NON ABELIAN EXTENSIONS AND HOMOTOPIES

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The motivation for this paper is the classical problem of topology to find simple algebraic models of homotopy categories of spaces. For example Kan [12] uses free simplicial groups as such models. Curtis [7] showed that simplicial groups of nilpotency degree m suffice to model the homotopy category of simply connected n-dimensional CW-spaces with $n \leq 1 + \{\log_2(m+1)\}$. Here $\{a\}$ is the least integer $\geq a$. Is a further simplification possible? We restrict to the homotopy category $\underline{\underline{CW}}(n, n+k)$ as a test case where $\underline{\underline{CW}}(n, n+k)$ consists of CW-complexes with cells only in dimension n and $n+k$. Using theorem (5.8) we show that an algebraic model of the category $\underline{\underline{CW}}(2, 4)$ can be given only in terms of groups of nilpotency degree 2 while Curtis needs nilpotency degree 4 in this case; see (5.11).

A CW-complex K in $\underline{\underline{CW}}(n, n+k)$ is the mapping cone of a map, $m = n+k-1$,

$$\tilde{a} : M(A, m) \to M(X, n)$$

where A, X are free abelian groups and $M(X, m)$ is the Moore space of X. The homotopy type of K is determined by the homomorphism

$$a : A \to \Gamma_n^{k-1}(X)$$

induced by $\tilde{a}$. Here the homotopy group

$$\Gamma_n^k(X) = \pi_{n+k} M(X, n) \tag{1}$$

is computable via the Hilton-Milnor theorem in terms of homotopy groups of spheres. For free abelian groups A, X, B, Y we consider homotopy commutative diagrams together with homotopies H, $m = n+k-1$,

$$\begin{array}{ccc} M(B,m) & \xrightarrow{\tilde{\xi}} & M(A,m) \\ {\scriptstyle \tilde{b}}\downarrow & \overset{H}{\Longrightarrow} & \downarrow{\scriptstyle \tilde{a}} \\ M(Y,n) & \xrightarrow[\tilde{\eta}]{} & M(X,n) \end{array} \quad . \tag{2}$$

The main result of this paper describes algebraic models of such diagrams. They are used to represent morphisms in the category $\underline{\underline{CW}}(n, n+k)$. The homotopy classes of $\tilde{\xi}$, $\tilde{\eta}$, $\tilde{a}$, $\tilde{b}$ are determined by the induced homomorphisms ξ, η, a, b in the commutative diagram

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$$\begin{array}{ccc} B & \xrightarrow{\xi} & A \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ \Gamma_n^{k-1}(Y) & \xrightarrow[\eta_*]{} & \Gamma_n^{k-1}(X) \end{array} \tag{3}$$

Two homotopies $H, H' : \tilde{\eta}\,\tilde{b} \simeq \tilde{a}\,\tilde{\xi}$ differ by a map $\alpha : M(B, m+1) \to M(X, n)$ which represents a homomorphism

$$\alpha \in Hom(B, \Gamma_n^k(X)).$$

Hence homotopies in (2) yield a connection between the functors Γ_n^{k-1} and Γ_n^k. We describe this connection algebraically by a 'non-abelian extension'

$$0 \to \Gamma_n^k(X) \to M(G) \xrightarrow{\delta_n^k(G)} N(G) \to \Gamma_n^{k-1}(X) \to 0 \tag{4}$$

where G is a free group with $G^{ab} = X$. Here δ_n^k is a functor which carries a free group to a crossed module; this functor can be described by use of the differential in the Moore chain complex of a simplicial group $G(X, n)$ representing the loop space $\Omega M(X, n)$. Using δ_n^k we are able to construct 'algebraic homotopies' which represent homotopies H in (2); see § 4 and § 5. This aim leads us to the algebraic concepts in § 1, § 2, § 3 where we introduce abelian groups and homomorphisms

$$Ext^2_{\underline{\underline{K}}}(A, B, \underline{\underline{cr}}) \to Pext^2_{\underline{\underline{K}}}(A, B, \underline{\underline{cr}}) \to H^2(\underline{\underline{gr}}(A, \underline{\underline{K}}), Hom(-, B)) \tag{5}$$

which are binatural for $\underline{\underline{K}}$ -modules A, B. Here $\underline{\underline{cr}}$ is an additive subcategory of the category of crossed modules. The group Ext^2, $Pext^2$ are generalizations of the classical functor Ext^2, and H^2 is the cohomology of a category. The natural transformation (5) yields as a special case a transformation of Jibladze-Pirashvili (3.11 [11]).

§ 1 Linear extensions of categories and the cohomology of categories

An extension of a group G by a G-module A is a short exact sequence of groups

$$(1.1) \qquad 0 \to A \underset{i}{\longrightarrow} E \underset{p}{\longrightarrow} G \to 0$$

where i is compatible with the action of G. Two such extensions E and E' are equivalent if there is an isomorphism $\epsilon : E \cong E'$ of groups with $p'\epsilon = p$ and $\epsilon i = i'$. It is well known that the equivalence classes of extensions are classified by the cohomology $H^2(G, A)$.

We now describe linear extensions of a small category $\underline{\underline{C}}$ by a "natural system" D. The equivalence classes of such extensions are equally classified by the cohomology $H^2(\underline{\underline{C}}, D)$. A natural system D on a category $\underline{\underline{C}}$ is the appropriate generalization of a G-module.

(1.2) Definition. Let $\underline{\underline{C}}$ be a category. The category of factorizations in $\underline{\underline{C}}$, denoted by $F\underline{\underline{C}}$, is given as follows. Objects are morphisms $f, g, \ldots$ in $\underline{\underline{C}}$ and morphisms $f \to g$ are pairs (α, β) for which

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A' \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ B & \xleftarrow{\beta} & B' \end{array}$$

commutes in $\underline{\underline{C}}$. Here $\alpha f \beta$ is factorization of g. Composition is defined by $(\alpha', \beta')(\alpha, \beta) = (\alpha'\alpha, \beta\beta')$. We clearly have $(\alpha, \beta) = (\alpha, 1)(1, \beta) = (1, \beta)(\alpha, 1)$. A natural system (of abelian groups) on $\underline{\underline{C}}$ is a functor $D : F\underline{\underline{C}} \to \underline{\underline{Ab}}$. The functor D carries the object f to $D_f = D(f)$ and carries the morphism $(\alpha, \beta) : f \to g$ to the induced homomorphism

$$D(\alpha, \beta) = \alpha_* \beta^* : D_f \to D_{\alpha f \beta} = D_g$$

Here we set $D(\alpha, 1) = \alpha_*$, $D(1, \beta) = \beta^*$.

We have a canonical forgetful functor $\pi : F\underline{\underline{C}} \to \underline{\underline{C}}^{op} \times \underline{\underline{C}}$ so that each bifunctor $D : \underline{\underline{C}}^{op} \times \underline{\underline{C}} \to \underline{\underline{Ab}}$ yields a natural system $D\pi$, as well denoted by D. Such a bifunctor is also called a $\underline{\underline{C}}$ -bimodule. In this case $D_f = D(B, A)$ depends only on the objects A, B for all $f \in \underline{\underline{C}}(B, A)$. Two functors F, $G : \underline{\underline{Ab}} \to \underline{\underline{Ab}}$ yield the $\underline{\underline{Ab}}$ -bimodule

$$Hom(F, G) : \underline{\underline{Ab}}^{op} \times \underline{\underline{Ab}} \to \underline{\underline{Ab}}$$

which carries (A, B) to the group of homomorphisms $Hom(FA, GB)$. If F is the identity functor we write $Hom(-, G)$.

For a group G and a G-module A the corresponding natural system D on the group G, considered as a category, is given by $D_g = A$ for $g \in G$ and $g_* a = g \cdot a$ for

$a \in A$, $g^*a = a$. If we restrict the following notion of a "linear extension" to the case $\underline{\underline{C}} = G$ and $D = A$ we obtain the notion of a group extension above.

(1.3) Definition. Let D be a natural system on $\underline{\underline{C}}$. We say that

$$D \overset{+}{\rightarrowtail} \underline{\underline{E}} \overset{p}{\twoheadrightarrow} \underline{\underline{C}}$$

is a linear extension of the category $\underline{\underline{C}}$ by D if (a), (b) and (c) hold.

(a) $\underline{\underline{E}}$ and $\underline{\underline{C}}$ have the same objects and p is a full functor which is the identity on objects.

(b) For each $f : A \to B$ in $\underline{\underline{C}}$ the abelian group D_f acts transitively and effectively on the subset $p^{-1}(f)$ of morphisms in $\underline{\underline{E}}$. We write $f_0 + \alpha$ for the action of $\alpha \in D_f$ on $f_0 \in p^{-1}(f)$.

