Formal group laws over Hopf algebras and their application to complex cobordism theory

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Abstract

It is well known that the formal product in complex cobordism theory describes the relation between first Conner-Floyd classes σ_1 of two complex line bundles with the σ_1 of their tensor product. In the present paper we generalize this construction to the case of virtual U-bundles of virtual dimension 1. To describe the relation between first Conner-Floyd classes in this case we need more general concept of a formal group law. In fact, we develop some algebraic technique related to generalized formal group laws with coefficients in a Hopf algebra. The obtained result may be considered as a straightforward solution of the problem, posed by S.P. Novikov: how does one compute the class $\sigma_1(\xi\otimes\eta)$ for two U-bundles (not necessarily U(1))? It is the first generalization of the formal product in complex cobordism theory.

Let $\mathcal{Z}$ (resp. $\widehat{\Xi}$) be the universal line bundle over $\mathbb{C}P^\infty$ (resp. the universal U-bundle over BU). Then $\sigma_1(\widehat{\Xi})=\mathfrak{h}(\sigma_1(\mathcal{Z}))$ for some power series $\mathfrak{h}(u)\in \mathrm{MU}^*(\mathrm{BSU})[[u]]$. Let $\mathfrak{H}(x,y)$ be the power series $\mathfrak{h}(F(\mathfrak{h}^{-1}(x),\mathfrak{h}^{-1}(y)))\in \mathrm{MU}^*(\mathrm{BSU})[[x,y]]$, where $F(u,v)\in \mathrm{MU}^*(\mathrm{pt})[[u,v]]$ is the formal product in complex cobordism theory. Then for any stable equivalence class $\{\xi\}\in \widetilde{\mathrm{K\,SU}}(X)$ of SU-bundles over a finite CW-complex X we can define a formal group law $(\varphi_\xi^*\mathfrak{H})(x,y)$ over the ring $\mathrm{MU}^*(X)$, where $\varphi_\xi\colon X\to \mathrm{BSU}$ is a classifying map for a representative ξ of $\{\xi\}$. Suppose ξ has virtual dimension 1. We have shown that $\sigma_1(\widehat{\xi}_1\otimes\widehat{\xi}_2\otimes\xi^{-1})=(\varphi_\xi^*\mathfrak{H})(\sigma_1(\widehat{\xi}_1),\sigma_1(\widehat{\xi}_2))$ for virtual U-bundles $\widehat{\xi}_1$, $\widehat{\xi}_2$ over X of virtual dimension 1, corresponding to the virtual SU-bundle ξ (i.e. differ from ξ by tensoring on a line bundle). In general the formal group law $(\varphi_\xi^*\mathfrak{H})(x,y)$ has coefficients in $\mathrm{MU}^*(X)$, but not $\mathrm{MU}^*(\mathrm{pt})$. In particular, for any stable equivalence class $\{\xi\}$ we obtain a direct summand subring $\varphi_\xi^*(K)\cong \mathrm{MU}^*(\mathrm{pt})$ in $\mathrm{MU}^*(X)$, generated by the coefficients of $(\varphi_\xi^*\mathfrak{H})(x,y)$.

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Introduction

It is well known that the set $\mathcal{L}in(X)$ of line bundles over a finite CW-complex X is a group with respect to the tensor product of bundles. Moreover, the assignment

$$\zeta \mapsto c_1(\zeta), \quad \mathcal{L}in(X) \to H^2(X;\mathbb{Z})$$

is an isomorphism of groups. The first Conner-Floyd class σ_1 (in other terminology, the first Chern class in complex cobordism theory) also defines an injection

$$\zeta \mapsto \sigma_1(\zeta), \quad \mathcal{L}in(X) \to \mathrm{MU}^2(X),$$

whose image consists of so-called "geometric cobordism classes", but the relation between $\sigma_1(\zeta_1)$, $\sigma_1(\zeta_2)$ and $\sigma_1(\zeta_1 \otimes \zeta_2)$ is more complicated. More precisely, there exists the formal group law ("formal product in complex cobordism theory") $F(u,v) \in \mathrm{MU}^*(\mathrm{pt})[[u,v]]$ such that $\sigma_1(\zeta_1 \otimes \zeta_2) = F(\sigma_1(\zeta_1), \sigma_1(\zeta_2))$ [6]. Moreover, the formal group law $F(u,v)$ is universal [7].

This paper is devoted to a generalization of the formal product in complex cobordism theory to the case of virtual U-bundles of virtual dimension 1. Let us recall that for a finite CW-complex X the tensor product of virtual bundles of virtual dimension 1 induces on $\widetilde{\mathrm{K}}_{\mathbb{C}}(X)$ the structure of a group. We shall denote this group by $\widetilde{\mathrm{K}}_{\mathbb{C}}(X)_\otimes$. We stress that $\widetilde{\mathrm{K}}_{\mathbb{C}}(X)$ consists of stable equivalence classes of virtual bundles over X of virtual dimension 0. Let $\widehat{\xi}$ be a virtual U-bundle over X of virtual dimension 1. By $\{\widehat{\xi}\}$ denote the stable equivalence class in $\widetilde{\mathrm{K}}_{\mathbb{C}}(X)$ of 0-dimensional virtual bundle $\widehat{\xi}-[1]$, where $[1]$ is the trivial line bundle. The tensor product of virtual bundles of virtual dimension 1 induces the following operation $\circ$ on the set $\widetilde{\mathrm{K}}_{\mathbb{C}}(X)$:

$$\{\widehat{\xi}_1\} \circ \{\widehat{\xi}_2\} = \{\widehat{\xi}_1\} + \{\widehat{\xi}_2\} + \{\widehat{\xi}_1\} \cdot \{\widehat{\xi}_2\},$$

where $+$, $\cdot$ are the sum and the product in the ring $\widetilde{\mathrm{K}}_{\mathbb{C}}(X)$ respectively. By $\mathcal{L}in(X)_0$ denote the set $\{\{\zeta\} \mid \zeta \text{ is a geometric line bundle over } X\} \subset \widetilde{\mathrm{K}}_{\mathbb{C}}(X)$ (i.e. every class of $\mathcal{L}in(X)_0$ has a representative of the form $\zeta - [1]$, where ζ is a geometric line bundle over X). Note that $\mathcal{L}in(X)_0$ is a subgroup in $\widetilde{\mathrm{K}}_{\mathbb{C}}(X)_\otimes$.

Let $\widehat{\Xi}_m$ be the universal m-dimensional $\mathrm{U}(m)$-bundle over $\mathrm{BU}(m)$. If $m \leq n$, then $\widehat{\Xi}_n|_{\mathrm{BU}(m)} = \widehat{\Xi}_m \oplus [n-m]$, where $[k]$ is the trivial k-dimensional bundle. Due to $(\widehat{\Xi}_n - [n-1])|_{\mathrm{BU}(m)} = \widehat{\Xi}_m - [m-1]$, we can define the universal virtual U-bundle $\widehat{\Xi}$ over BU of virtual dimension 1 such that $\widehat{\Xi}|_{\mathrm{BU}(n)} = \widehat{\Xi}_n - [n-1]$ for all $n \in \mathbb{N}$.

The functor $X \mapsto \widetilde{\mathrm{K}}_{\mathbb{C}}(X)_{\otimes}$ is represented by the H-group $\mathrm{BU}_{\otimes}$. By $\widehat{\phi}$ denote the product:

$$\widehat{\phi}\colon\ \mathrm{BU}_{\otimes}\times \mathrm{BU}_{\otimes}\to \mathrm{BU}_{\otimes}.$$

It is known that the H-group $\mathrm{BU}_{\otimes}$ is isomorphic to the product of H-groups $\mathrm{BSU}_{\otimes}\times \mathbb{C}P^{\infty}$ (see the Appendix). Suppose $x := \sigma_1(\widehat{\Xi}) \in \mathrm{MU}^2(\mathrm{BU})$, $R := \mathrm{MU}^*(\mathrm{pt})$, $H := \mathrm{MU}^*(\mathrm{BSU}_{\otimes})$. Then H is a Hopf algebra over R. Clearly the element $\widehat{\phi}^*(x)$ can be expressed as a power series

$$\mathfrak{F}(x\otimes 1, 1\otimes x)\in (H\widehat{\underset{R}{\otimes}}H)[[x\otimes 1, 1\otimes x]] \tag{1}$$

with the coefficients belonging to the tensor square of the Hopf algebra $\mathrm{MU}^*(\mathrm{BSU}_{\otimes})$. This power series has some properties, which generalize the corresponding properties of usual formal group laws (see Definition 1). So we will refer to it as a "formal group law over a Hopf algebra".

The series (1) is closely related to the formal product in complex cobordism theory $F(u,v)$. Put $u := \sigma_1(\mathcal{Z}) \in \mathrm{MU}^2(\mathbb{C}P^{\infty})$, where $\mathcal{Z}$ is the universal line bundle over $\mathbb{C}P^{\infty}$. Due to the isomorphism $\mathrm{BU}_{\otimes}\cong \mathrm{BSU}_{\otimes}\times \mathbb{C}P^{\infty}$ we see that x may be represented as a power series $x = \mathfrak{h}(u) \in H[[u]]$. It is not difficult to prove that

$$\mathfrak{F}(x\otimes 1, 1\otimes x) = (\Delta_H\mathfrak{h})(F((\mathfrak{h}^{-1}(x))\otimes 1, 1\otimes(\mathfrak{h}^{-1}(x)))),$$

i.e. (1) may be considered as a re-parameterization of $F(u,v)$ by $\mathfrak{h}(u)$. Thus to obtain the explicit formula for this power series it is sufficient to get the explicit form of the power series $\mathfrak{h}(u)$ and to describe the comultiplication Δ_H in the Hopf algebra H.

Let ξ be a virtual SU-bundle over X of virtual dimension 1. For stable equivalence class $\{\xi\} \in \widetilde{\mathrm{KSU}}(X) \subset \widetilde{\mathrm{K}}_{\mathbb{C}}(X)$ we construct the formal group law (over the ring $\mathrm{MU}^*(X)$) $(\varphi_{\xi}^*\mathfrak{H})(x,y)$ such that it defines the group structure on the set of first Conner-Floyd classes $\sigma_1(\widehat{\xi})$ of bundles $\widehat{\xi}$, whose stable equivalence classes $\{\widehat{\xi}\}$ belong to the coset $\{\xi\}\mathcal{L}in(X)_0 \subset \widetilde{\mathrm{K}}_{\mathbb{C}}(X)_{\otimes}$. For $\{\xi\} = \{[1]\}$ the corresponding formal group law $(\varphi_{\xi}^*\mathfrak{H})(x,y)$ coincides with the formal product in complex cobordism theory $F(u,v)$. Moreover, the formula $\sigma_1(\widehat{\xi}_1\otimes\widehat{\xi}_2\otimes\xi^{-1}) = (\varphi_{\xi}^*\mathfrak{H})(\sigma_1(\widehat{\xi}_1),\sigma_1(\widehat{\xi}_2))$ generalizes the corresponding formula $\sigma_1(\zeta_1\otimes\zeta_2) = F(\sigma_1(\zeta_1),\sigma_1(\zeta_2))$ for the formal product in complex cobordism theory. Here $\widehat{\xi}_1$, $\widehat{\xi}_2$ are virtual U-bundles of virtual dimension 1 such that $\{\widehat{\xi}_1\}$, $\{\widehat{\xi}_2\} \in \{\xi\}\mathcal{L}in(X)_0 \subset \widetilde{\mathrm{K}}_{\mathbb{C}}(X)_{\otimes}$ and ζ_1, ζ_2 are line bundles over X.

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1 General properties of formal group laws over Hopf algebras

1.1 Basic definitions

The aim of this section is to define some generalization of the notion of a formal group law. More precisely, we consider an analog of formal group laws with coefficients belonging to a Hopf algebra. In particular, we show that a formal group law over a Hopf algebra may be considered as an extension of a usual formal group law over a ring by a Hopf algebra.

Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a commutative topological Hopf algebra over a ring R (commutative, associative and with identity element $1 \in R$), where $\mu\colon H \mathop{\widehat{\otimes}}_R H \to H$ is the multiplication, $\eta\colon R \to H$ is the unit, $\Delta\colon H \to H \mathop{\widehat{\otimes}}_R H$ is the diagonal map (comultiplication), $\varepsilon\colon H \to R$ is the counit, and $S\colon H \to H$ is the antipode [5].

Definition 1. A power series $\mathfrak{F}(x \otimes 1, 1 \otimes y) \in (H \mathop{\widehat{\otimes}}_R H)[[x \otimes 1, 1 \otimes y]]$ is called *a formal group law over the Hopf algebra* $(H, \mu, \eta, \Delta, \varepsilon, S)$ if the following conditions hold:

1) (*associative*)

$$((\mathrm{id}_H \otimes \Delta)\mathfrak{F})(x \otimes 1 \otimes 1, 1 \otimes \mathfrak{F}(y \otimes 1, 1 \otimes z))$$
$$= ((\Delta \otimes \mathrm{id}_H)\mathfrak{F})(\mathfrak{F}(x \otimes 1, 1 \otimes y) \otimes 1, 1 \otimes 1 \otimes z).$$

2) (*identity*)

$$((\mathrm{id}_H \otimes \varepsilon)\mathfrak{F})(x \otimes 1, 0) = x \otimes 1,$$
$$((\varepsilon \otimes \mathrm{id}_H)\mathfrak{F})(0, 1 \otimes y) = 1 \otimes y.$$

3) (*formal inverse*) There is a power series $\Theta(x) \in H[[x]]$ such that

$$((\mu \circ (\mathrm{id}_H \otimes S))\mathfrak{F})(x, \Theta(x)) = 0 = ((\mu \circ (S \otimes \mathrm{id}_H))\mathfrak{F})(\Theta(x), x).$$

The formal group law $\mathfrak{F}(x \otimes 1, 1 \otimes y)$ is called *commutative* if $(\tau\mathfrak{F})(1 \otimes y, x \otimes 1) = \mathfrak{F}(x \otimes 1, 1 \otimes y)$, where $\tau\colon H \mathop{\widehat{\otimes}}_R H \to H \mathop{\widehat{\otimes}}_R H$ is the *switching map* $\tau(a \otimes b) = b \otimes a$.

Below we shall deal only with commutative formal group laws.

Remark 2. Note that a formal group law $\mathfrak{F}(x \otimes 1, 1 \otimes y)$ over a Hopf algebra H over a ring R defines the formal group law (in the usual sense) $F(x, y) \in R[[x, y]]$ over the ring R in the following way. Let us consider the series $((\varepsilon \otimes \varepsilon)\mathfrak{F})(x, y)$. If we identify $R \mathop{\otimes}_R R$ with R, we may assume that $((\varepsilon \otimes \varepsilon)\mathfrak{F})(x, y) = F(x, y) \in R[[x, y]]$. Note that for any coalgebra H the diagram

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \mathop{\widehat{\otimes}}_R H \\ \varepsilon \downarrow & & \downarrow \varepsilon\otimes\varepsilon \\ R & \xrightarrow{\cong} & R \otimes R \end{array} \tag{2}$$

commutes. Using (2) and 1) of Definition 1, we get $F(x,F(y,z))=F(F(x,y),z)$. Similarly, the conditions $F(x,0)=x$ and $F(0,y)=y$ may be verified. It is well known, that the existence of the formal inverse (in the case of usual formal group laws) follows from the proved conditions. However this may be deduced from 3) of Definition 1 in the standard way. Moreover, the formal inverse $\theta(x)$ for the formal group law $F(x,y)$ is equal to $(\varepsilon(\Theta))(x)$.

Therefore we may consider the formal group law $\mathfrak{F}(x\otimes 1,1\otimes y)$ over the Hopf algebra H as an extension of the usual formal group law $F(x,y)$ by the Hopf algebra H.

Remark 3. Let us define the homomorphisms

$$\widetilde{\Delta}\colon H[[x]]\overset{\widetilde{\Delta}}{\to} H[[x]]\widehat{\underset{R}{\otimes}}H[[x]],$$

$$\widetilde{\varepsilon}\colon H[[x]]\to R,\ \text{and}\ \widetilde{S}\colon H[[x]]\to H[[x]]$$

in the following way: $\widetilde{\Delta}(x)=\mathfrak{F}(x\otimes 1,1\otimes x)$, $\widetilde{\varepsilon}(x)=0$, $\widetilde{S}(x)=\Theta(x)$ and $\widetilde{\Delta}\mid_H=\Delta$, $\widetilde{\varepsilon}\mid_H=\varepsilon$, $\widetilde{S}\mid_H=S$. We claim that $(H[[x]],\widetilde{\mu},\widetilde{\eta},\widetilde{\Delta},\widetilde{\varepsilon},\widetilde{S})$ is the Hopf algebra (here $\widetilde{\mu}$, $\widetilde{\eta}$ are evident extensions of μ, η). Indeed, the commutativity of the diagram

$$\begin{array}{ccc}
H[[x]] & \xrightarrow{\widetilde{\Delta}} & H[[x]]\widehat{\underset{R}{\otimes}}H[[x]] \\
\widetilde{\Delta}\downarrow & & \downarrow \mathrm{id}_{H[[x]]}\otimes\widetilde{\Delta} \\
H[[x]]\widehat{\underset{R}{\otimes}}H[[x]] & \xrightarrow{\widetilde{\Delta}\otimes\mathrm{id}_{H[[x]]}} & H[[x]]\widehat{\underset{R}{\otimes}}H[[x]]\widehat{\underset{R}{\otimes}}H[[x]]
\end{array}$$

follows from the equations

$$(\mathrm{id}_{H[[x]]}\otimes\widetilde{\Delta})(\mathfrak{F}(x\otimes 1,1\otimes x))=((\mathrm{id}_H\otimes\Delta)\mathfrak{F})(x\otimes 1\otimes 1,1\otimes\mathfrak{F}(x\otimes 1,1\otimes x))$$

$$=((\Delta\otimes\mathrm{id}_H)\mathfrak{F})(\mathfrak{F}(x\otimes 1,1\otimes x)\otimes 1,1\otimes 1\otimes x)=(\widetilde{\Delta}\otimes\mathrm{id}_{H[[x]]})(\mathfrak{F}(x\otimes 1,1\otimes x)).$$

The commutativity of the diagram

$$\begin{tikzcd}
R\underset{R}{\otimes}H[[x]] & H[[x]]\widehat{\underset{R}{\otimes}}H[[x]] \arrow[l, "\widetilde{\varepsilon}\otimes\mathrm{id}_{H[[x]]}"'] \arrow[r, "\mathrm{id}_{H[[x]]}\otimes\widetilde{\varepsilon}"] & H[[x]]\underset{R}{\otimes}R \\
& H[[x]] \arrow[ul, "\cong"] \arrow[u, "\widetilde{\Delta}"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

follows from the equations

$$((\mathrm{id}_{H[[x]]}\otimes\widetilde{\varepsilon})\circ\widetilde{\Delta})(x)=((\mathrm{id}_H\otimes\varepsilon)\mathfrak{F})(x\otimes 1,1\otimes\widetilde{\varepsilon}(x))=x\otimes 1,$$

$$((\widetilde{\varepsilon}\otimes\mathrm{id}_{H[[x]]})\circ\widetilde{\Delta})(x)=((\varepsilon\otimes\mathrm{id}_H)\mathfrak{F})(\widetilde{\varepsilon}(x)\otimes 1,1\otimes x)=1\otimes x.$$

The axiom of antipode

$$(\widetilde{\mu}\circ(\mathrm{id}_{H[[x]]}\otimes\widetilde{S})\circ\widetilde{\Delta})(x)=(\widetilde{\mu}\circ(\widetilde{S}\otimes\mathrm{id}_{H[[x]]})\circ\widetilde{\Delta})(x)=(\widetilde{\eta}\circ\widetilde{\varepsilon})(x)=0$$

follows from 3) of Definition 1. Let us recall that the Hopf algebra has only one antipode. Therefore the power series $\Theta(x)$, satisfies 3) of Definition 1, is a unique.

Let $\mathfrak{F}_i(x\otimes 1, 1\otimes y),\ i=1,2$ be formal group laws over the Hopf algebra H.

