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On a Class of Representations of Quantum Groups

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Abstract

This paper is a short account of the construction of a new class of the infinite-dimensional representations of the quantum groups. The examples include finite-dimensional quantum groups $U_q(\mathfrak{g})$, Yangian $Y(\mathfrak{g})$ and affine quantum groups at zero level $U_q(\hat{\mathfrak{g}})_{c=0}$ corresponding to an arbitrary finite-dimensional semisimple Lie algebra $\mathfrak{g}$. At the intermediate step we construct the embedding of the quantum groups into the algebra of the rational functions on the quantum multi-dimensional torus. The explicit parameterization of the quantum groups used in this paper turns out to be closely related to the parameterization of the moduli spaces of the monopoles. As a result the proposed constructions of the representations provide a quantization of the moduli spaces of the monopoles on $\mathbb{R}^3$ and $\mathbb{R}^2 \times S^1$.

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1 Introduction

Constructions of explicit realizations of irreducible representations are not only interesting from the general point of view but also play a key role in the applications of the Representation theory to Number theory, Geometry and Physics. "Good" realizations often capture important features of the underlying object and thus lead to a better understanding of its properties.

Below we describe a construction of the infinite-dimensional representations of the quantum groups that may be considered as a generalization of the well-known construction of the finite-dimensional representations of the classical group due to Gelfand-Zetlin [GelZ], [GelG]. The authors came to this construction while trying to understand some explicit integral representations of the wave function of the quantum integrable theories [KL1], [KL2] obtained in the framework of Quantum Inverse Scattering Method (QISM) [F], [KS]. As it is in all other constructions of the irreducible representations, the crucial step is the choice of appropriate coordinates on Lie groups and maximal commutative subalgebras in (the skew-filed of fractions of) the universal enveloping algebra. Later, this construction was generalized to the case of the quantum groups $U_q(\mathfrak{gl}(N))$ [GKL3]. The obtained realization of the representation of the finite-dimensional quantum group possess very interesting properties. In particular, the representation space has a natural structure of $U_q(\mathfrak{g}) \otimes U_{q^\vee}(\mathfrak{g}^\vee)$ - bimodule where $U_{q^\vee}(\mathfrak{g}^\vee)$ is dual algebra. This duality turns out to be closely related to Langlands duality.

It would be natural to expect that further generalizations of the proposed constructions to the infinite-dimensional algebras are possible. Thus in [GKLO1] the construction of some class of the representations of the Yangian $Y(\mathfrak{g})$ for an arbitrary finite-dimensional semisimple Lie algebra $\mathfrak{g}$ was given. These representations arise as a quantization of the symplectic leaves of the classical counterpart of the Yangian. Surprisingly it turns out that the symplectic leaves of the classical Yangian coincide with the moduli spaces of the monopoles supplied with the appropriate symplectic structure. The corresponding explicit expressions for the symplectic structure on the moduli spaces of G -monopoles were derived in [AH] for $G = SU(2)$, in [B1] for $G = SU(N)$ and in [FKMM] for the general case. Thus the proposed in [GKLO1] construction of the representations of Yangian provides at the same time the quantization of the moduli space of monopoles. This connection between the variables arising in the context of QISM and the variables arising in the study of the monopoles using the twistor methods [AH] is quite remarkable and obviously is a particular manifestation of the deep relationship between these two subjects.

In this note we further generalize the construction of [GKL1], [GKL3], [GKLO1] to obtain a realization of a class of the infinite-dimensional representations of the finite-dimensional quantum groups $U_q(\mathfrak{g})$ and affine quantum groups $U_q(\hat{\mathfrak{g}})_{c=0}$ with zero level $c = 0$ for an arbitrary semi-simple Lie algebra $\mathfrak{g}$. As an intermediate step, we construct the embedding of quantum groups into the algebra of rational functions on the quantum multi-dimensional torus. We postpone the technical details to another occasion [GKLO2] and concentrate on the explicit expressions for the representations of the quantum groups.