(c) The action satisfies the linear distributivity law:

$$(f_0 + \alpha)(g_0 + \beta) = f_0 g_0 + f_* \beta + g^* \alpha.$$

Two linear extensions $\underline{\underline{E}}$ and $\underline{\underline{E}}'$ are equivalent if there is an isomorphism of categories $\epsilon : \underline{\underline{E}} \cong \underline{\underline{E}}'$ with $p'\epsilon = p$ and with $\epsilon(f_0 + \alpha) = \epsilon(f_0) + \alpha$ for $f_0 \in \mathrm{Mor}(\underline{\underline{E}})$, $\alpha \in D_{pf_0}$. The extension $\underline{\underline{E}}$ is split if there is a functor $s : \underline{\underline{C}} \to \underline{\underline{E}}$ with $ps = 1$. We obtain the canonical split linear extension

(d) $$D \overset{+}{\rightarrowtail} \underline{\underline{C}} \times D \twoheadrightarrow \underline{\underline{C}}$$

as follows. Objects in $\underline{\underline{C}} \times D$ are the same as in $\underline{\underline{C}}$ and morphisms $X \to Y$ in $\underline{\underline{C}} \times D$ are pairs (f, α) where $f : X \to Y \in \underline{\underline{C}}$ and $\alpha \in D(f)$. The composition law is given by

(e) $$(f, \alpha)(g, \beta) = (fg, f_* \beta + g^* \alpha)$$

Clearly the projection $\underline{\underline{C}} \times D \to \underline{\underline{C}}$ carries (f, α) to f and the action $D+$ is given by $(f, \alpha) + \alpha' = (f, \alpha + \alpha')$ for $\alpha' \in D(f)$. A splitting functor s yields the equivalence of linear extensions

(f) $$\epsilon : \underline{\underline{C}} \times D \cong \underline{\underline{E}}$$

given by $\epsilon(f, \alpha) = s(f) + \alpha$. We also consider the following maps between linear extensions

(1.4) $$\begin{array}{ccccc} D & \xrightarrow{+} & \underline{\underline{E}} & \xrightarrow{p} & \underline{\underline{F}} \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \varphi} \\ D' & \xrightarrow{+} & \underline{\underline{E}}' & \xrightarrow{p'} & \underline{\underline{F}}' \end{array}$$

Here ϵ and φ are functors with $p'\epsilon = \varphi p$ and $d : D_f \to D'_{\varphi f}$ is a natural transformation compatible with the action $+$, that is

$$\epsilon(f_0 + \alpha) = \epsilon(f_0) + d(\alpha)$$

for $\alpha \in D_f$. Let $\underline{\underline{C}}$ be a small category and let $M(\underline{\underline{C}}, D)$ be the set of equivalence classes of linear extensions of $\underline{\underline{C}}$ by $\underline{\underline{D}}$. Then there is a canonical bijection

$$(1.5) \qquad \psi : M(\underline{\underline{C}}, D) \cong H^2(\underline{\underline{C}}, D)$$

which maps the split extension to the zero element, see IV § 6 in Baues [2]. Here $H^n(\underline{\underline{C}}, D)$ denotes the <u>cohomology</u> of $\underline{\underline{C}}$ with coefficients in D which is defined below. We obtain a <u>representing cocycle</u> Δ_t of the cohomology class $\{\underline{\underline{E}}\} = \psi(\underline{\underline{E}}) \in H^2(\underline{\underline{C}}, D)$ as follows. Let t be a "splitting" function for p which associates with each morphism $f : A \to B$ in $\underline{\underline{C}}$ a morphism $f_0 = t(f)$ in $\underline{\underline{E}}$ with $pf_0 = f$. Then t yields a cocycle Δ_t by the formula

$$(1.6) \qquad t(gf) = t(g)t(f) + \Delta_t(g, f)$$

with $\Delta_t(g, f) \in D(gf)$. The cohomology class $\{\underline{\underline{E}}\} = \{\Delta_t\}$ is trivial if and only if $\underline{\underline{E}}$ is a split extension.

(1.7) <u>Definition</u>. Let $\underline{\underline{C}}$ be a small category and let $N_n(\underline{\underline{C}})$ be the set of sequences $(\lambda_1, \ldots, \lambda_n)$ of n composable morphisms in $\underline{\underline{C}}$ (which are the n-simplices of the <u>nerve</u> of $\underline{\underline{C}}$). For $n = 0$ let $N_0(\underline{\underline{C}}) = \mathrm{Ob}(\underline{\underline{C}})$ be the set of objects in $\underline{\underline{C}}$. The cochain group $F^n = F^n(\underline{\underline{C}}, D)$ is the abelian group of all functions

$$(1) \qquad c : N_n(\underline{\underline{C}}) \to \left(\bigcup_{g \in \mathrm{Mor}(\underline{\underline{C}})} D_g \right) = D$$

with $c(\lambda_1, \ldots, \lambda_n) \in D_{\lambda_1 \circ \ldots \circ \lambda_n}$. Addition in F^n is given by adding pointwise in the abelian groups D_g. The coboundary $\partial : F^{n-1} \to F^n$ is defined by the formula

$$(2) \qquad \begin{aligned} (\partial c)(\lambda_1, \ldots, \lambda_n) &= (\lambda_1)_* c(\lambda_2, \ldots, \lambda_n) \\ &+ \sum_{i=1}^{n-1} (-1)^i c(\lambda_1, \ldots, \lambda_i \lambda_{i+1}, \ldots, \lambda_n) \\ &+ (-1)^n (\lambda_n)^* c(\lambda_1, \ldots, \lambda_{n-1}) \end{aligned}$$

For $n = 1$ we have $(\partial c)(\lambda) = \lambda_* c(A) - \lambda^* c(B)$ for $\lambda : A \to B \in N_1(\underline{\underline{C}})$. One can check that $\partial c \in F^n$ for $c \in F^{n-1}$ and that $\partial\partial = 0$. Hence the <u>cohomology groups</u>

$$(3) \qquad H^n(\underline{\underline{C}}, D) = H^n(F^*(\underline{\underline{C}}, D), \delta)$$

are defined, $n \geq 0$. These groups are discussed in Baues [2]. By change of the universe cohomology groups $H^n(\underline{\underline{C}}, D)$ can also be defined if $\underline{\underline{C}}$ is not a small category. A functor $\phi : \underline{\underline{C}}' \to \underline{\underline{C}}$ induces the homomorphism

$$(4) \qquad \phi^* : H^n(\underline{\underline{C}}, D) \to H^n(\underline{\underline{C}}', \phi^* D)$$

where $\phi^* D$ is the natural system given by $(\phi^* D)_f = D_{\phi(f)}$. On cochains the map ϕ^* is given by the formula

$$(\phi^* f)(\lambda'_1, \ldots, \lambda'_n) = f(\phi\lambda'_1, \ldots, \phi\lambda'_n)$$

where $(\lambda', \ldots, \lambda'_n) \in N_n(\underline{\underline{C}}')$. A natural transformation $\tau : D \to D'$ between natural systems induces a homomorphism

$$\tau_* : H^n(\underline{\underline{C}}, D) \to H^n(\underline{\underline{C}}, D')$$

by $(\tau_* f)(\lambda_1, \ldots, \lambda_n) = \tau_\lambda f(\lambda_1, \ldots, \lambda_n)$ where $\tau_\lambda : D_\lambda \to D'_\lambda$ with $\lambda = \lambda_1 \circ \ldots \circ \lambda_n$ is given by the transformation τ.

§ 2 Extensions of $\underline{\underline{K}}$-modules

We introduce various generalizations of the classical group of 2-fold extensions Ext^2. For this we need the following notations; see also [3]. Let $\underline{\underline{Gr}}$ be the category of groups and N be a group. An N-group (or an action of N on a group M) is a homomorphism h from N to the group of automorphisms of M. For $x \in M$, $\alpha \in N$ we denote the action by $x^\alpha = h(\alpha^{-1})(x)$. The action is trivial if $x^\alpha = x$ for all x, α. For a homomorphism $\alpha : G \to N$ in $\underline{\underline{Gr}}$ an α -crossed homomorphism $g : G \to M$ is a function g satisfying

$$g(x \cdot y) = g(x)^{\alpha(y)} \cdot g(y) \tag{2.1}$$

for $x, y \in G$. For example given homomorphisms $g, h : G \to M$ the function $-g + h : G \to M$ defined by

$$(-g + h)(x) = g(x)^{-1} \cdot h(x)$$

for $x \in G$ is a g-crossed homomorphism where we use the action of M on M by inner automorphism. We define for functions $r, s : G \to M$ the sum $r + s$ by

$$(r + s)(x) = r(x) \cdot s(x) \tag{2.2}$$

where the right hand side is the product in M. A crossed module $\partial : M \to N$ is a homomorphism in $\underline{\underline{Gr}}$ together with an action of N on M such that for $x, y \in M$, $\alpha \in N$ we have

$$\begin{cases} \partial(x^\alpha) = \alpha^{-1} \cdot x \cdot \alpha, \\ x^{\partial y} = y^{-1} x\, y. \end{cases} \tag{2.3}$$

A morphism $\partial \to \partial'$ between crossed modules is a commutative diagram in $\underline{\underline{Gr}}$

$$\begin{array}{ccc} M & \xrightarrow{g} & M' \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ N & \xrightarrow{f} & N' \end{array}$$

where g is f-equivariant, that is $g(x^\alpha) = (gx)^{f(\alpha)}$. This is a weak equivalence if (f, g) induces isomorphisms $\pi_2(\partial) \cong \pi_i(\partial')$ for $i = 1, 2$ where $\pi_1(\partial) = \text{cokernel}(\partial)$ and $\pi_2(\partial) = \text{kernel}(\partial)$. For a crossed module ∂ the group $\pi_2(\partial)$ is abelian and central in M and $\pi_1(\partial)$ acts on $\pi_2(\partial)$ by $x^{\{\alpha\}} = x^\alpha$ for $x \in \pi_2(\partial)$, $\{\alpha\} \in \pi_1(\partial)$. Let $\underline{\underline{cross}}$ be the category of crossed modules and let $\underline{\underline{abcross}}$ be the full subcategory of all crossed modules ∂ for which $\pi_1(\partial)$ is abelian and acts trivially on $\pi_2(\partial)$.