Definition 4. *A homomorphism* $\mathfrak{h}\colon \mathfrak{F}_1 \to \mathfrak{F}_2$ *of formal group laws over the Hopf algebra* H is a power series $\mathfrak{h}(x) \in H[[x]]$ with constant term belonging to $\ker\varepsilon \subset H$ such that $(\Delta\mathfrak{h})(\mathfrak{F}_1(x\otimes 1, 1\otimes y)) = \mathfrak{F}_2(\mathfrak{h}(x)\otimes 1, 1\otimes \mathfrak{h}(y))$. The power series $\mathfrak{h}(x)$ defines *an isomorphism* if it is invertible (hence $\varepsilon(\mathfrak{h}'(0)) \in R$ is a unit, where $\mathfrak{h}'(0)$ is the coefficient of x). An isomorphism $\mathfrak{h}$ is *a strict isomorphism* if $\varepsilon(\mathfrak{h}'(0)) = 1$. Assume that $\mathfrak{F}_i(x\otimes 1, 1\otimes y),\ i=1,2$ are extensions of the same formal group law over R. Then a strict isomorphism $\mathfrak{h}$ is called to be a *strong isomorphism* if $(\varepsilon\mathfrak{h})(x) = x$.

Note that $\varepsilon(\mathfrak{h})\colon (\varepsilon\otimes\varepsilon)(\mathfrak{F}_1) \to (\varepsilon\otimes\varepsilon)(\mathfrak{F}_2)$ is the homomorphism of the formal group laws over R (where ε is the counit of the Hopf algebra H). We say that the homomorphism $\mathfrak{h}$ covers the homomorphism $\varepsilon(\mathfrak{h})$. In particular, a strong isomorphism covers the identity isomorphism of the formal group law over R. Moreover, $\varepsilon(\mathfrak{h})$ is an isomorphism (a strict isomorphism) if $\mathfrak{h}$ is an isomorphism (a strict isomorphism respectively).

Let us consider some examples of defined objects.

Example 5. (*Trivial extension*) Let $F(x,y)$ be a formal group law (in the usual sense) over a ring R, and $(H,\mu,\eta,\Delta,\varepsilon,S)$ be a Hopf algebra over the same ring R. Then $\mathfrak{F}(x\otimes 1, 1\otimes y) = ((\eta\otimes\eta)F)(x\otimes 1, 1\otimes y) \in (H\widehat{\underset{R}{\otimes}}H)[[x\otimes 1, 1\otimes y]]$ is the formal group law over the Hopf algebra H (recall that we identify $R\underset{R}{\otimes}R$ with R).

Example 6. A formal group law of the form $\mathfrak{c} + x\otimes 1 + 1\otimes y \in (H\widehat{\underset{R}{\otimes}}H)[[x\otimes 1, 1\otimes y]]$ over Hopf algebra H is an extension of the "linear group" $x\otimes 1 + 1\otimes y \in R[[x\otimes 1, 1\otimes y]]$. It follows from Definition 1 (axioms 1)-3)) that $\mathfrak{c}$ is a 2-cocycle in the cobar complex of H such that $(\mathrm{id}\otimes\varepsilon)\mathfrak{c} = 0 = (\varepsilon\otimes\mathrm{id})\mathfrak{c},\ \tau(\mathfrak{c}) = \mathfrak{c}$ (the last condition follows from the commutativity of considered formal group laws). Below we shall show that a formal group law $\mathfrak{c}+x\otimes 1+1\otimes y \in (H\widehat{\underset{R}{\otimes}}H)[[x\otimes 1, 1\otimes y]]$ over H is determined up to isomorphism by the cohomology class of 2-cocycle $\mathfrak{c}$ (see Lemma 15).

By $\mathfrak{G}(x,y)$ denote the power series $((\mu\circ(\mathrm{id}_H\otimes S))\mathfrak{F})(x,y) \in H[[x,y]]$.

One can check by a straightforward computation that $H[[x,y]]$ is a Hopf algebra with respect to a comultiplication

$$\widehat{\Delta}\colon H[[x,y]] \to H[[x,y]]\widehat{\underset{R}{\otimes}}H[[x,y]]$$

and an antipode

$$\widehat{S}\colon H[[x,y]] \to H[[x,y]]$$

defined as follows:

$$\widehat{\Delta}|_H = \Delta,\ \widehat{\Delta}(x) = \mathfrak{F}(x\otimes 1, 1\otimes x),\ \widehat{\Delta}(y) = ((S\otimes S)\mathfrak{F})(y\otimes 1, 1\otimes y) \tag{3}$$

and

$$\widehat{S}|_H = S,\ \widehat{S}(x) = \Theta(x),\ \widehat{S}(y) = (S\Theta)(y).$$

Let us define a ring homomorphism $(\eta)\colon R[[z]] \to H[[x,y]]$ as follows:

$$(\eta)|_R = \eta\colon R \to H \subset H[[x,y]], \quad (\eta)(z) = \mathfrak{G}(x,y).$$

We consider the ring $R[[z]]$ as a Hopf algebra over R with a comultiplication, induced by $z \mapsto F(z\otimes 1, 1\otimes z)$ and an antipode, induced by $z \mapsto \theta(z)$, where $F(x,y) \in R[[x,y]]$ is the formal group law $((\varepsilon\otimes\varepsilon)\mathfrak{F})(x,y)$ and $\theta(x)$ is its formal inverse.

Proposition 7. *The ring homomorphism $(\eta)\colon R[[z]] \to H[[x,y]]$ is a homomorphism of Hopf algebras.*

Proof. At first we must prove that

$$\widehat{\Delta}(\mathfrak{G}(x,y)) = (\Delta\mathfrak{G})(\mathfrak{F}(x\otimes 1, 1\otimes x), ((S\otimes S)\mathfrak{F})(y\otimes 1, 1\otimes y))$$

$$= F(\mathfrak{G}(x,y)\otimes 1, 1\otimes \mathfrak{G}(x,y)). \tag{4}$$

By $\widetilde{S}'$ denote the homomorphism $H[[x]] \to H[[x]]$ such that $\widetilde{S}'|_H = S\colon H \to H$ and $\widetilde{S}'(x) = x$. Let us consider the following composite homomorphism of Hopf algebras:

$$H[[z]] \overset{\widetilde{\Delta}}{\to} H[[x]]\widehat{\underset{R}{\otimes}}H[[y]] \overset{\mathrm{id}\otimes\widetilde{S}'}{\to} H[[x]]\widehat{\underset{R}{\otimes}}H[[y]] \overset{(\mu)}{\to} H[[x,y]],$$

where $\widetilde{\Delta}(z) = \mathfrak{F}(x\otimes 1, 1\otimes y)$, $\widetilde{\Delta}|_H = \Delta$, (μ) is the homomorphism such that

$$(\mu)\mid_{H\widehat{\otimes}_R H} = \mu\colon H\widehat{\underset{R}{\otimes}}H \to H$$

and

$$(\mu)(x\otimes 1) = x,\ (\mu)(1\otimes y) = y.$$

It follows from the axiom of antipode

$$\mu\circ(\mathrm{id}_H\otimes S)\circ\Delta = \eta\circ\varepsilon$$

that

$$(\mu)\circ(\mathrm{id}_{H[[x]]}\otimes\widetilde{S}')\circ\widetilde{\Delta}\mid_H = \eta\circ\varepsilon.$$

Hence there exists the homomorphism of Hopf algebras

$$(\eta)'\colon R[[z]] \to H[[x,y]]$$

such that the following diagram

$$\begin{tikzcd}
H[[z]] \arrow[r, "\widetilde{\Delta}"] \arrow[dr, "(\varepsilon)"'] & H[[x]]\widehat{\otimes}_R H[[y]] \arrow[r, "\mathrm{id}\otimes\widetilde{S}'"] & H[[x]]\widehat{\otimes}_R H[[y]] \arrow[d, "(\mu)"] \\
& R[[z]] \arrow[r, "(\eta)'"] & H[[x,y]]
\end{tikzcd}$$

commutes, where (ε) is the homomorphism such that $(\varepsilon)\mid_H = \varepsilon$, $(\varepsilon)(z) = z$. Since

$$(\eta)'(z) = \mathfrak{G}(x, y),$$

we obtain $(\eta)' = (\eta)$. This completes the proof of (4) that

$$\Delta_{R[[z]]}(z) = F(z \otimes 1, 1 \otimes z),$$

where $F(x, y) \in R[[x, y]]$ is the formal group law as above.

It remains to prove that

$$(\eta)(\theta(z)) = \widehat{S}(\mathfrak{G}(x, y)) = (S\mathfrak{G})(\Theta(x), (S\Theta)(y)).$$

Applying the operator $\widehat{\mu} \circ (\mathrm{id}_{H[[x,y]]} \otimes \widehat{S})$ (where $\widehat{\mu}\colon H[[x, y]] \widehat{\underset{R}{\otimes}} H[[x, y]] \to H[[x, y]]$ is the obvious multiplication) to the both sides of (4) and using $(\widehat{\mu} \circ (\mathrm{id}_{H[[x,y]]} \otimes \widehat{S}) \circ \widehat{\Delta})(\mathfrak{G}(x, y)) = 0$, we get

$$0 = F(\mathfrak{G}(x, y), (S\mathfrak{G})(\Theta(x), (S\Theta)(y))).$$

Therefore, $(S\mathfrak{G})(\Theta(x), (S\Theta)(y)) = \theta(\mathfrak{G}(x, y))$. $\square$

By $g(x) \in (R \underset{\mathbb{Z}}{\otimes} \mathbb{Q})[[x]]$ denote the logarithm of the formal group law $F(x, y) = ((\varepsilon \otimes \varepsilon)\mathfrak{F})(x, y)$.

Corollary 8. *The element* $\exp(g(\mathfrak{G}(x, y))) \in (H \widehat{\underset{\mathbb{Z}}{\otimes}} \mathbb{Q})[[x, y]]$ *is a grouplike element of the Hopf algebra* $(H \widehat{\underset{\mathbb{Z}}{\otimes}} \mathbb{Q})[[x, y]]$.

Proof. Let $\widehat{\Delta}_{\mathbb{Q}}$ be the comultiplication in the Hopf algebra $(H \widehat{\underset{\mathbb{Z}}{\otimes}} \mathbb{Q})[[x, y]]$, corresponding to (3). We have:

$$\widehat{\Delta}_{\mathbb{Q}}(\exp(g(\mathfrak{G}(x, y)))) = \exp(g(F(\mathfrak{G}(x, y) \otimes 1, 1 \otimes \mathfrak{G}(x, y))))$$

$$= \exp(g(\mathfrak{G}(x, y)) \otimes 1 + 1 \otimes g(\mathfrak{G}(x, y))) = \exp(g(\mathfrak{G}(x, y))) \otimes \exp(g(\mathfrak{G}(x, y))).$$

Moreover, we have

$$\widehat{S}(\exp(g(\mathfrak{G}(x, y)))) = \exp(g(\mathfrak{G}(x, y)))^{-1},$$

i.e. $\widehat{S}(g(\mathfrak{G}(x, y))) = g((S\mathfrak{G})(\Theta(x), (S\Theta)(y))) = -g(\mathfrak{G}(x, y))$. To prove this, we shall show that

$$F(\mathfrak{G}(x, y), (S\mathfrak{G})(\Theta(x), (S\Theta)(y))) = 0.$$

Indeed, applying the operator $\widehat{\mu} \circ (\mathrm{id}_{H[[x,y]]} \otimes \widehat{S})$ to the both sides of (4), we get:

$$F(\mathfrak{G}(x, y), (S\mathfrak{G})(\Theta(x), (S\Theta)(y)))$$

$$= ((\eta \circ \varepsilon)\mathfrak{G})((\mu \circ (\mathrm{id}_H \otimes S)\mathfrak{F})(x, \Theta(x)), (\mu \circ (S \otimes \mathrm{id}_H)\mathfrak{F})(y, (S\Theta)(y)))$$

$$= F(0, 0) = 0.$$

Now the required equality follows from

$$0 = g(0) = g(F(\mathfrak{G}(x, y), (S\mathfrak{G})(\Theta(x), (S\Theta)(y))))$$

$$= g(\mathfrak{G}(x, y)) + g((S\mathfrak{G})(\Theta(x), (S\Theta)(y))). \quad \square$$

1.2 Logarithms of formal group laws over Hopf algebras

In this section we generalize some useful construction from the case of formal group laws over rings to the case of formal group laws over Hopf algebras. More precisely, we show that there exists an analog of the logarithm in the general case. We also establish some condition of triviality of a formal group law over a Hopf algebra as an extension of a formal group law over a ring by a Hopf algebra.

In this section by $(H, \mu, \eta, \Delta, \varepsilon, S)$ we denote a commutative Hopf algebra over a ring R without torsion. In addition, suppose that H is complete with respect to the augmentation ideal $\ker(\varepsilon) \subset H$. Note that this assumption implies that for any $a \in H$, $\varepsilon(a) = 1 \in R$ there exists $a^{-1} \in H$. By $\mathfrak{F}(x \otimes 1, 1 \otimes x)$ denote a formal group law over the Hopf algebra H. By $H_{\mathbb{Q}}$ denote the Hopf algebra $H \widehat{\otimes}_{\mathbb{Z}} \mathbb{Q}$ over the ring $R_{\mathbb{Q}} := R \otimes_{\mathbb{Z}} \mathbb{Q}$. Clearly, that the formal group law $\mathfrak{F}(x \otimes 1, 1 \otimes y)$ may be considered as a formal group law over the Hopf algebra $H_{\mathbb{Q}}$. We shall write $\mu, \eta, \dots$ instead of $\mu_{\mathbb{Q}}, \eta_{\mathbb{Q}}, \dots$.

The main tool of this section is the following result.

Theorem 9. *Let $\mathfrak{F}(x \otimes 1, 1 \otimes x)$ be a commutative formal group law over H, considered as a formal group law over $H_{\mathbb{Q}}$. Then there exists a strict isomorphism of formal group laws over $H_{\mathbb{Q}}$ from $\mathfrak{F}(x \otimes 1, 1 \otimes x)$ to a commutative formal group law of the form $\mathfrak{c} + x \otimes 1 + 1 \otimes x$, where $\mathfrak{c} \in H_{\mathbb{Q}} \widehat{\otimes}_{R_{\mathbb{Q}}} H_{\mathbb{Q}}$.*

We recall that the definitions of homomorphism, isomorphism and strict isomorphism of formal group laws over a Hopf algebra were given in the previous section. Below we shall use notation of Section 1.1.

To prove the theorem, we need the following lemma.

Lemma 10. A series of the form $\mathfrak{c} + x \otimes 1 + 1 \otimes x \in (H_{\mathbb{Q}} \widehat{\otimes}_{R_{\mathbb{Q}}} H_{\mathbb{Q}})[[x \otimes 1, 1 \otimes x]]$ is a commutative formal group law over the Hopf algebra $H_{\mathbb{Q}}$ if and only if the following conditions hold:

(i) $(\mathrm{id} \otimes \Delta)\mathfrak{c} + 1 \otimes \mathfrak{c} - (\Delta \otimes \mathrm{id})\mathfrak{c} - \mathfrak{c} \otimes 1 = 0$;

(ii) $(\mathrm{id} \otimes \varepsilon)\mathfrak{c} = 0 = (\varepsilon \otimes \mathrm{id})\mathfrak{c}$;

(iii) $\tau(\mathfrak{c}) = \mathfrak{c}$, where τ is the switching map $\tau(a \otimes b) = b \otimes a$.

(Note that condition (i) means that $\mathfrak{c}$ is a 2-cocykle in the cobar complex of the Hopf algebra $H_{\mathbb{Q}}$.)

Proof of the lemma. Conditions (i), (ii), and (iii) are equivalent to the associativity axiom, the identity axiom and the commutativity axiom respectively. Let us show that the series $\Theta(x) := -(\mu \circ (\mathrm{id} \otimes S))\mathfrak{c} - x$ is a formal inverse. Indeed,

$$(\mu \circ (\mathrm{id} \otimes S))\mathfrak{c} + x + \Theta(x) = (\mu \circ (\mathrm{id} \otimes S))\mathfrak{c} + x - (\mu \circ (\mathrm{id} \otimes S))\mathfrak{c} - x = 0.$$

The symmetric condition follows from the equality $(\mu \circ (\mathrm{id} \otimes S))\mathfrak{c} = (\mu \circ (S \otimes \mathrm{id}))\mathfrak{c}$. $\square$

Proof of the theorem. By definition, put

$$\widetilde{\omega}(x) := (\mathrm{id}\otimes\widetilde{\varepsilon})\frac{\partial\mathfrak{F}(x,z)}{\partial z} \in H[[x]],$$

where $\widetilde{\varepsilon}\colon H[[z]] \to R$ is the map such that $\widetilde{\varepsilon}\mid_H = \varepsilon\colon H \to R$, $\widetilde{\varepsilon}(z) = 0$. Recall that $\widetilde{\Delta}\colon H[[x]] \to H[[x]]\widehat{\underset{R}{\otimes}}H[[x]] = (H\widehat{\underset{R}{\otimes}}H)[[x\otimes 1, 1\otimes x]]$ is the homomorphism such that $\widetilde{\Delta}\mid_H = \Delta$, $\widetilde{\Delta}(x) = \mathfrak{F}(x\otimes 1, 1\otimes x)$. We have:

$$(\Delta\widetilde{\omega})(\mathfrak{F}(x\otimes 1, 1\otimes x)) = \widetilde{\Delta}(\widetilde{\omega}(x)) = (\widetilde{\Delta}\circ(\mathrm{id}\otimes\widetilde{\varepsilon})\circ\frac{\partial}{\partial z})(\mathfrak{F}(x,z))$$

$$= ((\mathrm{id}\otimes\mathrm{id}\otimes\widetilde{\varepsilon})\circ(\widetilde{\Delta}\otimes\mathrm{id})\circ\frac{\partial}{\partial z})(\mathfrak{F}(x,z))$$

$$= ((\mathrm{id}\otimes\mathrm{id}\otimes\widetilde{\varepsilon})\circ\frac{\partial}{\partial z}\circ(\widetilde{\Delta}\otimes\mathrm{id}))(\mathfrak{F}(x,z))$$

$$= \left((\mathrm{id}\otimes\mathrm{id}\otimes\widetilde{\varepsilon})\circ\frac{\partial}{\partial z}\right)((\Delta\otimes\mathrm{id})\mathfrak{F})(\mathfrak{F}(x\otimes 1, 1\otimes x), z)$$

$$= \left((\mathrm{id}\otimes\mathrm{id}\otimes\widetilde{\varepsilon})\circ\frac{\partial}{\partial z}\right)((\mathrm{id}\otimes\Delta)\mathfrak{F})(x\otimes 1, \mathfrak{F}(1\otimes x, z))$$

$$= (\mathrm{id}\otimes\mathrm{id}\otimes\widetilde{\varepsilon})\left(\frac{\partial((\mathrm{id}\otimes\Delta)\mathfrak{F})(x\otimes 1, \mathfrak{F}(1\otimes x, z))}{\partial\mathfrak{F}(1\otimes x, z)}\right)\cdot 1\otimes\left((\mathrm{id}\otimes\widetilde{\varepsilon})\frac{\partial\mathfrak{F}(1\otimes x, z)}{\partial z}\right)$$

$$= \frac{\partial\mathfrak{F}(x\otimes 1, 1\otimes x)}{\partial(1\otimes x)}\cdot(1\otimes\widetilde{\omega})(1\otimes x).$$

Therefore, we have

$$(\Delta\widetilde{\omega})(\mathfrak{F}(x\otimes 1, 1\otimes x)) = \frac{\partial\mathfrak{F}(x\otimes 1, 1\otimes x)}{\partial(1\otimes x)}\cdot(1\otimes\widetilde{\omega})(1\otimes x). \tag{5}$$

If

$$\mathfrak{F}(x,z) = \sum_{i,j\geq 0} A_{i,j}x^i z^j \quad (A_{i,j}\in H\widehat{\underset{R}{\otimes}}H),$$

then

$$\widetilde{\omega}(x) = (\mathrm{id}\otimes\widetilde{\varepsilon})\sum_{i,j} A_{i,j}x^i j z^{j-1} = (\mathrm{id}\otimes\varepsilon)A_{0,1} + \sum_{i\geq 1}((\mathrm{id}\otimes\varepsilon)A_{i,1})x^i,$$

where $(\varepsilon\circ(\mathrm{id}\otimes\varepsilon))A_{0,1} = 1 \neq 0$. Therefore

$$\frac{1}{\widetilde{\omega}(x)} \in H[[x]] \quad \text{and}$$

$$\widetilde{\Delta}\left(\frac{1}{\widetilde{\omega}(x)}\right) = \frac{1}{\widetilde{\Delta}(\widetilde{\omega}(x))} = \frac{1}{(\Delta\widetilde{\omega})(\mathfrak{F}(x\otimes 1, 1\otimes x))} \in (H\widehat{\underset{R}{\otimes}}H)[[x\otimes 1, 1\otimes x]].$$

Therefore equality (5) may be rewritten in the form

$$\frac{d(1\otimes x)}{(1\otimes\widetilde{\omega})(1\otimes x)} = \frac{d\mathfrak{F}(x\otimes 1, 1\otimes x)}{(\Delta\widetilde{\omega})(\mathfrak{F}(x\otimes 1, 1\otimes x))}. \tag{6}$$

It is clear that
$$\frac{1}{\widetilde{\omega}(x)} = b_0 + b_1 x + \dots ,$$
where $b_i \in H,\ \varepsilon(b_0) = 1$. By $\mathfrak{g}(x)$ denote the power series
$$\int_0^x \frac{dt}{\widetilde{\omega}(t)} \in H_{\mathbb{Q}}[[x]].$$
Equality (6) implies
$$\mathfrak{c}' + (1 \otimes \mathfrak{g})(1 \otimes x) = (\Delta \mathfrak{g})(\mathfrak{F}(x \otimes 1, 1 \otimes x)), \tag{7}$$
where $\mathfrak{c}'$ is independent of $1 \otimes x$. Applying $\mathrm{id} \otimes \widetilde{\varepsilon}$ to the both sides of (7), we get
$$(\mathrm{id} \otimes \widetilde{\varepsilon})\mathfrak{c}' = (((\mathrm{id} \otimes \varepsilon) \circ \Delta)\mathfrak{g})(x \otimes 1) = (\mathfrak{g} \otimes 1)(x \otimes 1).$$
Applying $\widetilde{\varepsilon} \otimes \mathrm{id}$ to the both sides of (7), we get
$$(\widetilde{\varepsilon} \otimes \mathrm{id})\mathfrak{c}' + (1 \otimes \mathfrak{g})(1 \otimes x) = (1 \otimes \mathfrak{g})(1 \otimes x).$$
Hence
$$(\Delta \mathfrak{g})(\mathfrak{F}(x \otimes 1, 1 \otimes x)) = \mathfrak{c} + (\mathfrak{g} \otimes 1)(x \otimes 1) + (1 \otimes \mathfrak{g})(1 \otimes x), \tag{8}$$
where $\mathfrak{c}' = (\mathfrak{g} \otimes 1)(x \otimes 1) + \mathfrak{c},\ \ \mathfrak{c} \in H_{\mathbb{Q}} \mathop{\widehat{\otimes}}_{R_{\mathbb{Q}}} H_{\mathbb{Q}}$ and $(\mathrm{id} \otimes \varepsilon)\mathfrak{c} = 0 = (\varepsilon \otimes \mathrm{id})\mathfrak{c}$.

