ON GELFAND-KIRILLOV DIMENSION AND RELATED TOPICS

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INTRODUCTION

The purpose of this article is to contribute to the following two basic problems on Gelfand-Kirillov (GK-) dimension:

(I) Under what conditions is the GK-dimension of an algebra, or of a module, an integer (or ∞)?

(II) If $0 \to N \to M \to W \to 0$ is an exact sequence of modules over an algebra S, when does the equality $GK(M) = \max\{GK(N), GK(W)\}$ hold?

Recall that, in (II), one always has $GK(M) \geq \max\{GK(N), GK(W)\}$. If equality holds for <u>all</u> short exact sequences of S-modules, then GK-dimension is said to be exact for S-modules. In general, exactness fails quite drastically, even in situations which are otherwise considered to be well-behaved. For example, G. Bergman [3] has constructed an affine PI-algebra S having an ideal I of square 0 such that I is cyclic as right ideal of S, yet $GK(S) = 3 > GK(S/I) = 2$. Thus quite stringent conditions have to be imposed on the algebra or the modules in question for equality to hold in (II). On the other hand, the integrality question (I) presumably has a positive answer for many classes of algebras and modules that are of interest. Although the GK-dimension of an algebra (module) can be $0, 1, \infty$, or any real number ≥ 2 ($0, \infty$, or any real number ≥ 1 ; Warfield [19]), it tends to be an integer or ∞ in most cases which arise naturally. Thus no example of an affine algebra S which is either Noetherian or finitely presented but has finite non-integral GK-dimension seems to be known. However, positive results are rare. Some exceptions, where GK-dimension is known to be an integer or ∞ , are: finitely presented monomial algebras (Govorov [10]), almost commutative algebras (Tauvel [18]), and Noetherian PI-algebras (Lorenz-Small [14]).

GK-dimension, by its definition, measures the rate of growth of the steps in certain canonically defined filtrations on algebras and modules. Often, however, finer aspects of these filtrations are of interest in their own right, and some of these aspects are considered in detail in the present article.

Sections 1 and 2 form a unit and are devoted to studying the behaviour of the above filtrations under intersection with submodules (Section 1) or ideals (Section 2). The motivating problem here is the exactness problem (II) and, in Section 1, we prove a number of exactness results which use certain finiteness assumptions on the graded modules that are associated with the filtrations in question. Our methods in this section very nearly border on abstract nonsense. Yet, once the foundations have been laid in sufficient generality in Section 1, they then very easily yield a result (Proposition 2.8) proving the equality $GK(S) = GK(S/I)$ for nilpotent ideals $I \subset S$ which are finitely generated as right ideals under relatively mild assumptions on the algebra S/I. They are trivially satisfied, for example, if S/I is a finitely generated right module over a commutative subalgebra C/I, a situation which is crucial in the proof of integrality for the GK-dimension of Noetherian PI-algebras ([14], see also [12, Lemma 10.13]).

In Section 3, which is independent of the previous sections, we study Poincaré series of graded modules. By essentially paraphrasing the usual proof of the classical Hilbert-Serre Theorem in a suitable non-commutative setting, we obtain a rationality result for Poincaré series of Noetherian graded modules over an interesting class of graded algebras which includes, for example, positively graded affine PI-algebras. This result has applications to both the integrality problem (I) and the exactness problem (II).

The final Section 4 is again formally independent of the rest of the article. GK-dimension is not even explicitly mentioned in this section, but the motivation for the material presented here comes from the earlier results on GK-dimension and Poincaré series which all use certain finiteness assumptions on (associated) graded modules. We study a construction of associated graded rings and modules which is closely related to but different from the usual construction and seems to be more perspective in some respects. This construction is not new: When applied to the I-adic filtration of a ring R, it yields the so-called Rees ring of the ideal I, and it has also been used by Quillen in [15, proof of Theorem 7]. Thus some of the results discussed in Section 4 may very well be known, in some form, but the construction probably deserves to be popularized among non-commutative ring theorists.

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NOTATIONS AND CONVENTIONS

All rings considered in this article are associative and have a 1 which is interited by subrings. When not explicitely specified otherwise, modules will be understood to be right modules. Throughout, k denotes a commutative field which will be the base field for all algebras and vector spaces under consideration. Vector space dimensions, dim . , and GK-dimensions, GK(.), refer to k, and "finite-dimensional" will be abbreviated "f.d.". The subspace generated by a collection of elements $\alpha, \beta, \ldots$ in a given vector space will be denoted by $\langle\alpha, \beta, \ldots\rangle_k$. If V is a subspace of a k-algebra S, then we put

$$V^{(n)} = \langle v_1 \cdot v_2 \cdot \ldots \cdot v_\ell \mid \ell \leq n, v_i \in V\rangle_k \subset S ,$$

the subspace generated by all products of length at most n in S with factors taken from V. Here, a product of negative length is 0, and a product of length 0 is $1 \in S$. The subalgebra of S generated by V will be denoted by $k[V]$, so $k[V] = \bigcup_n V^{(n)} \subset S$. Finally, all filtrations considered in this article are understood to be exhaustive and increasing.

1. GENERALITIES ON EXACTNESS

Throughout this section, S will denote a k-algebra.

Definition. Let $N \subset M$ be S-modules. We will say that N is finitely controlled in M, and write $N \circledast M$, if the following condition is satisfied:

(1) For all finite-dimensional (f.d.) subspaces $E \subset M$ and $V \subset S$ there exist f.d. subspaces $E_1 \subset N$ and $V_1 \subset S$ such that, for all n, $N \cap E \cdot V^{(n)} \subset E_1 \cdot V_1{}^{(n)}$.

Remarks 1.1. If, in the above definition, M/N is finitely generated over S, say $M = E \cdot S + N$ with E a f.d. subspace of M, then it suffices to check (1) for this particular E alone in order to ensure that $N \circledast M$. To see this, note that if $F \subset M$ is any f.d. subspace then $F \subset E \cdot X + F_1$ for suitable f.d. subspaces $X \subset S$ and $F_1 \subset N$. Hence, for any subspace $V \subset S$,

$$N \cap F \cdot V^{(n)} \subset N \cap (E \cdot XV^{(n)} + F_1 \cdot V^{(n)}) = (N \cap E \cdot XV^{(n)}) + F_1 \cdot V^{(n)}$$
$$\subset (N \cap E \cdot V_1{}^{(n)}) + F_1 \cdot V^{(n)} \qquad (n \geq 1),$$

where $V_1 = V + XV \subset S$.

