further specification we mean a function $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ with $\omega(t) = 1$ near $t = 0$. We will also speak of cut-off functions on $\mathbb{B}$, asking that they vanish on the part of $\mathbb{B}$ not identified with the collar.

2.6 Mellin Sobolev spaces. For $s \in \mathbb{N}, \gamma \in \mathbb{R}$, the Mellin Sobolev space $\mathcal{H}^{s,\gamma}(X^\wedge)$ is the set of all $u \in \mathcal{D}'(X^\wedge)$ for which $t^{n/2-\gamma}(t\partial_t)^k Du(x,t) \in L^2(X^\wedge)$ whenever $k \le s$ and D is a differential operator of order $\le s-k$ on X. Interpolation and duality furnish $\mathcal{H}^{s,\gamma}(X^\wedge)$ for all $s, \gamma \in \mathbb{R}$. Note that duality is with respect to the pairing

$$(u,v) = \frac{1}{2\pi i}\int_{\Gamma_{\frac{n+1}{2}}} (Mu(z), Mv(z))_{L^2(X)}dz$$

and that $\mathcal{H}^{0,n/2}(X^\wedge) = L^2(X^\wedge)$.

These spaces make sense on $\mathbb{B}$, too: We pick a cut-off function ω on $\mathbb{B}$ and let $\mathcal{H}^{s,\gamma}(\mathbb{B}) = \{u : \omega u \in \mathcal{H}^{s,\gamma}(X^\wedge), (1-\omega)u \in H^s_{loc}(\operatorname{int}\mathbb{B})\}$.

2.7 Mellin Symbols and Mellin Operators. Let $\mu \in \mathbb{Z}, \gamma \in \mathbb{R}$. By $L^\mu(X;\mathbb{R})$ denote the space of parameter-dependent pseudodifferential operators of order μ on X with parameter space $\mathbb{R}$. $L^\mu(X;\Gamma_{1/2-\gamma})$ is the corresponding space with $\Gamma_{1/2-\gamma}$ identified with $\mathbb{R}$. Recall that we only use classical symbols!

Given $f \in C^\infty(\overline{\mathbb{R}}_+, L^\mu(X;\Gamma_{1/2-\gamma}))$ define the Mellin operator with (Mellin) symbol f and weight γ by

$$[\mathrm{op}_M^\gamma f]u(t) = \frac{1}{2\pi i}\int_{\Gamma_{1/2-\gamma}} t^{-z} f(t,z)[M_\gamma u](z)dz$$

for $u \in C_0^\infty(X^\wedge) = C_0^\infty(\mathbb{R}_+, C^\infty(X))$. It is easy to see that $\mathrm{op}_M^\gamma f : C_0^\infty(X^\wedge) \to C^\infty(X^\wedge)$ is continuous. Moreover,

$$\omega_1[\mathrm{op}_M^\gamma f]\omega_2 : \mathcal{H}^{s,\gamma+n/2}(X^\wedge) \to \mathcal{H}^{s-\mu,\gamma+n/2}(X^\wedge)$$

is bounded for all s.

We shall now turn to the analysis of asymptotics.

2.8 Example. Let ω be a cut-off function.
(a) $M(\omega) = z^{-1}M(-t\partial_t\omega)(z)$. Since $t\partial_t\omega \in C_0^\infty(\mathbb{R}_+)$, we obtain a meromorphic function with a single simple pole in $z = 0$; it is rapidly decreasing along each Γ_β, uniformly for β in compact intervals, including $\beta = 0$, provided we remove a neighborhood of $z = 0$ (by multiplication with a function which vanishes there and is 1 near infinity).
(b) Let $\mathrm{Re}\, p < 1/2, k \in \mathbb{N}$. Then $M(t^{-p}\ln^k t\omega(t))(z) = \frac{d^k}{dz^k}(M\omega)(z-p)$. This again is a meromorphic function with a single pole in $z = p$ of order $k+1$, it also is rapidly decreasing along each Γ_β, uniformly for β in compact intervals provided we remove a neighborhood of the pole itself.

2.9 Asymptotic types and Mellin Sobolev spaces with asymptotics. Fix $\gamma \in \mathbb{R}$. A weight datum g is a triple $\mathbf{g} = (\gamma + n/2,\ \gamma + n/2,\ (-1,0])$ consisting of two reals and an interval.
(a) An asymptotic type associated with g is a finite set $P = \{(p_j, m_j, C_j) : j = 1, \dots, J\}$ with $J \in \mathbb{N}$ (possibly $J = 0$, then P is the empty set), $p_j \in \mathbb{C}$ with $-1/2 - \gamma < \mathrm{Re}\, p_j < 1/2 - \gamma$, $m_j \in \mathbb{N}$, and C_j finite-dimensional subspaces of $C^\infty(X)$. We denote by $\pi_{\mathbb{C}}P$ the set $\{p_j : j = 1, \dots, J\}$.
(b) A Mellin asymptotic type is a sequence $P = \{(p_j, m_j, L_j) : j \in \mathbb{Z}\}$ with $p_j \in \mathbb{C}, \mathrm{Re}\, p_j \to \mp\infty$ as $j \to \pm\infty$, $m_j \in \mathbb{N}$, and L_j finite dimensional subspaces of finite rank operators in $L^{-\infty}(X)$. As before we write $\pi_{\mathbb{C}}P = \{p_j\}$.
(c) Given an asymptotic type P and $s, \gamma \in \mathbb{R}$, we let $\mathcal{H}_P^{s,\gamma+n/2}(\mathbb{B})$ be the space of all $u \in \mathcal{H}^{s,\gamma+n/2}(\mathbb{B})$ for which there exist $c_{jk} \in C_j, j = 1, \dots, J, k = 0, \dots, m_j$, such that, for all $\varepsilon > 0$,

$$u - \omega(t)\sum_{j=1}^{J}\sum_{k=0}^{m_j} c_{jk}t^{-p_j}\ln^k t \in \mathcal{H}^{s,\gamma+n/2+1-\varepsilon}(\mathbb{B}).$$

