Thin Discs and a Modera Theorem for CR Functions

Alexander Tumanov

Math. Department
University of Illinois
Urbana, IL 61821
USA

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn
GERMANY

MPI/95-86

THIN DISCS AND A MOREA THEOREM FOR CR FUNCTIONS

Alexander Tumanov*

Introduction

The classical Morera theorem says that a continuous function f in a domain $D \subset \mathbf{C}$ is holomorphic if $\int_\Gamma f(z)\,dz = 0$ for any closed curve Γ in D.

There are generalizations of this theorem to domains and real hypersurfaces in $\mathbf{C}^n$. See [A] and [GS] and references there.

In this paper, we obtain a version of the Morera theorem for CR functions on manifolds of higher codimension.

Let M be a smooth real manifold in $\mathbf{C}^n$. Recall that M is generic if the tangent space T_zM spans the whole space $\mathbf{C}^n$ for $z \in M$. A CR function on M is a continuous function that satisfies the weak tangential Cauchy-Riemann equations. An analytic disc is a continuous mapping $A : \bar{\Delta} \to \mathbf{C}^n$ holomorphic in the standard disc $\Delta = \{\zeta \in \mathbf{C} : |\zeta| < 1\}$. We say that the disc A is attached to M if it maps the circle $b\Delta$ to M, that is $A(b\Delta) \subset M$.

Let f be a continuous function on a real manifold $M \subset \mathbf{C}^n$. We say that f has the Morera property with respect to an analytic disc A attached to M if

$$\int_{b\Delta} f(A(\zeta))\,d\zeta = 0. \tag{1}$$

Note that for any analytic disc A attached to M there are also discs attached to M that differ from A by a change of variable in Δ only. Let $A_c(\zeta) = A(\frac{\zeta - c}{1 - c\zeta}), c \in \mathbf{R}, |c| < 1$. If a function f has the Morera property (1) with respect to all the discs A_c, then

$$f \circ A|_{b\Delta} \text{ extends holomorphically inside } \Delta. \tag{2}$$

Agranovsky and Val'sky [AV], Nagel and Rudin [NR], Stout [S], and others (see [A], [GS]) show that if a continuous function f on a real hypersurface $M \subset \mathbf{C}^n$ satisfies (2) or even (1) for discs obtained as sections of M by complex lines, then f is a CR function on M.

* Supported in part by NSF Grant DMS-9401652.

We prove the following theorem.

THEOREM. *Let f be a continuous function on a C^2 smooth generic manifold $M \subset \mathbf{C}^n$. Suppose f has the holomorphic extention property (2) for all analytic discs A attached to M. Then f is a CR function on M.*

Our goal is that the function f in Theorem is merely continuous. In case the function f is C^1 smooth, the conclusion of Theorem follows immediately by applying the holomorphic extendibility condition (2) to "infinitely small" discs attached to M.

In [T1] we prove an analogue of Theorem for $f \in C^1(M)$ without using arbitrarily small discs. Precisely, if (2) holds for all discs close to a given disc A, then f is CR in a neighborhood of $A(b\Delta)$. The author does not have a proof of this version for merely continuous functions.

Most of the results on the Morera theorem for hypersurfaces were obtained by applying harmonic analysis (see [A], [GS]). In particular, they use homogeneity of the family of complex lines in $\mathbf{C}^n$. In contrast, we do not use any group structure here. Our methods are based on polynomial approximations.

By Baouendi-Treves's approximation theorem [BT], a continuous CR function is locally a uniform limit of complex polynomials. We show that if a continuous function f satisfies (2) for a special $(d-1)$ parameter family of discs, where $d = \dim M$, then the approximation by polynomials still holds, whence f is a CR function.

To obtain the approximation result, we construct a family of discs that we call thin. These are discs stretched along complex tangential directions to M. We believe that the existence of such discs is of interest by itself.

Since our family of discs depends on $(d-1)$ parameters, we can raise the following question. Does Theorem still hold if (2) (respectively (1)) is assumed only for a given "generic" $(d-1)$ (respectively d) parameter family of discs? If this family does not admit a "nice" group stucture, this question is open even for $M = \mathbf{C}$.