Since $\mathfrak{g}(0) = 0$ and $\varepsilon(\mathfrak{g}'(0)) = 1$, the power series $\mathfrak{g}(x)$ is invertible (i.e. there is the power series $\mathfrak{g}^{-1}(x) \in H_{\mathbb{Q}}[[x]]$ such that $\mathfrak{g}(\mathfrak{g}^{-1}(x)) = x = \mathfrak{g}^{-1}(\mathfrak{g}(x))$). Therefore it defines a strict isomorphism of formal group laws over $H_{\mathbb{Q}}$.

To complete the proof we must verify condition (i) of the previous lemma. For this purpose we apply $\mathrm{id} \otimes \widetilde{\Delta}$ and $\widetilde{\Delta} \otimes \mathrm{id}$ to the both sides of (8). We have:
$$(((\mathrm{id} \otimes \Delta) \circ \Delta)\mathfrak{g})((\mathrm{id} \otimes \Delta)\mathfrak{F}(x \otimes 1 \otimes 1, 1 \otimes \mathfrak{F}(x \otimes 1, 1 \otimes x)))$$
$$= (\mathrm{id} \otimes \Delta)\mathfrak{c} + \mathfrak{g}(x) \otimes 1 \otimes 1 + 1 \otimes (\Delta \mathfrak{g})(\mathfrak{F}(x \otimes 1, 1 \otimes x))$$
$$= (((\Delta \otimes \mathrm{id}) \circ \Delta)\mathfrak{g})((\Delta \otimes \mathrm{id})\mathfrak{F}(\mathfrak{F}(x \otimes 1, 1 \otimes x) \otimes 1, 1 \otimes 1 \otimes x))$$
$$= (\Delta \otimes \mathrm{id})\mathfrak{c} + (\Delta \mathfrak{g})(\mathfrak{F}(x \otimes 1, 1 \otimes x)) \otimes 1 + 1 \otimes 1 \otimes \mathfrak{g}(x),$$
i. e.
$$(\mathrm{id} \otimes \Delta)\mathfrak{c} + \mathfrak{g}(x) \otimes 1 \otimes 1 + 1 \otimes \mathfrak{c} + 1 \otimes \mathfrak{g}(x) \otimes 1 + 1 \otimes 1 \otimes \mathfrak{g}(x)$$
$$= (\Delta \otimes \mathrm{id})\mathfrak{c} + \mathfrak{c} \otimes 1 + \mathfrak{g}(x) \otimes 1 \otimes 1 + 1 \otimes \mathfrak{g}(x) \otimes 1 + 1 \otimes 1 \otimes \mathfrak{g}(x).$$
This completes the proof. $\square$

Remark 11. Note that to any formal group law $\mathfrak{F}(x \otimes 1, 1 \otimes x)$ over H we assign some 2-cocycle $\mathfrak{c}$ in the cobar complex of the coalgebra $H_{\mathbb{Q}}$.

Remark 12. Note that $(\varepsilon \mathfrak{g})(x) \in R_{\mathbb{Q}}[[x]]$ is the logarithm of the formal group law $((\varepsilon \otimes \varepsilon)\mathfrak{F})(x, y) = F(x, y) \in R[[x, y]]$ over the ring R [4].

Remark 13. Since $\mathfrak{g}(x) = b_0x + b_1x^2 + \dots$ and $\varepsilon(b_0) = 1$, there exists the power series $(\Delta\mathfrak{g})^{-1}(x) = (\Delta(\mathfrak{g}^{-1}))(x) \in (H_{\mathbb{Q}} \widehat{\otimes}_{R_{\mathbb{Q}}} H_{\mathbb{Q}})[[x]]$. Using (8), we get

$$\mathfrak{F}(x \otimes 1, 1 \otimes x) = (\Delta\mathfrak{g})^{-1}(\mathfrak{c} + \mathfrak{g}(x) \otimes 1 + 1 \otimes \mathfrak{g}(x)). \tag{9}$$

Corollary 14. *Suppose $(\mu \circ (\mathrm{id}_H \otimes S))\mathfrak{c} = 0$, where $\mathfrak{c}$ is the cocycle in (8); then there exists a strong isomorphism $\mathfrak{f}$ from $\mathfrak{F}(x \otimes 1, 1 \otimes y)$ to the trivial extension of $((\varepsilon \otimes \varepsilon)\mathfrak{F})(x, y) = F(x, y)$ by H. Moreover, $\mathfrak{f}(0) = 0$.*

Proof. Recall that by $\mathfrak{G}(x, y)$ we denoted the power series $(\mu \circ (\mathrm{id}_H \otimes S)\mathfrak{F})(x, y)$. It follows from (9) under the condition $(\mu \circ (\mathrm{id}_H \otimes S))\mathfrak{c} = 0$ that $\mathfrak{G}(x, y) = g^{-1}(\mathfrak{g}(x) + (S\mathfrak{g})(y))$, where $g(x) = (\varepsilon\mathfrak{g})(x)$ is the logarithm of $F(x, y)$, see Remark 12 above. Since $\mathfrak{g}(0) = 0$, we have $\mathfrak{G}(x, 0) = g^{-1}(\mathfrak{g}(x)) \in H[[x]]$. The straightforward computation shows, that the power series $\mathfrak{f}(x) = g^{-1}(\mathfrak{g}(x))$ defines a homomorphism from $\mathfrak{F}(x \otimes 1, 1 \otimes y)$ to the trivial extension of $F(x, y)$ by H. Since $\mathfrak{f}(0) = 0$, $\varepsilon(\mathfrak{f}'(0)) = \varepsilon(\mathfrak{g}'(0)) = 1 \in R$, we see that $\mathfrak{f}(x)$ is invertible. Moreover, $(\varepsilon\mathfrak{f})(x) = g^{-1}(g(x)) = x$. □

Lemma 15. If cohomology classes $[\mathfrak{c}_1]$ and $[\mathfrak{c}_2]$ are equal, then the "linear" formal group laws $\mathfrak{c}_i + x \otimes 1 + 1 \otimes x$, $i = 1, 2$ over $H_{\mathbb{Q}}$ are isomorphic.

Proof. Suppose $[\mathfrak{c}_1] = [\mathfrak{c}_2]$; then there exists $\lambda \in H_{\mathbb{Q}}$ such that $\varepsilon(\lambda) = 0$ and $\mathfrak{c}_2 - \mathfrak{c}_1 = \Delta\lambda - \lambda \otimes 1 - 1 \otimes \lambda$. Hence $\Delta\lambda + \mathfrak{c}_1 + x \otimes 1 + 1 \otimes x = \mathfrak{c}_2 + (\lambda + x) \otimes 1 + 1 \otimes (\lambda + x)$. Therefore $\mathfrak{t}(x) := \lambda + x$ is an isomorphism from $\mathfrak{c}_1 + x \otimes 1 + 1 \otimes x$ to $\mathfrak{c}_2 + x \otimes 1 + 1 \otimes x$. □

We can make more precise the statement of the previous theorem.

Corollary 16. *If 2-cocycle $\mathfrak{c}$ in (8) is a coboundary, then there is a strict isomorphism from $\mathfrak{F}(x \otimes 1, 1 \otimes x)$, considered as a formal group law over $H_{\mathbb{Q}}$, to the "trivial" group $x \otimes 1 + 1 \otimes x$.*

Proof. We have

$$(\Delta\mathfrak{g})(\mathfrak{F}(x \otimes 1, 1 \otimes x)) = \mathfrak{c} + \mathfrak{g}(x) \otimes 1 + 1 \otimes \mathfrak{g}(x),$$

where $\mathfrak{c} = \lambda \otimes 1 - \Delta\lambda + 1 \otimes \lambda$. Let us consider the homomorphism $\mathfrak{h}(x) := \lambda + \mathfrak{g}(x)$. It follows from the completeness of the Hopf algebra H with respect to the augmentation ideal $\ker(\varepsilon) \subset H$ that there exists $\mathfrak{h}^{-1}(x) = \mathfrak{g}^{-1}(x - \lambda) \in H_{\mathbb{Q}}[[x]]$ (recall that $\varepsilon(\lambda) = 0$). Clearly $\varepsilon(\mathfrak{h}'(0)) = 1$. Hence the homomorphism $\mathfrak{h}(x)$ is a strict isomorphism. We have:

$$(\Delta\mathfrak{h})(\mathfrak{F}(x \otimes 1, 1 \otimes x)) = \Delta\lambda + (\Delta\mathfrak{g})(\mathfrak{F}(x \otimes 1, 1 \otimes x))$$

$$= \Delta\lambda + \mathfrak{c} + \mathfrak{g}(x) \otimes 1 + 1 \otimes \mathfrak{g}(x) = \Delta\lambda - \lambda \otimes 1 - 1 \otimes \lambda + \mathfrak{c} + \mathfrak{h}(x) \otimes 1 + 1 \otimes \mathfrak{h}(x)$$

$$= \mathfrak{h}(x) \otimes 1 + 1 \otimes \mathfrak{h}(x). \quad \square$$

Theorem 17. *Let $\mathfrak{F}(x \otimes 1, 1 \otimes y) \in (H \widehat{\otimes}_R H)[[x \otimes 1, 1 \otimes y]]$ be a formal group law over the Hopf algebra H and let $F(x, y) := ((\varepsilon \otimes \varepsilon)\mathfrak{F})(x, y) \in R[[x, y]]$ be the corresponding formal group law over R. Assume that 2-dimensional cohomology of the cobar complex of the Hopf algebra $H_{\mathbb{Q}}$ equals 0. Then there exists a strong isomorphism $\mathfrak{f}$ from $\mathfrak{F}(x \otimes 1, 1 \otimes y)$, considered as a formal group law over the Hopf algebra $H_{\mathbb{Q}}$, to the trivial extension of $F(x, y)$ by $H_{\mathbb{Q}}$.*

Proof. According to the previous corollary, there is an invertible power series $\mathfrak{h}(x) \in H_{\mathbb{Q}}[[x]]$ such that

$$(\Delta\mathfrak{h})(\mathfrak{F}(x \otimes 1, 1 \otimes y)) = \mathfrak{h}(x) \otimes 1 + 1 \otimes \mathfrak{h}(y) \tag{10}$$

and $(\varepsilon\mathfrak{h})(x) = g(x)$, where $g(x) \in R_{\mathbb{Q}}[[x]]$ is the logarithm of the formal group law $F(x, y)$. Let $\mathfrak{f}(z) \in H_{\mathbb{Q}}[[z]]$ be the power series $g^{-1}(\mathfrak{h}(z))$. We have:

$$(\Delta\mathfrak{f})(\mathfrak{F}(x \otimes 1, 1 \otimes y)) = g^{-1}((\Delta\mathfrak{h})(\mathfrak{F}(x \otimes 1, 1 \otimes y))) = g^{-1}(\mathfrak{h}(x) \otimes 1 + 1 \otimes \mathfrak{h}(y))$$

$$= g^{-1}(g(\mathfrak{f}(x)) \otimes 1 + 1 \otimes g(\mathfrak{f}(y))) = F(\mathfrak{f}(x) \otimes 1, 1 \otimes \mathfrak{f}(y)).$$

Clearly, $(\varepsilon\mathfrak{f})(x) = x$. □

Remark 18. Let us remark that under the hypothesis of the previous theorem the formal group law $\mathfrak{F}(x \otimes 1, 1 \otimes y)$ over the Hopf algebra H has the formal inverse of the form $\Theta(x) = (S\mathfrak{f})^{-1}(\theta(\mathfrak{f}(x)))$, where $\theta(x) \in R[[x]]$ is the formal inverse for the formal group law $F(x, y)$. Indeed, we have:

$$((\mu \circ (\mathrm{id}_H \otimes S))(\Delta\mathfrak{f})^{-1})(F(\mathfrak{f}(x), (S\mathfrak{f})((S\mathfrak{f})^{-1}(\theta(\mathfrak{f}(x))))))$$

$$= F(\mathfrak{f}(x), \theta(\mathfrak{f}(x))) = F(u, \theta(u)) = 0.$$

Example 19. Let us consider a formal group law of the form $\gamma + \alpha x \otimes 1 + (\tau\alpha) 1 \otimes x$, where $\alpha, \gamma \in H \widehat{\otimes}_R H$ and τ is the switching map $\tau(a \otimes b) = b \otimes a$. The axioms of associativity, identity and commutativity imply

$$\begin{gathered}(\mathrm{id} \otimes \varepsilon)\gamma = 0 = (\varepsilon \otimes \mathrm{id})\gamma, \quad (\mathrm{id} \otimes \varepsilon)\alpha = 1 = (\varepsilon \otimes \mathrm{id})(\tau\alpha), \\ (\mathrm{id} \otimes \varepsilon)(\tau\alpha) = (\varepsilon \otimes \mathrm{id})\alpha,\end{gathered} \tag{11}$$

$$(\mathrm{id} \otimes \Delta)\gamma + (\mathrm{id} \otimes \Delta)(\tau\alpha) \cdot 1 \otimes \gamma = (\Delta \otimes \mathrm{id})\gamma + (\Delta \otimes \mathrm{id})\alpha \cdot \gamma \otimes 1, \tag{12}$$

$$\begin{gathered}(\mathrm{id} \otimes \Delta)\alpha = (\Delta \otimes \mathrm{id})\alpha \cdot \alpha \otimes 1, \\ (\mathrm{id} \otimes \Delta)(\tau\alpha) \cdot 1 \otimes \alpha = (\Delta \otimes \mathrm{id})\alpha \cdot (\tau\alpha) \otimes 1, \\ (\mathrm{id} \otimes \Delta)(\tau\alpha) \cdot 1 \otimes (\tau\alpha) = (\Delta \otimes \mathrm{id})(\tau\alpha).\end{gathered} \tag{13}$$

The logarithm is the series

$$\mathfrak{g}(x) = \frac{x}{(\mathrm{id} \otimes \varepsilon)(\tau\alpha)} = \frac{x}{(\varepsilon \otimes \mathrm{id})\alpha}.$$

We shall verify the equality

$$(\Delta\mathfrak{g})(\gamma + \alpha x \otimes 1 + (\tau\alpha) 1 \otimes x) = \frac{\gamma}{\Delta(\varepsilon \otimes \mathrm{id})\alpha} + \frac{x}{(\mathrm{id} \otimes \varepsilon)(\tau\alpha)} \otimes 1 + 1 \otimes \frac{x}{(\mathrm{id} \otimes \varepsilon)(\tau\alpha)},$$

using (11)-(13). Since

$$(\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\alpha = \Delta(\varepsilon \otimes \mathrm{id})\alpha, \quad (\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\alpha = \alpha,$$

$$(\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})(\alpha \otimes 1) = ((\varepsilon \otimes \mathrm{id})\alpha) \otimes 1,$$

applying $\varepsilon \otimes \mathrm{id} \otimes \mathrm{id}$ to the both sides of the first equality (13), we get

$$\Delta(\varepsilon \otimes \mathrm{id})\alpha = \alpha \cdot (((\varepsilon \otimes \mathrm{id})\alpha) \otimes 1).$$

Since

$$(\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)(\Delta \otimes \mathrm{id})(\tau\alpha) = \Delta(\mathrm{id} \otimes \varepsilon)(\tau\alpha), \quad (\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)(\mathrm{id} \otimes \Delta)(\tau\alpha) = \tau\alpha,$$

$$(\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)(1 \otimes (\tau\alpha)) = 1 \otimes ((\mathrm{id} \otimes \varepsilon)(\tau\alpha)),$$

applying $\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon$ to the both sides of the third equality (13), we obtain

$$\Delta(\mathrm{id} \otimes \varepsilon)(\tau\alpha) = (\tau\alpha) \cdot (1 \otimes ((\mathrm{id} \otimes \varepsilon)(\tau\alpha)).$$

Now we must check that

$$\mathfrak{c} = \frac{\gamma}{\Delta(\varepsilon \otimes \mathrm{id})\alpha}$$

is a cocycle. To show this, we use (12), which is equivalent to

$$(\mathrm{id} \otimes \Delta)(\Delta(\varepsilon \otimes \mathrm{id})\alpha \cdot \mathfrak{c}) + (\mathrm{id} \otimes \Delta)(\tau\alpha) \cdot 1 \otimes (\Delta(\varepsilon \otimes \mathrm{id})\alpha \cdot \mathfrak{c})$$

$$= (\Delta \otimes \mathrm{id})(\Delta(\varepsilon \otimes \mathrm{id})\alpha \cdot \mathfrak{c}) + (\Delta \otimes \mathrm{id})\alpha \cdot (\Delta(\varepsilon \otimes \mathrm{id})\alpha \cdot \mathfrak{c}) \otimes 1.$$

It is easy to prove that

$$(\mathrm{id} \otimes \Delta)(\tau\alpha) \cdot 1 \otimes (\Delta(\varepsilon \otimes \mathrm{id})\alpha) = (\mathrm{id} \otimes \Delta)(\Delta(\varepsilon \otimes \mathrm{id})\alpha)$$

$$= (\Delta \otimes \mathrm{id})(\Delta(\varepsilon \otimes \mathrm{id})\alpha) = (\Delta \otimes \mathrm{id})\alpha \cdot (\Delta(\varepsilon \otimes \mathrm{id})\alpha) \otimes 1.$$

1.3 The functor, represented by a formal group law over a Hopf algebra

It is well known that a formal group law over a ring R defines a group structure on the nil-radical $\mathfrak{n}(A)$ of any associative commutative unitary R-algebra A [3]. We show that under some assumption of triviality a formal group law over a Hopf algebra defines a set of group structures on $\mathfrak{n}(A)$, which are different in general case. The technique, developed in this section, will be the main tool of proofs in Section 2.3.