2.10 Meromorphic Mellin symbols. (a) $M_O^\mu(X)$ is the space of all entire functions $h : \mathbb{C} \to L^\mu(X)$ such that $h|_{\Gamma_\beta} \in L^\mu(X;\Gamma_\beta)$ uniformly for β in compact intervals.
(b) Let P be a Mellin type. $M_P(X)$ is the space of all holomorphic $h : \mathbb{C} \setminus \pi_{\mathbb{C}}P \to L^\mu(X)$ with the following properties:

(i) In a neighborhood of p_j we have $h(z) = \sum_{k=0}^{m_j} \nu_{jk}(z-p_j)^{-k-1} + h_0(z)$ with $\nu_{jk} \in L_j$ and h_0 analytic near p_j;

(ii) for each interval $[c_1, c_2]$ we find elements ν_{jk} in L_j such that

$$h(\beta + i\tau) = \sum_{\{j:\mathrm{Re}\, p_j \in [c_1,c_2]\}} \sum_{k=0}^{m_j} \nu_{jk} M_{t\to z}(\omega(t)t^{-p_j} \ln^k t)(\beta + i\tau) \in L^\mu(X;\mathbb{R}_\tau)$$

uniformly for $\beta \in [c_1, c_2]$. We set $M_P^{-\infty}(X) = \bigcap_\mu M_P^\mu(X)$.

2.11 Theorem. *$M_P^\mu(X) = M_O^\mu(X) + M_P^{-\infty}(X)$ as a non-direct sum of Fréchet spaces.*

With these notions at hand we are ready to define the full algebra. Fix μ, γ and the weight datum $\mathbf{g} = (\gamma + n/2, \gamma + n/2, (-1, 0])$.

2.12 The residual elements: Green operators. $C_G(\mathbb{B}, \mathbf{g})$ is the space of all operators $G : C_0^\infty(\mathrm{int}\,\mathbb{B}) \to \mathcal{D}'(\mathrm{int}\,\mathbb{B})$ with continuous extensions

$$\begin{aligned} G : \mathcal{H}^{s,\gamma+n/2}(\mathbb{B}) &\to \mathcal{H}_{Q_1}^{\infty,\gamma+n/2}(\mathbb{B}) \quad \text{and} \\ G^* : \mathcal{H}^{s,-\gamma-n/2}(\mathbb{B}) &\to \mathcal{H}_{Q_2}^{\infty,-\gamma-n/2}(\mathbb{B}) \end{aligned}$$

for suitable asymptotic types Q_1, Q_2 and all s. Here, G^* is the adjoint with respect to the pairing $\mathcal{H}^{s,\gamma}, \mathcal{H}^{-s,-\gamma}$.
Note: $\mathcal{H}_{Q_1}^{\infty,\gamma+n/2} \hookrightarrow \mathcal{H}^{N,\gamma+n/2}(\mathbb{B})$ is compact for each N, hence $C_G(\mathbb{B}, \mathbf{g})$ consists of compact operators.

2.13 An ideal: The algebra $C_{M+G}(\mathbb{B}, \mathbf{g})$. $C_{M+G}(\mathbb{B}, \mathbf{g})$ is the space of all operators $R : C_0^\infty(\mathrm{int}\,\mathbb{B}) \to \mathcal{D}'(\mathrm{int}\,\mathbb{B})$ that can be written

$$R = \omega_1[\mathrm{op}_M^\gamma h]\omega_2 + G, \tag{1}$$

where

(i) $h_0 \in M_{P_0}^{-\infty}(X)$ for some Mellin asymptotic type P_0,

(ii) $\pi_{\mathbb{C}} P_0 \cap \Gamma_{1/2-\gamma} = \emptyset$,

(iii) ω_1, ω_2 cut-off functions, and

(iv) $G \in C_G(\mathbb{B}, \mathbf{g})$

Note: These operators form an algebra called the algebra of smoothing Mellin operators. It turns out to be an ideal in the final algebra, while the Green operators form an ideal in $C_{M+G}(\mathbb{B}, \mathbf{g})$. A change in the cut-off functions in (1) results in a Green operator.

2.14 The full algebra. $C^\mu(\mathbb{B}, \mathbf{g})$ is the space of all operators

$$A_M + A_\psi + R,$$

where $A_M = \omega_1[\mathrm{op}_M^\gamma h]\omega_2$, with $h \in C^\infty(\overline{\mathbb{R}}_+, M_O^\mu(X))$, is a Mellin operator supported close to the singularity, A_ψ is a pseudodifferential operator of order μ supported in the interior, and $R \in C_{M+G}(\mathbb{B}, \mathbf{g})$.
Note: $C^\mu(\mathbb{B}, \mathbf{g})$ is a Fréchet space with the natural topology. We let $C(\mathbb{B}, \mathbf{g}) = \cup_\mu C^\mu(\mathbb{B}, \mathbf{g})$. The intersection $\cap_\mu C^\mu(\mathbb{B}, \mathbf{g})$ coincides with $C_{M+G}(\mathbb{B}, \mathbf{g})$.