(2.4) Definition. Let $\partial \in \underline{\underline{abcross}}$ and let $f : \pi_1 \to \pi_1(\partial)$ and $g : \pi_2(\partial) \to \pi_2$ be homomorphisms in $\underline{\underline{Ab}}$. Then we define $f^*(\partial)$, $g_*(\partial) \in \underline{\underline{abcross}}$ by the following commutative diagram

$$\begin{array}{ccccccc} \pi_2(f^*\partial) & \longrightarrow & M & \xrightarrow{f^*(\partial)} & N' & \longrightarrow & \pi_1 \\ \| & & \| & & \downarrow \bar{f} & & \downarrow f \\ \pi_2(\partial) & \longrightarrow & M & \xrightarrow{\partial} & N & \longrightarrow & \pi_1(\partial) \\ g\downarrow & & \downarrow \bar{g} & & \| & & \| \\ \pi_2 & \longrightarrow & M' & \xrightarrow[g_*(\partial)]{} & N & \longrightarrow & \pi_1(g_*\partial) \end{array} \tag{1}$$

Here $(g, \bar{g})$ is a central push out diagram, that is $M' = \pi_2 x M/\sim$ where $(x + g(a), y) \sim (x, a + y)$ for $x \in \pi_2$, $a \in \pi_2(\partial)$, $y \in M$. The action of N on M' is given by $(x, y)^\alpha = (x, y^\alpha)$. Moreover $(\bar{f}, f)$ is a pull back diagram in $\underline{\underline{Gr}}$ and the action of $(\alpha, \beta) \in N'$ on M is defined by $y^{(\alpha,\beta)} = y^\alpha$, $\alpha \in N$, $\beta \in \pi_1$. Using the product of groups one gets for ∂, $\partial' \in \underline{\underline{abcross}}$ the object $\partial \times \partial' \in \underline{\underline{abcross}}$,

$$\begin{array}{ccccccc} \pi_2(\partial) \times \pi_2(\partial') & \longrightarrow & M \times M' & \xrightarrow{\partial \times \partial'} & N \times N' & \longrightarrow & \pi_1(\partial) \times \pi_1(\partial') \\ \| & & & & & & \| \\ \pi_2(\partial \times \partial') & & & & & & \pi_1(\partial \times \partial') \end{array}$$

with the action of $(\alpha, \beta) \in N \times N'$ on $(x, y) \in M \times M'$ given by $(x, y)^{(\alpha,\beta)} = (x^\alpha, y^\beta)$. We say that a subcategory $\underline{\underline{cr}} \subset \underline{\underline{abcross}}$ is <u>additive</u> if for $\partial, \partial' \in \underline{\underline{cr}}$ and maps f, g as above $f^*\partial \to \partial \to g_*\partial \in \underline{\underline{cr}}$ and $\partial \times \partial' \in \underline{\underline{cr}}$. These are the operations used for the definition of the 'Baer-sum' in (2.5) below.

We now describe examples of additive subcategories in $\underline{\underline{abcross}}$. A <u>central map</u> $\partial : M \to N$ is a homomorphism from an abelian group M to the center of a group N. This is the same as a crossed module for which the action of N on M is trivial. Let $\underline{\underline{cent}}$ be the category of central maps ∂ for which $\pi_1(\partial)$ is abelian. This is a full and additive subcategory of $\underline{\underline{abcross}}$. Moreover let $\underline{\underline{Pair}}(\underline{\underline{Ab}})$ be the <u>category of pairs</u> in $\underline{\underline{Ab}}$; objects are homomorphisms in $\underline{\underline{Ab}}$. This is a full and additive subcategory of $\underline{\underline{cent}}$. Further examples of additive subcategories in $\underline{\underline{abcross}}$ are given by the categories $\underline{\underline{rquad}}$ and $\underline{\underline{squad}}$ in (2.10) below.

Let $\underline{\underline{K}}$ be a category. A $\underline{\underline{K}}$ -<u>module</u> A is a functor $A : \underline{\underline{K}} \to \underline{\underline{Ab}}$. Morphisms between $\underline{\underline{K}}$ -modules are natural transformations. Let $\underline{\underline{Ab}}^{\underline{\underline{K}}}$ be the category of $\underline{\underline{K}}$ -modules.

(2.5) <u>*Definition*</u>. Let A, B be $\underline{\underline{K}}$ -modules and let $\underline{\underline{cr}}$ be an additive subcategory of $\underline{\underline{abcross}}$. We consider <u>extensions</u> δ in $\underline{\underline{cr}}$ which are natural exact sequences of groups

$$0 \to B(X) \to M(X) \xrightarrow{\delta(X)} N(X) \to A(X) \to 0 \tag{1}$$

where $\delta : \underline{\underline{K}} \to \underline{\underline{cr}}$ is a functor, $X \in \underline{\underline{K}}$. Here we have $A(X) = \pi_1\delta(X)$ and $B(X) = \pi_2\delta(X)$. An equivalence relation for such extensions is generated by the relation that $\delta \sim \partial'$ if there is a diagram

$$
(2) \qquad \begin{array}{ccccccccccc}
0 & \longrightarrow & B(X) & \longrightarrow & M(X) & \xrightarrow{\delta} & N(X) & \longrightarrow & A(X) & \longrightarrow & 0 \\
 & & \| & & \downarrow m & & \downarrow n & & \| & & \\
0 & \longrightarrow & B(X) & \longrightarrow & M'(X) & \xrightarrow{\delta'} & N'(X) & \longrightarrow & A(X) & \longrightarrow & 0
\end{array}
$$

which is natural in $X \in \underline{\underline{K}}$ where $(m, n) : \delta \to \delta'$ is a natural transformation in $\underline{\underline{cr}}$. Let

$$(3) \qquad Ext^2_{\underline{\underline{K}}}(A, B, \underline{\underline{cr}})$$

be the set of equivalence classes of such extensions (in general this is only a set in a suitable universe, compare the remark at the end of III.§ 5 of Mac Lane [13]). Morphisms $f : A' \to A$, $g : B \to B'$ between $\underline{\underline{K}}$ -modules <u>induce functions</u> f^*, g_* on (3) with $f^*\{\delta\} = \{f^*\delta\}$, $g_*\{\delta\} = \{g_*\delta\}$ where we apply (2.4) (1). We define the sum of equivalence classes $\{\delta\} + \{\delta'\} = \{\delta + \delta'\}$ by the <u>Baer sum</u>

$$(4) \qquad \delta + \delta' = (\nabla_B)_* \Delta_A^*(\delta \times \delta')$$

where $\nabla_B : B \oplus B \to B$ and $\Delta_A : A \to A \oplus A$ are the folding map and the diagonal respectively, $\nabla_B(b_0, b_1) = b_0 + b_1$, $\Delta_A(a) = (a, a)$. For the definition of $\delta + \delta'$ we use the additive structure of $\underline{\underline{cr}}$ in (2.4). A functor $\varphi : \underline{\underline{C}} \to \underline{\underline{K}}$ induces a homomorphism

$$(5) \qquad \varphi^* : Ext_{\underline{\underline{K}}}(A, B, \underline{\underline{cr}}) \to Ext_{\underline{\underline{C}}}(A\varphi, B\varphi, \underline{\underline{cr}})$$

and an inclusion $\psi : \underline{\underline{cr}} \subset \underline{\underline{cr}}'$ of additive subcategories induces a homomorphism

$$(6) \qquad \psi_* : \mathrm{Ext}_{\underline{\underline{K}}}(A, B, \underline{\underline{cr}}) \to \mathrm{Ext}_{\underline{\underline{K}}}(A, B, \underline{\underline{cr}}')$$

where $\varphi^*\{\delta\} = \{\delta\varphi\}$ and $\psi_*\{\delta\} = \{\psi\delta\}$.

In the next definition we generalize the concept of functors $\delta : \underline{\underline{K}} \to \underline{\underline{cr}}$ used in the definition of extensions above.

(2.6) *<u>Definition</u>.* A <u>pseudo functor</u>

$$(\delta, \theta) : \underline{\underline{K}} \to \underline{\underline{cr}}$$

carries each object X in $\underline{\underline{K}}$ to a crossed module $\delta_X = \delta : M(X) \to N(X) \in \underline{\underline{cr}}$ and carries each morphism $a : Y \to X$ in $\underline{\underline{K}}$ to a commutative diagram

$$(1)\qquad \begin{array}{ccc} M(Y) & \xrightarrow{\delta} & N(Y) \\ \downarrow{\scriptstyle a_*} & & \downarrow{\scriptstyle a_\sharp} \\ M(X) & \xrightarrow{\delta} & N(X) \end{array}$$

which is a morphism $(a_*, a_\sharp) : \delta_Y \to \delta_X$ in $\underline{\underline{cr}}$. Here M is a functor in $X \in \underline{\underline{K}}$ which induced a_*; but N is not a functor. The induced maps $a_\sharp$ satisfy for a composition $ab : Z \to Y \to X \in \underline{\underline{K}}$ the formula

$$(2)\qquad a_\sharp b_\sharp = (ab)_\sharp + \delta_X \theta(a,b) p_Z$$

where $p_Z : N(Z) \to \pi_1\delta(Z)$ is the quotient map and where $\theta(a,b) : \pi_1(\delta_Z) \to M(X)$ is an $(ab)_\sharp$ -crossed homomorphism and a central map satisfying the 2-cocycle condition $\partial(\theta) = 0$, see (1.7). That is, for $abc : W \to Z \to Y \to X \in \underline{\underline{K}}$ we have the equation

$$(3)\qquad 0 = a_*\theta(b,c) - \theta(ab,c) + \theta(a,bc) - \theta(a,b)c_*$$

where $a_* = M(a)$ is induced by M and where $c_* : \pi_1(\delta_W) \to \pi_1(\delta_Z)$ is induced by $c_\sharp$.

A <u>natural transformation</u> $(m, n, \varphi) : (\delta, \theta) \to (\delta', \theta')$ between pseudo functors carries each object X to a morphism $(m_X, n_X) : \delta_X \to \delta'_X$,

$$(4)\qquad \begin{array}{ccc} M(X) & \xrightarrow{\delta} & N(X) \\ \downarrow{\scriptstyle m_X} & & \downarrow{\scriptstyle n_X} \\ M'(X) & \xrightarrow{\delta'} & N'(X) \end{array}$$

in $\underline{\underline{cr}}$. Here m is a natural transformation $M \to M'$ between functors; but n satisfies for each $a : Y \to X \in \underline{\underline{K}}$ the equation

$$(5)\qquad n_X a_\sharp = a_\sharp n_Y + \delta'_X \varphi(a) p_Y$$

where $\varphi(a) : \pi_1(\delta_Y) \to M'(X)$ is an $a_\sharp n_Y$ -crossed homomoprhism and a central map satisfying the 1-cocycle condition $\partial\varphi = 0$, see (1.7). That is, for $ab : Z \to Y \to X \in K$ we have the equation

$$(6)\qquad \varphi(ab) = a_*\varphi(b) + \varphi(a)b_*$$

where $a_* = M(a)$ and where $b_* : \pi_1(\delta_z) \to \pi_1(\delta_4)$ is induced by $b_\sharp$. Moreover φ and θ, θ' satisfy the following compatibility relation

(7) $$0 = \theta'(a,b)(n_Z)_* + a_*\varphi(b) + \varphi(a)b_* - m_X\theta(a,b)$$

where $(n_Z)_* = \pi_1(n_Z)$, $a_* = M'(a)$, $b_* = \pi_1(b_\sharp)$. Clearly pseudo functors with $\theta = 0$ and natural transformations with $\varphi = 0$ are the same as functors and natural transformations between functors respectively.