Similar to the connection of the Yangian representations with the quantization of the

monopoles on $\mathbb{R}^3$ the proposed representations of the affine algebra are connected with the quantization of the periodic monopoles on $\mathbb{R}^2 \times S^1$. In particular the classification of the trigonometric r-matrices underlying the quantum affine algebras $U_q(\hat{\mathfrak{g}})$ [BD] corresponds to the classification of the special class of asymptotic boundary conditions on a monopole solutions on $\mathbb{R}^2 \times S^1$. It would be natural to make one step further and consider the quantization of the moduli space of the double-periodic monopoles on $\mathbb{R} \times S^1 \times S^1$. Presumably this should correspond to the quantum elliptic algebras and, indeed, the choice of the asymptotic boundary conditions may be associated with elliptic r-matrix [BD]. The detailed account of the relevant description of the moduli spaces of monopoles on $\mathbb{R}^2 \times S^1$ and $\mathbb{R} \times S^1 \times S^1$ and its relation to the representations of the quantum groups will be given elsewhere [GKLO2].

Finally note that the embedding of $U_q(\mathfrak{g})$ into non-commutative multi-dimensional torus proposed in this note differs from the known similar constructions [MV], [VK], [ANO]. Also there is an obvious similarity of our constructions with the constructions in [FO1], [FO2]. However results presented in this note seem to be new.

The plan of the paper is as follows. In Section 2 we recall the construction of a certain class of representations of the Yangian introduced in [GKLO1]. In Section 3 we give the generalization of the construction discussed in Section 2 to the case of the universal enveloping of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ at $c = 0$ for an arbitrary semisimple Lie algebra $\mathfrak{g}$. In Section 4 the explicit construction of the representations of the quantum groups $U_q(\mathfrak{g})$ for an arbitrary semisimple Lie algebra $\mathfrak{g}$ is given.

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2 A representation of $Y(\mathfrak{g})$

In this section we remind the explicit construction of a class of representations of the Yangian in terms of difference operators given in [GKLO1].

We start with the definition of the Yangian for a semisimple Lie algebra $\mathfrak{g}$ due to Drinfeld [Dr1]. Let $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ be a simple finite-dimensional Lie algebra $\mathfrak{g}$ of rank ℓ over $\mathbb{C}$ with a fixed Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b}$. Let $a = ||a_{ij}||$, $i, j = 1, \ldots, \ell$ be the Cartan matrix of $\mathfrak{g}$, Γ be the set of vertices of the Dynkin diagram of $\mathfrak{g}$, $\{\alpha_i \in \mathfrak{h}^*, i \in \Gamma\}$ be the set of simple roots and $\{\alpha_i^\vee, i \in \Gamma\}$ be the set of the corresponding coroots such that $a_{ij} = \alpha_i^\vee(\alpha_j)$. There exist coprime positive integers $d_1, \ldots, d_\ell$ such that the matrix $||d_i a_{ij}||$ is symmetric. Define the invariant bilinear form on $\mathfrak{h}^*$ by $(\alpha_i, \alpha_j) = d_i a_{ij}$, then $a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$.

Introduce the formal generating series $H_i(u)\,, E_i(u),\ F_i(u)\,,\ i \in \Gamma$:

$$
\begin{aligned}
H_i(u) &= 1 + \sum_{n=0}^{\infty} H_i^{(n)} u^{-n-1}\,, \\
E_i(u) = \sum_{n=0}^{\infty} E_i^{(n)} u^{-n-1}\,, &\qquad F_i(u) = \sum_{n=0}^{\infty} F_i^{(n)} u^{-n-1}\,.
\end{aligned}
\tag{2.1}
$$

Definition 2.1 *The Yangian $Y(\mathfrak{g})$ is the associative algebra with the elements $H_i^{(n)}, E_i^{(n)}$, $F_i^{(n)}$, $i \in \Gamma$; $n = 0, 1, 2 \ldots$ and the following defining relations*