The paper is organized as follows. In Section 1, we introduce a special space of functions on the unit circle. We need this space to estimate solutions to Bishop's [B] equation. In Section 2, we construct the family of discs. In section 3, we give a precise version of the main result.

The author would like to thank Mark Agranovsky for useful discussions.

This paper was written when the author was visiting the Max-Planck-Institute for

Mathematics in Bonn in summer 1995. The author wishes to thank the Institute for the support, hospitality, and stimulating atmosphere.

1. A special Lipschitz space

Let $C^{k,\alpha}$ denote the space of all functions with derivatives to order k satisfying a Lipschitz condition with exponent α. We denote the norm in this space by $||.||_{k,\alpha}$ and the norm in the space C^k by $||.||_k$.

In this section we introduce a suitable space of functions that are $C^{1,\alpha}$ except at finitely many points where they are only $C^{0,\alpha}$.

Let $a_1, \ldots, a_k$ be distinct points on $b\Delta$. For a function f on $b\Delta$, we set $f'(e^{i\theta}) = df(e^{i\theta})/d\theta$. Let $\chi(\zeta) = (\zeta - a_1)\ldots(\zeta - a_k)$.

DEFINITION 1.1. Let $0 < \alpha < 1$. Let $\mathcal{F}^\alpha(a_1, \ldots, a_k)$ be the set of all functions on $b\Delta$ for which the following norm in finite.

$$|||f||| = ||f||_0 + ||\chi f'||_{0,\alpha}.$$

We will write $\mathcal{F}^\alpha$ instead of $\mathcal{F}^\alpha(a_1, \ldots, a_k)$ if it is clear or unimportant what $a_1, \ldots, a_k$ are.

PROPOSITION 1.2. *Let* $0 < \alpha < 1$.

(i) *Let* $f \in \mathcal{F}^\alpha$. *Then* $|f'| \le \text{const}\,|||f||| \cdot |\chi|^{\alpha-1}$.

(ii) $\mathcal{F}^\alpha \subset C^{0,\alpha}(b\Delta)$; $||f||_{0,\alpha} \le \text{const}\,|||f|||$.

(iii) *Let* $H \in C^2$, $u \in \mathcal{F}^\alpha$, $H(0) = 0$, $dH(0) = 0$. *Then* $|||H \circ u||| \le \text{const}||H||_2|||u|||^2$.

(iv) *Let* T *denote the Hilbert transform on* $b\Delta$. *Then* $|||T||| < \infty$.

The constants above depend on $a_1, \ldots, a_k$ *and* α *only.*

PROOF. By partition of unity, $\mathcal{F}^\alpha(a_1, \ldots, a_k) = \sum_{j=1}^k \mathcal{F}^\alpha(a_j)$. Therefore, in proving (i)–(iv), we can assume that $k = 1$, $\chi(\zeta) = \zeta - a$, $a \in b\Delta$, $\mathcal{F}^\alpha = \mathcal{F}^\alpha(a)$.

(i) We have

$$|(\zeta - a)f'(\zeta) - (\zeta_1 - a)f'(\zeta_1)| \le ||\chi f'||_{0,\alpha}|\zeta - \zeta_1|^\alpha. \tag{1.1}$$

Let $\zeta_1 \to a$. Then $(\zeta_1 - a)f'(\zeta) \to 0$, otherwise f would be unbounded. Passing to the limit in (1.1) as $\zeta_1 \to a$, we get the needed estimate.

(ii) follows by integrating the estimate (i).

(iii) follows by straightforward estimates. Indeed, by definition, $|||H \circ u||| = ||H \circ u||_0 + ||\chi(H \circ u)'||_{0,\alpha}$. The first term $||H \circ u||_0$ obviously satisfies the needed estimate because $|H(u)| \leq \text{const}||H||_2 \cdot |u|^2$.