Similarly, if S is affine over k, then it suffices to verify (1) for any f.d. generating subspace V of S.

Part (i) of the following lemma explains the interest of finitely controlled submodules for our purposes. In part (ii), we list some formal properties of the relation $\circledast$ which are analogous to corresponding properties of the relation of being a direct summand. Indeed, if N is a direct summand of M, then obviously $N \circledast M$.

Lemma 1.2.

i. Suppose that $N \circledast M$. Then $GK(M) = \max\{GK(N), GK(M/N)\}$.

ii. Let $N \subset W \subset M$ be a chain of S-modules. Then

$$N \circledast M \Rightarrow N \circledast W ,$$
$$W \circledast M \Rightarrow W/N \circledast M/N ,$$

and

$$N \circledast W \text{ and } W \circledast M \Leftrightarrow N \circledast M \text{ and } W/N \circledast M/N .$$

Proof. In (i) and (ii), let $\bar{\ }: M \to M/N$ be the canonical map.

(i). If $E \subset M$ and $V \subset S$ are f.d. subspaces, then $\circledast$ yields

$$\begin{aligned}\dim E\cdot V^{(n)} &= \dim (N \cap E\cdot V^{(n)}) + \dim \bar{E}\cdot V^{(n)} \\ &\leq \dim E_1\cdot V_1{}^{(n)} + \dim \bar{E}\cdot V^{(n)} ,\end{aligned}$$

where $E_1 \subset N$ and $V_1 \subset S$ are as in (1). In view of [12, Lemma 2.1(a)], this shows that $GK(M) \leq \max \{GK(N), GK(M/N)\}$. Part (i) now follows from [12, Proposition 5.1(b)].

(ii). For the most part, this is routine. We only show that $W \circledast M$ follows from $N \circledast M$ and $\bar{W} \circledast \bar{M}$. Fix f.d. subspaces $E \subset M$ and $V \subset S$. Then $\bar{W} \circledast \bar{M}$ implies that $W \cap E\cdot V^{(n)} \subset F\cdot X^{(n)} + N$ for suitable f.d. subspaces $F \subset W$ and $X \subset S$. Put $G := E + F \subset M$ and $Y := V + X \subset S$. Then $N \cap G\cdot Y^{(n)} \subset G_1\cdot Y_1{}^{(n)}$ for suitable $G_1 \subset N$ and $Y_1 \subset S$, and so

$$\begin{aligned}W \cap E\cdot V^{(n)} &\subset (F\cdot X^{(n)} + N) \cap G\cdot Y^{(n)} = F\cdot X^{(n)} + (N \cap G\cdot Y^{(n)}) \\ &\subset F\cdot X^{(n)} + G_1\cdot Y_1{}^{(n)} \subset E_1\cdot V_1{}^{(n)}\end{aligned}$$

with $E_1 = F + G_1 \subset W$ and $V_1 = X + Y_1 \subset S$. ∎

For simplicity, the following results will be stated under the assumption that certain modules are finitely generated. This is justified by the fact that GK-dimension is defined locally.

We will use the following terminology. A filtration $\mathcal{F} = \{R^{(n)} \mid n \in \mathbb{Z}\}$ of a k-algebra R will be called standard if

$R^{(n)} = V^{(n)}$ for some f.d. subspace $V \subset R$. If $R^{(n)} \subset V^{(n)}$ holds for all n, then F will be called <u>substandard</u>. In either case, the corresponding subspace $V \subset R$ generates R as k-algebra so that R must be affine. On the other hand, if $F = \{R^{(n)} \mid n \in \mathbb{Z}\}$ is any filtration of R with $R^{(-1)} = 0$, $R^{(0)} = k$ and such that $gr_F(R) := \bigoplus_n R^{(n)}/R^{(n-1)}$ is affine over k, then F is easily seen to be substandard.

<u>Lemma 1.3.</u> Let $N \subset M$ be S-modules and suppose that $M = E\cdot S + N$ for some f.d. subspace $E \subset M$ which satisfies the following condition

(2) Every f.d. subspace $V \subset S$ is contained in a subalgebra $R = R(V) \subset S$ having a substandard filtration $F = \{R^{(n)}\}$ such that $\bigoplus_n N\cap E\cdot R^{(n)}/N\cap E\cdot R^{(n-1)}$ is finitely generated as $gr_F(R)$-module.

Then $N \circledast M$.

<u>Proof.</u> By Remark 1.1, it suffices to check (1) for the given subspace E. Fix $V \subset S$ and let R and F be as in (2) above. Then our assumption on $\bigoplus_n N\cap E\cdot R^{(n)}/N\cap E\cdot R^{(n-1)}$ implies that, for some s,

$$N \cap E\cdot R^{(n)} \subset (N \cap E\cdot R^{(s)})\cdot R^{(n-s)} \subset E_1\cdot R^{(n)} \qquad (n \geq 0),$$

where $E_1 = N \cap E\cdot R^{(s)} \subset N$. Moreover, $V \subset R^{(t)}$ for some t and $R^{(n)} \subset X^{(n)}$ for some f.d. subspace $X \subset R$. Hence

$$N \cap E\cdot V^{(n)} \subset N \cap E\cdot R^{(nt)} \subset E_1\cdot R^{(nt)} \subset E_1\cdot V_1^{(n)},$$

where $V_1 = X^{(t)} \subset R$. This shows that $N \circledast M$. ∎

<u>Proposition 1.4.</u> Let W be an S-module with a f.d. generating subspace $G \subset W$ satisfying the following condition

(3) Every f.d. subspace $V \subset S$ is contained in a subalgebra $R = R(V) \subset S$ having a substandard filtration $F = \{R^{(n)}\}$ such that $gr_{G,F}(W) := \bigoplus_n G\cdot R^{(n)}/G\cdot R^{(n-1)}$ is finitely presented over $gr_F(R)$.