$$
\begin{aligned}
[H_i(u), H_j(v)] &= 0\,, \\
[H_i(u), E_j(v)] &= -\frac{\imath\hbar}{2}\,(\alpha_i, \alpha_j)\,\frac{[H_i(u), E_j(u) - E_j(v)]_+}{u - v}\,, \\
[H_i(u), F_j(u)] &= \frac{\imath\hbar}{2}\,(\alpha_i, \alpha_j)\,\frac{[H_i(u), F_j(u) - F_j(v)]_+}{u - v}\,, \\
[E_i(u), F_j(v)] &= -\imath\hbar\,\frac{H_i(u) - H_i(v)}{u - v}\,\delta_{i,j}\,, \\
[E_i(u), E_i(v)] &= -\frac{\imath\hbar}{2}(\alpha_i, \alpha_i)\frac{(E_i(u) - E_i(v))^2}{u - v}\,, \\
[F_i(u), F_i(v)] &= \frac{\imath\hbar}{2}(\alpha_i, \alpha_i)\frac{(F_i(u) - F_i(v))^2}{u - v}\,, \\
[E_i(u), E_j(v)] &= \\
-\frac{\imath\hbar}{2}(\alpha_i, \alpha_j)\frac{[E_i(u), E_j(u) - E_j(v)]_+}{u - v} &- \frac{[E_i^{(0)}, E_j(u) - E_j(v)]}{u - v}\,, \qquad i \neq j\,, \\
[F_i(u), F_j(v)] &= \\
\frac{\imath\hbar}{2}(\alpha_i, \alpha_j)\frac{[F_i(u), F_j(u) - F_j(v)]_+}{u - v} &- \frac{[F_i^{(0)}, F_j(u) - F_j(v)]}{u - v}\,,
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
\sum_{\sigma \in \mathfrak{S}_m} [E_i(u_{\sigma(1)}), [E_i(u_{\sigma(2)}), \ldots, [E_i(u_{\sigma(m)}), E_j(v)] \ldots]] &= 0\,, \\
\sum_{\sigma \in \mathfrak{S}_m} [F_i(u_{\sigma(1)}), [F_i(u_{\sigma(2)}), \ldots, [F_i(u_{\sigma(m)}), F_j(u)] \ldots]] &= 0\,, \\
m = 1 - a_{ij} \quad \textit{for} \ \ i \neq j\,,&
\end{aligned}
\tag{2.3}
$$

where $[a, b]_+ := ab + ba$ and summation in (2.3) is performed over permutation group $\mathfrak{S}_m$.

Let $Y(\mathfrak{b}) \subset Y(\mathfrak{g})$ be the subalgebra generated by $H_i(u), E_i(u)$, $i \in \Gamma$.

The explicit description of the representation of the Yangian in terms of difference operators is based on the choice of a large enough commutative subalgebra. We shall use the

coefficients of the series $H_i(u)$ as the generators of this subalgebra. In the constructed representation $H_i^{(n)}$ will act by multiplication on the functions of some auxiliary variables. Thus to obtain representation of the Yangian we should find the representation of the other generators in terms of some difference operators acting on the same space of functions.

Let us introduce a set of variables $\{\gamma_{i,k}\,; i \in \Gamma;\, k = 1, \dots, m_i\}$, where $m_i \in \mathbb{N}$ and let $\mathcal{M}$ be the space of meromorphic functions in these variables. Define the following difference operators acting on $\mathcal{M}$: $\beta_{i,k} = e^{\imath\hbar d_i \frac{\partial}{\partial \gamma_{i,k}}}$. Below we use the convention $\prod_{s=j}^{k} f_s := 1$, for any f_s if $k < j$.

Consider the operators

$$H_i(u) = R_i(u) \frac{\prod\limits_{j \neq i} \prod\limits_{r=1}^{-a_{ji}} \prod\limits_{p=1}^{m_j} (u - \gamma_{j,p} - \frac{\imath\hbar}{2}(\alpha_i + r\alpha_j, \alpha_j))}{\prod\limits_{p=1}^{m_i} (u - \gamma_{i,p})(u - \gamma_{i,p} - \frac{\imath\hbar}{2}(\alpha_i, \alpha_i))}\,, \tag{2.4}$$

$$E_i(u) = d_i^{-1/2} \sum_{k=1}^{m_i} \frac{\prod\limits_{j=i+1}^{\ell} \prod\limits_{r=1}^{-a_{ji}} \prod\limits_{p=1}^{m_j} (\gamma_{i,k} - \gamma_{j,p} - \frac{\imath\hbar}{2}(\alpha_i + r\alpha_j, \alpha_j))}{(u - \gamma_{i,k}) \prod\limits_{p \neq k} (\gamma_{i,k} - \gamma_{i,p})}\, \beta_{i,k}^{-1}\,, \tag{2.5}$$