To estimate the second term, for brevity, we set $\Delta f = f(\zeta_1) - f(\zeta_2)$, where $\zeta_1, \zeta_2 \in b\Delta$. Then $\Delta(fg) = \Delta f\, g(\zeta_1) + f(\zeta_2)\Delta g$. Let H' denote $\partial H/\partial u$. We have $|H'(u)| \leq \text{const}||H||_2|u|$, $|\Delta u| \leq |||u||| \cdot |\zeta_1 - \zeta_2|^\alpha$. Therefore,

$$\begin{aligned} |\Delta(\chi(H \circ u)')| = |\Delta(\chi(H' \circ u)u')| &\leq |\Delta(\chi u')| \cdot |H'(u(\zeta_1))| + |(\chi u')(\zeta_2)| \cdot |\Delta(H' \circ u)| \\ &\leq |||u||| \cdot ||H' \circ u||_0 |\zeta_1 - \zeta_2|^\alpha + |||u||| \cdot ||H||_2 |\Delta u| \\ &\leq \text{const}||H||_2 |||u||| \cdot |\zeta_1 - \zeta_2|^\alpha, \end{aligned}$$

what we need.

(iv) Let Kf denote the inner limiting values of the Cauchy type integral of a function f on the unit circle:

$$(Kf)(z) = \lim_{r \to 1-0} \frac{1}{2\pi i} \int_{b\Delta} \frac{f(\zeta)\, d\zeta}{\zeta - rz}, \quad z \in b\Delta.$$

Since T can be expressed in terms of K, it suffices to show that K is bounded in $\mathcal{F}^\alpha$. Let D be the differentiation with respect to the complex variable, that is $(Df)(e^{i\theta}) = f'(e^{i\theta})/(ie^{i\theta})$. Let M_a be the multiplication by $\chi(\zeta) = \zeta - a$. We note that K commutes with D and almost commutes with M_a, that is $KD = DK$, $KM_a f = M_a K f + \int_{b\Delta} f(\zeta)\, d\zeta$. Hence, K commutes with $M_a D$. Since K is bounded in $C^{0,\alpha}$, by (ii) and definition of $\mathcal{F}^\alpha$, this implies immediately that K whence T is bounded in $\mathcal{F}^\alpha$.

Proposition 1.2 is proved.

2. Thin discs

Let M be a generic manifold in $\mathbf{C}^n$ and $M_0 \subset M$ a totally real submanifold of dimension n. Since we are interested in local questions, we can assume that M and M_0 are defined in a neighborhood of zero by a parametric equation

$$y = h(x, t), \tag{2.1}$$

where $x+iy = z \in \mathbf{C}^n$, $t \in \mathbf{R}^l$ is a parameter, h is a C^2 smooth function in a neighborhood of zero in $\mathbf{R}^n_x \times \mathbf{R}^l_t$. For $t = 0$, the equation (2.1) defines M_0. We assume that

$$\partial h(0,0)/\partial x = 0, \qquad \partial h_j(0,0)/\partial t_k = \delta_{jk}, \qquad 1 \le j \le n, \quad 1 \le k \le l, \tag{2.2}$$

where δ_{jk} is the Kronecker symbol. It is easy to see that $\dim M = d = n+l$, $\mathrm{CRdim} M = l$. We call x and t that satisfy (2.1) the x- and t-coordinates of the point $z = x+iy \in M$. The distance from $z \in M$ to M_0 is comparable to $|t|$, where t is the t-coordinate of z. The equation (2.1) is convenient to construct discs attached to M by Bishop's equation [B].

PROPOSITION 2.1. *For every $p \in M$ close to zero, and every small $\mathbf{R}^l$ valued function $t \in C^{0,\alpha}(b\Delta)$ $(0 < \alpha < 1)$ such that the t-coordinate of of p is $t(1)$, there exists a unique analytic disc $\zeta \mapsto A(\zeta)$ attached to M such that $A(1) = p$ and the t-coordinate of $A(\zeta)$ is $t(\zeta)$ for $\zeta \in b\Delta$.*

PROOF. Let $\zeta \mapsto A(\zeta) = x(\zeta)+iy(\zeta)$ be an analytic disc attached to M with $A(1) = p$. Since the disc is attached to M, we have $y(\zeta) = h(x(\zeta), t(\zeta))$ for $\zeta \in b\Delta$, where $t(\zeta)$ is the t-coordinate of $A(\zeta)$, $\zeta \in b\Delta$. Since the functions x and y are harmonic conjugates, the function x must satisfy the Bishop equation

$$x = -T_1 h(x, t) + x_0, \tag{2.3}$$

where x_0 is the x-coordinate of the given point p, T_1 denotes the harmonic conjugation operator on $b\Delta$ normalized by the condition $(T_1\phi)(1) = 0$. That is $T_1\phi = T\phi - (T\phi)(1)$, where T is the standard Hilbert transform.