(2.7) Definition. Let A, B be $\underline{\underline{K}}$ -modules. We call an exact sequence (2.5) (1) a pseudo extension in $\underline{\underline{cr}}$ if δ is given by a pseudo functor $(\delta, \theta) : \underline{\underline{K}} \to \underline{\underline{cr}}$. Equivalences between such pseudo extensions are defined by natural transformatioons between pseudo functors as in (2.5) (2). Let

(1) $$\mathrm{Pext}^2_{\underline{\underline{K}}}(A, B, \underline{\underline{cr}})$$

be the set of equivalence classes of pseudo extensions. Induced maps f^*, g_* for these sets are defined as in (2.5) and one obtains the Baer sum of pseudo extensions similarly as in (2.5) (4). Moreover functors φ, ψ induce φ^*, ψ_* as in (2.5) (5), (6). There is a natural transformation

(2) $$\phi : \mathrm{Ext}^2_{\underline{\underline{K}}}(A, B, \underline{\underline{cr}}) \to \mathrm{Pext}^2_{\underline{\underline{K}}}(A, B, \underline{\underline{cr}})$$

which carries $\{\delta\}$ to $\{\delta\}$.

(2.8) Proposition. *Via the Baer sum the sets* $Ext^2_{\underline{\underline{K}}}(A, B, \underline{\underline{cr}})$ *and* $Pext^2_{\underline{\underline{K}}}(A, B, \underline{\underline{cr}})$ *are abelian groups. Via induced maps* f^*, g_* *they yield functors*

$$(\underline{\underline{Ab}}^{\underline{\underline{K}}})^{op} \times \underline{\underline{Ab}}^{\underline{\underline{K}}} \to \underline{\underline{Ab}}$$

which are additive in the second variable B. *Moreover* φ^*, ψ_*, ϕ *are natural transformations in* $\underline{\underline{Ab}}$.

(2.9) Examples. The category $\underline{\underline{Ab}}^{\underline{\underline{K}}}$ is an abelian category so that $\mathrm{Ext}^2(A, B)$ is defined. It is clear that

$$\mathrm{Ext}^2_{\underline{\underline{K}}}(A, B, \underline{\underline{Pair}}(\underline{\underline{Ab}})) = \mathrm{Ext}^2(A, B)$$

Let $\underline{\underline{1}}$ be the trivial category consisting of one object and one morphism. Then

$$\mathrm{Ext}^2_{\underline{\underline{1}}}(A, B, \underline{\underline{abcross}}) = H^3(A, B)$$

where the right hand side is the cohomology of the abelian group A with coefficients in the abelian group B. For this compare for example [10]. More generally $Ext^2_{\underline{\underline{K}}}(A, B, \underline{\underline{abcross}})$ is a special case of a cohomology considered for example in [14], [15].

We also shall use the following examples of additive subcategories of $\underline{\underline{abcross}}$; compare [3].

(2.10) Definition. We define faithful functors

$$(1) \qquad \underline{\underline{squad}} \subset \underline{\underline{rquad}} \xrightarrow{\delta} \underline{\underline{abcross}}$$

as follows. An object $(\omega, \delta) \in \underline{\underline{rquad}}$ is called a reduced quadratic module; this is a crossed module $\delta : L \to M$ together with a 'quadratic map' $\omega : M^{ab} \otimes M^{ab} \to L$ such that the following properties are satisfied. Triple commutators in M are trivial and the quotient map $M \twoheadrightarrow M^{ab}$ to the abelianization M^{ab} of M is denoted by $x \longmapsto \{x\}$. The map ω is a homomorphism in $\underline{\underline{Gr}}$ with

$$(2) \qquad \begin{cases} a^x = a \cdot \omega(\{\delta a\} \otimes \{x\}) \\ \delta\omega(\{x\} \otimes \{y\}) = x^{-1}y^{-1}xy \\ \omega(\{\delta a\} \otimes \{x\} + \{x\} \otimes \{\delta a\}) = 0 \\ \omega(\{\delta a\} \otimes \{\delta b\}) = a^{-1}b^{-1}ab \end{cases}$$

for $x, y \in M$, $a, b \in L$. We say that (ω, δ) is stable if in addition

$$(3) \qquad \omega(\{x\} \otimes \{u\} + \{y\} \otimes \{x\}) = 0,$$

then (ω, δ) is an object in $\underline{\underline{squad}}$. A morphism $(l, m) : (\omega, \delta) \to (\omega', \delta')$ in $\underline{\underline{rquad}}$ or $\underline{\underline{squad}}$ is a morphism $(l, m) : \delta \to \delta'$ between crossed modules compatible with ω, ω', that is, $l\omega = \omega'(m^{ab} \otimes m^{ab})$. We obtain the additive structure of $\underline{\underline{rquad}}$ and $\underline{\underline{squad}}$ by $f^*(\omega, \delta) = (\omega, f^*\delta)$, $g_*(\omega, \delta) = (\bar{g}\omega, g_*\delta)$ where $(\bar{g}, 1) : \delta \to g_*\delta$ is the map in (2.4) (1). Moreover $(\omega, \delta) \times (\omega', \delta')) = (\bar{\omega}, \delta \times \delta')$ where

$$(4) \qquad \bar{\omega} : (M \times M')^{ab} \otimes (M \times M')^{ab} \xrightarrow{p} M^{ab} \otimes M^{ab} \times M'^{\mathrm{ab}} \otimes M'^{\mathrm{ab}} \to L \times L'$$

is the composition of the obvious quotient map p and the product $\omega \times \omega'$. The functor δ in (1) which carries (ω, δ) to δ is clearly compatible with the additive structures.

§ 3 Categories associated to extensions of $\underline{\underline{K}}$ -modules

We show that there is a natural transformation mapping the extension groups in § 2 to the cohomology of a certain category.

A group G has nilpotency degree n if all $(n+1)$ -fold iterated commutators in G are trivial. Let $\underline{\underline{Nil}}$ be the full subcategory of $\underline{\underline{Gr}}$ consisting of groups of nilpotency degree 2. Given a free abelian group A we obtain quotient maps

$$(3.1) \qquad G_A \twoheadrightarrow E_A \twoheadrightarrow A$$

Here G_A is a free group with abelianization A and E_A is the quotient group $G_A/\Gamma_3 G_A$ where $\Gamma_3 G_A$ is the subgroup of triple commutators in G_A. Let $\underline{\underline{ab}} \subset \underline{\underline{Ab}}$ be the full subcategory of free abelian groups A and let $\underline{\underline{nil}} \subset \underline{\underline{Nil}}$ and $\underline{\underline{gr}} \subset \underline{\underline{Gr}}$ be the full subcategories of groups E_A and G_A respectively with $A \in \underline{\underline{ab}}$. Then the quotient maps (3.1) yield the full functors

$$(3.2) \qquad \underline{\underline{gr}} \to \underline{\underline{nil}} \to \underline{\underline{ab}}$$

which carry G_A to E_A and E_A to A.

(3.3) *Definition.* Let $\underline{\underline{K}}$ be a category and let Γ be a $\underline{\underline{K}}$ -module. We define the functors

$$(1) \qquad \underline{\underline{gr}}(\Gamma, \underline{\underline{K}}) \to \underline{\underline{nil}}(\Gamma, \underline{\underline{K}}) \to \underline{\underline{ab}}(\Gamma, \underline{\underline{K}})$$

as follows. The objects in each of these categories are triple (X, A, a) with $X \in \underline{\underline{K}}$, $A \in \underline{\underline{ab}}$ and $a \in Hom(A, \Gamma(X))$. Morphisms are pairs

$$(2) \qquad (\xi, \eta) : (B, Y, b) \to (A, X, a)$$

where $\eta : Y \to X \in \underline{\underline{K}}$ and where ξ is a morphism $G_B \to G_A \in \underline{\underline{gr}}$, or $E_B \to E_A \in \underline{\underline{nil}}$ or $B \to A \in \underline{\underline{ab}}$ respectively such that the diagram

$$(3) \qquad \begin{array}{ccc} A & \xrightarrow{\xi_*} & B \\ a\big\downarrow & & \big\downarrow b \\ \Gamma(X) & \xrightarrow[\eta_*]{} & \Gamma(Y) \end{array}$$

commutes. The functors in (1) are the identity on objects and on morphisms they are defined by the functors in (3.2).

Now let $\underline{\underline{cr}}$ be an additive subcategory of $\underline{\underline{abcross}}$. We say that $\underline{\underline{cr}}$ is of $\underline{\underline{Ab}}$ -type, resp. of $\underline{\underline{Nil}}$ -type, if for all $\delta : M \to N \in \underline{\underline{cr}}$ we have $N \in \underline{\underline{Ab}}$ resp. $N \in \underline{\underline{Nil}}$. For example $\underline{\underline{cent}}$, $\underline{\underline{rquad}}$ and $\underline{\underline{squad}}$ are of $\underline{\underline{Nil}}$ -type.

(3.4) Definition. Let Γ and Λ be $\underline{\underline{K}}$ -modules and consider the following diagram with $(A, X, a) \in \underline{\underline{ab}}(\Gamma, \underline{\underline{K}})$.