$$\begin{aligned} F_i(u) = -d_i^{-1/2} \sum_{k=1}^{m_i} R_i(\gamma_{i,k} + \tfrac{\imath\hbar}{2}(\alpha_i, \alpha_i)) \times \\ \frac{\prod\limits_{j=1}^{i-1} \prod\limits_{r=1}^{-a_{ji}} \prod\limits_{p=1}^{m_j} (\gamma_{i,k} - \gamma_{j,p} - \frac{\imath\hbar}{2}(\alpha_i + r\alpha_j, \alpha_j) + \frac{\imath\hbar}{2}(\alpha_i, \alpha_i))}{(u - \gamma_{i,k} - \frac{\imath\hbar}{2}(\alpha_i, \alpha_i)) \prod\limits_{p \neq k} (\gamma_{i,k} - \gamma_{i,p})}\, \beta_{i,k}\,, \end{aligned} \tag{2.6}$$

where the rational functions $R_i(u)$ will be specified below.

Theorem 2.1 *[GKLO1] (i). For any set of positive integers $\{m_i,\ , i \in \Gamma\}$ satisfying the conditions $l_i := \sum_{j=1}^{\ell} m_j a_{ji} \in \mathbb{Z}_{\geq 0}$, introduce the polynomials $R_i(u) = \prod_{s=1}^{l_i} (u - \nu_{i,s})$, where $\{\nu_{i,s}\,, i \in \Gamma,\ s = 1, \dots, l_i\}$ is a set of arbitrary complex parameters. Then the operators (2.4)-(2.6) considered as formal power series in u^{-1}, define a representation of $Y(\mathfrak{g})$ in the space $\mathcal{M}$.*

(ii). Let $\{m_i,\, i \in \Gamma\}$ be an arbitrary set of positive integers and $R_i(u)$ be rational functions of the form $R_i(u) = \prod_{s=1}^{l_i^+} (u - \nu_{i,s}^+) / \prod_{s=1}^{l_i^-} (u - \nu_{i,s}^-)$, where $\{\nu_{i,s}^{\pm}\,, i \in \Gamma\,, s = 1, \dots, l_i^{\pm}\}$ is a set of arbitrary complex parameters and $l_i^+ - l_i^- = \sum_{j=1}^{\ell} m_j a_{ji}$. Then the operators (2.4), (2.5) considered as formal power series in u^{-1}, define a representation of $Y(\mathfrak{b})$ in the space $\mathcal{M}$.

Below we generalize this construction to the case of the quantum affine algebra at zero level.

3 A representation of $U_q(\hat{\mathfrak{g}})_{c=0}$

In this section we construct a representation of the quantum affine algebra at the zero level $U_q(\hat{\mathfrak{g}})_{c=0}$. The construction is the direct generalization of the construction for the Yangian described in the previous section. We start with the definition of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ at $c = 0$ for any semisimple Lie algebra $\mathfrak{g}$ in terms of generating series following [Dr1].

Let q be an undeterminate. Quantum affine algebra as an associative $\mathbb{C}(q)$-algebra may be present in terms of the elements $K_i^{\pm 1}$, $H_i^{(n)}$, $n \in \mathbb{Z}\backslash\{0\}$, $E_i^{(n)}, F_i^{(n)}, n \in \mathbb{Z}$, $i \in \Gamma$ and relations. Introduce the formal generating series $K_i^{\pm}(z)\,, E_i(z),$ and $F_i(z)\,,\ i \in \Gamma$:

$$
\begin{aligned}
K_i^{\pm}(z) &= K_i^{\pm 1} \exp\big(\pm (q_i - q_i^{-1}) \sum_{n\in\mathbb{N}} H_i^{(\pm n)} z^{\mp n}\big)\,, \\
E_i(z) &= \sum_{n\in\mathbb{Z}} E_i^{(n)} z^{-n}\,, \qquad F_i(z) = \sum_{n\in\mathbb{Z}} F_i^{(n)} z^{-n}\,,
\end{aligned}
\tag{3.1}
$$

where $q_i := q^{d_i}$.