One can check that for $h \in C^2$ with small $\partial h/\partial x$, small $t \in C^{0,\alpha}(b\Delta)$, the mapping $x \mapsto -T_1 h(x,t) + x_0$ is a contraction in a small ball in $C^{0,\alpha}(b\Delta)$. Thus, given sufficiently small $x_0 \in \mathbf{R}^n$ and $t \in C^{0,\alpha}(b\Delta)$, the equation (2.3) has a unique solution that defines the needed disc. The proof is complete.

We consider a thin disc on the complex plane $\mathbf{C}$. Let Δ_α be the domain in the upper half-plane bounded by the real axis and the arc of the circle $\{w \in \mathbf{C} : |\cos\alpha\pi - iw\sin\alpha\pi| =$

1}. This domain has corners at ± 1 with angles $\alpha\pi$. The domain Δ_α approaches the segment $[-1, 1]$ as $\alpha \to 0$. Let $\phi : \Delta \to \Delta_\alpha$ be the conformal mapping such that $\phi(1) = 0$, $\phi(\pm i) = \pm 1$. The mapping ϕ can be easily defined by an explicit formula. Note that $\phi \in C^{0,\alpha}(\bar{\Delta})$.

We set $\tau(\zeta) = \mathrm{Im}\phi(\zeta)$. Then $(T_1\tau)(\zeta) = -\mathrm{Re}\phi(\zeta)$. Note that $\tau(e^{i\theta}) \geq 0$, and $\tau(e^{i\theta}) = 0$ for $|\theta| < \pi/2$.

We consider discs attached to M with t-coordinate $t = \lambda\tau$, where $\lambda \in \mathbf{R}^l$. This leads to the equation

$$x = -T_1 h(x, \lambda\tau) + x_0. \tag{2.4}$$

The solution to this equation defines a family of discs $\zeta \mapsto A(\zeta, x_0, \lambda)$, where $x_0 \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}^l$ are small. The boundary of each disc of this family is split into two parts. The mapping A sends the right (respectively, left) half-circle to M_0 (respectively, $M \setminus M_0$). We claim these discs are thin as follows.

PROPOSITION 2.2. *The family $\zeta \mapsto A(\zeta, x_0, \lambda)$, $x_0 \in \mathbf{R}^n$, $\lambda \in \mathbf{R}^l$ has the following properties.*

(i) *The family A is $C^{0,\alpha}$ in ζ uniformly in x_0 and λ; A is C^1 in x_0 and λ uniformly in ζ.*

(ii) *For $\zeta_0 \in b\Delta$ with $\mathrm{Re}\zeta_0 < 0$, the evaluation map $(x_0, \lambda) \mapsto A(\zeta_0, x_0, \lambda)$ is a diffeomorphism from a neighborhood of zero in $\mathbf{R}^n \times \mathbf{R}^l$ to a neighborhood of zero in M. For $\lambda = 0$ or $\zeta_0 = 1$, this map sends (x_0, λ) to $x_0 + ih(x_0, 0) \in M_0$.*

(iii) *There are $\epsilon_0 > 0$, $c > 0$ such that for any $0 < \epsilon < \epsilon_0$, the set $\{A(\zeta, x_0, \lambda) : \zeta \in b\Delta,\ |x_0| \leq c,\ |\lambda| = \epsilon\}$ contains a neighborhood of zero in M.*

(iv) *The family A is C^1 in $\zeta \in b\Delta$ except at $\zeta = \pm 1$. Let $v = A' = \partial A(e^{i\theta}, x_0, \lambda)/\partial\theta$, $\tilde{\lambda} = (\lambda, 0) \in \mathbf{C}^n$. Then for small α, the direction of v is close to that of $\tilde{\lambda}$. Precisely, $\lim_{\alpha \to 0} \limsup_{(x_0,\lambda)\to(0,0)} \big| v/|v| \pm \tilde{\lambda}/|\lambda| \big| = 0$ for $\pm\mathrm{Re}\, e^{i\theta} < 0$.*