Then, for all exact sequences $0 \to N \to M \to W \to 0$ of S-modules terminating in W, one has $N \circledcirc M$ and so $GK(M) = \max\{GK(N), GK(W)\}$.

Proof. By Lemma 1.2(i), it suffices to prove $N \circledcirc M$. We check (2) in Lemma 1.3 for $E \subset M$ any f.d. subspace mapping onto G. So let $V \subset S$ be given and let $R = R(V)$ and $\mathcal{F}$ be as in (3). Then, setting $gr(N) = \bigoplus_n N \cap E \cdot R^{(n)} / N \cap E \cdot R^{(n-1)}$, we have an exact sequence of (graded) $gr_{\mathcal{F}}(R)$-modules

$$0 \to gr(N) \to gr_{E,\mathcal{F}}(M) \to gr_{G,\mathcal{F}}(W) \to 0 .$$

Here, the middle term is finitely generated and the end term is finitely presented over $gr_{\mathcal{F}}(R)$. Hence the initial term is finitely generated over $gr_{\mathcal{F}}(R)$, as required in (2). ∎

Note that, in (3) above, the existence of a finite presentation is automatic if $gr_{\mathcal{F}}(R)$ is right Noetherian. Thus we obtain the following result, due to Tauvel [18].

Corollary 1.5 (Tauvel). If S has a filtration $\mathcal{F} = \{S^{(n)}\}$, $S^{(-1)} = 0$, $S^{(0)} = k$, such that $gr_{\mathcal{F}}(S)$ is affine over k and right Noetherian, then GK is exact for S-modules.

The above results can sometimes be applied by first dropping to suitable subalgebras as described in the following lemma. Part (i) is probably well-known, but unrecorded as far as I know.

Lemma 1.6. Let $R \subset S$ be algebras such that S is finitely generated as right R-module.

i. If M is an S-module, then $GK(M_R) = GK(M_S)$.

ii. If $N \subset M$ are S-modules, then $N_R \circledcirc M_R \Leftrightarrow N_S \circledcirc M_S$.

Proof. Fix a f.d. subspace $G \subset S$ such that $1 \in G$ and $S = G \cdot R$. Then, for any f.d. subspace $V \subset S$, there exists a f.d. $V_1 \subset R$ with $V \cdot G \subset G \cdot V_1$. Hence $V^{(n)} \subset G \cdot V_1{}^{(n)}$ holds for all n and so, if $E \subset M$ is a f.d. subspace, then

$$E\cdot V^{(n)} \subset F\cdot V_1{}^{(n)} \quad \text{and} \quad N \cap E\cdot V^{(n)} \subset N \cap F\cdot V_1{}^{(n)} ,$$

where $F = E\cdot G \subset M$. As $E \subset M$ and $V \subset S$ are arbitrary, the first inclusion shows that $\mathrm{GK}(M_S) \le \mathrm{GK}(M_R)$, whereas the second inclusion proves that $N_R \circledast M_R$ implies $N_S \circledast M_S$. The rest is now clear. ∎

We end this section with a few comments on bimodules. Let R and T be k-algebras. Then (R,T)-bimodules ${}_RM_T$ (with identical k-operations on both sides, as usual) are right modules over $S = R^{op} \otimes_k T$ and so the foregoing applies. Part (i) of the following lemma is again a relatively straightforward but useful extension of well-known facts (e.g., [4,Lemma 2.3]) which is extracted from the proof of (ii).

<u>Lemma 1.7.</u> Let ${}_RN_T \subset {}_RM_T$ be (R,T)-bimodules and assume that M_T is finitely generated. Then

i. $\mathrm{GK}({}_RM_T) = \mathrm{GK}(M_T) \ge \mathrm{GK}({}_RM)$.

ii. $N_T \circledast M_T$ implies ${}_RN_T \circledast {}_RM_T$. If N_T is finitely generated, then the converse holds.

<u>Proof.</u> Let $E \subset M$ and $V \subset S = R^{op} \otimes_k T$ be f.d. subspaces. Choose a f.d. subspace $G \subset M$ with $E \subset G$ and $M = G\cdot T$, and choose $V_1 \subset R^{op}$, $V_2 \subset T$ f.d. with $V \subset V_1 \otimes V_2$ and $V_1\cdot G \subset G\cdot V_2$. Then

$$E\cdot V^{(n)} \subset V_1{}^{(n)}\cdot G\cdot V_2{}^{(n)} \subset G\cdot V_2{}^{(2n)} = G\cdot X^{(n)} ,$$

where $X = V_2{}^{(2)} \subset T$ is finite-dimensional. As E and V are arbitrary, this proves that $\mathrm{GK}(M_S) \le \mathrm{GK}(M_T)$, the non-trivial part of (i). Also, by considering intersections with N , we conclude that $N_T \circledast M_T$ implies $N_S \circledast M_S$. The proof of the converse, under the assumption that N_T is finitely generated, proceeds in a similar fashion. ∎

2. FACTORING OUT IDEALS

Throughout this section, S denotes a k-algebra.

We will study the relationship between $GK(S)$ and $GK(S/I)$, where I is an ideal of S, with special emphasis on the case where I is nilpotent and finitely generated as right ideal. In principle, this is of course a very special case of the exactness problem studied in Section 1. However, the following lemma which is implicit in work of Lenagan [13] shows that, in certain cases of interest, the general exactness problem can be reduced to the situation considered in this section.

Lemma 2.1. Consider the following conditions:

(a) For all ideals I of S and all $t \geq 1$, $GK(S/I) = GK(S/I^t)$.
(b) For all ideals I and J of S, $GK(S/IJ) = \max\{GK(S/I), GK(S/J)\}$.
(c) GK is exact for S-modules.

Then

i. (c) $\Rightarrow$ (b) $\Rightarrow$ (a).

ii. If S is right Noetherian and satisfies Gabriel's H-condition (see below), then also (b) $\Rightarrow$ (c).

Proof. (i) is straightforward.

