LOG CANONICAL THRESHOLDS OF SMOOTH FANO THREEFOLDS

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WITH AN APPENDIX BY JEAN-PIERRE DEMAILLY

Abstract. We compute global log canonical thresholds of some smooth Fano threefolds.

Contents

1. Introduction

The multiplicity of a polynomial $\phi \in \mathbb{C}[z_1, \ldots, z_n]$ in the origin $O \in \mathbb{C}^n$ is the number

$$\min\left\{ m \in \mathbb{Z}_{\geqslant 0} \ \Big|\ \frac{\partial^m \phi\big(z_1, \ldots, z_n\big)}{\partial^{m_1} z_1 \partial^{m_2} z_2 \ldots \partial^{m_n} z_n}\big(O\big) \neq 0 \right\} \in \mathbb{Z}_{\geqslant 0} \cup \big\{ +\infty \big\}.$$

There is a similar but more subtle invariant

$$c_0\big(\phi\big) = \sup\left\{ \varepsilon \in \mathbb{Q} \ \Big|\ \text{the function } \frac{1}{\big|\phi\big|^{2\varepsilon}} \text{ is locally integrable near } O \in \mathbb{C}^n \right\} \in \mathbb{Q}_{\geqslant 0} \cup \{+\infty\},$$

which is called the complex singularity exponent of the polynomial ϕ at the point O.

Example 1.1. Suppose that $n = 2$, and $\phi = 0$ defines an irreducible curve in $\mathbb{C}^2$. Then

$$c_0(\phi) = \min\left(1, \frac{1}{m} + \frac{1}{n}\right)$$

by [90], where (m, n) is the first pair of Puiseux exponents of ϕ. On the other hand, the equality

$$c_0\Big(z_1^{n_1} z_2^{n_2}\big(z_1^{km_1} + z_2^{km_2}\big)\Big) = \min\left(\frac{1}{n_1}, \frac{1}{n_2}, \frac{\frac{1}{m_1} + \frac{1}{m_2}}{k + \frac{n_1}{m_1} + \frac{n_2}{m_2}}\right)$$

holds (see [109]), where n_1, n_2, m_1, m_2, k are non-negative integers.

The first author was supported by the grants NSF DMS-0701465 and EPSRC EP/E048412/1, the second author was supported by the grants RFFI No. 08-01-00395-a, N.Sh.-1987.2008.1 and EPSRC EP/E048412/1.

Example 1.2. Let $m_1, \ldots, m_n$ be positive integers. Then

$$\min\left(1,\ \sum_{i=1}^{n}\frac{1}{m_i}\right) = c_0\left(\sum_{i=1}^{n} z_i^{m_i}\right) \geqslant c_0\left(\prod_{i=1}^{n} z_i^{m_i}\right) = \min\left(\frac{1}{m_1}, \frac{1}{m_2}, \ldots, \frac{1}{m_n}\right).$$

The set of complex singularity exponents has interesting properties. Put

$$\mathcal{H}_n = \Big\{c_0(\phi)\ \Big|\ \phi \in \mathbb{C}[z_1, \ldots, z_n]\Big\} \subset \mathbb{Q} \cup \{+\infty\}$$

which implies that $\mathcal{T}_n \subset [0, 1] \cup \{+\infty\}$. Then

- the set $\mathcal{H}_n$ is closed in $\mathbb{R} \cup \{+\infty\}$ (see [60]),
- we expect that $\mathcal{H}_n$ satisfies ascending chain condition (ACC) for every n (see [106]),
- the set $\mathcal{H}_n$ satisfies ACC for $n \leqslant 3$ (see [169], [1], [109], [141], [58]),
- it follows from [60] that the following assertions are equivalent:
  - the set $\mathcal{H}_n$ satisfies ACC for every $n \in \mathbb{Z}_{>0}$;
  - for every $n \in \mathbb{Z}_{>0}$, there is $\delta_n \in (0, 1)$ such that $\mathcal{H}_n \cap (\delta_n, 1) = \varnothing$;
- it follows from [106] that $\mathcal{H}_{n-1} \subset \mathcal{H}_n$ and

  $$\mathcal{H}_{n-1} \setminus \{1, +\infty\} \subseteq \partial\mathcal{H}_n \subseteq \mathcal{H}_{n-1} \setminus \{+\infty\},$$

  where $\partial\mathcal{H}_n$ is the set of all accumulation points of $\mathcal{H}_n$,
- it follows from [109] and [110] that the set $\mathcal{H}_2$ is the union

$$\left\{\frac{2}{m}\ \Big|\ 2 \leqslant m \in \mathbb{Z}_{>0}\right\} \bigcup \left\{\frac{m_1 + m_2}{km_1m_2 + n_1m_2 + n_2m_1}\ \middle|\ \begin{array}{c} -km_1 < n_1 - n_2 < km_2, \\ \gcd(m_1, m_2) = 1, \\ k, m_1, m_2, n_1, n_2 \in \mathbb{Z}_{\geqslant 0} \end{array}\right\} \bigcup \{0, +\infty\},$$