Definition 3.1 *Quantum affine algebra $U_q(\hat{\mathfrak{g}})$ at $c = 0$ is the associative algebra with elements $K_i^{\pm 1}, H_i^{(n)}$, $n \in \mathbb{Z}\backslash\{0\}$; $E_i^{(n)}, F_i^{(n)}$, $n \in \mathbb{Z}$; $i \in \Gamma$ and the following defining relations:*

$$
\begin{aligned}
K_i^{\pm}(z) K_j^{\pm}(w) &= K_j^{\pm}(w) K_i^{\pm}(z), \\
K_i^{+}(z) K_j^{-}(w) &= K_j^{-}(w) K_i^{+}(z)\,, \\
(z - q_i^{a_{ij}} w) K_i^{\pm}(z) E_j(w) &= (q_i^{a_{ij}} z - w) E_j(w) K_i^{\pm}(z)\,, \\
(z - q_i^{-a_{ij}} w) K_i^{\pm}(z) F_j(w) &= (q_i^{-a_{ij}} z - w) F_j(w) K_i^{\pm}(z)\,, \\
[E_i(z), F_j(w)] &= \frac{\delta_{i,j}}{q_i - q_i^{-1}}\, \delta(z/w)\big(K_i^{+}(w) - K_i^{-}(z)\big), \\
(z - q_i^{a_{ij}} w) E_i(z) E_j(w) &= (q_i^{a_{ij}} z - w) E_j(w) E_i(z), \\
(z - q_i^{-a_{ij}} w) F_i(z) F_j(w) &= (q_i^{-a_{ij}} z - w) F_j(w) F_i(z)\,,
\end{aligned}
\tag{3.2}
$$

$$
\sum_{\sigma\in\mathfrak{S}_m} \sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_{q_i} E_i(z_{\sigma(1)}) \dots E_i(z_{\sigma(k)}) E_j(w) E_i(z_{\sigma(k+1)}) \dots E_i(z_{\sigma(m)}) = 0\,,
$$

$$
\sum_{\sigma\in\mathfrak{S}m} \sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_{q_i} F_i(z_{\sigma(1)}) \dots F_i(z_{\sigma(k)}) E_j(w) F_i(z_{\sigma(k+1)}) \dots F_i(z_{\sigma(m)}) = 0\,,
$$

$$
m = 1 - a_{ij} \quad \textit{for} \ \ i \neq j\,.
$$

Here we use the standard notations $\begin{bmatrix} m \\ k \end{bmatrix}_q = \dfrac{[m]_q!}{[k]_q![m-k]_q!}$, $[k]_q! = \prod\limits_{1\le j\le k} \dfrac{q^j - q^{-j}}{q - q^{-1}}$. The formal delta-function is defined as $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$. For more details on the operator-valued formal series see, for example, [K].

To define a representation of $U_q(\hat{\mathfrak{g}})_{c=0}$ in terms of the difference operators, we start with construction of the embedding of the corresponding universal enveloping algebra into the algebra $\mathbb{T}_q$ of the rational functions of the non-commutative multi-dimensional torus.

Let $\mathbb{T}_q$ be the associative $\mathbb{C}(q)$-algebra of the rational functions of invertible elements $\mathbf{v}_{i,k}$, $\mathbf{u}_{i,k}$, $\mathbf{w}_{i,s}$, $i=1,\dots,\ell$; $k=1,\dots,m_i\in\mathbb{N}$; $s=1,\dots,l_i\in\mathbb{Z}_{\geq 0}$, subject to relations

$$\begin{gathered}\mathbf{v}_{i,k}\mathbf{v}_{j,l}=\mathbf{v}_{j,l}\mathbf{v}_{i,k}\,,\qquad \mathbf{u}_{i,k}\mathbf{u}_{j,l}=\mathbf{u}_{j,l}\mathbf{u}_{i,k}\,,\\ \mathbf{u}_{i,k}\mathbf{v}_{j,l}=q_i^{\delta_{i,j}\delta_{k,l}}\mathbf{v}_{j,l}\mathbf{u}_{i,k}\,,\end{gathered}\tag{3.3}$$

and $\mathbf{w}_{i,s}$ are central elements in $\mathbb{T}_q$. Chose the set of natural numbers $\{m_i\}$ satisfying the conditions

$$\sum_{j=1}^{\ell} m_j a_{ji}=l_i,\tag{3.4}$$

and consider the following formal generating functions $K_i^{\pm}$, $E_i(z)$ and $F_i(z)$ in variable z.