To avoid a confusion, in this section we shall write μ_H, ε_H instead of μ, ε respectively.

Let R be an associative commutative ring with identity element $1 \in R$. By $\mathfrak{Alg}_R$ denote the category of associative commutative unitary algebras over R. By $\mathfrak{n}(A)$ we denote the nil-radical of $A \in \mathfrak{Alg}_R$. Any commutative formal group law $F(x, y) \in R[[x, y]]$ defines a group operation $+_F$ on the set $\mathfrak{n}(A)$ by $\alpha +_F \beta := F(\alpha, \beta)$. This group we shall denote by $\mathfrak{n}(A)_F$.

Let H be a commutative topological Hopf algebra over R. As in Section 1.2 we suppose H is complete with respect to the augmentation ideal $\ker(\varepsilon_H) \subset H$. Let $\mathfrak{F}(x \otimes 1, 1 \otimes y) \in$

$(H\widehat{\underset{R}{\otimes}}H)[[x \otimes 1, 1 \otimes y]]$ be a formal group law over H such that it is an extension of $F(x, y)$ by H (i.e. $((\varepsilon_H \otimes \varepsilon_H)\mathfrak{F})(x, y) = F(x, y)$, where $\varepsilon_H\colon H \to R$ is the counit in H).

Let us remark that it follows from 2) of Definition 1 that $(\mathrm{id}_H \otimes \varepsilon_H)A_{0,0} = 0 = (\varepsilon_H \otimes \mathrm{id}_H)A_{0,0}$, where $\mathfrak{F}(x \otimes 1, 1 \otimes y) = \sum_{i,j} A_{i,j} x^i \otimes y^j$. The completeness of the Hopf algebra H with respect to the augmentation ideal $\ker(\varepsilon_H)$ implies that $(\mu_A \circ (\varphi \otimes \psi))A_{0,0} \in \mathfrak{n}(A)$ for all $\varphi, \psi \in \mathrm{Hom}_R(H, A)$. Hence $(\mu_A \circ (\varphi \otimes \psi)\mathfrak{F})(\alpha, \beta) \in \mathfrak{n}(A)$ for any $\alpha, \beta \in \mathfrak{n}(A)$, where $\mu_A\colon A \underset{R}{\otimes} A \to A$ is the multiplication in the algebra A.

Define the operation $\circ$ on the set $\mathfrak{n}(A) \times \mathrm{Hom}_R(H; A)$ as follows:

$$\begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \circ \begin{pmatrix} \beta \\ \psi \end{pmatrix} := \begin{pmatrix} (\mu_A \circ (\varphi \otimes \psi)\mathfrak{F})(\alpha, \beta) \\ \varphi \star \psi \end{pmatrix} \tag{14}$$

for $\alpha, \beta \in \mathfrak{n}(A)$, $\varphi, \psi \in \mathrm{Hom}_R(H, A)$, where

$$\varphi \star \psi = \mu_A \circ (\varphi \otimes \psi) \circ \Delta \in \mathrm{Hom}_R(H, A)$$

is the convolution of φ and ψ, $\Delta\colon H \to H\widehat{\underset{R}{\otimes}}H$ is the comultiplication in H.

Proposition 20. *The set $\mathfrak{n}(A) \times \mathrm{Hom}_R(H; A)$ is an Abelian group with respect to the operation $\circ$.*

Proof. Associativity of (14) follows from 1) of Definition 1. Indeed,

$$\begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \circ \left(\begin{pmatrix} \beta \\ \psi \end{pmatrix} \circ \begin{pmatrix} \gamma \\ \chi \end{pmatrix} \right) = \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \circ \begin{pmatrix} (\mu_A \circ (\psi \otimes \chi)\mathfrak{F})(\beta, \gamma) \\ \psi \star \chi \end{pmatrix}$$

$$= \begin{pmatrix} (\mu_A \circ (\varphi \otimes (\psi \star \chi))\mathfrak{F})(\alpha, (\mu_A \circ (\psi \otimes \chi)\mathfrak{F})(\beta, \gamma)) \\ \varphi \star \psi \star \chi \end{pmatrix}.$$

Similarly,

$$\left(\begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \circ \begin{pmatrix} \beta \\ \psi \end{pmatrix} \right) \circ \begin{pmatrix} \gamma \\ \chi \end{pmatrix} = \begin{pmatrix} (\mu_A \circ (\varphi \otimes \psi)\mathfrak{F})(\alpha, \beta) \\ \varphi \star \psi \end{pmatrix} \circ \begin{pmatrix} \gamma \\ \chi \end{pmatrix}$$

$$= \begin{pmatrix} (\mu_A \circ ((\varphi \star \psi) \otimes \chi)\mathfrak{F})((\mu_A \circ (\varphi \otimes \psi)\mathfrak{F})(\alpha, \beta), \gamma) \\ \varphi \star \psi \star \chi \end{pmatrix}.$$

It is not difficult to show that

$$(\mu_A \circ (\varphi \otimes (\psi \star \chi))\mathfrak{F})(\alpha, (\mu_A \circ (\psi \otimes \chi)\mathfrak{F})(\beta, \gamma))$$

is the result of application of

$$\mu_A \circ (\mathrm{id}_A \otimes \mu_A) \circ (\varphi \otimes \psi \otimes \chi)$$

to

$$((\mathrm{id}_H \otimes \Delta)\mathfrak{F})(x \otimes 1 \otimes 1, 1 \otimes \mathfrak{F}(y \otimes 1, 1 \otimes z))$$

and substitution of α, β, γ for x, y, z respectively. Similarly,

$$(\mu_A \circ ((\varphi \star \psi) \otimes \chi)\mathfrak{F})((\mu_A \circ (\varphi \otimes \psi)\mathfrak{F})(\alpha, \beta), \gamma)$$

is the result of application of

$$\mu_A \circ (\mu_A \otimes \mathrm{id}_A) \circ (\varphi \otimes \psi \otimes \chi)$$

to

$$((\Delta \otimes \mathrm{id}_H)\mathfrak{F})(\mathfrak{F}(x \otimes 1, 1 \otimes y) \otimes 1, 1 \otimes 1 \otimes z)$$

and substitution of α, β, γ for x, y, z respectively.

By ε denote the composition $\eta_A \circ \varepsilon_H \colon H \to A$, where $\eta_A \colon R \to A$ is the unit in A. It follows from 2) of Definition 1 that

$$\begin{pmatrix} 0 \\ \varepsilon \end{pmatrix}$$

is the identity element in the group.

Moreover, for any

$$\begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \in \mathfrak{n}(A) \times \mathrm{Hom}_R(H; A)$$

there exists the inverse element

$$\begin{pmatrix} \alpha \\ \varphi \end{pmatrix}^{-1} = \begin{pmatrix} (\varphi\Theta)(\alpha) \\ \varphi \circ S \end{pmatrix}, \tag{15}$$

where $\Theta(x) \in H[[x]]$ is the formal inverse in the formal group law $\mathfrak{F}(x \otimes 1, 1 \otimes y)$ and $S \colon H \to H$ is the antipode in H. Indeed,

$$(\mu_A \circ (\varphi \otimes (\varphi \circ S))\mathfrak{F})(\alpha, (\varphi\Theta)(\alpha)) = \varphi((\mu_H \circ (\mathrm{id}_H \otimes S)\mathfrak{F})(x, \Theta(x))|_{x=\alpha})$$

$$= \varphi(0) = 0$$

(we used 3) of Definition 1).

Commutativity follows from the commutativity of $\mathfrak{F}(x \otimes 1, 1 \otimes y)$ (recall that we assumed H to be commutative). □

The group $\mathfrak{n}(A) \times \mathrm{Hom}_R(H; A)$ with respect to the just defined operation will be denoted by $\mathrm{G}_{\mathfrak{F}}$. We have the following exact sequence of groups:

$$0 \to \mathfrak{n}(A)_F \xrightarrow{i} \mathrm{G}_{\mathfrak{F}} \xrightarrow{j} \mathrm{Hom}_R(H, A) \to 0, \tag{16}$$

where

$$i(\alpha) = \begin{pmatrix} \alpha \\ \varepsilon \end{pmatrix}, \quad j \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \varphi$$

(recall that ε is the identity element in the group (with respect to the convolution) $\mathrm{Hom}_R(H, A)$).

Proposition 21. *Suppose there exists a strong isomorphism from $\mathfrak{F}(x \otimes 1, 1 \otimes y)$ to the trivial extension of $F(x, y)$ by H (compare with Corollary 14); then sequence (16) splits.*

Proof. Recall that proposition's assumption means that there is an invertible power series $\mathfrak{f}(x) \in H[[x]]$ such that $(\Delta\mathfrak{f})(\mathfrak{F}(x \otimes 1, 1 \otimes y)) = F(\mathfrak{f}(x) \otimes 1, 1 \otimes \mathfrak{f}(y))$ and $(\varepsilon_H \mathfrak{f})(x) = x$. To prove the proposition we shall show that

$$\sigma \colon \mathrm{G}_{\mathfrak{F}} \to \mathfrak{n}(A)_F, \quad \sigma\begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = (\varphi\mathfrak{f})(\alpha)$$

is a splitting homomorphism.

1) σ is a group homomorphism:

$$\sigma\left(\begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \circ \begin{pmatrix} \beta \\ \psi \end{pmatrix}\right) = \sigma\begin{pmatrix} (\mu_A \circ (\varphi \otimes \psi)\mathfrak{F})(\alpha, \beta) \\ \varphi \star \psi \end{pmatrix}$$

$$= ((\varphi \star \psi)\mathfrak{f})((\mu_A \circ (\varphi \otimes \psi)\mathfrak{F})(\alpha, \beta)). \tag{17}$$

Applying $\mu_A \circ (\varphi \otimes \psi)$ to the both sides of the identity

$$(\Delta\mathfrak{f})(\mathfrak{F}(x \otimes 1, 1 \otimes y)) = F(\mathfrak{f}(x) \otimes 1, 1 \otimes \mathfrak{f}(y))$$

and substituting of α, β for x, y respectively, we obtain that (17) equals $F((\varphi\mathfrak{f})(\alpha), (\psi\mathfrak{f})(\beta))$.

2) σ is a section of i, i.e. $\sigma \circ i = \mathrm{id}_{\mathfrak{n}(A)_F}$. Indeed,

$$(\sigma \circ i)(\alpha) = \sigma\begin{pmatrix} \alpha \\ \varepsilon \end{pmatrix} = (\varepsilon\mathfrak{f})(\alpha) = \alpha. \quad \square$$

Note that for any $\alpha \in \mathfrak{n}(A)$ and $\varphi \in \mathrm{Hom}_R(H, A)$ the element

$$((\varepsilon_H \otimes \varphi)\mathfrak{F})(\alpha, 0) \in \mathfrak{n}(A)$$

may be considered as a *shift* of α by φ (let us remark that we use the identity $\mu_A \circ (\varepsilon \otimes \varphi) = \varepsilon_H \otimes \varphi$, which follows from the unital multiplication axiom $\mu_A \circ (\eta_A \otimes \mathrm{id}_A) \colon R \otimes A \cong A$ for the algebra A). Clearly that the function

$$\langle\alpha\rangle \colon \mathrm{Hom}_R(H, A) \to \mathfrak{n}(A), \ \langle\alpha\rangle(\varphi) := ((\varepsilon_H \otimes \varphi)\mathfrak{F})(\alpha, 0)$$

is a *section* of the projection $\pi_2 \colon \mathfrak{n}(A) \times \mathrm{Hom}_R(H, A) \to \mathrm{Hom}_R(H, A)$ such that $\langle\alpha\rangle(\varepsilon) = \alpha$. Therefore we obtain the map

$$\mathfrak{n}(A) \to \mathrm{Mor}_{Set}(\mathrm{Hom}_R(H, A); \mathfrak{n}(A)), \quad \alpha \mapsto \langle\alpha\rangle.$$

It follows from 2) of Definition 1 that for any $\varphi \in \operatorname{Hom}_R(H, A)$ $\langle 0\rangle(\varphi) = 0$.

If $\mathfrak{F}(x\otimes 1, 1\otimes y)$ is the trivial extension (see Example 5 above), then $\langle\alpha\rangle(\varphi) = \alpha$ for all $\varphi \in \operatorname{Hom}_R(H, A)$.

The set of sections $\{\langle\alpha\rangle\colon\ \alpha \in \mathfrak{n}(A)\}$ is a group with respect to the operation $\langle\alpha\rangle\langle\beta\rangle := \langle F(\alpha, \beta)\rangle$ (obviously that it is isomorphic to $\mathfrak{n}(A)_F$).

Now let $\varphi \in \operatorname{Hom}_R(H, A)$ be a fixed element. Since

$$\left\{\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}:\ \alpha \in \mathfrak{n}(A),\ \varphi \in \operatorname{Hom}_R(H, A)\right\}$$

is a group, the shift

$$\begin{pmatrix}\alpha\\ \varepsilon\end{pmatrix} \mapsto \begin{pmatrix}\alpha\\ \varepsilon\end{pmatrix}\circ\begin{pmatrix}0\\ \varphi\end{pmatrix} = \begin{pmatrix}((\varepsilon_H\otimes\varphi)\mathfrak{F})(\alpha, 0)\\ \varphi\end{pmatrix}$$

is a bijection from the set (more precisely, subgroup)

$$\left\{\begin{pmatrix}\alpha\\ \varepsilon\end{pmatrix}:\ \alpha \in \mathfrak{n}(A)\right\} \text{ to the (co)set } \left\{\begin{pmatrix}\beta\\ \varphi\end{pmatrix}:\ \beta \in \mathfrak{n}(A)\right\}.$$

Therefore for any $\varphi \in \operatorname{Hom}_R(H, A)$ the map $\rho_\varphi\colon \alpha \mapsto \langle\alpha\rangle(\varphi) = ((\varepsilon_H\otimes\varphi)\mathfrak{F})(\alpha, 0)$ defines a bijection $\mathfrak{n}(A) \to \mathfrak{n}(A)$.

Now we assume that $(\mu_H \circ (\mathrm{id}_H \otimes S))\mathfrak{c} = 0$, *where* $\mathfrak{c}$ *as in Corollary* 14. It follows from our assumption that

$$\begin{pmatrix}0\\ \varphi\circ S\end{pmatrix} = \begin{pmatrix}0\\ \varphi\end{pmatrix}^{-1}.$$

Given φ let us define a new operation $\circ_\varphi$ on the set $\mathfrak{n}(A)$ as follows:

$$\begin{pmatrix}\alpha\circ_\varphi\beta\\ \varphi\end{pmatrix} := \left(\left(\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}\circ\begin{pmatrix}0\\ \varphi\circ S\end{pmatrix}\right)\circ\left(\begin{pmatrix}\beta\\ \varphi\end{pmatrix}\circ\begin{pmatrix}0\\ \varphi\circ S\end{pmatrix}\right)\right)\circ\begin{pmatrix}0\\ \varphi\end{pmatrix}. \tag{18}$$

The operation $\circ_\varphi$ is compatible with the group structure on the set of sections $\{\langle\alpha\rangle\colon\ \alpha \in \mathfrak{n}(A)\}$ in the following sense.

Proposition 22. *If* $\alpha = \langle\alpha'\rangle(\varphi)$, $\beta = \langle\beta'\rangle(\varphi)$ *for some* $\alpha', \beta' \in \mathfrak{n}(A)$ *(obviously,* $\alpha' = (\mu_A\circ(\varphi\otimes(\varphi\circ S))\mathfrak{F})(\alpha, 0)$, $\beta' = (\mu_A\circ(\varphi\otimes(\varphi\circ S))\mathfrak{F})(\beta, 0)$*), then* $\alpha\circ_\varphi\beta = (\langle\alpha'\rangle\langle\beta'\rangle)(\varphi)$.

Proof. The proof is trivial. □

This proposition justifies the definition (18) of the operation $\circ_\varphi$.

Note that

$$\left(\left(\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}\circ\begin{pmatrix}0\\ \varphi\circ S\end{pmatrix}\right)\circ\left(\begin{pmatrix}\beta\\ \varphi\end{pmatrix}\circ\begin{pmatrix}0\\ \varphi\circ S\end{pmatrix}\right)\right)\circ\begin{pmatrix}0\\ \varphi\end{pmatrix}$$

$$= \begin{pmatrix}\alpha\\ \varphi\end{pmatrix}\circ\begin{pmatrix}\beta\\ \varphi\end{pmatrix}\circ\begin{pmatrix}0\\ \varphi\circ S\end{pmatrix} = \begin{pmatrix}\alpha\\ \varphi\end{pmatrix}\circ\begin{pmatrix}(\mu_A\circ(\varphi\otimes(\varphi\circ S))\mathfrak{F})(\beta, 0)\\ \varepsilon\end{pmatrix}$$

$$= \begin{pmatrix} ((\varphi\otimes\varepsilon_H)\mathfrak{F})(\alpha, (\mu_A\circ(\varphi\otimes(\varphi\circ S))\mathfrak{F})(\beta,0)) \\ \varphi \end{pmatrix}, \tag{19}$$

i.e. the product $\alpha\circ_\varphi\beta$ may be represented as follows

$$((\varphi\otimes\varepsilon_H)\mathfrak{F})(\alpha, (\mu_A\circ(\varphi\otimes(\varphi\circ S))\mathfrak{F})(\beta,0)).$$

For $\varphi=\varepsilon$ the operation $\circ_\varepsilon$ on $\mathfrak{n}(A)$ coincides with $+_F$, i.e. the set $\mathfrak{n}(A)$ with respect to the operation, corresponding to ε, is the group $\mathfrak{n}(A)_F$.

Proposition 23. *For any $\varphi\in\mathrm{Hom}_R(H,A)$ the corresponding operation $\circ_\varphi$ induces the structure of an Abelian group on the set $\mathfrak{n}(A)$.*

Proof. It follows from Proposition 20 and from the definition that the operation $\circ_\varphi$ is associative and commutative. We claim that $0\in\mathfrak{n}(A)$ is its identity element. Indeed, it follows from the assumption $(\mu_H\circ(\mathrm{id}_H\otimes S))\mathfrak{c}=0$ that $(\mu_A\circ(\varphi\otimes(\varphi\circ S))\mathfrak{F})(0,0)=0$. Then using the identity axiom 2) of Definition 1, we get

$$\alpha\circ_\varphi 0 = ((\varphi\otimes\varepsilon_H)\mathfrak{F})(\alpha, (\mu_A\circ(\varphi\otimes(\varphi\circ S))\mathfrak{F})(0,0)) = ((\varphi\otimes\varepsilon_H)\mathfrak{F})(\alpha,0)=\alpha.$$

To complete the proof we must verify that for any $\alpha\in\mathfrak{n}(A)$ there is the inverse element $\beta=(\alpha)_\varphi^{-1}\in\mathfrak{n}(A)$ (with respect to the operation $\circ_\varphi$). It follows from the definition of $\circ_\varphi$ that

$$\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}\circ\begin{pmatrix}\beta\\ \varphi\end{pmatrix}\circ\begin{pmatrix}0\\ \varphi\circ S\end{pmatrix}=\begin{pmatrix}0\\ \varphi\end{pmatrix}.$$

Therefore

$$\begin{pmatrix}(\mu_A\circ(\varphi\otimes(\varphi\circ S))\mathfrak{F})(\alpha,0)\\ \varepsilon\end{pmatrix}\circ\begin{pmatrix}\beta\\ \varphi\end{pmatrix}=\begin{pmatrix}0\\ \varphi\end{pmatrix}.$$

Using (15), we get

$$\begin{pmatrix}\beta\\ \varphi\end{pmatrix}=\begin{pmatrix}\theta((\mu_A\circ(\varphi\otimes(\varphi\circ S))\mathfrak{F})(\alpha,0))\\ \varepsilon\end{pmatrix}\circ\begin{pmatrix}0\\ \varphi\end{pmatrix}$$

$$=\begin{pmatrix}((\varepsilon_H\otimes\varphi)\mathfrak{F})(\theta((\mu_A\circ(\varphi\otimes(\varphi\circ S))\mathfrak{F})(\alpha,0)),0)\\ \varphi\end{pmatrix},$$

where $\theta(x)=(\varepsilon_H\Theta)(x)\in R[[x]]$ is the formal inverse in the formal group law $F(x,y)$. Hence,

$$(\alpha)_\varphi^{-1}=((\varepsilon_H\otimes\varphi)\mathfrak{F})(\theta((\mu_A\circ(\varphi\otimes(\varphi\circ S))\mathfrak{F})(\alpha,0)),0). \quad\square \tag{20}$$

Using the compatibility of the operation $\circ_\varphi$ with the group structure on the set of sections $\{\langle\alpha\rangle:\ \alpha\in\mathfrak{n}(A)\}$, one can establish a canonical isomorphism between $(\mathfrak{n}(A),\circ_\varphi)$ and $\mathfrak{n}(A)_F=(\mathfrak{n}(A),\circ_\varepsilon)$.