(1)
$$\begin{array}{ccccccccc} & & & & \tilde{A} & \longrightarrow & A & & \\ & & & & \big\downarrow \tilde{a} & & \big\downarrow a & & \\ 0 \longrightarrow & \Lambda(X) & \longrightarrow & M(X) & \xrightarrow{(\delta,\theta)} & N(X) & \longrightarrow & \Gamma(X) & \longrightarrow 0 \end{array}$$

where the bottom row is a pseudo extension in $\underline{\underline{cr}}$. We set

(2)
$$\tilde{A} = \begin{cases} A & \text{if } \underline{\underline{cr}} \text{ is of } \underline{\underline{Ab}}\text{-type} \\ E_A & \text{if } \underline{\underline{cr}} \text{ is of } \underline{\underline{Nil}}\text{-type} \\ G_A & \text{otherwise} \end{cases}$$

Hence we can choose a homomorphism $\tilde{a}$ such that the diagram commutes. Using these data we define a <u>linear extension of categories</u>

(3)
$$Hom(-, \Lambda) \overset{+}{\rightarrowtail} \underline{\underline{T}}(\delta, \theta) \twoheadrightarrow \begin{cases} \underline{\underline{ab}}(\Gamma, \underline{\underline{K}}) & \text{if } \underline{\underline{cr}} \text{ is of } \underline{\underline{Ab}}\text{-type} \\ \underline{\underline{nil}}(\Gamma, \underline{\underline{K}}) & \text{if } \underline{\underline{cr}} \text{ is of } \underline{\underline{Nil}}\text{-type} \\ \underline{\underline{gr}}(\Gamma, \underline{\underline{K}}) & \text{otherwise} \end{cases}$$

as follows. The natural system $Hom(-, \Lambda)$ is the bifunctor which carries the pair of objects $((B, Y, b), (A, X, a))$ to the abelian group $Hom(B, \Lambda(X))$; induced maps for this bifunctor are defined by (3.2) in the obvious way.

The objects in $\underline{\underline{T}}(\delta, \theta)$ are the same as $\underline{\underline{ab}}(\Gamma, \underline{\underline{K}})$. A morphism

(3)
$$(\xi, \eta, H) : (B, Y, b) \to (A, X, a)$$

in $\underline{\underline{T}}(\delta, \theta)$ is given by a morphism (ξ, η) in $\underline{\underline{gr}}(\Gamma, \underline{\underline{K}})$, $\underline{\underline{nil}}(\Gamma, \underline{\underline{K}})$, and $\underline{\underline{ab}}(\Gamma, \underline{\underline{K}})$ respectively and by a function

$$H : \tilde{B} \to M(X)$$

which is a $(\eta_\sharp \tilde{b})$ -crossed homomorphism. Here $\tilde{B}$ is given by B as in (2). Moreover H satisfies

(4)
$$\delta_X H(e) = -\eta_\sharp \tilde{b} + \tilde{a}\,\xi$$

in $N(X)$. The composition of morphisms in $\underline{\underline{T}}(\delta,\theta)$ is defined by

$$(\xi,\eta,H)(\xi',\eta',H') = (\xi\xi',\eta\eta',H*H') : (Z,C,c) \to (Y,B,b) \to (X,A,a) \quad \text{with}$$

$$H*H' = \theta(\eta,\eta')\tilde{c} + \eta_* H' + H\,\xi' \tag{5}$$

The projection functor p is the identity on objects and carries (ξ,η,H) to (ξ,η). Finally the action of $\alpha \in Hom(B,\Lambda(X))$ is given by

$$(\xi,\eta,H) + \alpha = (\xi,\eta,H+\alpha)$$

with $(H+\alpha)(e) = H(e)\cdot(i_X\alpha p_B(e)) \in M(X)$ where $p_B : \tilde{B} \to B$ is the quotient map and $i_X : \Lambda(X) \subset M(X)$ is the inclusion.

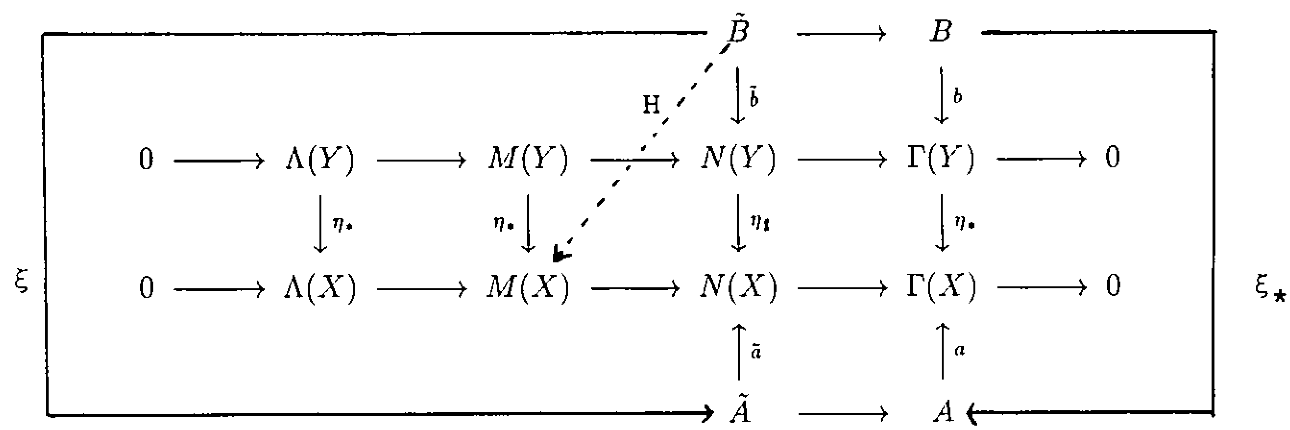
(3.5) Proposition. *The linear extension for the category $\underline{\underline{T}}(\delta,\theta)$ above is well defined and one obtains well defined binatural homomorphisms as in the following commutative diagram*

$$\begin{array}{ccc}
Pext^2_{\underline{\underline{K}}}(\Gamma,\Lambda,\underline{\underline{cr}}) & \xrightarrow{\ \phi\ } & H^2(\underline{\underline{gr}}(\Gamma,\underline{\underline{K}}),Hom(-,\Lambda)) \\
\uparrow \| & & \uparrow \\
Pext^2_{\underline{\underline{K}}}(\Gamma,\Lambda,\underline{\underline{cr}}) & \xrightarrow{\ \phi_1\ } & H^2(\underline{\underline{nil}}(\Gamma,\underline{\underline{K}}),Hom(-,\Lambda)) \\
\uparrow \| & & \uparrow \\
Pext^2_{\underline{\underline{K}}}(\Gamma,\Lambda,\underline{\underline{cr}}) & \xrightarrow{\ \phi_2\ } & H^2(\underline{\underline{ab}}(\Gamma,\underline{\underline{K}}),Hom(-,\Lambda))
\end{array}$$

Here ϕ_1 is defined if $\underline{\underline{cr}}$ is of $\underline{\underline{Nil}}$ -type and ϕ_2 is defined if $\underline{\underline{cr}}$ is of $\underline{\underline{Ab}}$ -type.

The homomorphisms carry the equivalence class of the extension (δ,θ) to the equivalence class of the linear extension $\underline{\underline{T}}(\delta,\theta)$; see (1.5). The right hand side of the diagram is induced by the functors in (3.3) (1). The cohomology groups in the diagram are also additive functors in $\Lambda \in \underline{\underline{Ab}}^{\underline{\underline{K}}}$.

Proof of (3.5). We obtain for the morphism (ξ,η,H) the following diagram



with $\delta_X H = -\tilde{a}\xi + \tilde{b}\eta_\sharp$; see (3.4) (4). If $\bar{H}$ is given with $\delta_X \bar{H} = -\tilde{a}\xi + \tilde{b}\eta_\sharp$ then the exactness of the rows shows that there is a unique homomorphism α with $\bar{H} = H + \alpha$. Now (3.4) (5) shows that this action of α satisfies the linear distributivity law. Moreover the 2-cocycle condition for θ shows that the composition (3.4) (5) is associative, here we use also the assumption that θ is central. Now given a natural transformation

$$(m, n, \varphi) : (\delta, \theta) \to (\delta', \theta')$$

we obtain an induced equivalence of linear extensions

(1) $$(m, n, \varphi)_* : \underline{\underline{T}}(\delta, \theta) \to \underline{\underline{T}}(\delta', \theta')$$

which is the identity on objects and which carries (ξ, η, H) to (ξ, η, H_φ) with

(2) $$H_\varphi = \varphi(\eta)\tilde{b} + m_X H$$

Here we choose the lift $\tilde{\tilde{a}} = n_X \tilde{a}$ for a and $\tilde{\tilde{b}} = n_Y \tilde{b}$ for b. We point out that different choices of lifts $\tilde{a}$ for a yield equivalent categories. In fact, let $\tilde{\tilde{a}}$, $\tilde{a}$ be both lifts of a as in (3.4) (1). Since $\tilde{A}$ has a freeness property there exists a crossed homomorphism $H_a : \tilde{A} \to M(X)$ with $\delta_X H_a = -\tilde{a} + \tilde{\tilde{a}}$. The equivalence then carries (ξ, η, H) to $(\xi, \eta, \bar{H})$ with

(3) $$\bar{H} = -\eta_* H_b + H + H_a \xi.$$

This completes the proof that the functions in (3.5) are well defined. The additivity in Λ at both sides shows that the functions are homomorphisms. Naturality is also easy to check.

q.e.d.

