# RELATING THE ASSOCIAHEDRON AND THE PERMUTOHEDRON


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## INTRODUCTION

Recently it was shown by Kapranov [4] that the combinatorics of the permutohedra and associahedra can be combined to give a 'hybrid' family of polytopes, the permutoassociahedra. In this short note we put forward a slightly different point of view: the associahedra can themselves be seen as retracts of the permutohedra. We construct a natural cellular quotient map from the permutohedron $P_n$ to the associahedron $K_{n+1}$. In dimension 3 we also give $K_5$ as the convex hull of a particular subset of the usual vertices of $P_4$.

## 1. THE QUOTIENT MAP

We begin by recalling the definitions of the permutohedra and the associahedra. See [4] and the references there for more details.

The *permutohedron* [5, 8] (or *zilchgon* [2], or *parallelohedron* [1]) $P_n$ is the convex hull of the $n!$ vertices $(\pi^{-1}(1), \pi^{-1}(2), \ldots, \pi^{-1}(n)) \in \mathbb{R}^n$, for permutations $\pi \in S_n$. As a cellular complex $P_n$ is the realization of the poset $\mathcal{P}_n$ of partitions of $\underline{n} = \{1, 2, \ldots, n\}$. That is, an $(n-r)$-cell of $P_n$ is labelled by a tuple $(A_1, A_2, \ldots, A_r)$ of non-empty disjoint subsets of $\underline{n}$ with $\bigcup A_i = \underline{n}$. A permutation $\pi \in S_n$ gives a 0-cell of $P_n$ via $A_i = \{\pi(i)\}$, and the 1-skeleton of $P_n$ is just the Cayley graph of $S_n$. An $r$-cell $(A_i)_{i=1}^r$ is isomorphic to the product $P_{a_1} \times P_{a_2} \times \cdots \times P_{a_r}$, where $a_i = |A_i|$, and its boundary consists of those cells given by further partitioning the $A_i$. Note that $P_n$ is $(n-1)$-dimensional.

The *associahedron* [9, 10] (or *Stasheff polytope*) $K_n$ is the realization of the poset $\mathcal{K}_n$ of bracketings of $n$ variables, or equivalently of rooted trees with $n$ leaves or of certain subdivisions of the $(n+1)$-gon. It has dimension $n-2$. An $(n-r)$-cell of $K_n$ corresponds to a (meaningful) insertion of $r-2$ pairs of parentheses into the expression $x_1 x_2 \ldots x_n$, or to a rooted tree with $n$ leaves and $r-1$ internal nodes. The boundary consists of the cells obtained by inserting further parentheses into the expression. By [3, 6], the associahedron $K_n$ may also be obtained as the convex hull of a particular collection of $c_{n-1}$ points in $\mathbb{R}^{n+1}$, where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the $n$th Catalan number. These vertices correspond to the complete bracketings, the binary trees, or the triangulations of the $(n+1)$-gon.

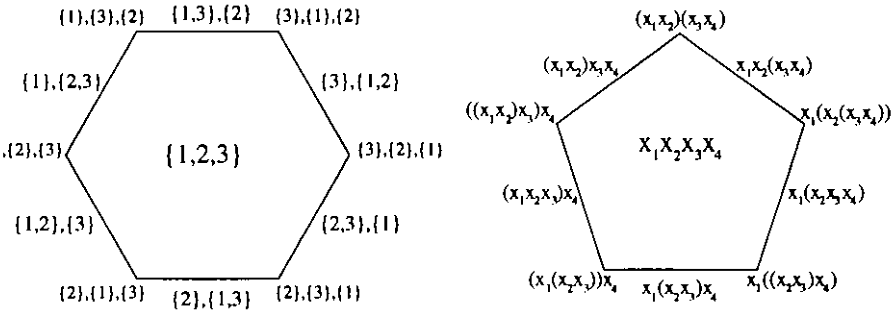


FIGURE 1. The permutohedron and associahedron of dimension 2.