PROOF. The proof of (i) and (ii) is quite standard. The existence of the solution of class $C^{0,\alpha}$ has been already proved in Proposition 1. Estimates in $C^{0,\alpha}$ show that $A(\zeta, x_0, \lambda)$ is $C^{0,1}$ in x_0 and λ. Therefore one can differentiate the equation (2.4) with respect to x_0 and λ. Estimates show again that the derivatives still have Lipschitz regularity with respect to all the variables (see, e. g., [T2]). Plugging $x_0 = 0$, $\lambda = 0$ in the equations for the derivatives, by (2.2), we get the Jacobian matrix at zero of the evaluation map in the (x, t)-coordinates:

$$\begin{pmatrix} \partial x/\partial x_0 & \partial x/\partial\lambda \\ \partial t/\partial x_0 & \partial t/\partial\lambda \end{pmatrix} = \begin{pmatrix} \mathbf{1} & * \\ 0 & \tau(\zeta_0)\mathbf{1} \end{pmatrix},$$

where **1** is the identity matrix, and the asterisk denotes unimportant matrix elements. Since $\tau(\zeta_0) > 0$ for $\mathrm{Re}\zeta_0 < 0$, the first statement of (ii) follows. Plugging $\lambda = 0$ or $\zeta_0 = 1$ in (2.4), by uniqueness, we get the second statement of (ii).

The property (iii) follows from (i) and (ii) by a simple continuity argument. Indeed, the family of discs A is also a one parameter deformation of maps $A_{\zeta_0} : B^n_c \times S^{l-1}_\epsilon \to M$, where $B^n_c \subset \mathbf{R}^n_{x_0}$ is the ball of radius c and $S^{l-1}_\epsilon \subset \mathbf{R}^l_\lambda$ is the sphere of radius ϵ, the parameter is $\zeta_0 \in b\Delta$. By (ii), the image of, say A_{-1} is a smooth tube-like hypersurface around a piece of M_0 in M. On the other hand, A_1 maps (x_0, λ) to M_0. The deformation is "small" since the discs $\zeta \mapsto A(\zeta, x_0, \lambda)$ have diameters comparable to $|\lambda| = \epsilon$ (see (2.9) below). Therefore, choosing ϵ much smaller then c, we get what we need.

The most important part of Proposition 2 is (iv) which expresses the fact that the discs are thin. One cannot prove (iv) by estimates in the standard Lipschitz spaces since the discs are merely $C^{0,\alpha}$.

We show that the solution to (2.4) is in $\mathcal{F}^\alpha = \mathcal{F}^\alpha(-i, i)$. We first note that $\phi = -T_1\tau + i\tau \in \mathcal{F}^\alpha$, which can be seen from an explicit formula for ϕ. We also note that there is $C > 0$ such that

$$|(T_1\tau)'| \geq C|\chi|^{\alpha-1}. \tag{2.5}$$

Let $x = x_0 + u$. We rewrite (2.4) in the form

$$u = F(u), \qquad \text{where} \quad F(u) = -T_1 h(x_0 + u, \lambda\tau). \tag{2.6}$$

Consider the successive approximations:

$$u_0 = 0, \qquad u_{j+1} = F(u_j).$$

One can see that F is a contraction in $C^{0,\alpha}$. Hence, the sequence converges in $C^{0,\alpha}$. It suffices to show that the sequence is bounded in $\mathcal{F}^\alpha$. Then by the Arzela lemma, there is a subsequence that converges to the solution in $\mathcal{F}^\alpha$.