§ 4 Extension for spherical modules and homotopies

For a pointed topological space X let $H_n X$ and $\pi_n X$ be the n-th homology group and homotopy group of X respectively. For a free abelian group A we choose a one point union of 1-spheres

$$M_A = \bigvee_Z S^1$$

such that $H_1 M_A = A$ and $\pi_1 M_A = G_A$. The $(n-1)$ -fold suspension of M_A,

(4.1) $$M(A,n) = \Sigma^{n-1} M_A = \bigvee_Z S^n,$$

is a 'Moore space' of A. Let $[X,Y]$ be the set of homotopy classes of basepoint preserving maps $X \to Y$; this is the set of morphisms in the homotopy category $\underline{\underline{Top}}^*/\simeq$. The homology functor H_n yields an identification, $n \geq 2$,

(1) $$Hom(A,B) = [M(A,n), M(B,n)]$$

so that we get a full and faithful functor $\underline{\underline{ab}} \to \underline{\underline{Top}}^*/\simeq$ which carries A to $M(A,n)$. Moreover we have by use of π_n, $n \geq 2$,

(2) $$Hom(A, \pi_n X) = [M(A,n), X]$$

(4.2) Definition. For $n \geq 2$ and $k \geq 0$ let

$$\Gamma_n^k : \underline{\underline{ab}} \xrightarrow{M(-,n)} \underline{\underline{Top}}^*/\simeq \xrightarrow{\pi_n} \underline{\underline{Ab}}$$

be the functor which carries the free abelian group A to the homotopy group $\pi_{n+k} M(A,n)$. We call Γ_n^k a <u>spherical</u> $\underline{\underline{ab}}$ -<u>module</u>. For $k = 0$ we clearly have $\Gamma_n^0(A) = A$.

(4.3) Remark. The spherical $\underline{\underline{ab}}$ -modules can be described algebraically only in terms of homotopy groups of spheres $\pi_m S^n$ and primary operations; compare [8]. For example in the stable range $k < n-1$ we get $\Gamma_n^k(A) = A \otimes \pi_{n+k}(S^n)$. In the metastable range $k < 2n-2$ we have $\Gamma_n^k(A) = A \otimes \pi_{n+k}\{S^n\}$. Here the right hand side is the quadratic tensor product in [4] and $\pi_{n+k}\{S^n\}$ is the quadratic $\mathbb{Z}$ -module

$$\pi_{n+k}\{S^n\} = \left(\pi_{n+k} S^n \xrightarrow{H} \pi_{n+k} S^{2n-1} \xrightarrow{P} \pi_{n+k} S^n\right)$$

given by the Hopf invariant H and the map P induced by the Whitehead product $[i_n, i_n] : S^{2n-1} \to S^n$. As a special case we obtain

$$\pi_3\{S^2\} = (\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z})$$

so that $\Gamma_2^1(A) = A \otimes \pi_3\{S^2\}$ is Whitehead's quadratic functor. Moreover $\Gamma_n^1(A) = A \otimes \mathbb{Z}/2$ for $n \geq 3$. Further examples are described in Table 2 of [4].

(4.4) *Notation*. Given an $\underline{\underline{ab}}$ -module Γ (like for example $\Gamma = \Gamma_n^k$) we obtain the $\underline{\underline{nil}}$ -module and the $\underline{\underline{gr}}$ -module

(1) $$\begin{cases} \underline{\underline{nil}} \to \underline{\underline{ab}} \xrightarrow{\Gamma} \underline{\underline{Ab}}, \\ \underline{\underline{gr}} \to \underline{\underline{ab}} \xrightarrow{\Gamma} \underline{\underline{Ab}} \end{cases}$$

where both compositions are also denoted by Γ. There are canonical functors

(2) $$\underline{\underline{gr}}(\Gamma, \underline{\underline{gr}}) \to \underline{\underline{nil}}(\Gamma, \underline{\underline{nil}}) \to \underline{\underline{ab}}(\Gamma, \underline{\underline{ab}})$$

with categories defined as in (3.3). An object (A, X, Γ) in $\underline{\underline{ab}}(\Gamma, \underline{\underline{ab}})$ is given by free abelian groups X, A and a homomorphism $a : A \to \Gamma(X)$. This yields the corresponding objects (A, E_X, a) and (A, G_X, a) which we also denote by (A, X, a) so that the functors in (2) are the identity on objects. On morphisms (ξ, η) the functors in (2) are given by the functors $\underline{\underline{gr}} \to \underline{\underline{nil}}$ and $\underline{\underline{nil}} \to \underline{\underline{ab}}$ respectively.

We now consider homotopies between certain maps. Let $I \subset \mathbb{R}$ be the unit interval and let $IX = I \times X / I \times *$ be the cylinder of a pointed CW-complex X. We have the inclusion and projection

$$X \vee X \xrightarrow{(i_0, i_1)} IX \xrightarrow{pr} X$$

where $i_t(x) = \{t, x\}$ and $pr\{t, x\} = x$ for $t \in I$, $x \in X$. Here (i_0, i_1) is a cofibration in $\underline{\underline{Top}}$. A homotopy between pointed maps $f, g : X \to Y$ is a map $H : IX \to Y$ with $H\, i_0 = f$ and $H\, i_1 = g$. A track $H : f \simeq g$ is a homotopy class relative $X \vee X$ of such homotopies. Let

(4.5) $$T(f, g) = [IX, Y]^{(f,g)}$$

be the set of tracks $f \simeq g$. If $X = \Sigma X'$ is a suspension we have a canonical isomorphism of abelian groups

(1) $$\sigma_f : T(f, f) \cong [\Sigma X, Y]$$

We use σ_f for the definition of the transitive and effective action

(2) $$\begin{cases} T(f, g) \times [\Sigma X, Y] \xrightarrow{+} T(f, g) \\ H + \alpha = H + \sigma_g(\alpha) = \sigma_f(\alpha) + H \end{cases}$$

Here the right hand side is defined by addition of homotopies. The properties of this action are described in VI.3.13 of [3] and in [1].

(4.6) Definition. Let $k \geq 1$, $n \geq 2$. We associate with spherical modules Γ_n^{k-1}, Γ_n^k a linear extension of categories $\underline{\underline{H}}(n, n+k)$ which we call a track extension:

$$Hom(-, \Gamma_n^k) \overset{+}{\rightarrowtail} \underline{\underline{H}}(n, n+k) \overset{p}{\twoheadrightarrow} \underline{\underline{gr}}(\Gamma_n^{k-1}, \underline{\underline{gr}}) \tag{1}$$

The objects in $\underline{\underline{H}}(n, n+k)$ are the same as in $\underline{\underline{ab}}(\Gamma_n^{k-1}, \underline{\underline{ab}})$ and the functor p is the identity on objects. A morphism

$$(\xi, \eta, H) : (B, Y, b) \to (A, X, a) \tag{2}$$

in $\underline{\underline{H}}(n, n+k)$ is obtained by a morphism (ξ, η) in $\underline{\underline{gr}}(\Gamma_n^{k-1}, \underline{\underline{gr}})$ and the functor p carries (ξ, η, H) to (ξ, η). Here H is a track as in diagram (4) below. For each object (X, A, a) we choose a map

$$\tilde{a} : M(A, m) \to M(X, n) \tag{3}$$

representing the homotopy class a with $m = n+k-1$. Compare (4.1) (2). Moreover we choose a diagram in $\underline{\underline{Top}}^*$

$$\begin{array}{ccc} M(B,m) & \xrightarrow{\Sigma^{m-1} t\xi} & M(A,m) \\ {\scriptstyle \tilde{b}}\downarrow & \overset{H}{\Longrightarrow} & \downarrow{\scriptstyle \tilde{a}} \\ M(Y,n) & \xrightarrow[\Sigma^{n-1} t\eta]{} & M(X,n) \end{array} \tag{4}$$

where $t\xi : M_B \to M_A$, $t\eta; M_Y \to M_X$ are maps which induce $\xi = \pi_1 t\xi$ and $\eta = \pi_1 t\eta$; for this recall that $G_A = \pi_1 M_A$. The diagram is homotopy commutative since $\eta_* b = a\xi_*$ so that there exists a track H. The action of $\alpha \in Hom(B, \Gamma_n^k X)$ in (1) is defined by

$$(\xi, \eta, H) + \alpha = (\xi, \eta, H + \alpha) \tag{5}$$

where $H + \alpha$ is given by (4.5) (2). Finally composition of morphism as in (2) is obtained by pasting the tracks in the following diagram.

$$\begin{array}{ccccc} & & \overset{t(\xi\xi')_*}{\frown} \;\; \Uparrow O & & \\ M(C,m) & \xrightarrow{(t\xi')_*} & M(B,m) & \xrightarrow{(t\xi)_*} & M(A,m) \\ {\scriptstyle \tilde{c}}\downarrow & \overset{H'}{\Longrightarrow} & \downarrow & \overset{H}{\Longrightarrow} & \downarrow \\ M(Z,n) & \xrightarrow[(t\eta')_*]{} & M(Y,n) & \xrightarrow[(t\eta)_*]{} & M(X,n) \\ & & \underset{t(\eta\eta')_*}{\smile} \;\; \Downarrow O & & \end{array} \tag{6}$$

Here the canonical tracks $\mathcal{O}$ are suspensions of the unique tracks $(t\xi)(t\xi') \simeq t(\xi\xi')$ and $(t\eta)(t\eta') \simeq t(\eta\eta')$. One readily checks that $\underline{\underline{H}}(n, n+k)$ is a well defined category and that (1) is a well defined linear extension.

The next result yields an algebraic description of the topological track extension $H(n, n+k)$.

(4.7) Theorem. *Let $n \geq 2$, $k \geq 1$. Then there exists an extension*

$$(*) \qquad \{\delta_n^k\} \in Ext^2_{\underline{\underline{gr}}}(\Gamma_n^{k-1}, \Gamma_n^k, \underline{\underline{abcross}})$$

of spherical modules such that the linear extensions

$$\underline{\underline{H}}(n, n+k) \sim \underline{\underline{T}}(\delta_n^k)$$

are equivalent. Moreover we may replace $\underline{\underline{abcross}}$ in (*) by $\underline{\underline{rquad}}$ for $n + k = 3$ and by $\underline{\underline{squad}}$ for $n + k > 3$.