2.15 Theorem. *The composition of operators yields a continuous map*

$$C^{\mu}(\mathbb{B},\mathrm{g}) \times C^{\mu'}(\mathbb{B},\mathrm{g}) \to C^{\mu+\mu'}(\mathbb{B},\mathrm{g}).$$

We have the ideal structure:

$$C_G(\mathbb{B},\mathrm{g}) \trianglelefteq C_{M+G}(\mathbb{B},\mathrm{g}) \trianglelefteq C^{\mu}(\mathbb{B},\mathrm{g}).$$

2.16 Mellin quantization. For $h \in C^{\infty}(\overline{\mathbb{R}}_+, M_O^{\mu}(X))$ there is a $p \in C^{\infty}(\overline{\mathbb{R}}_+, \widetilde{L}^{\mu}(X;\mathbb{R}))$ such that

$$\operatorname{op} p \equiv \operatorname{op}_M^{\gamma} h \mod L^{-\infty}(X^{\wedge}).$$

Here $C^{\infty}(\mathbb{R}_+, \widetilde{L}^{\mu}(X;\mathbb{R}))$ denotes the space of *totally characteristic symbols* (also Fuchs type symbols), i.e. the elements of $C^{\infty}(\mathbb{R}_+, L^{\mu}(X;\mathbb{R}))$ that can be written $p(t,\tau) = q(t,t\tau)$ for some $q \in C^{\infty}(\overline{\mathbb{R}}_+, L^{\mu}(X;\mathbb{R}))$. The symbol p has the asymptotic expansion

$$p(t,\tau) \sim \sum_{k=0}^{\infty} \frac{1}{k!}\partial_{t'}^k D_\tau^k \{h(t,-iT(t,t')\tau)\frac{T(t,t')}{t'}\}|_{t'=t}$$

with $T(t,t') = \frac{t-t'}{\ln t - \ln t'}$. Note that $T(t,t') = t$.

2.17 Symbols. To an operator in the cone algebra we can therefore associate two important symbols, namely

(i) the interior pseudodifferential symbol which is in fact defined up to the boundary with a totally characteristic degeneracy, and

(ii) the operator family $\{h(0,z) + h_0(z) : H^s(X) \to H^{s-\mu}(X) : z \in \Gamma_{1/2-\gamma}\}$, the so-called *conormal symbol* .

The conormal symbol plays a central role in the Fredholm theory on manifolds with conical singularities. The Fredholm property for an operator is equivalent to the invertibility of the interior principal symbol and the invertibility of the conormal symbol on $\Gamma_{1/2-\gamma}$.

2.18 Notes and Remarks. This is a simplified and comprehensive version of the cone calculus. I used the material in the joint work [23, 24]. Other good sources are Egorov and Schulze [6] and Schulze [28].

Lecture 3: Noncommutative Residues on Manifolds with Conical Singularities

We start with a negative result:

3.1 Example. Wodzicki's residue does not extend to cone algebra. In order to see this recall first that $\mathbb{B}$ is $(n+1)$-dimensional. Suppose $h \in C^{\infty}(\overline{\mathbb{R}}_+, M_O^{-n-1}(X))$, and h vanishes for $t \geq 1$. For $\gamma = 1/2$ we consider the operator $\operatorname{op}_M^{1/2} h$. According to 2.16 we can find a pseudodifferential symbol: $\operatorname{op}_M^{1/2} h \equiv \operatorname{op} p \mod L^{-\infty}(X^{\wedge})$ with $p_{-n-1}(x,t,\xi,\tau) = h(t)_{-n-1}(x,\xi,-it\tau)$. In order to distinguish it from the densities we shall analyze below, we now write W-res for the Wodzicki density introduced in Theorem 1.7. We then have

$$\begin{aligned}
&\text{W-res}_{(x,t)}\operatorname{op} p\\
&= \left(\int_{S^n} p_{-n-1}(x,t,\xi,\tau)\sigma(\xi,\tau)\right) dx\,dt\\
&= \left(\int_{S^n} h(t)_{-n-1}(x,\xi,-it\tau)\sigma(\xi,\tau)\right) dx\,dt\\
&= \left(\int_{S^{n-1}} \int_{-\infty}^{\infty} h(t)_{-n-1}(x,\xi,-it\tau)d\tau\,\sigma(\xi)\right) dx\,dt\\
&= t^{-1}\left(\int_{S^{n-1}} \int_{-\infty}^{\infty} h(t)_{-n-1}(x,\xi,-is)ds\,\sigma(\xi)\right) dx\,dt;
\end{aligned}$$

here $\sigma(\xi,\tau)$ is the n-form corresponding to the $n-1$-form σ used in Section 1. For the third equality we have used that the integrand is a closed form, hence we can shift the contour.

In order to compute the noncommutative residue we would have to integrate the density over the collar $X \times [0,1)$. This, however, is not possible in general; it is possible if $h(t)_{-n-1}$ vanishes for $t=0$.

We shall now define a different density:

3.2 Definition. Let A be as in 2.14. Near $x \in X$ let $h(0)(x,\xi,i\tau)$ be the local symbol of $h(0,i\tau)$. The subscript $-n-1$ in the notation $h(0)_{-n-1}(x,\xi,i\tau)$, below, indicates the term of homogeneity $-n-1$ with respect to (ξ,τ). Define

$$\operatorname{res}_x A = \left(\int_{S^{n-1}} \int_{-\infty}^{\infty} \operatorname{Tr} h(0)_{-n-1}(x,\xi,i\tau)d\tau\,\sigma(\xi)\right) dx_1 \wedge \ldots \wedge dx_n.$$

Since the operators may have values in a vector bundle E, we also took the trace Tr on $\mathcal{L}(E)$ in the integral above. For $n=1$, the sphere S^{n-1} consists of two points, and we replace integration over it by taking $h(0)_{-2}(x,1,i\tau) + h(0)_{-2}(x,-1,i\tau)$.