(ii). Recall that S satisfies the H-condition, by definition, if for every finitely generated right S-module M there exists a positive integer n such that $S/\mathrm{ann}_S(M)$ embeds into M^n. Consequently, $GK(M) = GK(S/\mathrm{ann}_S(M))$ holds for all finitely generated S-modules M. Also, since S is right Noetherian, exactness for finitely generated S-modules implies exactness in general. It is now clear how to derive (c) from (b). ∎

Right Noetherian rings satisfying the H-condition are identical with the so-called right FBN-rings [8, Section 7]. Prominent examples are right Noetherian PI-algebras, for which the exactness question is open. (GK is however known to be exact for two-sided Noetherian PI-algebras [12,Section 10].)

Lemma 2.2. Let I be an ideal of S . Then

$$GK(S) \geq \max\{GK(S/I), GK({}_SI_S)\} .$$

If ${}_SI_S \circledast {}_SS_S$, then equality holds.

Proof. Consider the exact sequence $0 \to I \to S \to S/I \to 0$ of (S,S)-bimodules and use the fact that $GK(S) = GK({}_SS_S)$ and $GK(S/I) = GK({}_S(S/I)_S)$, by Lemma 1.7(i). The last assertion follows from Lemma 1.2(i). ∎

Remark 2.3. Using Remark 1.1, one easily verifies that ${}_SI_S \circledast {}_SS_S$ is equivalent to

(4) For all f.d. subspaces $V \subset S$ there exist f.d.subspaces $I_1 \subset I$ and $W \subset S$ such that, for all $n \geq 0$,

$$I \cap V^{(n)} \subset \sum_{i+j\leq n} W^{(i)}.I_1.W^{(j)} .$$

Of course, equality also holds in Lemma 2.2 whenever I contains a right (or left) regular element of S . For then S_S embeds into I_S and so $GK(S) \leq GK(I_S) \leq GK({}_SI_S)$. But, in view of Lemma 2.1(a), our main interest lies in the case where I is nilpotent.

Examples 2.4. (a) Assume that $S = I \oplus T$, where T is a subalgebra of S and I is an ideal of square 0 . Then ${}_SI_S \circledast {}_SS_S$. For, if $V \subset S$ is a f.d. subspace, then $V \subset U \oplus W$ for suitable f.d. subspaces $U \subset I$ and $W \subset T$, and $I^2 = 0$ implies

$$I \cap V^{(n)} \subset \sum_{i+j\leq n-1} W^{(i)}.U.W^{(j)} .$$

This applies in particular to algebras of the form $S = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} =$

$M \oplus (A \times B)$, where A and B are k-algebras and ${}_A M_B$ is a bimodule. We conclude that

$$GK(\begin{bmatrix} A & M \\ 0 & B \end{bmatrix}) = \max \{GK(A), GK(B), GK({}_A M_B)\} .$$

This sharpens [12, Proposition 5.8].

(b) Set $S = k\{X,Y\}/(Y)^4 = k[x,y]$, where $\{X,Y\}$ is free on $\{X,Y\}$, and take $I = (y)^2 \subset S$. Then $GK(S) = 4$, $GK(S/I) = 2$, and $GK({}_S I_S) = 3$. The former two equalities are well-known and easy to prove. To compute $GK({}_S I_S)$ note that the monomials $\mu_{u,v,r} := x^u y x^v y x^r$ and $\mu_{u,v,r,s} := x^u y x^v y x^r y x^s$ (u,v,r,s non-negative integers) form a k-basis of I . Moreover, $V := \langle x,y\rangle_k \otimes 1 + 1 \otimes \langle x,y\rangle_k$ generates $S^{op} \otimes_k S$ as k-algebra. If $E \subset I$ is a f.d. subspace, then there exist $p,q \geq 0$ such that $E \subset \sum_{t \leq p} yx^t y \cdot V^{(q)}$. Moreover,

$$yx^t y \cdot V^{(n)} = \langle \mu_{u,t,v} \mid u+v \leq n\rangle_k + \langle \mu_{u,t,r,s} \mid u+r+s \leq n-1\rangle_k + \langle \mu_{u,v,t,s} \mid u+v+s \leq n-1\rangle_k .$$

Thus $\dim yx^t y \cdot V^{(n)}$ grows like n^3 , which proves that $GK({}_S I_S) = 3$. Therefore, the inequality in Lemma 2.2 is strict in this case.

Further examples with ${}_S I_S \circledast {}_S S_S$ can be obtained from Lemma 2.7 below. We now turn to the case where I is finitely generated as a right ideal of S .

Lemma 2.5. Let I be an ideal of S .

i. If I_S is finitely generated and ${}_S I_S \circledast {}_S S_S$, then $I_S \circledast S_S$.

ii. If $I_S \circledast S_S$ then, for any integer $t \geq 1$,

$$GK(S) = \max \{GK(S/I), GK(I^t{}_S)\} .$$

In particular, if I is also nilpotent, then $GK(S) = GK(S/I)$.

Proof. (i) is a special case of Lemma 1.7(ii).

(ii). Fix $t \geq 1$ and let V be a f.d. subspace of S. Using our assumption $I_S \circledast S_S$, we find f.d. subspaces $G \subset I$ and $V = V_0 \subset V_1 \subset \ldots \subset V_t =: W \subset S$ such that, for all $n \geq 0$,

$$I \cap V_i^{(n)} \subset G \cdot V_{i+1}^{(n)} .$$

For each n, fix subspaces $X_{i,n}$ $(i=0,\ldots,t)$ of $V_i^{(n)}$ with $(I \cap V_i^{(n)}) \oplus X_{i,n} = V_i^{(n)}$. Then

$$V_i^{(n)} \subset X_{i,n} + G \cdot V_{i+1}^{(n)}$$

and so

$$V^{(n)} \subset X_{0,n} + G \cdot X_{1,n} + \ldots + G^{t-1} \cdot X_{t-1,n} + G^t \cdot W^{(n)} .$$

Letting $\bar{\ } : S \to S/I$ denote the canonical map, we have $\dim X_{i,n} = \dim \overline{V_i}^{(n)} \leq \dim \overline{W}^{(n)}$ for all n and i. Therefore, setting $g := 1 + \dim G + \ldots + \dim G^{t-1}$ and $F := G^t \subset I^t$, we obtain

$$\dim V^{(n)} \leq g \cdot \dim \overline{W}^{(n)} + \dim F \cdot W^{(n)} .$$

This proves that $GK(S) \leq \max \{GK(S/I), GK(I^t{}_S)\}$, and hence (ii). ∎

The rest of this section is devoted to a particular class of examples with ${}_S I_S \circledast {}_S S_S$.