Although for all φ the corresponding group structures on $\mathfrak{n}(A)$ are isomorphic to each other (in particular, all of them are isomorphic to the group, corresponding to ε, i.e. $\mathfrak{n}(A)_F$), in general they do not coincide. Moreover, we have a homomorphism from the group $\mathrm{Hom}_R(H, A)$ to the "group of group structures" on the set $\mathfrak{n}(A)$.

Hence a nontrivial extension $\mathfrak{F}(x \otimes 1, 1 \otimes y)$ of a formal group law $F(x, y)$ in general defines a set of isomorphic group structures on the nil-radical $\mathfrak{n}(A)$.

By $\mathfrak{H}(x, y) \in H[[x, y]]$ denote the power series $((\mathrm{id}_H \otimes \varepsilon_H)\mathfrak{F})(x, \mathfrak{G}(y, 0))$, where $\mathfrak{G}(x, y) = (\mu_H \circ (\mathrm{id}_H \otimes S)\mathfrak{F})(x, y)$ as above.

Lemma 24. The power series $\mathfrak{H}(x, y)$ may be obtained by applying

$$(\mu_H \circ (\mathrm{id}_H \otimes \mu_H) \circ (\mathrm{id}_H \otimes \mathrm{id}_H \otimes S)) \tag{21}$$

to the power series

$$((\mathrm{id}_H \otimes \Delta)\mathfrak{F})(x \otimes 1 \otimes 1, 1 \otimes \mathfrak{F}(y \otimes 1, 1 \otimes z)) \tag{22}$$

and putting $z = 0$.

Proof. Lemma's assertion may be verified by a straightforward computation. Indeed, the result of application of (21) to (22) is equal to

$$((\mathrm{id}_H \otimes \varepsilon_H)\mathfrak{F})(x, (\mu_H \circ (\mathrm{id}_H \otimes S)\mathfrak{F})(y, 0)) = ((\mathrm{id}_H \otimes \varepsilon_H)\mathfrak{F})(x, \mathfrak{G}(y, 0)). \quad \square$$

Note that $(\varepsilon_H \mathfrak{H})(x, y) = F(x, y)$.

By $\mathfrak{I}(x) \in H[[x]]$ denote the power series $((\varepsilon_H \otimes \mathrm{id}_H)\mathfrak{F})(\theta(\mathfrak{G}(x, 0)), 0)$, where $\theta(x)$ is the formal inverse for $F(x, y)$.

Lemma 25. The power series $\mathfrak{I}(x)$ may be obtained by applying

$$(\mu_H \circ (\mu_H \otimes \mathrm{id}_H) \circ ((\mathrm{id}_H \otimes S) \otimes \mathrm{id}_H))$$

to the power series

$$((\Delta \otimes \mathrm{id}_H)\mathfrak{F})(\theta(\mathfrak{F}(x \otimes 1, 1 \otimes y)) \otimes 1, 1 \otimes 1 \otimes z)$$

and putting $y = z = 0$.

Proof. Straightforward computation. $\square$

Note that $(\varepsilon_H \mathfrak{I})(x) = \theta(x)$.

Proposition 26. *Let A be an R-algebra and let φ be an R-homomorphism $H \to A$. Then*

$$\alpha \circ_\varphi \beta = (\varphi \mathfrak{H})(\alpha, \beta)$$

for all α, $\beta \in \mathfrak{n}(A)$ and $(\alpha)_\varphi^{-1} = (\varphi \mathfrak{I})(\alpha)$ for any $\alpha \in \mathfrak{n}(A)$.

Proof follows from the comparison of (19) and (20) with the lemmas above.$\square$

Remark 27. Let us remember that the set $\mathrm{Hom}_R(H, A)$ is a group with respect to the convolution $\star$: $\varphi \star \psi = \mu_A \circ (\varphi \otimes \psi) \circ \Delta \in \mathrm{Hom}_R(H, A)$ for all $\varphi, \psi \in \mathrm{Hom}_R(H, A)$. The convolution induces a group operation $\diamond$ on the set of formal group laws $\{(\varphi\mathfrak{H})(x, y) \in A[[x, y]] \mid \varphi \in \mathrm{Hom}_R(H, A)\}$ in the following way:

$$((\varphi\mathfrak{H})(x, y)) \diamond ((\psi\mathfrak{H})(x, y)) := ((\varphi \star \psi)\mathfrak{H})(x, y).$$

Note that the series $(\varepsilon\mathfrak{H})(x, y) = (\eta_A F)(x, y)$ is the identity element in considered group and $((\varphi \circ S)\mathfrak{H})(x, y) = (\varphi^{-1}\mathfrak{H})(x, y)$ is the inverse element for the given one $(\varphi\mathfrak{H})(x, y)$.

Let us recall that we have assumed that $(\mu_H \circ (\mathrm{id}_H \otimes S))\mathfrak{c} = 0$. It follows from Corollary 14 that

$$(\Delta\mathfrak{f})(\mathfrak{F}(x \otimes 1, 1 \otimes y)) = F(\mathfrak{f}(x) \otimes 1, 1 \otimes \mathfrak{f}(y)))$$

for an invertible power series $\mathfrak{f}(x) \in H[[x]]$ with zero constant term such that $(\varepsilon_H\mathfrak{f})(x) = x$.

Proposition 28. *We have:*

$$\mathfrak{H}(x, y) = \mathfrak{f}^{-1}(F(\mathfrak{f}(x), \mathfrak{f}(y))), \quad \mathfrak{I}(x) = \mathfrak{f}^{-1}(\theta(\mathfrak{f}(x))).$$

The power series $\mathfrak{H}(x, y)$ is a formal group law (in the usual sense) over the ring H with the formal inverse $\mathfrak{I}(x)$.

Remark 29. Note that the formal group law $\mathfrak{H}(x, y)$ is isomorphic to $F(x, y)$, considered as a formal group law over $H \supset R$.

Proof. We have:

$$\mathfrak{G}(x, y) = (\varepsilon_H\mathfrak{f}^{-1})(F(\mathfrak{f}(x), (S\mathfrak{f})(y))) = F(\mathfrak{f}(x), (S\mathfrak{f})(y)),$$

hence

$$\mathfrak{G}(y, 0) = F(\mathfrak{f}(y), 0) = \mathfrak{f}(y).$$

Also we have:

$$((\mathrm{id}_H \otimes \varepsilon_H)\mathfrak{F})(x, y) = \mathfrak{f}^{-1}(F(\mathfrak{f}(x), y)).$$

By substituting of $\mathfrak{G}(y, 0)$ for y in the latter formula, we obtain

$$\mathfrak{H}(x, y) = \mathfrak{f}^{-1}(F(\mathfrak{f}(x), \mathfrak{f}(y))),$$

as required.

Similarly,

$$((\varepsilon_H \otimes \mathrm{id}_H)\mathfrak{F})(x, y) = \mathfrak{f}^{-1}(F(x, \mathfrak{f}(y))).$$

Therefore,

$$\mathfrak{I}(x) = \mathfrak{f}^{-1}(F(\theta(\mathfrak{f}(x)), 0) = \mathfrak{f}^{-1}(\theta(\mathfrak{f}(x))). \quad \square$$

2 A generalization of the formal product in complex cobordism theory

2.1 The basic formula

By $\mathrm{BU}_\otimes$, $\mathrm{BSU}_\otimes$ we denote the classifying spaces BU, BSU with H-group structures related to the tensor product of virtual bundles of virtual dimension 1. Note that the H-group $\mathrm{BU}_\otimes$ (resp. $\mathrm{BSU}_\otimes$) represents the homotopy functor $X \mapsto \widetilde{\mathrm{K}}_{\mathbb{C}}(X)_\otimes$ (resp. $X \mapsto \widetilde{\mathrm{K}\,\mathrm{SU}}(X)_\otimes$) to the category of Abelian groups. Let $\widehat{\Xi}$ (resp. Ξ) be *the universal* U-*bundle* over BU *of virtual dimension* 1 (resp. *the universal* SU-*bundle* over BSU *of virtual dimension* 1).

It is known that there is an isomorphism of H-groups

$$\mathrm{BU}_\otimes \cong \mathrm{BSU}_\otimes \times \mathbb{C}P^\infty, \tag{23}$$

where the space $\mathbb{C}P^\infty$ is considered with respect to the usual H-group structure, related to the tensor product of line bundles. The proof of (23) will be given in the Appendix. Isomorphism of H-groups (23) corresponds to the following isomorphism of bundles

$$\widehat{\Xi} \cong \Xi \boxtimes \mathcal{Z},$$

where $\mathcal{Z}$ is the canonical (universal) line bundle over $\mathbb{C}P^\infty$ and $\boxtimes$ denotes the exterior tensor product of bundles.

We shall use the notation σ_r for the *r-th Conner-Floyd characteristic class* [2]. By s_r we denote the *r-th Newton characteristic class in complex cobordism theory*, corresponding to the symmetric polynomial $u_1^r + \ldots + u_n^r$ (note that $\sigma_1 = s_1$).

Let R be the cobordism ring $\mathrm{MU}^*(\mathrm{pt})$ and let H be the Hopf algebra $\mathrm{MU}^*(\mathrm{BSU}_\otimes)$ over R. Put $x = \sigma_1(\widehat{\Xi}) \in \mathrm{MU}^2(\mathrm{BU})$, $u = \sigma_1(\mathcal{Z}) \in \mathrm{MU}^2(\mathbb{C}P^\infty)$. It is well known that $\mathrm{MU}^*(\mathrm{BU}) = R[[\sigma_1(\widehat{\Xi}), \sigma_2(\widehat{\Xi}), \ldots]]$ and $H = \mathrm{MU}^*(\mathrm{BSU}) = R[[\sigma_2(\Xi), \sigma_3(\Xi), \ldots]]$. On the other hand, it follows from (23) that $\mathrm{MU}^*(\mathrm{BU}) = H[[u]]$. Hence $x = \mathfrak{h}(u)$ for some power series $\mathfrak{h}(u) \in H[[u]]$. The main goal of this section is to prove the explicit formula for this power series.

By definition, put $\widetilde{s}_i := s_i(\Xi) \in \mathrm{MU}^{2i}(\mathrm{BSU})$ (clearly $\widetilde{s}_1 = 0$).

Proposition 30.

$$\mathfrak{h}(u) = 1 \otimes u + \sum_{i,j \geq 1} \alpha_{i,j} \widetilde{s}_i \otimes u^j, \tag{24}$$

where $\alpha_{i,j} \in R$ are the coefficients of the formal product $F(u,v) = u + v + \sum \alpha_{i,j} u^i v^j$ in complex cobordism theory [1].

Proof. Let $\widehat{\Xi}_k$ (Ξ_k) be the universal k-dimensional $\mathrm{U}(k)$-bundle over $\mathrm{BU}(k)$ (the universal k-dimensional $\mathrm{SU}(k)$-bundle over $\mathrm{BSU}(k)$ respectively). Consider the restriction of the virtual bundle $\Xi_k \otimes [k] - [k(k-1)]$ (where by $[n]$ we denote the trivial n-dimensional bundle) to the $2k$-skeleton $\mathrm{BSU}(k)^{(2k)}$. There exists a (unique up to isomorphism) geometric k-dimensional representative $\Xi_k^\bullet$ of the stable equivalence class $\{\Xi_k \otimes [k] - [k(k-1)]\} \in \mathrm{K}_{\mathbb{C}}(\mathrm{BSU}(k)^{(2k)})$.

Let

$$\lambda_k\colon\ \mathrm{BSU}(k)^{(2k)} \to \mathrm{BU}(k)$$

be a classifying map for $\Xi_k^\bullet$ (i.e. $\lambda_k^*(\widehat{\Xi}_k) = \Xi_k^\bullet$). It follows easily that

$$\lambda_k^*(s_i(\widehat{\Xi}_k)) = s_i(\lambda_k^*(\widehat{\Xi}_k)) = k\widetilde{s}_i^{(k)},$$

where $\widetilde{s}_i^{(k)} := s_i(\Xi_k - [k-1]) \in \mathrm{MU}^{2i}(\mathrm{BSU}(k)^{(2k)})$.

Now let us apply the splitting principle to the tensor product $\widehat{\Xi}_k \boxtimes \mathcal{Z}$:

$$\begin{tikzcd}
\widehat{\Xi}_k \boxtimes \mathcal{Z} \arrow[d] & (\mathcal{Z}_1 \oplus \ldots \oplus \mathcal{Z}_k) \boxtimes \mathcal{Z} \arrow[l] \arrow[d] \\
\mathrm{BU}(k) \times \mathbb{C}P^\infty & \underbrace{\mathbb{C}P^\infty \times \ldots \times \mathbb{C}P^\infty}_{k\ times} \times \mathbb{C}P^\infty. \arrow[l, "\widehat{\mu}_k \times \mathrm{id}_{\mathbb{C}P}"']
\end{tikzcd}$$

We have:

$$(\widehat{\mu}_k \times \mathrm{id}_{\mathbb{C}P})^*(\widehat{\Xi}_k \boxtimes \mathcal{Z}) = (\mathcal{Z}_1 \oplus \ldots \oplus \mathcal{Z}_k) \boxtimes \mathcal{Z} = (\mathcal{Z}_1 \boxtimes \mathcal{Z}) \oplus \ldots \oplus (\mathcal{Z}_k \boxtimes \mathcal{Z}) := \Theta_k,$$

$$s_r(\Theta_k) = s_1(\mathcal{Z}_1 \boxtimes \mathcal{Z})^r + \ldots + s_1(\mathcal{Z}_k \boxtimes \mathcal{Z})^r = F(v_1 \otimes 1, 1 \otimes u)^r + \ldots + F(v_k \otimes 1, 1 \otimes u)^r.$$

In particular,

$$s_1(\Theta_k) = v_1 \otimes 1 + 1 \otimes u + \sum_{i,j\geq 1} \alpha_{i,j} v_1^i \otimes u^j + \ldots + v_k \otimes 1 + 1 \otimes u + \sum_{i,j\geq 1} \alpha_{i,j} v_k^i \otimes u^j$$

$$= (\widehat{\mu}_k \times \mathrm{id}_{\mathbb{C}P})^*(s_1(\widehat{\Xi}_k) \otimes 1 + k \otimes u + \sum_{i,j\geq 1} \alpha_{i,j} s_i(\widehat{\Xi}_k) \otimes u^j),$$

hence

$$s_1(\widehat{\Xi}_k \boxtimes \mathcal{Z}) = s_1(\widehat{\Xi}_k) \otimes 1 + k \otimes u + \sum_{i,j\geq 1} \alpha_{i,j} s_i(\widehat{\Xi}_k) \otimes u^j.$$

By $x^{(k)}$ denote the first Conner-Floyd class

$$\sigma_1((\Xi_k - [k-1]) \boxtimes \mathcal{Z}) = s_1((\Xi_k - [k-1]) \boxtimes \mathcal{Z}) \in \mathrm{MU}^2(\mathrm{BSU}(k)^{(2k)} \times \mathbb{C}P^\infty).$$

Consider the following diagram:

$$\begin{tikzcd}
\Xi_k^\bullet \boxtimes \mathcal{Z} \arrow[r] \arrow[d] & \widehat{\Xi}_k \boxtimes \mathcal{Z} \arrow[d] \\
\mathrm{BSU}(k)^{(2k)} \times \mathbb{C}P^\infty \arrow[r, "\lambda_k \times \mathrm{id}_{\mathbb{C}P}"] & \mathrm{BU}(k) \times \mathbb{C}P^\infty,
\end{tikzcd}$$

where λ_k is the classifying map for $\Xi_k^\bullet$, as above. We have:

$$s_1(\Xi_k^\bullet \boxtimes \mathcal{Z}) = \lambda_k^*(s_1(\widehat{\Xi}_k)) \otimes 1 + k \otimes u + \sum_{i,j\geq 1} \alpha_{i,j} \lambda_k^*(s_i(\widehat{\Xi}_k)) \otimes u^j$$

$$= k \otimes u + k \sum_{i,j\geq 1} \alpha_{i,j} \widetilde{s}_i^{(k)} \otimes u^j = k(1 \otimes u + \sum_{i,j\geq 1} \alpha_{i,j} \widetilde{s}_i^{(k)} \otimes u^j)$$

(let us recall that $\lambda_k^*(s_1(\widehat{\Xi}_k)) = s_1(\Xi_k^\bullet) = 0$, because $\Xi_k^\bullet$ is an $\mathrm{SU}(k)$-bundle). Since

$$\Xi_k^\bullet \boxtimes \mathcal{Z} = ((\Xi_k - [k-1]) \otimes [k]) \boxtimes \mathcal{Z} = ((\Xi_k - [k-1]) \boxtimes \mathcal{Z}) \otimes [k],$$

we get

$$x^{(k)} = 1 \otimes u + \sum_{i,j \geq 1} \alpha_{i,j} \widetilde{s}_i^{(k)} \otimes u^j. \tag{25}$$

Using the isomorphism of the bundles $\widehat{\Xi} \cong \Xi \boxtimes \mathcal{Z}$, it can easily be checked that the sequence

$$\{x^{(k)}\}_{k \in \mathbb{N}},\ x^{(k)} \in \mathrm{MU}^2(\mathrm{BSU}(k)^{(2k)} \times \mathbb{C}P^\infty)$$

defines the element $x = \sigma_1(\widehat{\Xi}) \in \mathrm{MU}^2(\mathrm{BSU} \times \mathbb{C}P^\infty) = \mathrm{MU}^2(\mathrm{BU})$. Therefore the required formula (24) may be obtained as the limit of (25), as k tends to ∞. $\square$

Remark 31. Let us remark that the series $\mathfrak{h}(u)$ is invertible, i.e. there exists the series $\mathfrak{h}^{-1}(x) \in H[[x]]$ such that $\mathfrak{h}^{-1}(\mathfrak{h}(u)) = u$, $\mathfrak{h}(\mathfrak{h}^{-1}(x)) = x$. Moreover, $(\varepsilon_H \mathfrak{h})(u) = u$.

The remainder of this section is devoted to the description of the comultiplication in the Hopf algebra $\widehat{H} := \mathrm{MU}^*(\mathrm{BU}_\otimes)$ (consequently, in the Hopf algebra $H = \mathrm{MU}^*(\mathrm{BSU}_\otimes)$).