3.3 Remark.

(a) The decomposition $h+h_0$ is not unique, but h_0 is of order $-\infty$ and therefore gives no contribution.

(b) $\operatorname{res}_x A = (\int_{S^n} \operatorname{Tr} h(0)_{-n-1}(x,\xi,i\tau)\sigma(\xi,\tau))\, dx_1 \wedge \ldots \wedge dx_n$ in view of the fact that $h(0)_{-n-1}(x,\xi,i\tau)\sigma(\xi,\tau)$ is a closed form.

3.4 Lemma. $\operatorname{res}_x A$ *defines a density on* X.

Proof. We fixed t as a global coördinate. So changes of coordinates are of the form $(x,t) \mapsto (\chi(x),t)$. Hence the lemma follows as in the standard case. $\triangleleft$

3.5 Definition. For $A \in C^\mu(\mathbb{B},\mathrm{g})$ let

$$\operatorname{res} A = \int_X \operatorname{res}_x A = \int_X \int_{S^{n-1}} \int_{-\infty}^{\infty} \operatorname{Tr} h(0)_{-n-1}(x,\xi,i\tau)d\tau\,\sigma(\xi)dx_1 \wedge \ldots \wedge dx_n.$$

We may write

$$\operatorname{res} A = \text{W-res} \int_{-\infty}^{\infty} h(0)_{-n-1}(\cdot,\cdot,i\tau)d\tau$$

with Wodzicki's residue of the $(-n)$-homogeneous $\int_{-\infty}^{\infty} h(0)_{-n-1}d\tau$.

3.6 Example. Let Δ be the Laplace-Beltrami operator for a two-dimensional manifold with a geometrical conical singularity. Close to the singularity a computation shows that $t^2\Delta = c^2\partial_x^2 + (t\partial_t)^2$, so it has the Mellin symbol $g(x,t,\xi,z) = -c^2|\xi|^2 + z^2$. Here c is a suitable constant depending on the opening angle of the cone. A parametrix A to $t^2\Delta$ therefore has the Mellin symbol $(-c^2|\xi|^2 + z^2)^{-1}$ modulo lower order terms. This is the desired component of order -2, 'integration' over the two points of $S_x^*S^1$ gives $2(-c^2 + z^2)^{-1}$. Thus

$$\operatorname{res} A = -2\int_{S^1}\int_{-\infty}^{\infty}(c^2+\tau^2)^{-1}d\tau\, dx = -4\pi^2/c.$$

We shall now produce an extension of Wodzicki's residue. On the collar we consider the algebra of all Mellin operators with vanishing Mellin symbol in $t = 0$.

3.7 Operators on the collar. Consider the operators in the cone algebra that can be written in the form

$$A = \omega_1[\operatorname{op}_M^\gamma h]\omega_2 + R$$

with

(i) $h \in C^\infty(\overline{\mathbb{R}}_+, M_O^\mu(X)), h(0) \in M_O^{-\infty}(X)$,

(ii) ω_1, ω_2 cut-off functions with $\omega_1\omega_2 = \omega_1$,

(iii) $R \in C_{M+G}(\mathbb{B}, \mathbf{g})$.

What is important is that we may choose $h(0) \in M_O^{-\infty}(X)$ and that $\omega_1\omega_2 = \omega_1$. The latter condition normalizes the representation in a certain sense.

The conormal symbol of the composition of two operators is the product of the conormal symbols. Hence the operators of this type form an algebra.

For $(x,t) \in X \times (0,1)$ define

$$\operatorname{res}^0_{x,t} A = \left(\int_{S^{n-1}}\int_{-\infty}^{\infty}\omega_1(t)h(t)_{-n-1}(x,\xi,i\tau)d\tau\,\sigma(\xi)\right)dx_1\wedge\ldots\wedge dx_n \wedge \frac{dt}{t}.$$

3.8 Lemma.

$$\operatorname{res}^0_{x,t} A = \int_{S^n_{\xi,\tau}} \operatorname{Tr}\omega_1(t)h(t)_{-n-1}(x,\xi,i\tau)\sigma(\xi,\tau)dx_1\wedge\ldots dx_n\wedge\frac{dt}{t}.$$

It is a density on $X \times \mathbb{R}_+$.

Proof. The identity follows from the fact that $h(t)_{-n-1}(x,\xi,i\tau)\sigma(\xi,\tau)$ is closed. That it is a density can then be proven as before. ◁

3.9 Theorem. *For A as above let*

$$\operatorname{res}^0 A = \int_0^1\int_X \operatorname{res}^0_{x,t} A.$$

This makes sense, since $h(0)_{-n-1} = 0$. Moreover, res^0 is a trace on operators of this form.