Definition. A k-algebra S will be called <u>strongly finitely presented</u> if there exists a f.d. subspace $V \subset S$ such that

(a) $S = k[V]$, i.e., V generates S as k-algebra,

and

(b) $gr_V(S) := \bigoplus_n V^{(n)}/V^{(n-1)}$ is a finitely presented k-algebra.

The algebra S will be called <u>locally strongly finitely presented</u> if every affine subalgebra of S is contained in a strongly finitely presented subalgebra of S.

Remarks and Examples 2.6. (a) Strongly finitely presented algebras are finitely presented. Indeed, if $V \subset S$ is as in the definition, then there is an obvious exact sequence $0 \to J \to k\{V\} \to S \to 0$, where $k\{V\}$ is free on a k-basis of V . Hence $\mathrm{gr}_V(S) \cong \mathrm{gr}_V(k\{V\})/\mathrm{gr}_V(J) \cong k\{V\}/\mathrm{gr}_V(J)$, where $\mathrm{gr}_V(J) := \bigoplus_n J \cap V^{(n)}/J \cap V^{(n-1)}$. By (b), $\mathrm{gr}_V(J)$ is finitely generated as ideal of $k\{V\}$, and hence so is J .

(b) Recall that an algebra S is called almost commutative if there exists a f.d. generating subspace $V \subset S$ such that $\mathrm{gr}_V(S)$ is commutative. In this case, $\mathrm{gr}_V(S)$ is affine and commutative, and hence finitely presented. Thus almost commutative algebras are strongly finitely presented.

(c) Using the Artin-Tate Lemma, one easily shows that any affine k-algebra which is finitely generated as module over its center is strongly finitely presented. Somewhat more generally, if $S = \Sigma_{i=1}^{t} Ry_i$ for some subalgebra R and if $R = k[W]$ with $W \subset R$ f.d. such that $\langle y_j W , y_p y_q \mid \text{all } j,p,q\rangle_k \subset \Sigma_{i=1}^{t} Wy_i$ and $\mathrm{gr}_W(R)$ is left Noetherian and finitely presented, then $\mathrm{gr}_V(S)$ is left Noetherian and finitely presented for $V = \langle W, y_1, \dots, y_t\rangle_k$.

(d) If S is strongly finitely presented, then $\mathrm{gr}_V(S)$ is not automatically finitely presented, for any generating subspace V , but V must be carefully chosen. For example, take $S = k\{X,B\}/(X^2-1, XBXB-BXBX) = k[x,b]$, where $k\{X,B\}$ is free on $\{X,B\}$. Then S is a skew group ring, $S \cong k[a,b]*C_2$, with $k[a,b]$ the polynomial ring in two variables a and b which are interchanged by the action of $C_2 = \langle x\rangle$. Thus S is an affine Noetherian PI-algebra which is in fact a finitely generated module over its center.

Taking $V = \langle x,b\rangle_k \subset S$ we have

$$T := \mathrm{gr}_V(S) \cong k\{X,B\}/(X^2, XB^nXB-BXB^nX;\ n \geq 1)$$

so that T is not finitely presented. For later use, we also note that T is not Noetherian. Indeed, if ξ and β denote the images of X and B in T and $I = (\xi)$ is the ideal generated by ξ, then $I^3 = 0$ and I^2 is free as left and right T/I-module on $\{\xi\beta^n\xi \mid n \geq 1\}$. Thus T has infinite uniform dimension (left and right).

On the other hand, (c) above implies that S is strongly finitely presented. In fact, if $W = \langle x,b,a := xbx\rangle_{k} \subset S$, then

$$\mathrm{gr}_W(S) \cong k\{X,A,B\}/(X^2, AB-BA, XA-BX, AX-XB)$$

which is finitely presented, and Noetherian.

The following lemma is analogous to Proposition 1.4.

<u>Lemma 2.7.</u> Let R be a k-algebra which is locally strongly finitely presented. Then, for all exact sequences of k-algebras $0 \to I \to S \overset{\pi}{\to} R \to 0$, one has ${}_SI_S \otimes {}_SS_S$.

<u>Proof.</u> Let $V \subset S$ be a f.d. subspace. By assumption on R, there is a f.d. subspace $W \subset S$ such that $V \subset S_0 := k[W]$ and $R_0 := \pi(S_0)$ is strongly finitely presented, with $\pi(W)$ being the required subspace. Using the filtrations $W^{(n)}$ on S_0, $\pi(W^{(n)})$ on R_0, and $I \cap W^{(n)}$ on $I_0 := I \cap S_0$, we have an isomorphism of associated graded algebras

$$\mathrm{gr}(R_0) \cong \mathrm{gr}(S_0)/\mathrm{gr}(I_0) .$$

Since $\mathrm{gr}(R_0)$ is finitely presented and $\mathrm{gr}(S_0)$ is affine, $\mathrm{gr}(I_0)$ is finitely generated as ideal of $\mathrm{gr}(S_0)$. It follows that there exists an $s \geq 1$ with

$$I \cap W^{(n)} \subset \sum_{i+j\leq n} W^{(i)}\cdot(I \cap W^{(s)})\cdot W^{(j)}$$

for all n. Since $V \subset W^{(d)}$ for some d, we conclude that ${}_SI_S \otimes {}_SS_S$ (see Remark 2.3). ∎

We summarize our discussion in the following proposition which generalizes [14,Lemma 3].

<u>Proposition 2.8.</u> Let I be an ideal of S which is finitely generated as right ideal and suppose that S/I is a finitely generated right module over some subalgebra which is locally strongly finitely presented.

Then, for all integers $t \geq 1$,

$$GK(S) = \max\{GK(S/I), GK(I^t{}_S)\} .$$

In particular, if I is also nilpotent, then $GK(S) = GK(S/I)$.