Consider the following diagram

$$\begin{array}{ccc}
(\mathbb{C}P^\infty)^k \times (\mathbb{C}P^\infty)^l & \xrightarrow{\Psi_{k,l}} & (\mathbb{C}P^\infty)^{kl} \\
\Big\downarrow{\scriptstyle \widehat{\mu}_k \times \widehat{\mu}_l} & & \Big\downarrow{\scriptstyle \widehat{\mu}_{kl}} \\
\mathrm{BU}(k) \times \mathrm{BU}(l) & \xrightarrow{\widehat{\Phi}_{k,l}} & \mathrm{BU}(kl),
\end{array}$$

where the map $\widehat{\Phi}_{k,l}$ such that $\widehat{\Phi}_{k,l}^*(\widehat{\Xi}_{kl}) = \widehat{\Xi}_k \boxtimes \widehat{\Xi}_l$. We have the corresponding diagram for characteristic classes:

$$\begin{array}{ccc}
\sum_{i,j} F(u_i \otimes 1, 1 \otimes v_j)^r & \xleftarrow{\Psi_{k,l}^*} & \sum_{i,j} w_{ij}^r \\
\Big\uparrow{\scriptstyle \widehat{\mu}_k^* \otimes \widehat{\mu}_l^*} & & \Big\uparrow{\scriptstyle \widehat{\mu}_{kl}^*} \\
s_r(\widehat{\Xi}_k \boxtimes \widehat{\Xi}_l) = \widehat{\Phi}_{k,l}^*(s_r(\widehat{\Xi}_{kl})) & \xleftarrow{\widehat{\Phi}_{k,l}^*} & s_r(\widehat{\Xi}_{kl}),
\end{array}$$

where $1 \leq i \leq k$, $1 \leq j \leq l$, $r \geq 1$. Clearly the power series

$$\sum_{1 \leq i \leq k, 1 \leq j \leq l} F(u_i \otimes 1, 1 \otimes v_j)^r$$

may be expressed in terms of $s_m(u_1, \dots, u_k) := u_1^m + \ldots + u_k^m$ and $s_n(v_1, \dots, v_l) := v_1^n + \ldots + v_l^n$, $m, n \geq 1$. Let us define $\gamma_{m,n}^{(r)} \in R$ as follows:

$$F(u,v)^r = u^r + v^r + \sum_{m,n \geq 1,\, m+n \geq r} \gamma_{m,n}^{(r)} u^m v^n.$$

In particular, $\gamma_{m,n}^{(1)} = \alpha_{m,n} \in R$ are the coefficients of the formal product in complex cobordism theory. It is easily shown that

$$\widehat{\Phi}_{k,l}^*(s_r(\widehat{\Xi}_{kl})) = s_r(\widehat{\Xi}_k) \otimes l + k \otimes s_r(\widehat{\Xi}_l) + \sum_{m,n \geq 1} \gamma_{m,n}^{(r)} s_m(\widehat{\Xi}_k) \otimes s_n(\widehat{\Xi}_l). \tag{26}$$

In particular, if $r = 1$, we have:

$$\widehat{\Phi}^*_{k,l}(s_1(\widehat{\Xi}_{kl})) = s_1(\widehat{\Xi}_k) \otimes l + k \otimes s_1(\widehat{\Xi}_l) + \sum_{m,n\geq 1} \alpha_{m,n} s_m(\widehat{\Xi}_k) \otimes s_n(\widehat{\Xi}_l). \tag{27}$$

Let $\widehat{\Xi}^\bullet_n$ be the geometric representative of the stable equivalence class

$$\{\widehat{\Xi}_n \otimes [n] - [n(n-1)]\} \in \mathrm{K}_{\mathbb{C}}(\mathrm{BU}(n)^{(2n)}).$$

Let $\widehat{\lambda}_n\colon \mathrm{BU}(n)^{(2n)} \to \mathrm{BU}(n)$ be the classifying map of $\widehat{\Xi}^\bullet_n$, i.e. $\widehat{\lambda}^*_n(\widehat{\Xi}_n) = \widehat{\Xi}^\bullet_n$. Clearly,

$$\widehat{\lambda}^*_n(s_r(\widehat{\Xi}_n)) = s_r(\widehat{\Xi}^\bullet_n) = n s_r(\widehat{\Xi}_n - [n-1]),\ r \geq 0. \tag{28}$$

The bundle $\widehat{\Xi}^\bullet_n$ over $\mathrm{BU}(n)^{(2n)}$ has the following universal property.

Lemma 32. Let X be a finite CW-complex, $\dim(X) \leq 2n$. Let η_n be an n-dimensional complex vector bundle over X such that $\{\eta_n\} = n\{\theta_1\} \in \mathrm{K}_{\mathbb{C}}(X)$, where θ_1 is a virtual bundle of virtual dimension 1. Then there exists a (unique up to homotopy) map $\tau\colon X \to \mathrm{BU}(n)^{(2n)}$ such that $\eta_n = \tau^*(\widehat{\Xi}^\bullet_n)$.

Proof. The proof is trivial. □

Applying the previous lemma to the bundle $\widehat{\Xi}^\bullet_k \boxtimes \widehat{\Xi}^\bullet_l$ over $\mathrm{BU}(k)^{(2k)} \times \mathrm{BU}(l)^{(2l)}$, we get a classifying map

$$\widehat{\phi}_{k,l}\colon \mathrm{BU}(k)^{(2k)} \times \mathrm{BU}(l)^{(2l)} \to \mathrm{BU}(kl)^{(2kl)}$$

such that $\widehat{\phi}^*_{k,l}(\widehat{\Xi}^\bullet_{kl}) = \widehat{\Xi}^\bullet_k \boxtimes \widehat{\Xi}^\bullet_l$ and the following diagram

$$\begin{array}{ccc} \mathrm{BU}(k) \times \mathrm{BU}(l) & \xrightarrow{\widehat{\Phi}_{k,l}} & \mathrm{BU}(kl) \\ \Big\uparrow{\scriptstyle \widehat{\lambda}_k \times \widehat{\lambda}_l} & & \Big\uparrow{\scriptstyle \widehat{\lambda}_{kl}} \\ \mathrm{BU}(k)^{(2k)} \times \mathrm{BU}(l)^{(2l)} & \xrightarrow{\widehat{\phi}_{k,l}} & \mathrm{BU}(kl)^{(2kl)} \end{array}$$

commutes (up to homotopy). Using (26), (28) and the previous diagram, we get:

$$\widehat{\phi}^*_{k,l}(s_r(\widehat{\Xi}_{kl} - [kl-1])) = s_r(\widehat{\Xi}_k - [k-1]) \otimes 1 + 1 \otimes s_r(\widehat{\Xi}_l - [l-1])$$
$$+ \sum_{m,n\geq 1} \gamma^{(r)}_{m,n} s_m(\widehat{\Xi}_k - [k-1]) \otimes s_n(\widehat{\Xi}_l - [l-1]).$$

Thus, passing to the limit as $k, l \to \infty$, we get:

$$\Delta_{\widehat{H}}(s_r(\widehat{\Xi})) = s_r(\widehat{\Xi}) \otimes 1 + 1 \otimes s_r(\widehat{\Xi}) + \sum_{m,n\geq 1} \gamma^{(r)}_{m,n} s_m(\widehat{\Xi}) \otimes s_n(\widehat{\Xi}),\ r \geq 1. \tag{29}$$

In particular, using (27), we obtain:

$$\widehat{\phi}^*_{k,l}(s_1(\widehat{\Xi}_{kl} - [kl-1])) = s_1(\widehat{\Xi}_k - [k-1]) \otimes 1 + 1 \otimes s_1(\widehat{\Xi}_l - [l-1])$$
$$+ \sum_{m,n\geq 1} \alpha_{m,n} s_m(\widehat{\Xi}_k - [k-1]) \otimes s_n(\widehat{\Xi}_l - [l-1]),$$

i.e. (passing to the limit as $k, l \to \infty$)

$$\Delta_{\widehat{H}}(x) = x \otimes 1 + 1 \otimes x + \sum_{m,n \geq 1} \alpha_{m,n} s_m(\widehat{\Xi}) \otimes s_n(\widehat{\Xi})$$

(compare with (32)).

Since the rings $\mathrm{MU}^*(\mathrm{BSU})$, $\mathrm{MU}^*(\mathrm{BU})$ have no torsion and $\mathrm{MU}^*(\mathrm{BSU})\widehat{\underset{\mathbb{Z}}{\otimes}}\mathbb{Q} = R_{\mathbb{Q}}[[s_2(\Xi), s_3(\Xi), \dots]]$, $\mathrm{MU}^*(\mathrm{BU})\widehat{\underset{\mathbb{Z}}{\otimes}}\mathbb{Q} = R_{\mathbb{Q}}[[s_1(\widehat{\Xi}), s_2(\widehat{\Xi}), \dots]]$, formula (29) gives the complete description of the comultiplication of considered Hopf algebras.

2.2 The formal group law over $H = \mathrm{MU}^*(\mathrm{BSU}_\otimes)$ related to virtual bundles of virtual dimension 1

The aim of this section is to obtain a generalization of the formal product in complex cobordism theory from the case of complex *line bundles* to the case of all complex *vector bundles of virtual dimension* 1.

Since $\mathrm{BU}_\otimes$ is an H-group, we have the product

$$\widehat{\phi}\colon\ \mathrm{BU}_\otimes \times \mathrm{BU}_\otimes \to \mathrm{BU}_\otimes$$

and the inversion $\widehat{\iota}\colon\ \mathrm{BU}_\otimes \to \mathrm{BU}_\otimes$.

As above, by H we denote the Hopf algebra $\mathrm{MU}^*(\mathrm{BSU}_\otimes)$ over the ring $R = \mathrm{MU}^*(\mathrm{pt})$.

Proposition 33.

$$\mathrm{MU}^*(\mathrm{BU}) = H[[x]], \quad \mathrm{MU}^*(\mathrm{BU} \times \mathrm{BU}) = (H\widehat{\underset{R}{\otimes}}H)[[x \otimes 1, 1 \otimes x]], \tag{30}$$

where $x = \sigma_1(\widehat{\Xi})$ *as above. Moreover,*

$$\widehat{\phi}^*(x) = \mathfrak{F}(x \otimes 1, 1 \otimes x) \in (H\widehat{\underset{R}{\otimes}}H)[[x \otimes 1, 1 \otimes x]] \tag{31}$$

is a commutative formal group law over the Hopf algebra H *with the formal inverse* $\Theta(x) = \widehat{\iota}^*(x) \in H[[x]]$.

Proof. It follows from (23) that $\mathrm{MU}^*(\mathrm{BU}) = H[[u]]$. On the other hand, the explicit formula

$$x = \mathfrak{h}(u) = u + \sum_{i,j \geq 1} \alpha_{i,j} \widetilde{s}_i u^j \in H[[u]],$$

obtained in the previous section (see also Remark 31), shows that $H[[u]] = H[[x]]$. This verifies equalities (30). Hence the element $\widehat{\phi}^*(x)$ is the power series $\mathfrak{F}(x \otimes 1, 1 \otimes x) \in (H\widehat{\underset{R}{\otimes}}H)[[x \otimes 1, 1 \otimes x]]$.

To verify the associative axiom 1) of Definition 1 note that the maps $\widehat{\phi} \circ (\widehat{\phi} \otimes \mathrm{id}_{\mathrm{BU}})$ and $\widehat{\phi} \circ (\mathrm{id}_{\mathrm{BU}} \otimes \widehat{\phi})$ are classifying maps for the bundle $\widehat{\Xi} \boxtimes \widehat{\Xi} \boxtimes \widehat{\Xi}$ over $\mathrm{BU} \times \mathrm{BU} \times \mathrm{BU}$ and hence they are homotopic. Therefore

$$((\Delta_H \otimes \mathrm{id}_H)\mathfrak{F})(\mathfrak{F}(x \otimes 1, 1 \otimes x) \otimes 1, 1 \otimes 1 \otimes x)$$

$$= (\widehat{\phi} \otimes \mathrm{id}_{\mathrm{BU}})^* \circ \widehat{\phi}^*(x) = (\mathrm{id}_{\mathrm{BU}} \otimes \widehat{\phi})^* \circ \widehat{\phi}^*(x)$$

$$= ((\mathrm{id}_H \otimes \Delta_H)\mathfrak{F})(x \otimes 1 \otimes 1, 1 \otimes \mathfrak{F}(x \otimes 1, 1 \otimes x)).$$

To verify the identity axiom 2) of Definition 1 note that

$$((\mathrm{id}_H \otimes \varepsilon)\mathfrak{F})(x \otimes 1, 0), \quad ((\varepsilon \otimes \mathrm{id}_H)\mathfrak{F})(0, 1 \otimes x)$$

is the image of x under the composite map $i_1^* \circ \widehat{\phi}^*$, $i_2^* \circ \widehat{\phi}^*$ respectively, where $i_k\colon \mathrm{BU} \to \mathrm{BU} \times \mathrm{BU}$ by $i_1(y) = (y, *)$, $i_2(y) = (*, y)$. Clearly that both $\widehat{\phi} \circ i_1$ and $\widehat{\phi} \circ i_2$ are homotopic to the identity map.

To verify the formal inversion axiom 3) of Definition 1 note that

$$\widehat{\phi} \circ (\mathrm{id}_{\mathrm{BU}} \times \widehat{\iota}) \circ \mathrm{diag}_{\mathrm{BU}} \simeq *, \quad \widehat{\phi} \circ (\widehat{\iota} \times \mathrm{id}_{\mathrm{BU}}) \circ \mathrm{diag}_{\mathrm{BU}} \simeq *,$$

where $\mathrm{diag}_{\mathrm{BU}}\colon \mathrm{BU} \to \mathrm{BU} \times \mathrm{BU}$ is the diagonal map. Therefore

$$\mathrm{diag}_{\mathrm{BU}}^* \circ (\mathrm{id}_{\mathrm{BU}} \times \widehat{\iota})^* \circ \widehat{\phi}^*(x) = ((\mu \circ (\mathrm{id}_H \otimes S))\mathfrak{F})(x, \Theta(x)) = 0$$

$$= ((\mu \circ (S \otimes \mathrm{id}_H))\mathfrak{F})(\Theta(x), x) = \mathrm{diag}_{\mathrm{BU}}^* \circ (\widehat{\iota} \times \mathrm{id}_{\mathrm{BU}})^* \circ \widehat{\phi}^*(x). \quad \square$$

The following proposition asserts that the formal group law $\mathfrak{F}(x \otimes 1, 1 \otimes x)$ is strongly isomorphic to the trivial extension of the formal product $F(u, v)$ in complex cobordism theory by the Hopf algebra H. Moreover, the explicit form of the change of variable is given.

Proposition 34.

$$\mathfrak{F}(x \otimes 1, 1 \otimes x) = (\Delta_H \mathfrak{h})(F((\mathfrak{h}^{-1}(x)) \otimes 1, 1 \otimes (\mathfrak{h}^{-1}(x)))). \tag{32}$$

Moreover $\Theta(x) = (S_H \mathfrak{h})(\theta(\mathfrak{h}^{-1}(x)))$, *where* $\Theta(x) \in H[[x]]$ *is the formal inverse for the formal group law* $\mathfrak{F}(x \otimes 1, 1 \otimes x)$.

Remark 35. It follows from (24) that the formal group law $\mathfrak{F}(x \otimes 1, 1 \otimes x)$ is strongly isomorphic to the trivial extension of $F(u, v)$ by H. Indeed, $(\varepsilon_H \mathfrak{h})(u) = u$, in particular, $\varepsilon_H(\mathfrak{h}'(0)) = \varepsilon_H(1 + \sum_{i \geq 2} \alpha_{i,1} \widetilde{s}_i) = 1$.

Proof. Note that (23) implies that the Hopf algebra $\mathrm{MU}^*(\mathrm{BU}_\otimes)$ is isomorphic to the tensor product of Hopf algebras

$$H[[u]] = H \widehat{\otimes}_R R[[u]], \tag{33}$$

i.e. the comultiplication

$$\widehat{\Delta}\colon H[[u]] \to (H \widehat{\otimes}_R H)[[u \otimes 1, 1 \otimes u]]$$

is following: $\widehat{\Delta}|_H = \Delta_H$, $\widehat{\Delta}(u) = F(u \otimes 1, 1 \otimes u)$, where $F(u, v)$ is the formal product in complex cobordism theory; and the antipode

$$\widehat{S}\colon H[[u]] \to H[[u]]$$

is following: $\widehat{S}|_H = S_H$, $\widehat{S}(u) = \theta(u)$, where $\theta(u)$ is the formal inverse in the formal group law $F(u, v)$.

Using isomorphism (33), we get:

$$\begin{aligned}\widehat{\phi}^*(x) = \widehat{\Delta}(x) = \widehat{\Delta}(\mathfrak{h}(u)) &= (\Delta_H \mathfrak{h})(F(u \otimes 1, 1 \otimes u)) \\ &= (\Delta_H \mathfrak{h})(F((\mathfrak{h}^{-1}(x)) \otimes 1, 1 \otimes (\mathfrak{h}^{-1}(x)))).\end{aligned}$$

Using (33) again, we get:

$$\widehat{\iota}^*(x) = \Theta(x) = (S_H \mathfrak{h})(\theta(u)) = (S_H \mathfrak{h})(\theta(\mathfrak{h}^{-1}(x))). \quad \square$$

Remark 36. Note that $((\varepsilon_H \otimes \varepsilon_H)\mathfrak{F})(u, v) = F(u, v)$ and $(\varepsilon_H \Theta)(u) = \theta(u)$ (see Remark 31).

Remark 37. Now we mention briefly the way to obtain another extensions of the formal product $F(u, v)$ by H. To this purpose we can consider cohomology operations. For the Adams operation Ψ^k let us define the endomorphism ρ_k of the Hopf algebra H by setting $\rho_k(\sigma)(\Xi) = \sigma(\Psi^k \Xi)$ for any $\sigma \in H$, where Ξ is the universal SU-bundle over BSU. Put $x_k := \sigma_1((\Psi^k \Xi) \boxtimes \mathcal{Z}) \in \mathrm{MU}^2(\mathrm{BSU} \times \mathbb{C}P^\infty) = \mathrm{MU}^2(\mathrm{BU})$. Clearly

$$x_k = (\rho_k \mathfrak{h})(u) = u + \sum_{i,j \geq 1} \alpha_{i,j} \rho_k(\widetilde{s}_i) u^j.$$

By $\mathfrak{h}_k(u)$ denote the power series $(\rho_k \mathfrak{h})(u) \in H[[u]]$. If we replace $\mathfrak{h}(u)$ by $\mathfrak{h}_k(u)$ in (32), we get some other extension

$$\begin{aligned}\mathfrak{F}_k(x_k \otimes 1, 1 \otimes x_k) &:= (\Delta_H \mathfrak{h}_k)(F((\mathfrak{h}_k^{-1}(x_k)) \otimes 1, 1 \otimes (\mathfrak{h}_k^{-1}(x_k)))) \\ &\in (H \mathop{\widehat{\otimes}}_R H)[[x_k \otimes 1, 1 \otimes x_k]] = \mathrm{MU}^*(\mathrm{BU}) \mathop{\widehat{\otimes}}_R \mathrm{MU}^*(\mathrm{BU})\end{aligned}$$

of the formal group law $F(u, v)$ by the Hopf algebra H (note that $\mathfrak{F}_1(x_1 \otimes 1, 1 \otimes x_1) = \mathfrak{F}(x \otimes 1, 1 \otimes x)$). We claim that $\mathfrak{F}_k(x_k \otimes 1, 1 \otimes x_k) = ((\rho_k \otimes \rho_k)\mathfrak{F})(x_k \otimes 1, 1 \otimes x_k)$. Indeed, it follows from the equality $(\rho_k \otimes \rho_k) \circ \Delta_H = \Delta_H \circ \rho_k$.

Applying the endomorphisms of the Hopf algebra H, corresponding to the operations $\Xi \mapsto \Xi^k$, $k \in \mathbb{Z}$, to the coefficients of the power series $\mathfrak{h}(u)$, we obtain another extensions.

Let $g(u) = u + \sum_{j \geq 1} \beta_j u^{j+1} \in R_{\mathbb{Q}}[[u]]$ be the logarithm of the formal product $F(u, v)$ in complex cobordism theory [6]. By $\overline{\mathfrak{g}}(x)$ denote the power series $g(\mathfrak{h}^{-1}(x)) \in H_{\mathbb{Q}}[[x]]$. Note that $g(u) = \overline{\mathfrak{g}}(x)$ as elements of $\mathrm{MU}^2(\mathrm{BU}) \mathop{\widehat{\otimes}}_{\mathbb{Z}} \mathbb{Q}$, because of $x = \mathfrak{h}(u)$.