Proof. This can be shown just like in Theorem 1.7. ◁

3.10 Lemma. *Let p be the totally characteristic pseudodifferential symbol associated with $A \bmod L^{-\infty}(X^\wedge)$, cf. 2.16,*

$$p(t,\tau) \sim \omega_1(t) \sum_{k=0}^{\infty} \frac{1}{k!} \partial_{t'}^k D_\tau^k \{h(t, -iT(t,t')\tau) T(t,t')/t'\}|_{t'=t}$$

with $T(t,t') = (t-t')/(\ln t - \ln t')$. Then

$$\mathrm{res}^0 A = \text{W-res}\, p.$$

Proof. The terms in the asymptotic expansion may be rewritten in the form $D_\tau^k h(t, -it\tau)\varphi_k(t,\tau)$, where φ_k is a polynomial in τ of degree $\le k/2$ and $\varphi_0 = 1$.

$$\begin{aligned} & \int_{S^n_{\xi,\tau}} \mathrm{Tr}\, p(t)_{-n-1}(x,\xi,\tau) dx \wedge dt \\ = & \int_{S^n_{\xi,\tau}} \mathrm{Tr} \left(\sum_{k=0}^{\infty} \frac{1}{k!} \partial_{t'}^k D_\tau^k \{h(t)(x,\xi,-iT(t,t')\tau)T(t,t')/t'\}|_{t'=t} \right)_{-n-1} \\ = & \int_{S^n_{\xi,\tau}} \mathrm{Tr}\, h(t)_{-n-1}(x,\xi,-it\tau)\sigma(\xi,\tau) dx \wedge dt. \end{aligned}$$

We deduce that

$$\begin{aligned} \text{W-res}\, p &= \int_0^1 \int_X \int_{S^n_{\xi,\tau}} \mathrm{Tr}\, h(t)_{-n-1}(x,\xi,-it\tau)\sigma(\xi,\tau) dx \wedge dt \\ &= \int_0^1 \int_X \int_{S^{n-1}_\xi} \int_{-\infty}^{\infty} \mathrm{Tr}\, h(t)_{-n-1}(x,\xi,-it\tau) d\tau\, \sigma(\xi) dx \wedge dt \\ &= \mathrm{res}^0 A \end{aligned}$$

provided we choose compatible orientations on S^{n-1} and S^n. ◁

3.11 The extended cone algebra. On $\mathbb{B}$ fix a smooth function $\underline{t}$ which coincides with the geodesic distance to the boundary near the boundary of $\mathbb{B}$ and which is strictly positive in the interior. Let $C(\mathbb{B}, \mathbf{g})^+$ be the space of finite sums of operators of the form $\underline{t}^m B$ with $\mathrm{Re}\, m \ge 0$ and $B \in C(\mathbb{B}, \mathbf{g})$. Similarly define $C_{M+G}(\mathbb{B}, \mathbf{g})^+$. We obtain the subspaces

$$\begin{aligned} C(\mathbb{B}, \mathbf{g})_0^+ &= \mathrm{span}\, \{t^m B : B \in C(\mathbb{B}, \mathbf{g}), \mathrm{Re}\, m > 0\} \text{ and} \\ C_{M+G}(\mathbb{B}, \mathbf{g})_0^+ &= \mathrm{span}\, \{t^m B : B \in C_{M+G}(\mathbb{B}, \mathbf{g}), \mathrm{Re}\, m > 0\}. \end{aligned}$$

Why is this an algebra? Multiplication of a Green operator by $\underline{t}^m$ from the left or from the right yields a Green operator, since $\mathrm{Re}\, m \ge 0$. Similarly there is no problem for pseudodifferential operators in the interior. For the Mellin operators we use the following computation. Note that we may assume $\underline{t} = t$, since we are close to the boundary.

$$\begin{aligned} \mathrm{op}_M^\gamma h(t^m u)(t) &= \frac{1}{2\pi i} \int_{\Gamma_{1/2-\gamma}} \int_0^\infty (t/t')^{-z} h(t,z)(t'^m u(t')) \frac{dt'}{t'} dz \\ &= \frac{t^m}{2\pi i} \int_{\Gamma_{1/2-\gamma}} \int_0^\infty (t/t')^{-(z+m)} h(t,z) u(t) \frac{dt'}{t'} dz \\ &= \frac{t^m}{2\pi i} \int_{\Gamma_{1/2-\gamma+m}} \int_0^\infty (t/t')^{-\zeta} h(t,\zeta - m) u(t')) \frac{dt'}{t'} d\zeta. \end{aligned}$$

If h is holomorphic we may shift the contour and immediately obtain that the last expression equals $t^m[\mathrm{op}_M^\gamma T^{-m}h]u(t)$. In case h has poles between $\Gamma_{1/2-\gamma}$ and $\Gamma_{1/2-\gamma+m}$, say at p_j, then Cauchy's theorem says that we pick up the residues. Those result in terms of the form $c(u)t^{-p_j}\ln^k t$ with $1/2-\gamma \le \mathrm{Re}\, p_j \le 1/2-\gamma+m$. Hence $\omega_1[\mathrm{op}_M^\gamma h]t^m\omega_2 = t^m\omega_1[\mathrm{op}_M^\gamma T^{-m}h]\omega_2 + G$ for a Green operator G. There is a minor difficulty if h is smoothing and $T^{-m}h$ has a singularity on $\Gamma_{1/2-\gamma}$. Then we write

$$t^m \mathrm{op}_M^\gamma h = t^m \mathrm{op}_M^{\gamma_0} h + G'$$

with $\gamma - \mathrm{Re}\, m \le \gamma_0 \le \gamma$ and a Green operator G'; in fact we shall adopt this slightly revised interpretation of elements in $C_{M+G}(\mathbb{B},\mathbf{g})$ (which is standard in the cone calculus).

We can now state the theorem on uniqueness. A full proof is given in [26].