<u>Proof.</u> By Lemma 1.6(i), we can assume that S/I is locally strongly finitely presented. The result now follows from Lemmas 2.7 and 2.5 . ∎

3. A NON-COMMUTATIVE HILBERT-SERRE THEOREM

In this section, we prove a rationality theorem for Poincaré series of graded modules which extends the classical Hilbert-Serre Theorem to certain non-commutative graded algebras. Our proof is a modification of the usual proof of the Hilbert-Serre Theorem as given, for example, in [1,Theorem 11.1].

We will be concerned with graded algebras $S = \bigoplus_{n\geq 0} S_n$ which have the following three properties:

(1) S is affine over k and connected (i.e., $S_0 = k$), and $GK(S) < \infty$.

(2) Every graded ideal I of S contains a finite product of primes $P \supset I$. These can be chosen to be graded, because if P is prime then so is $\bigoplus_n P \cap S_n$.

(3) If P is a graded prime ideal of S, $P \neq \bigoplus_{n>0} S_n$, then $\bar{S} = S/P$ contains a non-zero normal element which is homogeneous of positive degree. Here, an element $x \in \bar{S}$ is called normal if $\bar{S}x = x\bar{S}$.

The main examples we have in mind are as follows. Note that, in these examples, a stronger form of (3) holds: $x \in \bar{S}$ can even be chosen to be central in $\bar{S}$. Note also that the center of a graded algebra is a graded subalgebra.

Examples 3.1. (a) Affine graded (connected) PI-algebras. For these, $GK(S) < \infty$ is due to Berele [2] (or see [12,Corollary 10.7]), (2) is a consequence of A. Braun's theorem [7], and (3) follows from the fact that any non-zero ideal of a prime (or even semiprime) PI-algebra S intersects the center of S non-trivially [16,Theorem 1.6.27].

(b) Enveloping algebras $U = U(\mathfrak{g})$ of finite-dimensional graded Lie-algebras $\mathfrak{g} = \bigoplus_{i=1}^{t} \mathfrak{g}_i$, where the $\mathfrak{g}_i$ are subspaces with $[\mathfrak{g}_i,\mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ ($= 0$ for $i+j > t$) and U is graded as in [11,Section 8.2]. Here, $GK(U) = \dim \mathfrak{g}$ ([12,Theorem 6.10]), (2) is clear, since U is Noetherian, and (3) follows from the fact that U is polycentral, because $\mathfrak{g}$ is nilpotent ([9, Proposition 4.7.1]). Our theorem thus in particular implies [11,Satz 8.1], except for the precise form of $\text{l.c.m.}\{k_i\}$.

Assumption (1) above implies that any finitely generated graded S-module $M = \bigoplus_{n\geq 0} M_n$ satisfies $\dim M_n < \infty$ for all n. Thus the Poincaré series $P_M(t)$ of M can be defined by

$$P_M(t) = \sum_{n\geq 0} \dim M_n t^n \in \mathbb{Z}[[t]].$$

Theorem 3.2. Assume that $S = \bigoplus_{n\geq 0} S_n$ satisfies (1),(2),(3). Then, for any Noetherian graded S-module $M = \bigoplus_{n\geq 0} M_n$, $P_M(t)$ is a rational function of the form

$$P_M(t) = f(t)/\prod_{i=1}^{s} (1 - t^{k_i}) ,$$

where $f(t) \in \mathbb{Z}[t]$.

Proof. If $GK(S) = 0$, then S and M are finite-dimensional and so $P_M(t) \in \mathbb{Z}[t]$.

Suppose that $GK(S) > 0$ and that the assertion is true for all graded homomorphic images $\bar{S}$ of S which satisfy $GK(\bar{S}) \leq GK(S) - 1$. By assumption (2), there are graded prime ideals P_i $(i=1,\ldots,r)$ of S, not necessarily distinct, so that $M\cdot P_1P_2\cdot\ldots\cdot P_r = 0$. Setting $M_0 = M$, $M_i = M_{i-1}\cdot P_i$ $(i \geq 1)$ we obtain a decreasing sequence of graded submodules of M, with $M_r = 0$. Since $P_M(t) = \Sigma_{i=1}^{r} P_{M_{i-1}/M_i}(t)$, it suffices to show that each $P_{M_{i-1}/M_i}(t)$ has the required form. Thus, after replacing M by M_{i-1}/M_i and S by S/P_i, we may assume that S is a prime ring.

Now let $x \in S$ be normal, as in (3), with $x \in S_m$ $(m > 0)$, say. Then xS is a graded ideal of S with $GK(S/xS) \leq GK(S) - 1$, since x is regular in S ([12,Proposition 3.15]). Moreover, $K := \mathrm{ann}_M(x)$ and $M\cdot x$ are graded submodules of M, and hence K and $L := M/M\cdot x$ are graded Noetherian S-modules which are in fact modules over S/xS. By assumption, $P_K(t)$ and $P_L(t)$ have the desired form. Furthermore, for each n, we have an exact sequence of vector spaces

$$0 \to K_n = K\cap M_n \to M_n \xrightarrow{\cdot x} M_{n+m} \to L_{n+m} = M_{n+m}/M_n\cdot x \to 0 .$$

We deduce that

$$\dim M_{n+m} - \dim M_n = \dim L_{n+m} - \dim K_n$$

holds for all n and, multiplying with t^{n+m} and summing over all $n \geq 0$ in $\mathbb{Z}[t]$, we obtain

$$(1 - t^m) P_M(t) = P_L(t) - t^m P_K(t) + h(t)$$

for some polynomial $h(t) \in \mathbb{Z}[t]$ of degree $< m$. This implies the result. ∎

Let us record the following (standard) consequence which formed our original motivation for proving Theorem 3.2.

<u>Corollary 3.3.</u> If $S = \bigoplus_{n\geq 0} S_n$ satisfies (1),(2),(3) and $M = \bigoplus_{n\geq 0} M_n$ is a Noetherian graded S-module, then $GK(M_S)$ is an integer. Moreover, GK is exact for Noetherian graded S-modules.