Proposition 38. *The power series $\overline{\mathfrak{g}}(x)$ is the logarithm of the formal group law $\mathfrak{F}(x \otimes 1, 1 \otimes x)$ over the Hopf algebra H in the sense of Theorem 9.*

Proof. Let us recall that in Theorem 9 the power series

$$\mathfrak{g}(x) := \int_0^x \frac{dt}{\widetilde{\omega}(t)} \in H_{\mathbb{Q}}[[x]]$$

was considered, where

$$\widetilde{\omega}(x) := (\mathrm{id} \otimes \widetilde{\varepsilon}) \frac{\partial \mathfrak{F}(x \otimes 1, 1 \otimes z)}{\partial (1 \otimes z)} \in H[[x]]$$

(here $\widetilde{\varepsilon}\colon H[[z]] \to R$ is the homomorphism such that $\widetilde{\varepsilon}\mid_H = \varepsilon_H\colon H \to R$, $\widetilde{\varepsilon}(z) = 0$). We must prove that $\mathfrak{g}(x) = \overline{\mathfrak{g}}(x)$. We have:

$$\frac{\partial \mathfrak{F}(x\otimes 1, 1\otimes z)}{\partial(1\otimes z)} = \frac{d(\Delta_H\mathfrak{h})(F(\mathfrak{h}^{-1}(x)\otimes 1, 1\otimes \mathfrak{h}^{-1}(z)))}{dF(\mathfrak{h}^{-1}(x)\otimes 1, 1\otimes \mathfrak{h}^{-1}(z))}\cdot$$

$$\frac{\partial F(\mathfrak{h}^{-1}(x)\otimes 1, 1\otimes \mathfrak{h}^{-1}(z))}{\partial(1\otimes \mathfrak{h}^{-1}(z))}\cdot\left(1\otimes\frac{d(\mathfrak{h}^{-1}(z))}{dz}\right).$$

Applying the operator $(\mathrm{id}\otimes\widetilde{\varepsilon})$ to the both sides of the previous equation, we get:

$$\widetilde{\omega}(x) = \mathfrak{h}'(\mathfrak{h}^{-1}(x))\cdot\omega(\mathfrak{h}^{-1}(x))\cdot\frac{1}{\varepsilon_H(\mathfrak{h}'(0))}$$

(we used $\mathfrak{h}^{-1}(0) = 0$), where $\omega(y) := \frac{\partial F(y,t)}{\partial t}|_{t=0}$. Recall that $\varepsilon_H(\mathfrak{h}'(0)) = 1$. Therefore we obtain:

$$\mathfrak{g}'(x) = \frac{dg(\mathfrak{h}^{-1}(x))}{dx}.$$

Using $\mathfrak{g}(0) = 0 = g(\mathfrak{h}^{-1}(0))$, we obtain the required identity $\overline{\mathfrak{g}}(x) = \mathfrak{g}(x)$. $\square$

Due to this coincidence, we shall use the notation $\mathfrak{g}(x)$ instead of $\overline{\mathfrak{g}}(x)$ for the series $g(\mathfrak{h}^{-1}(x))$.

Reminder. We have proved in Theorem 9 that the power series $\mathfrak{g}(x)$ has the following properties:

(i) $(\Delta\mathfrak{g})(\mathfrak{F}(x\otimes 1, 1\otimes x)) = \mathfrak{c} + \mathfrak{g}(x)\otimes 1 + 1\otimes\mathfrak{g}(x)$;

(ii) $\mathfrak{g}(0) = 0$;

(iii) $(\varepsilon_H\mathfrak{g})(u) = g(u)$.

One can check the property (i) by the direct computation, using $\mathfrak{g}(x) = g(\mathfrak{h}^{-1}(x)) = g(u) \in \mathrm{MU}^2(\mathrm{BU})\widehat{\otimes}_{\mathbb{Z}}\mathbb{Q}$. Indeed, we have:

$$\widehat{\Delta}(g(u)) = g(F(u\otimes 1, 1\otimes u)) = g(u)\otimes 1 + 1\otimes g(u)$$

$$= \mathfrak{g}(x)\otimes 1 + 1\otimes\mathfrak{g}(x) = \widehat{\Delta}(\mathfrak{g}(x))$$

$$= (\Delta\mathfrak{g})(\mathfrak{F}(x\otimes 1, 1\otimes x))$$

(in particular we see that the cocycle $\mathfrak{c}$ equals 0). The property $\mathfrak{g}(0) = 0$ easily follows from the explicit formula (24) for $\mathfrak{h}(u)$ and $g(0) = 0$. $\square$

Now let us return to isomorphism (23). Let $p\colon \mathrm{BU}_\otimes \to \mathrm{BSU}_\otimes$ be the projection onto the first factor. Then p is a morphism of H-groups. Let ξ be a virtual vector U-bundle of virtual dimension 1 over a finite CW-complex X. Let $\varphi_\xi\colon X \to \mathrm{BU}$ be a classifying map for ξ. Hence the composite map $\varphi_{\overline{\xi}} := p\circ\varphi_\xi\colon X \to \mathrm{BSU}$ is the classifying map for the corresponding virtual SU-bundle $\overline{\xi}$ of virtual dimension 1 (let us remark that for any ξ there exists a unique line bundle ζ such that $c_1(\xi\otimes\zeta) = 0$, where c_1 is the first Chern class; clearly $\overline{\xi} = \xi\otimes\zeta$). By μ_X denote the multiplication $\mathrm{MU}^*(X)\otimes_R \mathrm{MU}^*(X) \to \mathrm{MU}^*(X)$.

Theorem 39. *For any virtual complex vector bundles ξ, η of virtual dimension 1 over X we have*

$$\sigma_1(\xi\otimes\eta)=(\mu_X\circ(\varphi^*_{\overline{\xi}}\otimes\varphi^*_{\overline{\eta}})\mathfrak{F})(\sigma_1(\xi),\ \sigma_1(\eta)). \tag{34}$$

Let ξ^{-1} be the inverse vector bundle for ξ in sense of H-group structure $\mathrm{BU}_{\otimes}$ (i.e. $\xi\otimes\xi^{-1}=[1]$ is the trivial bundle). Then

$$\sigma_1(\xi^{-1})=(\varphi^*_{\overline{\xi}}\Theta)(\sigma_1(\xi)) \tag{35}$$

and we have

$$(\mu_X\circ(\varphi^*_{\overline{\xi}}\otimes(\varphi^*_{\overline{\xi}}\circ S_H))\mathfrak{F})(\sigma_1(\xi),\sigma_1(\xi^{-1}))=0 \tag{36}$$

(note that $\varphi^*_{\overline{\xi}}\circ S_H=\varphi^*_{\overline{\xi}^{-1}}$, $\overline{\xi}\otimes\overline{\xi}^{-1}=[1]$ is the trivial bundle).

Remark 40. If ξ, η are line bundles (i.e. geometric), then (34) reduces to the well-known formula $\sigma_1(\xi\otimes\eta)=F(\sigma_1(\xi),\sigma_1(\eta))$, where $F(u,v)\in R[[u,v]]$ is the formal product in complex cobordism theory (note that in the last case $\varphi_{\overline{\xi}}$, $\varphi_{\overline{\eta}}$ are maps to a point). Moreover, in this case $\sigma_1(\xi^{-1})=(\varphi^*_{\overline{\xi}}\Theta)(\sigma_1(\xi))=\theta(\sigma_1(\xi))$ (see Remark 36).

Proof. The proof is based on the following observation. Since the diagram

$$\begin{tikzcd}
 & \mathrm{BU}_{\otimes} \arrow[d, "p"] \\
X \arrow[ur, "\varphi_{\xi}"] \arrow[r, "\varphi_{\overline{\xi}}"'] & \mathrm{BSU}_{\otimes}
\end{tikzcd}$$

commutes, $\varphi^*_{\xi}|_H=\varphi^*_{\overline{\xi}}\colon H\to\mathrm{MU}^*(X)$ (we consider H as a Hopf subalgebra in $\mathrm{MU}^*(\mathrm{BU}_{\otimes})$). More precisely, consider the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{BU}_{\otimes}\times\mathrm{BU}_{\otimes} \arrow[rr, "\widehat{\phi}"] \arrow[d, "p\times p"] & & \mathrm{BU}_{\otimes} \arrow[d, "p"] \\
 & \mathrm{BSU}_{\otimes}\times\mathrm{BSU}_{\otimes} \arrow[rr, "\phi"'] & & \mathrm{BSU}_{\otimes} \\
X\times X \arrow[uur, "\varphi_{\xi}\times\varphi_{\eta}"] \arrow[ur, "\varphi_{\overline{\xi}}\times\varphi_{\overline{\eta}}"'] & & X, \arrow[ll, "\mathrm{diag}_X"] \arrow[uur, "\dagger"] \arrow[ur, "\varphi_{\overline{\xi}\otimes\overline{\eta}}"'] &
\end{tikzcd}$$

where by $\dagger$ we denote the classifying map $\varphi_{\xi\otimes\eta}$ and ϕ is the product in the H-group $\mathrm{BSU}_{\otimes}$. We have:

$$\begin{aligned}
\sigma_1(\xi\otimes\eta)&=\varphi^*_{\xi\otimes\eta}(x)=(\mu_X\circ(\varphi^*_{\xi}\otimes\varphi^*_{\eta})\circ\widehat{\phi}^*)(x)\\
&=(\mu_X\circ(\varphi^*_{\xi}\otimes\varphi^*_{\eta}))(\mathfrak{F}(x\otimes 1,1\otimes x))=(\mu_X\circ(\varphi^*_{\overline{\xi}}\otimes\varphi^*_{\overline{\eta}})\mathfrak{F})(\varphi^*_{\xi}(x),\varphi^*_{\eta}(x))\\
&=(\mu_X\circ(\varphi^*_{\overline{\xi}}\otimes\varphi^*_{\overline{\eta}})\mathfrak{F})(\sigma_1(\xi),\sigma_1(\eta)),
\end{aligned}$$

because the coefficients of the power series $\mathfrak{F}(x\otimes 1,1\otimes x)$ belong to $H\mathop{\widehat{\otimes}}_R H$.

To verify (35), consider the following commutative diagram:

$$
\begin{tikzcd}
 & & \mathrm{BU}_\otimes \arrow[r, "\widehat{\iota}"] \arrow[d, "p"] & \mathrm{BU}_\otimes \arrow[d, "p"] \\
 & & \mathrm{BSU}_\otimes \arrow[r, "\ddagger", "\iota"'] & \mathrm{BSU}_\otimes \\
X \arrow[uurr, "\varphi_\xi"] \arrow[urr, "\varphi_{\overline{\xi}}"'] & X, \arrow[l, "\mathrm{id}_X"] \arrow[uurr] \arrow[urr, "\varphi_{\overline{\xi}^{-1}}"'] & &
\end{tikzcd}
$$

where by $\ddagger$ we denote the classifying map $\varphi_{\xi^{-1}}$ and ι is the inversion in the H-group $\mathrm{BSU}_\otimes$. We have:

$$\sigma_1(\xi^{-1}) = \varphi_\xi^*(\widehat{\iota}^*(x)) = \varphi_\xi^*(\Theta(x)) = (\varphi_{\overline{\xi}}^*\Theta)(\sigma_1(\xi)).$$

Now (36) follows from axiom 3) of Definition 1. $\square$

Remark 41. Let us note the analogy between the formulas (34), (14) and (35), (15).

Let (ξ, η) be a couple of virtual U-bundles over X of virtual dimension 1 such that the stable equivalence class $\{\xi \otimes \eta\} \in \mathrm{K}_{\mathbb{C}}(X)$ has a geometric line bundle as a representative. One can extend the stable equivalence relation to the set of such couples over X. The set of stable equivalence classes $\{(\xi, \eta)\}$ of couples is a group (isomorphic to $\widetilde{\mathrm{K\,SU}}(X)_\otimes \times \mathcal{L}in(X) \times \mathcal{L}in(X) = \widetilde{\mathrm{K}}_{\mathbb{C}}(X)_\otimes \times \mathcal{L}in(X)$) with respect to the operation $\circ$, induced by the tensor product: $\{(\xi_1, \eta_1)\} \circ \{(\xi_2, \eta_2)\} := \{(\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2)\}$. For two virtual U-bundles ϑ_1, ϑ_2 over X of virtual dimension 1 by $\sigma_1(\vartheta_1) \odot \sigma_1(\vartheta_2)$ denote $(\mu_X \circ (\varphi_{\overline{\vartheta}_1}^* \otimes \varphi_{\overline{\vartheta}_2}^*)\mathfrak{F})(\sigma_1(\vartheta_1), \sigma_1(\vartheta_2)) \in \mathrm{MU}^2(X)$. Using (4), one can prove the following proposition.

Proposition 42.

$$((\varphi_{\overline{\xi}_1}^* \star \varphi_{\overline{\xi}_2}^*)\mathfrak{G})(\sigma_1(\xi_1) \odot \sigma_1(\xi_2), \sigma_1(\eta_1) \odot \sigma_1(\eta_2))$$

$$= F((\varphi_{\overline{\xi}_1}^*\mathfrak{G})(\sigma_1(\xi_1), \sigma_1(\eta_1)), (\varphi_{\overline{\xi}_2}^*\mathfrak{G})(\sigma_1(\xi_2), \sigma_1(\eta_2))),$$

where $\varphi_{\overline{\xi}_1}^* \star \varphi_{\overline{\xi}_2}^* := \mu_X \circ (\varphi_{\overline{\xi}_1}^* \otimes \varphi_{\overline{\xi}_2}^*) \circ \Delta_H$. *Note that for a couple* (ξ, η) *we have* $(\varphi_{\overline{\xi}}^*\mathfrak{G})(\sigma_1(\xi), \sigma_1(\eta)) = \sigma_1(\zeta)$, *where* ζ *is the line bundle over* X *such that* $\zeta = \xi \otimes \eta$. *Moreover,* $\varphi_{\overline{\xi}}^* \circ S_H = \varphi_{\overline{\eta}}^*\colon H \to \mathrm{MU}^*(X)$, *where* S_H *is the antipode in the Hopf algebra* H.

Remark 43. Now let us give a "topological interpretation" of the series $\mathfrak{G}(x, y)$. Let us consider the bundle $\Xi^{-1} \boxtimes \mathcal{Z}$ over $\mathrm{BSU} \times \mathbb{C}P^\infty \cong \mathrm{BU}$, where Ξ^{-1} is the virtual SU-bundle over BSU of virtual dimension 1 such that $\Xi \otimes \Xi^{-1} = [1]$. Suppose $y := \sigma_1(\Xi^{-1} \boxtimes \mathcal{Z}) \in \mathrm{MU}^2(\mathrm{BU})$. Then $y = (S_H\mathfrak{h})(u)$, where $u = \sigma_1(\mathcal{Z})$ as above. Let $\tau\colon \mathrm{BSU} \times \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ be a classifying map for the line bundle $[1] \boxtimes \mathcal{Z} \boxtimes \mathcal{Z}$, i.e. $\tau^*(\mathcal{Z}) = [1] \boxtimes \mathcal{Z} \boxtimes \mathcal{Z}$. Clearly $\tau^*(w) = F(u, v)$, where $w := \sigma_1(\mathcal{Z}) \in \mathrm{MU}^2(\mathbb{C}P^\infty)$, $u := \sigma_1([1] \boxtimes \mathcal{Z} \boxtimes [1])$, $v := \sigma_1([1] \boxtimes [1] \boxtimes \mathcal{Z})$. But the bundle $[1] \boxtimes \mathcal{Z} \boxtimes \mathcal{Z}$ can be represented as the tensor product $(\Xi \boxtimes \mathcal{Z} \boxtimes [1]) \otimes (\Xi^{-1} \boxtimes [1] \boxtimes \mathcal{Z})$. This corresponds to the representation $\tau^*(w)$ as a power series of x, y. More precisely, $\tau^*(w) = F(u, v) = F(\mathfrak{h}^{-1}(x), (S_H\mathfrak{h})^{-1}(y)) = \mathfrak{G}(x, y)$ (compare with (4)).

2.3 Formal group laws related to stable equivalence classes of SU-bundles

In this section we obtain another generalization of the formal product in complex cobordism theory. More precisely, for any stable equivalence class of SU-bundles $\{\xi\} \in \widetilde{\mathrm{K\,SU}}(X)$ we construct the formal group law $(\varphi^*_\xi \mathfrak{H})(x,y)$ such that it defines the group structure on the set of first Conner-Floyd classes $\sigma_1(\widehat{\xi})$ of bundles $\widehat{\xi}$, whose stable equivalence classes $\{\widehat{\xi}\} \in \widetilde{\mathrm{K}}_{\mathbb{C}}(X)$ belong to the coset $\{\xi\}\mathcal{L}in(X)_0 \subset \widetilde{\mathrm{K}}_{\mathbb{C}}(X)_\otimes$, where $\mathcal{L}in(X)_0$ is the subgroup in $\widetilde{\mathrm{K}}_{\mathbb{C}}(X)_\otimes$, corresponding to line bundles over X (see the Introduction), and the symbol $\otimes$ in $\widetilde{\mathrm{K}}_{\mathbb{C}}(X)_\otimes$ denotes the group structure on $\widetilde{\mathrm{K}}_{\mathbb{C}}(X)$, induced by the tensor product of virtual bundles of virtual dimension 1. It follows from isomorphism (23) that $\widetilde{\mathrm{K}}_{\mathbb{C}}(X)_\otimes \cong \widetilde{\mathrm{K\,SU}}(X)_\otimes \times \mathcal{L}in(X)$. Moreover, for $\{\xi\} = \{[1]\}$ the corresponding formal group law $(\varphi^*_\xi \mathfrak{H})(u,v)$ coincides with the formal product in complex cobordism theory $F(u,v)$. It is well known that this power series describes the relation between first Conner-Floyd classes of two line bundles ζ_i, $i=1,2$ and the first Conner-Floyd class of their tensor product $\zeta_1 \otimes \zeta_2$, more precisely, $\sigma_1(\zeta_1 \otimes \zeta_2) = F(\sigma_1(\zeta_1), \sigma_1(\zeta_2))$ [6]. Below we shall obtain a generalization of this formula.

By $\mathfrak{H}(x,y)$ denote the power series $\mathfrak{h}(F(\mathfrak{h}^{-1}(x), \mathfrak{h}^{-1}(y))) \in H[[x,y]]$, where $\mathfrak{h}(u) \in H[[u]]$ as in (24). Obviously $\mathfrak{H}(x,y)$ is a formal group law (in the usual sense) over the ring H, with the formal inverse $\mathfrak{I}(x) := \mathfrak{h}(\theta(\mathfrak{h}^{-1}(x)))$. Moreover, the formal group law $\mathfrak{H}(x,y)$ is isomorphic (not strictly, because of $\mathfrak{h}'(0) \neq 1$) to the formal product in complex cobordism theory $F(u,v)$.

The power series $\mathfrak{H}(x,y)$ may be obtained in the following way. Recall that

$$\mathrm{MU}^*(\mathrm{BU}) = H[[u]] = H \widehat{\underset{R}{\otimes}} R[[u]].$$

Let us define a homomorphism $\kappa\colon H[[w]] \to H[[u,v]]$ as follows: $\kappa|_H = \mathrm{id}_H$, $\kappa(w) = F(u,v) = u+v+\sum_{i,j\geq 1} \alpha_{i,j} u^i v^j$. Instead of w, u, v we take another generators $z = \mathfrak{h}(w)$, $x = \mathfrak{h}(u)$, and $y = \mathfrak{h}(v)$ respectively. Then

$$\kappa(z) = \kappa(\mathfrak{h}(w)) = \mathfrak{h}(\kappa(w)) = \mathfrak{h}(F(u,v)) = \mathfrak{h}(F(\mathfrak{h}^{-1}(x), \mathfrak{h}^{-1}(y))) = \mathfrak{H}(x,y).$$

Define a homomorphism $\varsigma\colon H[[u]] \to H[[u]]$ as follows: $\varsigma|_H = \mathrm{id}_H$, $\varsigma(u) = \theta(u)$. We have: $\varsigma(x) = \varsigma(\mathfrak{h}(u)) = \mathfrak{h}(\varsigma(u)) = \mathfrak{h}(\theta(\mathfrak{h}^{-1}(x))) = \mathfrak{I}(x)$.

Put $\mathfrak{H}(x,y) = x + y + \sum_{i,j\geq 1} \widetilde{\alpha}_{i,j} x^i y^j$. Let us recall that the formal group law $F(u,v) \in R[[u,v]]$ is the universal one and the ring R is the Lasard ring [7]. Hence there exists a unique ring homomorphism $\lambda\colon R \to H$ such that $\lambda(\alpha_{i,j}) = \widetilde{\alpha}_{i,j}$, $i,j \geq 1$. Clearly, $\varepsilon_H(\widetilde{\alpha}_{i,j}) = \alpha_{i,j}$, $i,j \geq 1$. By P denote the composite homomorphism $\lambda \circ \varepsilon_H\colon H \to H$.

Let $K \subset H$ be the subring, generated by $\{\widetilde{\alpha}_{i,j},\ i,j \geq 1\}$.