3.12 Theorem.

(a) *The dimension of the space of continuous traces on the quotient* $C(\mathbb{B},\mathbf{g})^+/C_{M+G}(\mathbb{B},\mathbf{g})^+$ *equals the number of conical points.*

(b) *On* $C(\mathbb{B},\mathbf{g})_0^+/C_{M+G}(\mathbb{B},\mathbf{g})_0^+$ *there is a unique continuous trace, namely the extension of Wodzicki's residue.*

Of course, *every* point of $\mathbb{B}$ may be considered a (fictitious) conical point. We understand (a) in the sense that we only count those points where $C(\mathbb{B},\mathbf{g})$ has the cone algebra structure described in Section 2. The continuity requirement is that convergence of the Mellin symbols implies convergence of the traces of the associated operators.

Lecture 4: The Noncommutative Residue and Dixmier's Trace

Dixmier's paper [4] settled a longstanding question: Is every completely additive trace proportional to the standard operator trace on the set where it is finite? Dixmier showed that the answer is 'no' by explicitly constructing counter-examples. We start this section by reviewing his result, following Connes [3] in presentation and terminology.

4.1 The spaces $\mathcal{L}^{(1,\infty)}(H)$ and $\mathcal{L}_0^{(1,\infty)}(H)$. Let H be an (infinite-dimensional) Hilbert space, $T \in \mathcal{K}(H)$, and $|T| = (T^*T)^{1/2}$. Let $\mu_0(T) \ge \mu_1(T) \ge \ldots$ be the sequence of eigenvalues of $|T|$, repeated according to their multiplicity. It is well-known that

$$\mu_j(T) = \inf\{\|T-F\| : \mathrm{rank}\, F = j\} = \min\{\|T|_{E^\perp}\| : \dim E = j\}. \qquad (1)$$

We define $\sigma_N(T) = \sum_{j=0}^N \mu_j(T)$ and let $\mathcal{L}^{(1,\infty)}(H) = \{T \in \mathcal{K}(H) : \sigma_N(T) = O(\ln N)\}$, endowed with the norm

$$\|T\|_{1,\infty} = \sup_{N\ge 2} \frac{\sigma_N(T)}{\ln N}.$$

We have a natural subspace $\mathcal{L}_0^{(1,\infty)}(H) = \{T \in \mathcal{K}(H) : \sigma_N(T) = o(\ln N)\}$.

4.2 Lemma. *Let σ_N be as in 4.1, and let T, T_1, and T_2 be compact.*

(a) $\sigma_N(T) = \max\{\|TP_E\|_1 : \dim E = N\}$, *with the $\mathcal{L}^1$-norm $\|\cdot\|_1$ and P_E denoting the projection on E.*

(b) $\sigma_N(T) = \max\{\mathrm{trace}(TP_E) : \dim E = N\}$ *for* $T \ge 0$.

(c) $\sigma_N(T_1+T_2) \le \sigma_N(T_1) + \sigma_N(T_2)$.

(d) $\sigma_N(T_1) + \sigma_N(T_2) \le \sigma_{2N}(T_1 + T_2)$ *if* $T_1, T_2 \ge 0$.

(e) $\mathcal{L}_0^{(1,\infty)}, \mathcal{L}^{(1,\infty)}$ *are two-sided ideals in* $\mathcal{L}(H)$.

Proof. For (a) use 4.1(1); the maximum is attained by choosing E the eigenspace with respect to the first N eigenvalues of $|T|$. (a) implies (b); the maximum is attained for the same E. (c) is immediate from (a). Since the dimension is subadditive one gets (d) from (b). Finally (e) is a consequence of the estimate $\mu_j(TA) \le \mu_j(T)\|A\|$ valid for bounded A. $\triangleleft$

4.3 Cesàro Mean. We define the Cesàro mean Mf for $f \in L^\infty(1,\infty)$ by

$$(Mf)(t) = \frac{1}{\ln t} \int_1^t f(s) \frac{ds}{s}.$$

The function Mf is continuous and bounded; $M : L^\infty(1,\infty) \to C_b(1,\infty)$ is continuous. Moreover, $M1 = 1$ and $M(f(\lambda\cdot)) - Mf \in C_{b(0)}(1,\infty)$. Here, $\lambda > 0$, and the subscript (0) indicates that the function vanishes at infinity.

4.4 The 'limit' $\lim_\omega$. Let $T_1, T_2 \in \mathcal{L}^{(1,\infty)}$ be positive and

$$\alpha_N = \frac{\sigma_N(T_1)}{\ln N}, \ \beta_N = \frac{\sigma_N(T_2)}{\ln N}, \ \gamma_N = \frac{\sigma_N(T_1 + T_2)}{\ln N}.$$

Then $\{\alpha_N\}, \{\beta_N\}$, and $\{\gamma_N\}$ are bounded sequences. By 4.2 we have

$$\gamma_N \le \alpha_N + \beta_N \le (\ln 2N / \ln N)\, \gamma_{2N}, \tag{1}$$

but in general no convergence. We embed $\mathcal{L}^\infty$ into $L^\infty(1,\infty)$ in the canonical way by associating to the sequence $\{a_N\}$ the function $f_{\{a_N\}}$ which has the value a_j on the interval $[j, j+1[$, $j = 1, 2, \ldots$. Next we choose a linear form ω on $C_b(1,\infty)$ with (i) $\omega \ge 0$, (ii) $\omega(1) = 1$, and (iii) $\omega(f) = 0$ for $f \in C_{b(0)}$. Then we define $\lim_\omega\{a_N\} = \omega(Mf_{\{a\}})$ with the help of Cesàro's mean.