By Taylor's formula,

$$\begin{aligned} &h(x_0 + u, t) = h(x_0, 0) + h_t(x_0, 0)t + h_x(x_0, 0)u + H(x_0, u, t), \\ &||H(x_0, ., .)||_2 \leq \mathrm{const}||h||_2, \qquad H(x_0, u, t) = O(|u|^2 + |t|^2). \end{aligned} \tag{2.7}$$

Plugging (2.7) in (2.6) yields

$$F(u) = -h_t(x_0, 0)\lambda T_1\tau - h_x(x_0, 0)T_1 u - T_1 H(x_0, u, \lambda\tau). \tag{2.8}$$

By Proposition 1.2 and (2.7), we have

$$|||T_1 u||| \le \text{const}|||u|||, \qquad |||T_1 H(x_0, u, \lambda\tau)||| \le \text{const}||h||_2(|||u|||^2 + |\lambda|^2).$$

Therefore, estimating (2.8) yields

$$|||F(u)||| = O(|\lambda|) + o(1)|||u||| \qquad \text{as} \quad x_0 \to 0, \quad \lambda \to 0.$$

Hence, the solution to (2.4) is in $\mathcal{F}^\alpha$ and

$$|||x - x_0||| = |||u||| = O(|\lambda|) \tag{2.9}$$

For $t, \lambda \in \mathbf{R}^l$, we set $\tilde{t} = (t, 0) \in \mathbf{R}^n$, $\tilde{\lambda} = (\lambda, 0) \in \mathbf{R}^n$. By (2.2), we have $h(x,t) = \tilde{t} + O(|x|^2 + |t|^2)$. The equation (2.4) is a perturbation of the trivial "equation" obtained by neglecting the big "O". Therefore, the discs $\zeta \mapsto A(\zeta, x_0, \lambda)$ are perturbations of the flat discs $\zeta \mapsto A_0(\zeta, x_0, \lambda) = -\tilde{\lambda} T_1 \tau + x_0 + i\tilde{\lambda}\tau$. The direction of $A_0' = -\tilde{\lambda}(T_1\tau)' + i\tilde{\lambda}\tau' = \tilde{\lambda}\phi'$ is close to that of $\tilde{\lambda}$ just because the domain Δ_α is close to the segment $[-1, 1]$.

We compare the discs A and A_0 in the space $\mathcal{F}^\alpha$. By (2.8), we get

$$|||x - (x_0 - h_t(x, 0)\lambda T_1\tau)||| = |||u + h_t(x, 0)\lambda T_1 \tau||| = o(|\lambda|)$$

as $x_0 \to 0$, $\lambda \to 0$. Note that by (2.2), $|h_t(x,0)\lambda - \tilde{\lambda}| = o(|\lambda|)$, also. Hence,

$$|||x - (x_0 - \tilde{\lambda} T_1 \tau)||| = o(|\lambda|).$$

Therefore, by Proposition 1.2 (i),

$$|x' + \tilde{\lambda}(T_1\tau)'| = |\chi|^{\alpha - 1} o(|\lambda|). \tag{2.10}$$

By (2.2), the direction of $v = A' = x' + ih(x, \lambda\tau)'$ is close to that of $x' + i\tilde{\lambda}\tau'$. The latter, by (2.5) and (2.10), is close to the direction of $A_0' = -\tilde{\lambda}(T_1\tau)' + i\tilde{\lambda}\tau' = \tilde{\lambda}\phi'$, whence to that of $\tilde{\lambda}$.

The proof is complete.

3. Main result

Let $\zeta \mapsto A(\zeta, x_0, \lambda)$ be the family of discs from Proposition 2.2 and let $\epsilon > 0$ and $c > 0$ satisfy (iii) of this proposition. We need (iii) only to reduce by 1 the number of parameters. Recall $\dim M = d = n + l$.

THEOREM 3.1. *Let f be a continuous function on M defined by (2.1). Suppose $f \circ A|_{b\Delta}$ extends to be holomorphic in Δ for the $(d-1)$ parameter family of discs $\zeta \mapsto A(\zeta, x_0, \lambda)$, where $|x_0| \le c$, $|\lambda| = \epsilon$. Then f is a CR function in a neighborhood of zero in M.*

PROOF. We follow the proof of the Baouendi-Treves approximation theorem. We set

$$f_N(z) = (N/\pi)^{n/2} \int_{M_0} f(w) e^{-N(z-w)^2} dw_1 \wedge \cdots \wedge dw_n, \tag{3.1}$$

where $(z-w)^2 = (z_1 - w_1)^2 + \cdots + (z_n - w_n)^2$. If M is small, then by (2.2), there is $\kappa < 1$ such that

$$|\mathrm{Im}(z-w)| \le \kappa \, |\mathrm{Re}(z-w)| \tag{3.2}$$

for $z, w \in M_0$. This condition ensures that $K_N(z,w) = (N/\pi)^{n/2} e^{-N(z-w)^2} dw_1 \wedge \cdots \wedge dw_n$ forms a $\delta(z-w)$-shaped sequence as $N \to \infty$, $z, w \in M_0$. Thus, f_N converges to f uniformly on M_0.