<u>Proof.</u> Using the formula $GK(M_S) = \overline{\lim_n} \log_n d_M(n)$, where $d_M(n) = \Sigma_{m\leq n} \dim M_m$ ([12,Lemma 6.1]), and the explicit form of $P_M(t)$ as given in Theorem 3.2, it follows that $GK(M_S)$ is equal to $\mathrm{ord}_{t=1} P_M(t)$, the order of the pole of $P_M(t)$ at $t = 1$. This proves integrality of $GK(M_S)$. Exactness also follows, because if $0 \to N \to M \to W \to 0$ is an exact sequence of Noetherian graded

S-modules (respecting degrees), then $P_M(t) = P_N(t) + P_W(t)$ and so $\mathrm{ord}_{t=1} P_M(t) \leq \max \{\mathrm{ord}_{t=1} P_N(t), \mathrm{ord}_{t=1} P_W(t)\}$. ∎

Theorem 3.2 can also be used, in the usual fashion, to define a notion of multiplicity for Noetherian graded S-modules by considering Hilbert-Samuel polynomials. Refined versions of the above exactness statement can then be derived as in [11,Kapitel 8] or [12,Chap. 7], for example.

The familiar examples of affine PI-algebras with non-integral GK-dimension ([5,Satz 2.10] or [12,Theorem 1.8]) show that a fairly strong assumption on the module M is needed for $P_M(t)$ to have the form described in Theorem 3.2. Unfortunately, our assumption on M to be Noetherian limits the usefulness of the result in dealing with an a priori ungraded affine algebra by first passing to an associated graded algebra. Noetherianness tends to get lost in the process. An explicit example is given by Example 2.6(d) and an even simpler example follows.

<u>Example 3.4.</u> Let $G = \langle x,y \mid x^2 = (xy)^2 = 1\rangle$ be the infinite dihedral group and let $S = kG$ be the group algebra of G. Then S is a finitely generated module over its center and $S = k[V]$ with $V = \langle x,y\rangle_k$. But, setting $T = \mathrm{gr}_V(S)$, one easily checks that

$$T \cong k\{X,Y\}/(X^2, YXY) = k[\xi,\eta] ,$$

where ξ and η denote the images of X and Y, respectively. If $I = (\xi)$ is the ideal of T generated by ξ, then $I^3 = 0$ and $I^2 = \bigoplus_{j\geq 1} I_j$ where $I_j = \xi\eta^j\xi \cdot T = \xi\eta^j\xi \cdot k = T \cdot \xi\eta^j\xi$ is an ideal of T. Thus T has infinite uniform dimension. If, instead of V, the subspace $W = \langle x,y,y^{-1}\rangle_k$ of S is used, then

$$\mathrm{gr}_W(S) \cong k\{X,Y,Y_1\}/(YY_1, Y_1Y, X^2, YX-XY_1, Y_1X-XY)$$

is a finitely generated module over the commutative subalgebra generated by the images of Y and Y_1. Hence $\mathrm{gr}_W(S)$ is Noetherian.

These examples make it clear that Theorem 3.2 does not immediately yield the known fact that (affine) Noetherian PI-algebras have integral GK-dimension (Lorenz-Small [14]). What is required here is the existence of "good" generating subspaces in the sense of the following

<u>Question.</u> If S is an affine (right) Noetherian PI-algebra, does there exist a f.d. generating subspace V of S such that $gr_V(S)$ is (right) Noetherian ?

It may be worthwhile to investigate, quite generally, the class of all k-algebras S having the property that for all f.d. subspaces $V \subset S$ there exists a f.d. subspace $W = W(V) \subset S$ with $V \subset k[W]$ and $gr_W(k[W])$ (right) Noetherian. If such a W exists, then it can even be chosen so that $V \subset W$ (cf. Lemma 4.2).

We also remark that, in both examples discussed above, the Poincaré seies $P(V;t)$ of $T = gr_V(S)$ is readily shown to be rational, even though T is not Noetherian. But J. T. Stafford [17] has constructed affine PI-algebras S having f.d. generating subspaces $V \subset W$ so that the Poincaré series $P(V;t)$ of $gr_V(S)$ is rational whereas $P(W;t)$ is irrational.

4. ON ASSOCIATED GRADED RINGS AND MODULES

In this section, we will study certain graded rings $G(R)$ and modules $G(M)$ which can be associated with a given filtration on a ring R, or module M, and which have some advantages over the usual associated graded rings and modules.

We consider a C-algebra R, where C is a central subring of R, together with a filtration $F = \{R^{(n)} \mid n \in \mathbb{Z}\}$ such that

$$0 = R^{(-1)} \subset \ldots \subset R^{(n)} \subset R^{(n+1)} \subset \ldots \subset R = \bigcup_n R^{(n)},$$
$$C \subset R^{(0)}, \quad R^{(n)}R^{(m)} \subset R^{(n+m)}.$$

Furthermore, let M be a filtered R-module with filtration $G = \{M^{(n)} \mid n \in \mathbb{Z}\}$ satisfying, for some m_0,

$$0 = M^{(m_0)} \subset \ldots \subset M^{(n)} \subset M^{(n+1)} \subset \ldots \subset M = \bigcup_n M^{(n)},$$
$$M^{(n)} \cdot R^{(m)} \subset M^{(n+m)}.$$

In addition to the usual associated graded rings and modules $gr(R) = gr_F(R) = \bigoplus_n R^{(n)}/R^{(n-1)}$ and $gr(M) = gr_G(M) = \bigoplus_n M^{(n)}/M^{(n-1)}$, we will consider the graded subring

$$G(R) = G_F(R) := \bigoplus_n R^{(n)}X^n$$

of the polynomial ring $R[X]$ and the graded $G(R)$-module

$$G(M) = G_G(M) := \bigoplus_n M^{(n)}X^n.$$

Of course, $G(R)$ and $G(M)$ could have been defined without explicitly refering to the "variable" X, in analogy with $gr(R)$ and $gr(M)$, but X will be a convenient notational device in the following. Note that X is a central element of $G(R)$.