**Definition 1.1.** Consider a relation $\sim$ on $\mathcal{P}_n$ as follows. For a partition $(A_i)_1^r$ we say that $A_{k-1}$ and $A_k$ are *independent* if there exists $x \in \bigcup_{i>k} A_i$ such that $\max A_{k-1} < x < \min A_k$ or $\max A_k < x < \min A_{k-1}$. Then $\sim$ is the equivalence relation generated by

$$(A_1, A_2, \ldots, A_n) \sim (A_1, \ldots, A_{k-2}, A_{k-1} \cup A_k, A_{k+1}, \ldots, A_n)$$

if $A_{k-1}$ and $A_k$ are independent.

To give the motivation for this definition, consider a composite of $n+1$ variables $x_1x_2\ldots x_{n+1}$ which is to be evaluated. There are $n$ composition operations to be performed, and so $n!$ ways of carrying out the evaluation, which we can think of as the vertices of $P_n$. Similarly we interpret a general face of $P_n$, given by a partition $(A_i)_{i=1}^r$, as the following evaluation procedure: carry out simultaneously ("in parallel") the composition operations between $x_i$ and $x_{i+1}$ for $i \in A_1$, then on the resulting terms carry out the composition operations indicated by $A_2$, then for $A_3$, and so on. An $(n-r)$-dimensional face of $P_n$ gives a procedure for evaluating the composite $x_1x_2\ldots x_{n+1}$ in $r$ stages.

To any such evaluation procedure there is an associated tree, with $n+1$ leaves labelled by the variables $x_i$ and at least $r$ internal nodes labelled by the compositions. Thus we have constructed a function from partitions of $\underline{n}$ to trees with $n+1$ leaves:

$$\theta : \mathcal{P}_n \to \mathcal{K}_{n+1}\,.$$

This respects the poset structures since taking a finer partition gives further parentheses or extra internal nodes. The function is also surjective: for any tree, choose an ordering of the internal nodes which respects the natural partial order. Such an ordering defines a composition procedure and hence a partition which under $\theta$ gives the original tree. There is a choice of ordering when two nodes in the tree correspond to terms which are to be composed later; the composition may be carried out first at one node then at the other, or both simultaneously. As in definition 1.1 we call such nodes *independent*. It is clear that $\theta$ maps two partitions to the same tree if and only if they are equivalent under the relation $\sim$. Thus $\mathcal{K}_{n+1} \cong \mathcal{P}_n/\!\sim$ and $\theta$ is the quotient map $\mathcal{P}_n \to \mathcal{P}_n/\!\sim$.

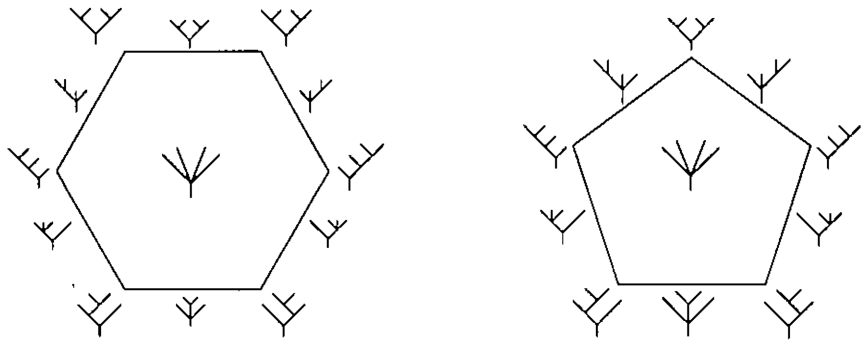

FIGURE 2. The trees associated to $P_3$ and $K_4$.

Taking the realization of the map $\theta$ gives:

**Proposition 1.2.** There is a natural cellular quotient map of $(n-1)$-dimensional complexes

$$P_n \xrightarrow{\theta} K_{n+1}$$

from the permutohedron to the associahedron.

The restriction of $\theta$ to the vertices is (essentially) the function from $S_n$ to binary trees used by Loday [7].

In dimension two, $\theta$ consists of quotienting one of the edges of the hexagon to give a pentagon. We can see this arising quite naturally in homotopy theory, as follows. We consider the hexagon as the space of paths through the cube: the vertices of the former correspond to the six paths through the edges of the latter, with edges corresponding to the homotopies between paths given by the six square faces. But the cube is in turn the path space of a 4-simplex $\sigma$. Five of the faces of the cube correspond to actual homotopies of homotopies of paths, given by the faces of $\sigma$. The sixth, however, is the product of the homotopies given by $\sigma(012)$ and $\sigma(234)$. It is this square which corresponds to the "degenerate" edge of the hexagon.