The functions $K_i^+(z)$ and $K_i^-(z)$ are defined as infinite series expansion in z^{-1} and z of the same rational function

$$K_i(z)=c_i\prod_{j=1}^{\ell}\prod_{p=1}^{m_j}\mathbf{v}_{j,p}^{a_{ji}}\cdot\prod_{s=1}^{l_i}(z\mathbf{w}_{i,s}^{-1}-\mathbf{w}_{i,s})\,\frac{\prod\limits_{j\neq i}\prod\limits_{r=1}^{-a_{ji}}\prod\limits_{p=1}^{m_j}(z-q_j^{a_{ji}+2r}\mathbf{v}_{j,p}^2)}{\prod\limits_{p=1}^{m_i}(z-\mathbf{v}_{i,p}^2)(z-q_i^2\mathbf{v}_{i,p}^2)}\,,\tag{3.5}$$

where $c_i=\prod_{j=1}^{\ell}q_j^{m_ja_{ji}/2}$. Due to conditions (3.4) the degrees of the polynomials in the numerator and in the denominator in (3.5) coincide and thus $K_i^{\pm}(z)$ as the power series in $z^{\mp 1}$ have the form (3.1) where $K_i=c_i\prod_{s=1}^{l_i}\mathbf{w}_{i,s}^{-1}\cdot\prod_{j=1}^{\ell}\prod_{p=1}^{m_j}\mathbf{v}_{j,p}^{a_{ji}}$.

Let $R^{(\pm)}(z)$ be the polynomials such that $R_i^{(+)}(z)R_i^{(-)}(z)=\prod_{s=1}^{l_i}(z\mathbf{w}_{i,s}^{-1}-\mathbf{w}_{i,s})$. The other generating series are defined as follows

$$\begin{aligned}E_i(z)&=\frac{c_i}{q_i-q_i^{-1}}\prod_{p=1}^{m_i}\mathbf{v}_{i,p}\cdot\prod_{j=i+1}^{\ell}\prod_{p=1}^{m_j}\mathbf{v}_{j,p}^{a_{ji}}\cdot\\ &\sum_{k=1}^{m_i}\delta(z/\mathbf{v}_{i,k}^2)\,\mathbf{v}_{i,k}^{-2}R_i^{(+)}(\mathbf{v}_{ik}^2)\frac{\prod\limits_{j=i+1}^{\ell}\prod\limits_{r=1}^{-a_{ji}}\prod\limits_{p=1}^{m_j}(\mathbf{v}_{i,k}^2-q_j^{a_{ji}+2r}\mathbf{v}_{j,p}^2)}{\prod\limits_{p\neq k}(\mathbf{v}_{i,k}^2-\mathbf{v}_{i,p}^2)}\,\mathbf{u}_{i,k}^{-1}\,,\\ F_i(z)&=-\frac{q_i^{-2m_i}}{q_i-q_i^{-1}}\prod_{p=1}^{m_i}\mathbf{v}_{i,p}\cdot\prod_{j=1}^{i-1}\prod_{p=1}^{m_j}\mathbf{v}_{j,p}^{a_{ji}}\cdot\\ &\sum_{k=1}^{m_i}\delta(z/q_i^2\mathbf{v}_{i,k}^2)\,\mathbf{v}_{i,k}^{-2}R_i^{(-)}(q_i^2\mathbf{v}_{i,k}^2)\frac{\prod\limits_{j=1}^{i-1}\prod\limits_{r=1}^{-a_{ji}}\prod\limits_{p=1}^{m_j}(q_i^2\mathbf{v}_{i,k}^2-q_j^{a_{ji}+2r}\mathbf{v}_{j,p}^2)}{\prod\limits_{p\neq k}(\mathbf{v}_{i,k}^2-\mathbf{v}_{i,p}^2)}\,\mathbf{u}_{i,k}\,.\end{aligned}\tag{3.6}$$

Theorem 3.1 *For any set of positive integers* $\{m_i, i \in \Gamma\}$ *obeying the conditions (3.4), the generating series* $K_i^{\pm}(z)$, $E_i(z)$, *and* $F_i(z)$ *defined by (3.5), (3.6), satisfy the relations (3.2) and therefore define an embedding* $\pi : U_q(\hat{\mathfrak{g}})_{c=0} \hookrightarrow \mathbb{T}_q$.

The proof will be given in [GKLO2].