Theorem 44. *λ defines the canonical ring isomorphism $\lambda\colon R \to K \subset H$, $\alpha_{i,j} \overset{\lambda}{\mapsto} \widetilde{\alpha}_{i,j}$. P is a projector (i.e. $P^2 = P$ and $H = \operatorname{im} P \oplus \ker P$) and $\operatorname{im} P = K \subset H$.*

Proof. The proof is trivial. Note that $\varepsilon_H \circ \lambda = \mathrm{id}_R$. □

Proposition 45. *The subrings K, $R \subset H$ do not coincide.*

Proof. It is sufficient to prove that $\widetilde{\alpha}_{i,j} \neq \alpha_{i,j}$ for some $i, j \geq 1$. Indeed, we have: $\varepsilon_H|_R = \mathrm{id}_R$. Therefore if $\widetilde{\alpha}_{i,j} \neq \varepsilon_H(\widetilde{\alpha}_{i,j}) = \alpha_{i,j}$, then $\widetilde{\alpha}_{i,j} \notin R$.

To prove this proposition we need some more information on the logarithm $g_{\mathfrak{H}}(x)$ (more precisely, about its coefficients) of the formal group law $\mathfrak{H}(x, y)$. By c denote the element $1 + \sum_{i\geq 2} \alpha_{i,1}\widetilde{s}_i \in H$. We claim that

$$g_{\mathfrak{H}}(x) = cg_F(\mathfrak{h}^{-1}(x)) \tag{37}$$

(where $g_F(u)$ is the logarithm of the formal group law $F(u, v)$). Indeed, it can easily be checked by a direct computation that $g_{\mathfrak{H}}(\mathfrak{H}(x, y)) = g_{\mathfrak{H}}(x) + g_{\mathfrak{H}}(y)$ and $g'_{\mathfrak{H}}(0) = 1$.

Assume that $F(x, y) = \mathfrak{H}(x, y)$, i.e. $\alpha_{i,j} = \widetilde{\alpha}_{i,j}$ for all $i, j \geq 1$. This implies $g_F(x) = g_{\mathfrak{H}}(x)$. Let us recall that $\mathfrak{h}^{-1}(x) = u$. Substituting u for $\mathfrak{h}^{-1}(x)$ in (37), we get:

$$g_F(x) = cg_F(u),$$

where $x = \mathfrak{h}(u)$. Hence

$$\mathfrak{h}(u) + \beta_1(\mathfrak{h}(u))^2 \equiv c(u + \beta_1 u^2) \mod u^3$$

(recall that $g_F(u) = u + \sum_{i\geq 1} \beta_i u^{i+1}$). Let us compare the coefficients of u^2. Using (24), we get:

$$\sum_{i\geq 2} \alpha_{i,2}\widetilde{s}_i + \beta_1 \sum_{i\geq 2} \alpha_{i,1}\widetilde{s}_i + \beta_1(\sum_{i\geq 2} \alpha_{i,1}\widetilde{s}_i)^2 = 0. \tag{38}$$

It is well known that the characteristic classes $\widetilde{s}_i := s_i(\Xi)$, $i \geq 2$, are algebraically independent. Therefore the equation (38) contradicts well-known properties of the universal formal group law (see, for example, [1]). $\square$

Since $\kappa(z) = \mathfrak{H}(x, y) \in K[[x, y]]$, we have:

$$H[[z]] \supset K[[z]] \stackrel{\kappa|_{K[[z]]}}{\longrightarrow} K[[x, y]] \subset H[[x, y]],$$

i.e. $\kappa(K[[z]]) \subset K[[x, y]]$. It follows from the existence and the uniqueness of a formal inverse in a formal group law that $\mathfrak{I}(x) \in K[[x]] \subset H[[x]]$. Therefore the subring $K[[x]] \subset H[[x]]$ may be considered as an analog of the ring $\mathrm{MU}^*(\mathbb{C}P^\infty) = R[[u]]$, where the formal product in complex cobordism theory "lives".

Now let X be a finite CW-complex. Let us fix a virtual SU-bundle ξ over X of virtual dimension 1. Let $\varphi := \varphi_\xi \colon X \to \mathrm{BSU}$ be its classifying map. Then we have the formal group law $(\varphi^*\mathfrak{H})(x, y) \in \mathrm{MU}^*(X)[[x, y]]$ over the ring $\mathrm{MU}^*(X)$. Its coefficients $\varphi^*(\widetilde{\alpha}_{i,j})$, $i, j \geq 1$ generate the subring $\varphi^*(K) \subset \mathrm{MU}^*(X)$, which is isomorphic to the ring R. Moreover, it is a direct summand in $\mathrm{MU}^*(X)$.

We say that a virtual U-bundle $\widehat{\xi}$ over X of virtual dimension 1 is a lift of ξ, if $\overline{\widehat{\xi}} = \xi$. Let us recall that $\overline{\widehat{\xi}}$ is the virtual SU-bundle of virtual dimension 1, classifying by the map $p \circ \varphi_{\widehat{\xi}}$, where $\varphi_{\widehat{\xi}} \colon X \to \mathrm{BU}$ is a classifying map for $\widehat{\xi}$ and $p \colon \mathrm{BU}_\otimes \to \mathrm{BSU}_\otimes$ is the canonical homomorphism of H-groups. Let $\mathcal{L}(\xi) \subset \mathrm{K}_{\mathbb{C}}(X)$ be the set of stable equivalence classes of lifts of the bundle ξ. Consider the following operation $\circ$

$$\widehat{\xi}_1 \circ \widehat{\xi}_2 := \widehat{\xi}_1 \otimes \widehat{\xi}_2 \otimes \xi^{-1} \tag{39}$$

on the set of lifts of fixed SU-bundle ξ.

Proposition 46. *The set $\mathcal{L}(\xi)$ is an Abelian group with respect to the operation, induced by the operation (39) on lifts. Moreover, for all ξ the corresponding groups $\mathcal{L}(\xi)$ are isomorphic to each other (in particular, they are isomorphic to the group $H^2(X;\mathbb{Z})$ of line bundles over X with respect to the tensor product).*

Proof. The proof is trivial. □

Theorem 47. *Let $\widehat{\xi}_1$, $\widehat{\xi}_2$ be two lifts of ξ. Then*

$$\sigma_1(\widehat{\xi}_1 \circ \widehat{\xi}_2) = \sigma_1(\widehat{\xi}_1 \otimes \widehat{\xi}_2 \otimes \xi^{-1}) = (\varphi^*\mathfrak{H})(\sigma_1(\widehat{\xi}_1), \sigma_1(\widehat{\xi}_2)). \tag{40}$$

Moreover, $\sigma_1(\widehat{\xi}^{-1}) = (\varphi^*\mathfrak{I})(\sigma_1(\widehat{\xi}))$, where $\widehat{\xi}^{-1}$ is the lift of ξ such that its stable equivalence class $\{\widehat{\xi}^{-1}\} \in \mathcal{L}(\xi)$ is the inverse element for $\{\widehat{\xi}\}$ in the group $\mathcal{L}(\xi)$ (i.e. $\{\widehat{\xi} \otimes \widehat{\xi}^{-1} \otimes \xi^{-1}\} = \{\xi\}$). The assignment $\widehat{\xi} \mapsto \sigma_1(\widehat{\xi})$ defines an isomorphism from the group $\mathcal{L}(\xi)$ to the group $\{\sigma_1(\widehat{\xi}) \mid \{\widehat{\xi}\} \in \mathcal{L}(\xi)\} \subset \mathrm{MU}^2(X)$ with respect to the operation $\sigma_1(\widehat{\xi}_1) \circ \sigma_1(\widehat{\xi}_2) := (\varphi^*\mathfrak{H})(\sigma_1(\widehat{\xi}_1), \sigma_1(\widehat{\xi}_2))$.

Remark 48. We stress that the bundle $\widehat{\xi}^{-1}$ in the statement of the theorem in general does not coincide with the inverse bundle for $\widehat{\xi}$ in the sense of H-group structure on $\mathrm{BU}_{\otimes}$, i.e. in general $\{\widehat{\xi} \otimes \widehat{\xi}^{-1}\} \neq \{[1]\} \in \mathrm{K}_{\mathbb{C}}(X)$.

Remark 49. Note, that (40) may be considered as a generalization of the formal product in complex cobordism theory. Indeed, the following conditions are equivalent:

(i) the bundle ξ is (stable equivalent to) the trivial bundle $[1]$;

(ii) its classifying map φ is (homotopic to) the map into a point;

(iii) the bundles $\widehat{\xi}_1$, $\widehat{\xi}_2$ are geometric line bundles (more precisely, their stable equivalence classes have line bundles as representatives).

The condition (i) implies $\sigma_1(\widehat{\xi}_1 \otimes \widehat{\xi}_2 \otimes \xi^{-1}) = \sigma_1(\widehat{\xi}_1 \otimes \widehat{\xi}_2)$. It follows from (ii) that

$$(\varphi^*\mathfrak{H})(u,v) = (\varepsilon_H\mathfrak{H})(u,v) = F(u,v).$$

Therefore in the case $\{\xi\} = \{[1]\}$ we obtain the well-known formula $\sigma_1(\widehat{\xi}_1 \otimes \widehat{\xi}_2) = F(\sigma_1(\widehat{\xi}_1), \sigma_1(\widehat{\xi}_2))$ for first Conner-Floyd classes of line bundles.

Proof. If ζ is a line bundle over X, then we say that the element $\sigma_1(\zeta) \in \mathrm{MU}^2(X)$ is a *geometric cobordism.*

It follows from the isomorphism (23) that for any lift $\widehat{\xi}$ of ξ there is a unique line bundle ζ such that $\widehat{\xi} = \zeta \otimes \xi$. Applying Theorem 39, we get:

$$\sigma_1(\widehat{\xi}) = (\mu_X \circ (\varepsilon \otimes \varphi^*)\mathfrak{F})(\sigma_1(\zeta), 0), \tag{41}$$

where $\varepsilon = \eta_X \circ \varepsilon_H \colon H \to \mathrm{MU}^*(X)$, here $\eta_X \colon R \to \mathrm{MU}^*(X)$ is the unit of the R-algebra $\mathrm{MU}^*(X)$ (in other words, $\varepsilon = \varphi^*_{[1]}$, where $\varphi_{[1]} \colon X \to \mathrm{BSU}$ is a classifying map for the trivial bundle). Hence $\sigma_1(\widehat{\xi})$ is the *shift* (see page 17) of the geometric cobordism $\sigma_1(\zeta)$ by $\varphi^* = \varphi^*_{\xi} \in \mathrm{Hom}_R(H, \mathrm{MU}^*(X))$. Moreover, we have a bijection from the set $\{\sigma_1(\zeta) \mid \zeta \text{ is a line bundle over X}\}$ to the set $\{\sigma_1(\widehat{\xi}) \mid \widehat{\xi} \text{ is a lift of } \xi\}$.

Using the notation of Section 1.3, we have:

$$\begin{pmatrix} \sigma_1(\xi_1) \\ \varphi^*_{\overline{\xi}_1} \end{pmatrix} \circ \begin{pmatrix} \sigma_1(\xi_2) \\ \varphi^*_{\overline{\xi}_2} \end{pmatrix} = \begin{pmatrix} (\mu_X \circ (\varphi^*_{\overline{\xi}_1} \otimes \varphi^*_{\overline{\xi}_2})\mathfrak{F})(\sigma_1(\xi_1), \sigma_1(\xi_2)) \\ \varphi^*_{\overline{\xi}_1} \star \varphi^*_{\overline{\xi}_2} \end{pmatrix}$$

for two virtual U-bundles ξ_1, ξ_2 over X of virtual dimension 1 with the corresponding SU-bundles $\overline{\xi}_1, \overline{\xi}_2$ (compare with (14)). It follows from Theorem 39 that $(\mu_X \circ (\varphi^*_{\overline{\xi}_1} \otimes \varphi^*_{\overline{\xi}_2})\mathfrak{F})(\sigma_1(\xi_1), \sigma_1(\xi_2)) = \sigma_1(\xi_1 \otimes \xi_2)$. Therefore for two lifts $\widehat{\xi}_1$, $\widehat{\xi}_2$ of the virtual SU-bundle ξ as above, using $\sigma_1(\xi^{-1}) = 0$ and $\varphi^*_{\xi^{-1}} = \varphi^*_{\xi} \circ S_H$, we obtain:

$$\begin{pmatrix} \sigma_1(\widehat{\xi}_1) \\ \varphi^*_{\xi} \end{pmatrix} \circ \begin{pmatrix} \sigma_1(\widehat{\xi}_2) \\ \varphi^*_{\xi} \end{pmatrix} \circ \begin{pmatrix} 0 \\ \varphi^*_{\xi} \circ S_H \end{pmatrix} = \begin{pmatrix} \sigma_1(\widehat{\xi}_1 \otimes \widehat{\xi}_2 \otimes \xi^{-1}) \\ \varphi^*_{\xi} \end{pmatrix}$$

(compare with (19)), hence $\sigma_1(\widehat{\xi}_1 \otimes \widehat{\xi}_2 \otimes \xi^{-1}) = \sigma_1(\widehat{\xi}_1) \circ_{\varphi^*_{\xi}} \sigma_1(\widehat{\xi}_2)$ (see (18)). Applying the results of Section 1.3 (Propositions 22, 23, 26, and 28), we obtain the required assertion. Note that it follows from Proposition 38 that the condition $(\mu_H \circ (\mathrm{id}_H \otimes S))\mathfrak{c} = 0$ (see page 18) is satisfied. $\square$

Remark 50. Note that $\sigma_1(\widehat{\xi}) = (\mu_X \circ (\varepsilon \otimes \varphi^*_{\xi})\mathfrak{F})(\sigma_1(\zeta), 0) = (\varphi^*_{\xi}\mathfrak{h})(\sigma_1(\zeta))$.

Thus we have the following situation: for any stable equivalence class $\{\xi\} \in \widetilde{\mathrm{KSU}}(X)$ we get the formal group law $(\varphi^*_{\xi}\mathfrak{H})(x, y)$ over the ring $\varphi^*_{\xi}(K) \subset \mathrm{MU}^*(X)$ (recall that the ring $\varphi^*_{\xi}(K)$ is a direct summand in $\mathrm{MU}^*(X)$). This formal group law defines the group structure on the set $\{\sigma_1(\widehat{\xi}) \mid \widehat{\xi} \text{ is a lift of } \xi\}$ of first Conner-Floyd classes of lifts of ξ (here ξ is a representative of the class $\{\xi\}$ as above). Note that for all $\{\xi\}$ the corresponding formal group laws are isomorphic to each other (and the corresponding rings $\varphi^*_{\xi}(K)$ are isomorphic to each other). If $\{\xi\} = \{[1]\}$ is the stable equivalence class of the trivial bundle, then $(\varphi^*_{[1]}\mathfrak{H})(u, v) = F(u, v)$ is the usual formal product in complex cobordism theory, the corresponding ring $\varphi^*_{[1]}(K)$ is $R = \mathrm{MU}^*(\mathrm{pt})$, and the set $\{\sigma_1(\widehat{[1]}) \mid \widehat{[1]} \text{ is a lift of } [1]\}$ is the set of *geometric cobordisms* $\{\sigma_1(\zeta) \mid \zeta \text{ is a line bundle}\}$ in $\mathrm{MU}^2(X)$. Note also that we have $(\varphi^*_{\xi\otimes\eta}\mathfrak{H})(x, y) = ((\varphi^*_{\xi} \star \varphi^*_{\eta})\mathfrak{H})(x, y)$ for virtual SU-bundles ξ, η over X of virtual dimension 1, where $\star$ denotes the convolution $\mu_X \circ (\varphi^*_{\xi} \otimes \varphi^*_{\eta}) \circ \Delta_H$ of φ^*_{ξ}, φ^*_{η} in $\mathrm{Hom}_R(H, \mathrm{MU}^*(X))$ (compare with Remark 27).

3 Appendix: An isomorphism of some H-groups

Lemma 51. There exists an isomorphism of H-groups: $\mathrm{BU}_\otimes \cong \mathrm{BSU}_\otimes \times \mathbb{C}P^\infty$.

Proof. Put

$$\mathrm{G}_{n,2n} = \mathrm{U}(2n)/\,\mathrm{U}(n) \times \mathrm{U}(n), \quad \mathrm{G}^+_{n,2n} = \mathrm{U}(2n)/\,\mathrm{U}(n) \times \mathrm{SU}(n).$$

Recall that $\mathrm{BU} = \varinjlim_n \mathrm{G}_{n,2n}$ and $\mathrm{BSU} = \varinjlim_n \mathrm{G}^+_{n,2n}$.

Below we shall use the notation of Section 2.1.

By $\widehat{\xi}_n$ denote the restriction $\widehat{\Xi}_n|_{G_{n,2n}}$ of the universal $\mathrm{U}(n)$-bundle $\widehat{\Xi}_n$ to the subspace $\mathrm{G}_{n,2n} \subset \mathrm{BU}(n)$. Let $\lambda_n\colon\ \mathrm{G}_{n,2n} \to \mathbb{C}P^\infty$ be the classifying map for $c_1(\widehat{\xi}_n) \in H^2(\mathrm{G}_{n,2n};\mathbb{Z})$. If $i_n^{n+1}\colon\ \mathrm{G}_{n,2n} \hookrightarrow \mathrm{G}_{n+1,2n+2}$ is the standard embedding, then $\lambda_n = \lambda_{n+1} \circ i_n^{n+1}$. Hence we get the map $\lambda := \varinjlim_n \lambda_n\colon\ \mathrm{BU} \to \mathbb{C}P^\infty$. Consider the virtual bundle $\omega_n := (\widehat{\xi}_n - [n-1]) \otimes \lambda_n^*(\overline{\mathcal{Z}})$ over $\mathrm{G}_{n,2n}$ of virtual dimension 1, where $\overline{\mathcal{Z}} = \mathcal{Z}^{-1}$. We claim that it is an SU-bundle. Indeed, $c_1(\omega_n) = c_1(\widehat{\xi}_n - [n-1]) - c_1(\lambda_n^*(\mathcal{Z})) = 0$. Let $\kappa_n\colon\ \mathrm{G}_{n,2n} \to \mathrm{BSU}$ be the classifying map for the bundle ω_n, i.e. $\kappa_n^*(\Xi) = \omega_n$. It can easily be checked that $(i_n^{n+1})^*(\omega_{n+1}) = \omega_n$. Therefore $\kappa := \varinjlim_n \kappa_n\colon\ \mathrm{BU} \to \mathrm{BSU}$ is the classifying map for the virtual SU-bundle ω over BU such that $\omega|_{\mathrm{G}_{n,2n}} = \omega_n$ for all $n \geq 0$.

Now let us consider the standard embedding $j_n\colon\ \mathrm{G}^+_{n,2n} \hookrightarrow \mathrm{G}_{n,2n}$. It is easy to prove that $j_n^*(\widehat{\xi}_n) = \xi_n$, where $\xi_n := \Xi_n|_{\mathrm{G}^+_{n,2n}}$ (here we consider $\mathrm{G}^+_{n,2n}$ as a subspace in $\mathrm{BSU}(n)$). Hence

$$j_n^*(\omega_n) = j_n^*((\widehat{\xi}_n - [n-1]) \otimes \lambda_n^*(\overline{\mathcal{Z}})) = \xi_n - [n-1]$$

(where we used $j_n^*(\lambda_n^*(\overline{\mathcal{Z}})) = [1]$). Therefore, $(\kappa_n \circ j_n)^*(\Xi) = \xi_n - [n-1]$. This implies that

$$\varinjlim_n(\kappa_n \circ j_n)\colon\ \varinjlim_n \mathrm{G}^+_{n,2n} = \mathrm{BSU} \to \mathrm{BSU} \tag{42}$$

is a homotopy equivalence. Applying Whitehead's theorem (using (42)), we get that the map $\kappa \times \lambda\colon\ \mathrm{BU} \to \mathrm{BSU} \times \mathbb{C}P^\infty$ is a homotopy equivalence.

Now we consider $\kappa_n \times \lambda_n\colon\ \mathrm{G}_{n,2n} \to \mathrm{BSU} \times \mathbb{C}P^\infty$. We have:

$$(\kappa_n \times \lambda_n)^*(\Xi \boxtimes \mathcal{Z}) = \omega_n \otimes \lambda_n^*(\mathcal{Z}) = (\widehat{\xi}_n - [n-1]) \otimes \lambda_n^*(\overline{\mathcal{Z}}) \otimes \lambda_n^*(\mathcal{Z}) = \widehat{\xi}_n - [n-1].$$

Therefore $\kappa \times \lambda$ is the homotopy equivalence such that $(\kappa \times \lambda)^*(\Xi \otimes \mathcal{Z}) = \widehat{\Xi}$. Now it is easy to prove that $\kappa \times \lambda$ induces the required isomorphism of H-groups. $\square$

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