We will show that f_N converge to f on boundaries of the discs $\zeta \mapsto A(\zeta, x_0, \lambda)$, $|x_0| \le c$, $|\lambda| = \epsilon$. By Proposition 2.2 (iii), they fill a neighborhood of zero in M. Hence, the function f will be a CR function in that neighborhood as a limit of a sequence of the entire functions f_N, completing the proof.

Without a loss of generality, we choose the disc with $\lambda = (\epsilon, 0, \ldots, 0)$, $x_0 = 0$ and show that f_N converge to f on the boundary of this disc. We consider a $(n-1)$ parameter subfamily of $\zeta \mapsto A(\zeta, x_0, \lambda)$ with fixed $\lambda = (\epsilon, 0, \ldots, 0)$ and fixed $x_{01} = 0$, the first component of x_0. Using this subfamily, we perturb M_0 by replacing for each disc the part of its boundary in M_0 by the other part. We set

$$K_{\pm} = \{A(\zeta, x_0, \lambda) : |\zeta| = 1, \pm \mathrm{Re}\zeta > 0, |x_0| < c/2, x_{01} = 0\}.$$

K_+ is an open subset of M_0. The set K_- is C^1 smooth, but presumably, merely $C^{0,\alpha}$ up to the boundary. Nevertheless, for small α, the tangent planes $T_z K_-$ are close to $\mathbf{R}^n_x$. Indeed, $T_z K_-$ at $z = A(e^{i\theta}, x_0, \lambda)$ is spanned by $v_1 = \partial A(e^{i\theta}, x_0, \lambda)/\partial\theta$ and $v_j = \partial A(e^{i\theta}, x_0, \lambda)/\partial x_{0j}$, where $1 < j \le n$. By Proposition 2.2 (ii, iv), the direction of v_j is close to the x_j-axis for $1 \le j \le n$.

Let $M_1 = (M \setminus K_+) \cup K_-$. Since f extends holomorphically to the discs, the integrand in (3.1) also extends there. Therefore, we can replace integration over M_0 by integration over M_1. For small α, the condition (3.2) still holds for $z \in M_1$ near $A(\zeta, 0, \lambda)$ and every $w \in M_1$ by the above remark regarding $T_z K_-$. Hence, f_N converges to f at $z = A(\zeta, 0, \lambda)$, what we need.

The proof is complete.

References

[A] M. L. AGRANOVSKY, *Invariant function spaces on homogeneos manifolds of Lie groups and applications*, Transl. Math. Monographs, **126** (1993), 131 p.

[AV] M. L. AGRANOVSKY AND R. E. VAL'SKY, *Maximality of invariant algebras of functions*, Siberian Math. J. **12** (1971), 1–7.

[BT] M. S. BAOUENDI AND F. TREVES, *A property of the functions and distributions annihilated by a locally integrable system of complex vector fields*, Ann. of Math. **114** (1981), 387–421.

[B] E. BISHOP, *Differentiable manifolds in complex Euclidean space*, Duke Math. J. **32** (1965), 1–21.

[GS] J. GLOBEVNIK AND E. L. STOUT, *Boundary Morera theorems for holomorphic functions of several complex variables*, Duke Math.J. **64** (1991), 571-615

[NR] A. NAGEL AND W. RUDIN, *Moebius-invariant functions on balls and spheres*, Duke Math. J. **43** (1976), 841–865.

[S] E. L. STOUT, *The boundary values of holomorphic functions of several complex variables*, Duke Math. J. **44** (1977), 105–108.

[T1] A. E. TUMANOV, *Extending CR functions on a manifold of finite type over a wedge*, Mat. Sbornik **136** (1988), 129–140.

[T2] ————, *On the propagation of extendibility of CR functions*, "Complex analysis and geometry", Lect. Notes in Pure and Appl. Math., Marcel-Dekker, to appear.