Note that $\lim_\omega$ coincides with the usual limit on convergent sequences by (ii) and (iii). Furthermore, $\lim_\omega a_{2N} = \lim_\omega a_N$.

4.5 Dixmiers trace. For a positive operator $T \in \mathcal{L}^{(1,\infty)}$ let

$$\mathrm{Tr}_\omega(T) = \lim_\omega \frac{1}{\ln N} \sum_{n=0}^{N} \mu_n(T).$$

As Proposition 4.6(a) shows, Tr_ω is additive. We can therefore extend it uniquely to a linear map on $\mathcal{L}^{(1,\infty)}$, also denoted Tr_ω.

4.6 Proposition. *Let* $T, T_1, T_2 \in \mathcal{L}^{(1,\infty)}(H), S \in \mathcal{L}(H)$.

(a) $\mathrm{Tr}_\omega(T_1 + T_2) = \mathrm{Tr}_\omega(T_1) + \mathrm{Tr}_\omega(T_2)$ *for positive* T_1, T_2.

(b) $\mathrm{Tr}_\omega(T) \ge 0$ *if* $T \ge 0$.

(c) *If S is invertible, then* $\mathrm{Tr}_\omega(STS^{-1}) = \mathrm{Tr}_\omega(T)$. *In particular,* Tr_ω *is independent of the inner product in* H.

(d) $\mathrm{Tr}_\omega(ST) = \mathrm{Tr}_\omega(TS)$.

(e) $\mathrm{Tr}_\omega \equiv 0$ *on* $\mathcal{L}_0^{(1,\infty)}$, *so it vanishes on trace class operators.*

Proof. (a) follows from 4.4(1) together with the last remark in 4.4. We only have to check (c) for positive T. Then use 4.2(a)/(b). Finally (c) implies (d) first for invertible S, then for arbitrary S by adding a large multiple of the identity and using linearity. $\triangleleft$

4.7 Example. Consider the operator $(1-\Delta)^{-n/2} : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$, where Δ is the Laplacian. The eigenvalues of Δ are known to be the lengths $|k|^2$ as k varies over $\mathbb{Z}^n$, so the eigenvalues of $(1-\Delta)^{-n/2}$ are $(1-|k|^2)^{-n/2}$.

Let us show that $(1-\Delta)^{-n/2} \in \mathcal{L}^{(1,\infty)}$ and $\mathrm{Tr}_\omega(1-\Delta)^{-n/2} = \Omega_n/n$, independent of ω with $\Omega_n = \mathrm{vol}\, S^{n-1}$: We let N_R denote the number of lattice points in B_R, the ball of radius R. Clearly, $N_R \sim \mathrm{vol}\, B_R$, hence $\ln N_R \sim n \ln R$. Moreover,

$$\begin{aligned}\sum_{|k|\le R} (1+|k|)^{n/2} &\sim \Omega_n \int_0^R (1+r^2)^{-n/2} r^{n-1} dr \\ &\sim \Omega_n \int_1^R r^{-1} dr = \Omega_n \ln R.\end{aligned}$$

We conclude that

$$(\ln N_R)^{-1} \sum_{|k|\le R} (1+|k|)^{n/2} \sim \frac{\Omega_n \ln R}{n \ln R} = \frac{\Omega_n}{n}.$$

Recall that for $(1-\Delta)^{-n/2} : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ we had computed in 1.9 that

$$\mathrm{res}\,(1-\Delta)^{-n/2} = \mathrm{vol}\, S^{n-1} \mathrm{vol}\, \mathbb{T}^n = \Omega_n (2\pi)^n.$$

In one special case we therefore have proven the following result:

4.8 Theorem (Connes 1988). Let M be closed, compact, n-dimensional, E a vector bundle over M, and $P : L^2(M,E) \to L^2(M,E)$ a pseudodifferential operator of order $-n$. Then

(a) $P \in \mathcal{L}^{(1,\infty)}(L^2(M,E))$.

(b) $\mathrm{res}\, P = (2\pi)^n n \mathrm{Tr}_\omega P$, independent of ω.

Proof. We start with the observation that both res and Tr_ω are local: If $\{\varphi_1, \dots, \varphi_J\}$ is a partition of unity on M and if $\{\psi_1, \dots, \psi_J\}$ are smooth functions with $\varphi_j \psi_j = \varphi_j$, then

$$\mathrm{res}\, P = \sum \mathrm{res}\, \varphi_j P \psi_j \quad \text{and} \quad \mathrm{Tr}_\omega P \;=\; \sum \mathrm{Tr}_\omega \varphi_j P \psi_j,$$

since $\mathrm{res}\,(\varphi_j P(1-\psi_j)) = \mathrm{res}\,((1-\psi_j)\varphi_j P) = 0$, similarly for Tr_ω. Thus we may assume that $M = \mathbb{T}^n$.

Part (a) now follows from writing $P = (P(1-\Delta)^{n/2})(1-\Delta)^{-n/2}$: The first factor on the right hand side is bounded, $(1-\Delta)^{-n/2} \in \mathcal{L}^{(1,\infty)}$, and $\mathcal{L}^{(1,\infty)}$ is an ideal.