## 2. DIMENSION THREE

Consider the function $\phi$ given by the restriction of $\theta : P_4 \to K_5$ to the vertices of the permutohedron $P_4$. We define a right inverse $\iota : K_5 \to P_4$ to $\phi$ with the property that for any face $F$ of $K_5$ the vertices $\{\iota(v) : v \text{ a vertex of } F\}$ are coplanar.

For eight of the vertices $v \in K_5$ there is a unique vertex $\iota(v) \in P_4$ such that $\phi\iota(v) = v$. For the remaining vertices we make the following choices:

$$\begin{array}{lll} (x_1x_2)(x_3(x_4x_5)) \mapsto 4312 & (x_1x_2)((x_3x_4)x_5) \mapsto 3412 & ((x_1x_2)(x_3x_4))x_5 \mapsto 3124 \\ ((x_1x_2)x_3)(x_4x_5) \mapsto 1243 & (x_1(x_2x_3))(x_4x_5) \mapsto 2143 & x_1((x_2x_3)(x_4x_5)) \mapsto 2431 \end{array}$$

We check the coplanarity of the vertices $\{\iota(v) : v \text{ a vertex of } F\}$ for the faces $F$ of the associahedron. For vertices $v$ of the pentagon $F = (x_1x_2)x_3x_4x_5$ we note that the $\iota(v)$ all lie in the plane $\lambda_1 + 1 = \lambda_2$ (and of course $\sum \lambda_i = 10$) in $\mathbb{R}^4 = \{(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\}$. Similarly $\iota$ maps the vertices of $F = x_1x_2x_3(x_4x_5)$ to the plane $\lambda_4 + 1 = \lambda_3$. For the remaining pentagonal and square faces the vertices are mapped to vertices of original faces of the permutohedron. In fact we have

**Proposition 2.1.** The associahedron $K_5$ may be defined as the convex hull of subset $\{\iota(v) : v \text{ a vertex of } K_5\}$ of the usual vertices of the permutohedron $P_4$ in $\mathbb{R}^4$. Furthermore $K_5$ may be obtained from $P_4$ by intersection with the region $\lambda_1+1 \geqslant \lambda_2$, $\lambda_4 + 1 \geqslant \lambda_3$.

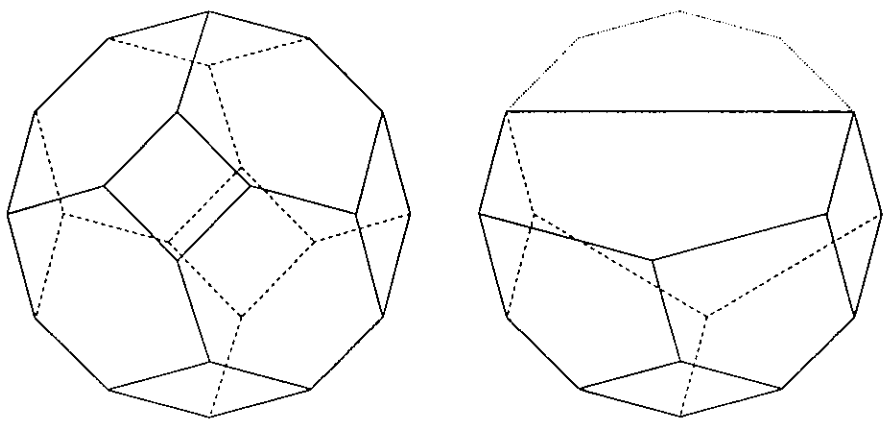

FIGURE 3. $K_5$ obtained from $P_4$ by two perpendicular cuts.

**Remark 2.2.** There is no corresponding result for $K_6$ and $P_5$. The vertices of the faces $x_1(x_2x_3)x_4x_5x_6$ and $x_1x_2x_3(x_4x_5)x_6$ of $K_6$ would have to be mapped to the hyperplanes $\lambda_2 = 1$ and $\lambda_4 = 1$ respectively. But then $\iota$ must map the vertices of the intersection $x_1(x_2x_3)(x_4x_5)x_6$ to points with $\lambda_2 = \lambda_4 = 1$, which is clearly not the case for any vertices of $P_5$.

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