Let us introduce a set of variables $\{\gamma_{i,k} \in \mathbb{C}\,; i \in \Gamma;\, k = 1, \dots, m_i\}$, and let $\mathcal{M}$ be the space of meromorphic functions in these variables. We also fix a set of the complex numbers $\nu_{i,s}$, $i = 1, \dots, \ell$, $s = 1, \dots, l_i$. Define the following representation of $\mathbb{T}_q$ in terms of difference operators acting on $\mathcal{M}$:

$$\mathbf{u}_{i,k} = e^{\imath\omega_1 d_i \frac{\partial}{\partial\gamma_{i,k}}}, \quad \mathbf{v}_{i,k} = e^{\frac{2\pi\gamma_{ik}}{\omega_2}}, \quad \mathbf{w}_{i,s} = e^{\frac{2\pi\nu_{i,s}}{\omega_2}}, \quad q = e^{\frac{2\pi\imath\omega_1}{\omega_2}}, \tag{3.7}$$

where ω_1 and ω_2 are arbitrary complex parameters. The following proposition is the simple consequence of Theorem 3.1.

Proposition 3.1 *Let us given the representation of* $\mathbb{T}_q$ *defined by (3.7). The coefficients* $K_i^{\pm 1}$, $H_i^{(n)}$, $n \in \mathbb{Z}\backslash\{0\}$, $E_i^{(n)}, F_i^{(n)}, n \in \mathbb{Z}$, $i \in \Gamma$ *of the formal power series (3.5), (3.6) define a representation of* $U_q(\hat{\mathfrak{g}})_{c=0}$ *in* $\mathcal{M}$.

4 A representation of $U_q(\mathfrak{g})$

The finite-dimensional quantum groups $U_q(\mathfrak{g})$ may be naturally considered as the subalgebras of the affine quantum groups $U_q(\hat{\mathfrak{g}})_{c=0}$. Indeed, the generators $K_i^{\pm 1}$, $E_i^{(0)}$, and $F_i^{(0)}$ introduced in the previous section obey the commutation relations of the finite-dimensional quantum group $U_q(\mathfrak{g})$. We shall omit the superscript (0) and write down simply E_i and F_i. Thus the representations of the affine quantum groups introduced above automatically provide the representations of the finite-dimensional quantum groups. Below we give the explicit expressions for the generators in this representations. However we would prefer to work in this section in a slightly more general case and consider the various rational forms of the quantum groups. Let Q and P be the root and weight lattices of the Lie algebra $\mathfrak{g}$. There are different rational forms of the quantum group not isomorphic as $\mathbb{C}(q)$ algebras which may be enumerated by the choice of a sublattice M such that $Q \subseteq M \subseteq P$ (see [CP], [L] and references therein). Thus for example the adjoint (the smallest) rational form $U_q^Q(\mathfrak{g})$ is the associative $\mathbb{C}(q)$-algebra generated by E_i, F_i and $K_i^{\pm}$, $i \in \Gamma$ with the defining relations:

$$\begin{gathered}
K_i K_i^{-1} = K_i^{-1} K_i = 1\,, \qquad K_i K_j = K_j K_i\,, \\
K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j\,, \qquad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j\,, \\
E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}\,, \\
\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} E_i^{1-a_{ij}-r} E_j E_i^r = 0\,, \qquad i \neq j\,, \\
\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} F_i^{1-a_{ij}-r} F_j F_i^r = 0\,, \qquad i \neq j\,.
\end{gathered} \tag{4.1}$$

On the other hand, the largest, simply-connected, rational from $U_q^P(\mathfrak{g})$ is obtained by adjoining to $U_q^Q(\mathfrak{g})$ the invertible elements $L_i\,, i \in \Gamma$, such that $K_i = \prod_j L_j^{a_{ji}}$ and obeying the relations $L_i E_j L_i^{-1} = q_i^{\delta_{i,j}} E_j$, $L_i F_j L_i^{-1} = q_i^{-\delta_{i,j}} F_j$. Given any intermediate lattice $Q \subseteq M \subseteq P$, the corresponding rational form $U_q^M(\mathfrak{g})$ is obtained by adjoining to $U_q^Q(\mathfrak{g})$ the elements $K_{\beta_i} = \prod_j L_j^{n_{ji}}$ for any basis $\beta_i = \sum_j n_{ji}\lambda_j \in M$, $i \in \Gamma$, where λ_j are the fundamental weights. For any simple root α_i let $\alpha_i = \sum_j m_{ji}\beta_j$. Then the generators K_i are expressed by K_{β_j} as $K_i = \prod_j K_{\beta_j}^{m_{ji}}$. Let $||M_{ij}||$ be the matrix inverse to $||m_{ij}||$ and $d = \det||a_{ij}||$. Let $\mathbb{T}_q$ be the quantum torus defined in the previous section. The following Theorem is a generalization of the embedding of the finite-dimensional quantum group into quantum torus obtained by the restriction of embedding of the affine quantum group described in Section 3.