In order to see (b) we first note that we proved it for $(1-\Delta)^{-n/2}$, see 4.7. By linearity it is enough to consider $T = P + \lambda(1-\Delta)^{-n/2}$ for $P \ge 0$ and large positive λ. In that case, $T : L^2(M,E) \to H^n(M,E)$ is invertible, and $T = A^{-1}$ with a pseudodifferential operator A of order n satisfying the assumptions of Seeley's theorem. By Wodzicki's formula 1.11(2),

$$\frac{\mathrm{res}\, T}{n(2\pi)^n} = \frac{\mathrm{res}\, A^{-1}}{(2\pi)^n \mathrm{ord}\, A} = \mathrm{Res}_{s=1}\zeta_A = -\mathrm{Res}_{s=-1}\zeta_T = \mathrm{Res}_{s=1} \sum_{j=0}^{\infty} \lambda_j^s,$$

where $\lambda_0 \geq \lambda_1 \geq \ldots$ are the eigenvalues of T, so the λ_j^{-1} are the eigenvalues of A.

Let $\lambda_0 \geq \ldots \geq \lambda_{k_0-1} \geq 1 > \lambda_{k_0} \geq \lambda_{k_0+1}$, denote by θ the characteristic function of $\mathbb{R}_+$, and define $\mu(x) = \sum_{k=0}^{\infty} \theta(x + \ln \lambda_{k+k_0})$. This is a positive measure. Its Laplace transform is

$$\int_0^\infty e^{-sx} d\mu(x) = \sum_{k=k_0}^{\infty} \lambda_k^s = \zeta_A(s) - \sum_{k=0}^{k_0-1} \lambda_k^s.$$

According to Seeley's result, this function is analytic for $\operatorname{Re} s > 1$ and extends to $\{\operatorname{Re} s < 1-\varepsilon\}$ with a simple pole in $s = 1$. We can therefore apply Ikehara's Tauberian theorem (see e.g. [5, Section 47]) and conclude that $\operatorname{Res}_{s=1}\zeta_A(s) = \lim e^{-x}\mu(x) =: c$.

Now $\mu(x) = \sum_{\{k: x \geq -\ln \lambda_{k+k_0}\}} 1 = \sum_{\{k: e^{-x} \leq \lambda_{k+k_0}\}} 1$ so that $\mu(x) = j$ iff $\lambda_{k_0+j+1} < e^{-x} \leq \lambda_{k_0+j}$. From this we derive that $j\lambda_{k_0+j+1} < \mu(x)e^{-x} \leq j\lambda_{k_0+j}$, hence $\lambda_{k_0+j} \sim c/j$ with above c. We conclude that $T \in \mathcal{L}^{(1,\infty)}$ and $\operatorname{Tr}_\omega(T) = \lim_{N\to\infty} \sigma_N/\ln N = c$. Note that the limit exists and therefore is independent of ω. ◁

4.9 Corollary: Weyl's theorem. Let M be closed, compact, n-dimensional, let Δ be the Laplace–Beltrami operator on M with respect to some Riemannian metric. For the eigenvalues λ_j of $-\Delta$ we then get the asymptotics

$$\lambda_j \sim 4\pi^2 \left(\frac{n}{\Omega_n}\right)^{2/n} \left(\frac{j}{\operatorname{vol} M}\right)^{2/n}.$$

Proof. We deduce this from the last part of the proof of the previous theorem rather than from the assertion. Consider $(1-\Delta)^{-n/2}$. Its inverse A satisfies the assumptions of Seeley; ζ_A is analytic on $\{\operatorname{Re} s > 1\}$ and extends to a larger half-plane with a simple pole. We know from 1.9 that $\operatorname{res}(I-\Delta)^{-n/2} = \Omega_n \operatorname{vol} M$, hence $\operatorname{Tr}_\omega(I-\Delta)^{-n/2} = (2\pi)^{-n}\operatorname{res}(I-\Delta)^{-n/2}/n = (2\pi)^{-n}\Omega_n \operatorname{vol} M/n =: c_n$. An application of Ikehara's Tauberian theorem as above implies that the eigenvalues μ_j of $(I-\Delta)^{-n/2}$ satisfy $\mu_j \sim c_n/j$. The identity $\lambda_j = \mu_j^{-2/n}$ then proves the result. ◁

4.10 Notes and Remarks. The idea of the proof of Theorem 4.8 was adapted from Várilly and Gracia-Bondia [31]. Corollary 4.9 follows already from Seeley's results and was stated in [30] more generally for positive definite pseudodifferential operators. The Laplacian is a nice example in that we have the explicit form of the constants.

In the article [2], Connes used the coincidence of the noncommutative residue and Dixmier's trace in the following way:

For an algebra $\mathcal{A}$, a p-summable Fredholm module $(\mathcal{H}, F)$ over $\mathcal{A}$, and a finite projective module $\mathcal{E}$ over $\mathcal{A}$ with an $\mathcal{A}$-valued inner product, one can introduce the notion of connections ∇ and curvature θ.

He then considers the case of a 4-dimensional smooth compact Riemannian Spin^c manifold. The Fredholm module $(\mathcal{H}, F)$ consists of the Hilbert space $\mathcal{H}$ of L^2-spinors, and $F = D|D|^{-1}$, where D is the Dirac operator. Under a compatibility assumption he can show that the (abstractly defined) curvature θ is an element of $\mathcal{L}^{(2,\infty)}$ so that the value of the Dixmier trace $\operatorname{Tr}_\omega(\theta^2) = I(\theta)$ defines a positive functional independent of ω.

Moreover, given a classical connection A, the classical Yang-Mills action $YM(A)$ of A can be recovered by

$$YM(A) = 16\pi^2 \inf I(\theta)$$

with the infimum taken over a suitable class of connections related to A.

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