Here $\underline{\underline{T}}(\delta_n^k)$ is defined as in (3.4) with $\theta = 0$. Hence for $N + k > 3$ the cohomology class $\{\underline{\underline{H}}(n, n+k)\}$ of the track extension is in the image of

$$Ext^2_{\underline{\underline{gr}}}(\Gamma_n^{k-1}, \Gamma_n^k, \underline{\underline{squad}}) \to H^2(\underline{\underline{nil}}(\Gamma_n^{k-1}, \underline{\underline{gr}}), Hom(-, \Gamma_n^k)) \to H^2(\underline{\underline{gr}}(\Gamma_n^{k-1}, \underline{\underline{gr}}), Hom(-, \Gamma_n^k))$$

where we use the homomorphisms in (3.5).

(4.9) Remark. For a free abelian group A with basis Z let G_A be the free group with basis Z and let $G(A, n)$ be the free simplicial group generated by the set Z in degree n. Then $G_A = G(A, n)_n$ and each homomorphism $\xi : G_B \to G_A$ induces a homomorphism of simplicial groups $G(B, n) \to G(A, n)$. One has a natural homotopy equivalence

$$| G(A, n) | \simeq \Omega M(A, n)$$

where the left hand side is the realization of the simplicial group. Let $N = NG(A, n)$ be the Moore chain complex of $G(A, n)$ given by

$$\begin{cases} N_m = \bigcap_{i<m} \text{kernel}\, d_i^* \\ (\partial_m : N_m \to N_{m-1}) = \text{restriction of } \; d_m^* \end{cases}$$

Then ∂_m induces the extension $\tilde{\delta}_n^k = (\partial_m)_*$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_m G(A, n) & \longrightarrow & \text{cokernel}\, \partial_{m+1} & \xrightarrow{\tilde{\delta}_n^k} & \ker \partial_{m-1} & \longrightarrow & \pi_{m-1} G(A, n) & \longrightarrow & 0 \\ & & \| & & & & & & \| & & \\ & & \Gamma_n^k(A) & & & & & & \Gamma_n^{k-1}(A) & & \end{array}$$

where $n+k=m+1$. Here $\tilde{\delta}_n^k$ has the natural structure of a crossed module so that we obtain this way an element

$$\{\tilde{\delta}_n^k\} \in Ext^2_{\underline{\underline{gr}}}(\Gamma_n^{k-1}, \Gamma_n^k, \underline{\underline{abcross}})$$

In fact this class coincides with $\{\delta_n^k\}$ in theorem (4.7); compare [5]. In the proof below we construct δ_n^k by using the 2-type of an iterated loop space.

Proof of (4.7). Let $\underline{\underline{CW}}$ be the category of CW-complexes X with $X^0 = *$ and of cellular maps. The crossed chain complex $\rho(X)$ is given by the boundary maps

$$\ldots \to \pi_3(X^3, X^2) \xrightarrow{d_3} \pi_2(X^2, X^1) \xrightarrow{d_2} \pi_1(X^1)$$

We obtain a functor

$$\lambda : \underline{\underline{CW}} \to \underline{\underline{cross}} \tag{1}$$

which carries X to the crossed module

$$\lambda(X) : \pi_2(X^2, X^1)/\mathrm{image}\, d_3 \to \pi_1(X^1)$$

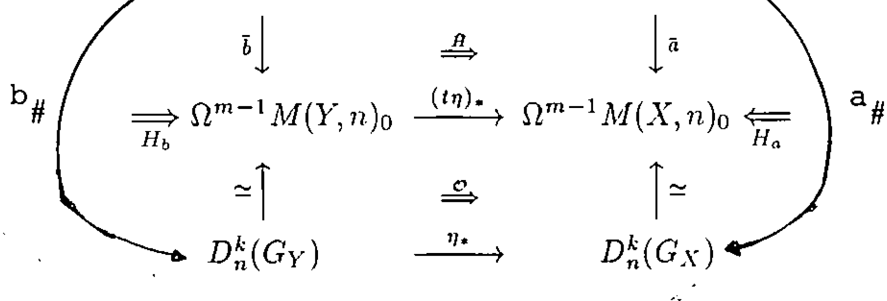
given by the boundary d_2, so that there is a natural quotient map $\rho(X) \to \lambda(X)$; see III. § 2 in [3]. Here $\lambda(X)$ represents the 2-type of X with $\pi_1\lambda(X) = \pi_1 X$ and $\pi_2\lambda(X) = \pi_2 X$. We define the functor

$$\begin{cases} D_k^n : \underline{\underline{gr}} \to \underline{\underline{CW}} \\ D_k^n(G_A) = \mid S\Omega^{m-1}\Sigma^{n-1}BG_A \mid \end{cases} \tag{2}$$

Here BG_A is the classifying space of G_A and $\Omega^{m-1}\Sigma^{n-1}$ denotes the iterated loop space of an iterated suspension, $m = n+k-1$. Moreover SX is the singular set of simplices $\Delta^n \to X$ which carry the 0-skeleton of Δ^n to the basepoint in X and $\mid SX \mid$ is the realization. Then the composition of D_k^n and λ yield a functor ($n \geq 2$)

$$\delta_k^n = \lambda D_k^n : \underline{\underline{gr}} \to \underline{\underline{abcross}} \tag{3}$$

which is an extension of spherical modules since $\pi_1 D_k^n(G_A) = \pi_m \Sigma^{n-1} BG_A = \Gamma_n^{k-1}(A)$ is abelian and acts trivially on $\pi_2 D_k^n(G_A) = \pi_{m+1}\Sigma^{n-1}BG_A = \Gamma_n^k(A)$. We choose for each object (A, X, a) a cellular map $a_\sharp$ and a track H_a as in the following diagram where $\Omega(X)_0$ is the path component of $*$ in the loop space $\Omega(X)$.



Here $\tilde{a}, \bar{H}, \bar{b}$ are adjoints of $\tilde{a}, H, \tilde{b}$ in (4.6) (4). Moreover $\mathcal{O}$ denotes the canonical track induced by the unique track in the diagram

$$(5) \qquad \begin{array}{ccc} M_Y & \xrightarrow{t\eta} & M_X \\ \simeq\uparrow & \Longrightarrow & \uparrow\simeq \\ B(G_Y) & \xrightarrow[\eta_*]{} & B(G_X) \end{array}$$

Let $\tilde{H}$ be the track obtained by pasting the tracks in (4), that is

$$(6) \qquad \tilde{H} : \eta_* b_\sharp \simeq a_\sharp(t\xi)$$

Then $\tilde{H}$ induces a track $\rho\tilde{H} : \rho(\eta_* b_\sharp) \simeq \rho(a_\sharp t\xi)$ in the category of crossed chain complexes and the quotient map $q : \rho(D_n^k X) \to \lambda(D_n^k X)$ yields the track $q\rho\tilde{H}$ in the category of crossed modules. Here $q\rho\tilde{H}$ corresponds to a $(\eta_* b_\sharp)_*$ -crossed homomorphism $\tilde{\tilde{H}}$ (see III.2.6 in [3]) and the equivalence $\underline{\underline{H}}(n, n+k) \to T(\delta_n^k)$ carries (ξ, η, H) to $(\xi, \eta, \tilde{\tilde{H}})$. This completes the proof of (4.7). Using (3.4) in [3] we see that δ_n^k is equivalent to an extension in $\underline{\underline{rquad}}$ for $n + k = 3$ and in $\underline{\underline{squad}}$ for $n + k > 3$.

q.e.d.

§ 5 The extension for the spherical modules Γ_n^0, Γ_n^1

For each free abelian group A we have the exact sequence of groups

$$(5.1) \qquad 0 \to \Gamma A \to \otimes^2 A \xrightarrow{\partial_2^1} E_A \to A \to 0$$

Here ΓA is the subgroup of $\otimes^2 A = A \otimes A$ generated by the element $a \otimes a$, $a \in A$. The map $p : E_A \to A$ is the abelianization in (3.1) and ∂_2^1 is the commutator homomorphism which carries $a \otimes b$ to $x^{-1}y^{-1}xy$ where $x, y \in E_A$ are elements with $px = a$, $py = b$. One readily checks that ∂_2^1 is a central map and that (5.1) is exact. Let $\hat{\otimes}^2 A$ be the quotient of $\otimes^2 A$ by the relations $a \otimes b + b \otimes a \sim 0$. Then one obtains by (5.1) the induced exact sequence, $n \geq 3$,

$$(5.2) \qquad 0 \to A \otimes \mathbb{Z}/2 \to \hat{\otimes}^2 A \xrightarrow{\partial_n^1} E_A \to A \to 0$$

with $\partial_n^1 = \sigma_* \partial_2^1$ where $\sigma : \Gamma A \to A \otimes \mathbb{Z}/2$ carries $a \otimes a$ to $a \otimes 1$. We point out that $(\omega, \partial_2^1) \in \underline{\underline{rquad}}$ where ω is the identity of $\otimes^2 A$ and that $(\omega', \partial_n^1) \in \underline{\underline{squad}}$ where ω' is the quotient map $\otimes^2 A \to \hat{\otimes}^2 A$.

For the spherical $\underline{\underline{nil}}$ -modules Γ_n^0, Γ_n^1 we have $\Gamma_n^0(A) = A$ and $\Gamma_2^1(A) = \Gamma(A)$ and $\Gamma_n^1(A) = A \otimes \mathbb{Z}/2$ for $n \geq 3$ so that ∂_n^1 in (5.1), (5.2) is an extension of $\underline{\underline{nil}}$ -modules in $\underline{\underline{cent}}$, or in $\underline{\underline{rquad}}$ for $n = 2$ and in $\underline{\underline{squad}}$ for $n > 2$.

(5.3) Theorem. *Let $n \geq 2$. Then there is an equivalence of linear extensions*

$$\underline{\underline{H}}(n, n+1) \sim \underline{\underline{T}}(\partial_n^1)$$

Here the right hand side is the pull back of $\underline{\underline{T}}(\partial_n^1)$ via $\underline{\underline{gr}} \to \underline{\underline{nil}}$. Moreover the pull back of the class $\{\partial_n^1\}$ via $\underline{\underline{gr}} \to \underline{\underline{nil}}$ yields $\{\delta_n^1\}$ in (4.7).