The following lemma shows that, as far as finiteness conditions and Poincaré series are concerned, the constructions

$G(M)$ and $gr(M)$ are essentially equivalent. However, as the proof will demonstrate, the relationship of $G(M)$ to both M and $gr(M)$ is quite perspective, thereby making $G(M)$ a useful link between M and $gr(M)$.

<u>Lemma 4.1</u> (notation as above).

i. $G(M)$ is finitely generated (fin. presented; Noetherian) as $G(R)$-module if and only if $gr(M)$ is finitely generated (fin. presented; Noetherian) as $gr(R)$-module. In this case, M is finitely generated (fin. presented; Noetherian) as R-module.

ii. $G(R)$ is finitely generated (fin. presented; right Noetherian) as C-algebra if and only if the same holds for $gr(R)$. In this case, R is finitely generated (fin. presented; right Noetherian) as C-algebra.

iii. Assume that all $M^{(n)}$ are finitely generated as C-modules and let λ be an additive integer-valued function on the class of all finitely generated C-modules. Put

$$P_{G(M)}(t) := \sum_n \lambda(M^{(n)})\, t^n \quad , \quad P_{gr(M)}(t) := \sum_n \lambda(M^{(n)}/M^{(n-1)})\, t^n .$$

Then, in $\mathbb{Z}(t)$, we have

$$(1 - t)\cdot P_{G(M)}(t) = P_{gr(M)}(t) .$$

<u>Proof.</u> Part (iii) is obvious, and (ii) is similar to (i) and, for the most part, follows from the proof of (i). So we will concentrate on (i).

We will use the following trivial fact: If S is any ring, I is an ideal of S, and M is an S-module which is finitely generated (fin. presented; Noetherian), then $M/M\cdot I \cong M \otimes_S S/I$ is likewise, as module over S/I.

Consider the following ideals I and J of $G(R)$:

$$I := G(R)\cdot X = \bigoplus_n R^{(n-1)}X^n \quad , \quad J := G(R)\cdot(X - 1) .$$

Then $G(M)\cdot I = G(M)\cdot X = \bigoplus_n M^{(n-1)}X^n$, $G(R)/I \cong gr(R)$, and

$G(M)/G(M)\cdot I \cong gr(M)$. In view of the above remark, this proves the implications "$\Rightarrow$" in (i). Similarly, $G(R)/J \cong R$ via $\Sigma r_n X^n \mapsto \Sigma r_n$, and $G(M)/G(M)\cdot J \cong M$ via $\Sigma m_n X^n \mapsto \Sigma m_n$, which proves the assertions about M as R-module in (i).

As to the implications "$\Leftarrow$", we consider the filtrations

$$G(M)^{(n)} := \bigoplus_m M^{(\min(m,n))} X^m, \quad G(R)^{(n)} := \bigoplus_m R^{(\min(m,n))} X^m$$

of $G(M)$ and $G(R)$. It is readily checked that $G(M)^{(n)}\cdot G(R)^{(m)} \subset G(M)^{(n+m)}$ and, clearly,

$$G(M)^{(n)}/G(M)^{(n-1)} \cong \bigoplus_{m\geq n} \frac{M^{(n)}}{M^{(n-1)}} X^m \cong \frac{M^{(n)}}{M^{(n-1)}} \otimes_C C[X].$$

Therefore,

$$gr(G(M)) \cong gr(M) \otimes_C C[X]$$

and similarly

$$gr(G(R)) \cong gr(R) \otimes_C C[X].$$

Using the Hilbert Basis Theorem and [6,§2 n°9,Cor. 1], we obtain the following implications:

$gr(M)$ fin. generated (fin. presented; Noetherian) over $gr(R)$

$\Downarrow$

$gr(G(M))$ fin. gen. (fin. pres.; Noetherian) over $gr(G(R))$

$\Downarrow$

$G(M)$ fin. gen. (fin. pres.; Noetherian) over $G(R)$.

This completes the proof of (i). ∎

Another advantage of $G(M)$ over $gr(M)$, besides its more obvious connection with M, comes from the fact that variations of the filtration G are more easily dealt with using $G_G(M)$ rather than $gr_G(M)$. The proof of the lemma below should serve as an example. We will use the following notation. If $V \subset R$ is a C-submodule generating R as C-algebra, then we write $G_V(R) = G_F(R)$ where $F = \{V^{(n)}\}$ is the V-filtration of R.

Lemma 4.2. Let V be a finitely generated C-submodule of R generating R as C-algebra and put $W = V^{(d)}$, for some fixed $d \geq 1$.

Then $G_V(R)$ is (right) Noetherian if and only if $G_W(R)$ is (right) Noetherian.

Proof. Since $V^{(n)} \subset W^{(n)} = V^{(dn)}$ holds for all n, we have inclusions of C-algebras

$$S := \bigoplus_n W^{(n)} X^{dn} \subset G_V(R) = \bigoplus_n V^{(n)} X^n \subset G_W(R) = \bigoplus_n W^{(n)} X^n .$$

Here, S is isomorphic to $G_W(R)$ via $X \mapsto X^d$. Moreover, setting $A := \bigoplus_{0 \leq m \leq d-1} V^{(m)} X^m \subset G_V(R)$, we have

$$G_V(R) = S \cdot A = A \cdot S .$$

So $G_V(R)$ is finitely generated as (left and right) S-module. This proves "$\Leftarrow$" .

For the converse, note that

$$G_V(R) = S \oplus {}_S B_S \quad \text{with } B = \bigoplus_{d \nmid m} V^{(m)} X^m .$$

Thus, for any right ideal I of S,

$$I \cdot G_V(R) \cap S = I ,$$

and similarly for left ideals. ∎

The examples in 2.6(d) and 3.4 show that Lemma 4.2 fails to hold in general if V is more freely varied. Note that the last part of the above argument works for general filtrations $F = \{R^{(n)}\}$ and shows that if $G(R) = G_F(R)$ is (right) Noetherian then so is $G_d(R) = G_{F^d}(R)$, where $F^d = \{R^{(dn)}\}$. Moreover, assuming the Poincaré series $P_{G(R)}(t)$ is defined as in Lemma 4.1(iii), it is not hard to show that if $P_{G(R)}(t)$ is rational then $P_{G_d(R)}(t)$ is also rational.

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