Theorem 4.1 *(i). Let $\mathfrak{g}$ be an arbitrary semisimple Lie algebra. Let $K_{\beta_i}^{\pm 1}, E_i, F_i$, $i \in \Gamma$ be the generators of the rational form of quantum group $U_q^M(\mathfrak{g})$, associated with the lattice M such that $Q \subseteq M \subseteq P$. The following expressions define the embedding $\pi : U_q^M(\mathfrak{g}) \hookrightarrow \mathbb{T}_q$:*

$$\pi(K_{\beta_i}) = c_{\beta_i} \prod_{j=1}^{\ell} \Big(\prod_{s=1}^{l_j} \mathbf{w}_{j,s}^{-dM_{ji}} \cdot \prod_{p=1}^{m_j} \mathbf{v}_{j,p}^{n_{ji}} \Big)\,,$$

$$\pi(E_i) = \frac{c_{\alpha_i}}{q_i - q_i^{-1}} \prod_{p=1}^{m_i} \mathbf{v}_{i,p} \cdot \prod_{j=i+1}^{\ell} \prod_{p=1}^{m_j} \mathbf{v}_{j,p}^{a_{ji}} \cdot$$

$$\sum_{k=1}^{m_i} \mathbf{v}_{i,k}^{-2} R_i^{(+)}(\mathbf{v}_{ik}^2) \frac{\prod_{j=i+1}^{\ell} \prod_{r=1}^{-a_{ji}} \prod_{p=1}^{m_j} (\mathbf{v}_{i,k}^2 - q_j^{a_{ji}+2r} \mathbf{v}_{j,p}^2)}{\prod_{p \neq k} (\mathbf{v}_{i,k}^2 - \mathbf{v}_{i,p}^2)}\, \mathbf{u}_{i,k}^{-1}\,, \tag{4.2}$$

$$\pi(F_i) = -\frac{q_i^{-2m_i}}{q_i - q_i^{-1}} \prod_{p=1}^{m_i} \mathbf{v}_{i,p} \cdot \prod_{j=1}^{i-1} \prod_{p=1}^{m_j} \mathbf{v}_{j,p}^{a_{ji}} \cdot$$

$$\sum_{k=1}^{m_i} \mathbf{v}_{i,k}^{-2} R_i^{(-)}(q_i^2 \mathbf{v}_{i,k}^2) \frac{\prod_{j=1}^{i-1} \prod_{r=1}^{-a_{ji}} \prod_{p=1}^{m_j} (q_i^2 \mathbf{v}_{i,k}^2 - q_j^{a_{ji}+2r} \mathbf{v}_{j,p}^2)}{\prod_{p \neq k} (\mathbf{v}_{i,k}^2 - \mathbf{v}_{i,p}^2)}\, \mathbf{u}_{i,k}\,,$$

where $R_i^{(\pm)}(z)$ be the polynomials such that $R_i^{(+)}(z) R_i^{(-)}(z) = \prod_{s=1}^{l_i} (z\mathbf{w}_{i,s}^{-d} - \mathbf{w}_{i,s}^d)$ and $c_{\beta_i} = \prod_{j=1}^{\ell} q_j^{m_j n_{ji}/2}$.
(ii). Let the elements of $\mathbb{T}_q$ are represented by (3.7). Then the operators (4.2) define a representation of $U_q^M(\mathfrak{g})$ in $\mathcal{M}$.

Remark 4.1 *In the case $\mathfrak{g} = \mathfrak{sl}(\ell+1)$ the Theorem 4.1 is a natural generalization of the Theorem 3.1 in [GKL3].*

Note that there are known several embeddings of $U_q(\mathfrak{g})$ into the algebra of quantum torus (e.g. [ANO], [MV], [VK]). Our construction is the new one and has deep relations with the

Quantum Inverse Scattering Method [GKL1] as well as with the natural parameterization of the moduli spaces of monopoles [GKLO1]. The detailed discussion of the connection with the monopoles on $\mathbb{R}^2 \times S^1$ will be described in [GKLO2].

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