The theorem shows that the complicated crossed extension $\{\delta_n^1\}$ in (4.7) can be replaced by the simple central extension $\{\partial_n^1\}$ above. The theorem is proved in VI. § 4 of [3]. Now recall that E_A is a quotient of $G_A = \pi_1(M_A)$.

(5.4) Corollary. *Let $a, b : M_B \to M_A$ be maps which induce the same homomorphism $\pi_1(a)_* = \pi_1(b)_* : E_B \to E_A$. Then there is a canonical track ($n \geq 2$)*

$$\mathcal{O}_{a,b} : \Sigma^{n-1} a \simeq \Sigma^{n-1} b$$

satisfying $\mathcal{O}_{a,b} + \mathcal{O}_{b,c} = \mathcal{O}_{a,c}$, $d_* \mathcal{O}_{a,b} = \mathcal{O}_{da,db}$ and $e^* \mathcal{O}_{a,b} = \mathcal{O}_{ae,be}$ for maps $d : M_A \to M_D$, $e : M_E \to M_B$, $c : M_B \to M_A$.

Proof. Let $\mathcal{O}_{a,b} = \mathcal{O}$ be given by the morphism

$$\begin{array}{ccc} M(B,n) & \xrightarrow{\Sigma^{n-1}a} & M(A,n) \\ {\scriptstyle 1}\downarrow & \overset{\mathcal{O}}{\Longrightarrow} & \downarrow{\scriptstyle 1} \\ M(B,n) & \xrightarrow[\Sigma^{n-1}b]{} & M(A,n) \end{array}$$

in $\underline{\underline{H}}(n, n+1)$ which via the equivalence in (5.3) corresponds to the morphism $(\xi, \xi, 0)$ in $\underline{\underline{T}}(\partial_n^1)$ where $\xi = \pi_1(a)_* = \pi_1(b)_* : E_A \to E_B$.

q.e.d.

We now replace the canonical tracks $\mathcal{O}$ in (4.6) (6) by the canonical tracks defined in (5.4). This leads to the following definition.

(5.5) Definition. Let $k \geq 1, n \geq 2$. We associate with the spherical modules Γ_n^{k-1}, Γ_n^k a linear extension of categories $\underline{\underline{T}}(n, n+k)$ which we call the $\underline{\underline{nil}}$-track extension:

$$(1) \qquad Hom(-, \Gamma_n^k) \stackrel{+}{\rightarrowtail} \underline{\underline{T}}(n, n+k) \stackrel{p}{\twoheadrightarrow} \underline{\underline{nil}}(\Gamma_n^{k-1}, \underline{\underline{nil}})$$

The objects of $\underline{\underline{T}}(n, n+k)$ are the same as in $\underline{\underline{ab}}(\Gamma_n^{k-1}, \underline{\underline{ab}})$ and the functor p is the identity on objects. A morphism

$$(2) \qquad (\xi, \eta, H) : (B, Y, b) \to (A, X, a)$$

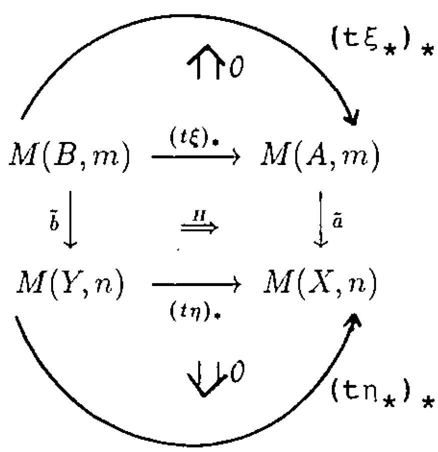
in $\underline{\underline{T}}(n, n+k)$ is obtained by a morphism (ξ, η) in $\underline{\underline{nil}}(\Gamma_n^{k-1}, \underline{\underline{nil}})$ and the functor p carries (ξ, η, H) to (ξ, η). Let $t\xi : M_B \to M_A$, $t\eta : M_Y \to M_X$ be maps which induce $\xi = \pi_1(t\xi)_* : E_B \to E_A$ and $\eta = \pi_1(t\eta)_* : E_Y \to E_X$. Then H in (2) is a track as in (4.6) (4) and composition of morphisms in (2) is defined as in (4.6) (6) where we replace $\mathcal{O}$ by the canonical tracks in (5.4). Then (5.4) shows that $\underline{\underline{T}}(n, n+k)$ is a well defined category. The action of $Hom(-, \Gamma_n^k)$ is defined as in (4.6) (5).

(5.6) Corollary. *Let $n \geq 2$, $k \geq 1$. There is an equivalence of linear extensions*

$$\underline{\underline{H}}(n, n+k) \sim \underline{\underline{T}}(n, n+k)$$

where the right hand side is the pull back of $\underline{\underline{T}}(n, n+k)$ via the functor $\underline{\underline{gr}}(\Gamma_n^{k-1}, \underline{\underline{gr}}) \to \underline{\underline{nil}}(\Gamma_n^{k-1}, \underline{\underline{nil}})$.

Proof. The equivalence carries (ξ, η, H) to $(\xi_*, \eta_*, \bar{H})$ where $\xi_* : E_B \to E_A$, $\eta_* : E_Y \to E_X$ are induced by ξ and η respectively and where $\bar{H}$ is obtained by pasting the tracks in the following diagram where $\mathcal{O}$ is given by (5.4).



q.e.d.

Using (5.6) and (4.7) we see that we have for $n+k>3$, $n\geq 2$ compatible elements in the following groups with $D=Hom(-,\Gamma_n^k)$

$$\begin{array}{ccc} & & \{\underline{\underline{T}}(n,n+k)\}\in H^2(\underline{\underline{nil}}(\Gamma_n^{k-1},\underline{\underline{nil}}),D) \\ & & \downarrow \\ \{\delta_n^k\}\in Ext^2_{\underline{\underline{gr}}}(\Gamma_n^{k-1},\Gamma_n^k,\underline{\underline{squad}}) & \longrightarrow & H^2(\underline{\underline{nil}}(\Gamma_n^{k-1},\underline{\underline{gr}}),D) \\ & & \downarrow \\ & & \{\underline{\underline{H}}(n,n+k)\}\in H^2(\underline{\underline{gr}}(\Gamma_n^{k-1},\underline{\underline{gr}}),D) \end{array}$$

Here the problem arises of constructing the 'common refinement' of the elements $\{\delta_n^k\}$ and $\{\underline{\underline{T}}(n,n+k)\}$. In Baues [6] we consider in detail the case $n=2$, $k=2$ where $\Gamma_2^1(A)=\Gamma(A)$ and $\Gamma_2^2(A)=\Gamma(A)\otimes\mathbb{Z}/2\oplus L(A,1)_3$. Using pseudo extensions as in (2.7) we show:

(5.8) Theorem. *There is an extension*

$$\{\partial_2^2\}\in Pext^2_{\underline{\underline{nil}}}(\Gamma,\Gamma\otimes\mathbb{Z}/2,\underline{\underline{cent}})$$

such that the linear extensions

$$\underline{\underline{T}}(2,4)\sim\underline{\underline{T}}(i_*\partial_2^2)$$

are equivalent. Here $i:\Gamma(A)\otimes\mathbb{Z}/2\subset\Gamma_2^2(A)$ *is the inclusion.*

An explicit formula for ∂_2^2 is given in [6].

Let $n\geq 2$, $k\geq 1$ and let $\underline{\underline{CW}}(n,n+k)$ be the full homotopy category consisting of CW-complexes K with cells only in dimension n and $n+k$. We may assume that K is the mapping cone of a map $\tilde{a}:M(A,n+k-1)\to M(X,n)$ with $A=H_{n+k}K$, $X=H_nK\in\underline{\underline{ab}}$. There is a linear extension of categories

$$D\overset{+}{\rightarrowtail}\underline{\underline{CW}}(n,n+k)\overset{p}{\twoheadrightarrow}\underline{\underline{ab}}(\Gamma_n^{k-1},\underline{\underline{ab}}) \tag{5.9}$$

where p carries K to the object (A,X,a) with a induced by $\tilde{a}$ (this is a special case of the extension $PRIN(\mathfrak{X})$ in V.3.12 and V.7.14 of [2]). The natural system D on a morphism $(\xi,\eta):(B,Y,b)\to(A,X,a)$ is defined by the quotient

$$D(\xi,\eta)=Hom(B,\Gamma_n^kX)/I(b,\eta,a)$$

where the subgroup $I(b,\eta,a)$ can be computed as in V.7.17 of [2]; see also 5.12 [1].

(5.10) Theorem. *There is a commutative diagram of linear extensions:*

$$\begin{array}{ccccc} Hom(-,\Gamma_n^k) & \overset{+}{\longrightarrow} & \underline{\underline{T}}(n,n+k) & \longrightarrow & \underline{\underline{nil}}(\Gamma_n^{k-1},\underline{\underline{nil}}) \\ \downarrow j & & \downarrow\lambda & & \downarrow \\ D & \overset{+}{\longrightarrow} & \underline{\underline{CW}}(n,n+k) & \longrightarrow & \underline{\underline{ab}}(\Gamma_n^{k-1},\underline{\underline{ab}}) \end{array}$$

Here λ carries (ξ, η, H) to the principal map $C(\xi, \eta, H)$ between mapping cones; see V. § 2 in [2]. The functor λ is a quotient functor.

(5.11) Corollary. *There is a natural equivalence relation $\simeq$ on the category $\underline{\underline{T}}(n, n+k)$ so that*

$$\underline{\underline{CW}}(n, n+k) = \underline{\underline{T}}(n, n+k)/\simeq$$

Hence an algebraic model $\underline{\underline{T}}(\partial_n^k) \sim \underline{\underline{T}}(n, n+k)$ as in § 4, § 5 will also lead to an algebraic model for the homotopy category $\underline{\underline{CW}}(n, n+k)$. In [6] we compute this way explicitly $\underline{\underline{CW}}(2, 4)$ by use of ∂_2^2 in (5.8).

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