  which implies that $\partial\mathcal{H}_2 = \mathcal{H}_1 \setminus \{1, +\infty\}$, where $\mathcal{H}_1 = \{1/n \mid n \in \mathbb{Z}_{\geqslant 0}\} \cup \{0\}$,
- it follows from [110] that the intersection $\mathcal{H}_3 \cap [\frac{5}{6}, 1)$ is the union

$$\left\{\frac{5}{6} + \frac{1}{m}\ \Big|\ m \geqslant 6\right\} \bigcup \left\{\frac{5}{6} + \frac{2}{3m}\ \Big|\ m \geqslant 4\right\} \bigcup \left\{\frac{5}{6} + \frac{4}{9m+6}\ \Big|\ m \geqslant 2\right\} \bigcup \left\{\frac{19}{20}, \frac{15}{16}, \frac{12}{13}, \frac{25}{28}, \frac{15}{17}, \frac{5}{6}\right\},$$

  where $m \in \mathbb{Z}_{>0}$, which implies that $5/6$ is the largest accumulation point of $\mathcal{H}_3$ (cf. [145], [146]),
- it follows from [124] that $\partial\mathcal{H}_3 = \mathcal{H}_2 \setminus \{1, +\infty\}$ (cf. [106]),
- it follows from [104] that $41/42$ is the maximal element of the set $\mathcal{H}_3 \cap [0, 1)$.

Remark 1.3. For a non-constant ϕ, the complex singularity exponent $c_0(\phi)$ coincides with the absolute value of the biggest root of the Bernstein–Sato polynomial of ϕ (see [10], [105]).

Let X be a variety[1] with at most log canonical singularities (see [102]), let $Z \subseteq X$ be a closed subvariety, and let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on the variety X. Then the number

$$\mathrm{lct}_Z(X, D) = \sup\Big\{\lambda \in \mathbb{Q}\ \Big|\ \text{the log pair } (X, \lambda D) \text{ is log canonical along } Z\Big\} \in \mathbb{Q} \cup \{+\infty\}$$

is called a log canonical threshold of the divisor D along Z. It follows from [105] that

$$\mathrm{lct}_O\Big(\mathbb{C}^n, (\phi = 0)\Big) = c_0(\phi),$$

so that $\mathrm{lct}_Z(X, D)$ is an algebraic counterpart of the number $c_0(\phi)$. One has

$$\mathrm{lct}_X(X, D) = \inf\Big\{\mathrm{lct}_P(X, D)\ \Big|\ P \in X\Big\} = \sup\Big\{\lambda \in \mathbb{Q}\ \Big|\ \text{the log pair } (X, \lambda D) \text{ is log canonical}\Big\},$$

and, for simplicity, we put $\mathrm{lct}(X, D) = \mathrm{lct}_X(X, D)$.

[1]All varieties are assumed to be complex, algebraic, projective and normal if the opposite is not stated explicitly.

Example 1.4. Suppose that $X=\mathbb{P}^2$ and $D\in|\mathcal{O}_{\mathbb{P}^2}(3)|$. Then

$$\mathrm{lct}\big(X,D\big)=\begin{cases}1 \text{ if } D \text{ is a smooth curve,}\\ 1 \text{ if } D \text{ is a curve with ordinary double points,}\\ 5/6 \text{ if } D \text{ is a curve with one cuspidal point,}\\ 3/4 \text{ if } D \text{ consists of a conic and a line that are tangent,}\\ 2/3 \text{ if } D \text{ consists of three lines intersecting at one point,}\\ 1/2 \text{ if } \mathrm{Supp}\big(D\big) \text{ consists of two lines,}\\ 1/3 \text{ if } \mathrm{Supp}\big(D\big) \text{ consists of one line.}\end{cases}$$

Example 1.5. Suppose that $X=\mathbb{P}^2$ and $D\in|\mathcal{O}_{\mathbb{P}^2}(d)|$ for $d\geqslant 3$. The papers [103] and [112] show that the curve D is semistable (stable, respectively) if $\mathrm{lct}(X,D)\geqslant 3/d$ ($>3/d$, respectively).

The set of log canonical thresholds of Weil divisors has interesting properties, which are similar to the properties of the set $\mathcal{H}_n$ (cf. [8]). Put

$$\mathcal{T}_n=\left\{\mathrm{lct}\big(X,D\big)\ \middle|\ \begin{array}{l}\text{the variety } X \text{ has at most log canonical singularities,}\\ \dim(X)=n \text{ and } D \text{ is effective } \mathbb{Q}\text{-Cartier Weil divisor}\end{array}\right\}\subset\mathbb{Q}\cup\big\{+\infty\big\},$$

which implies that $\mathcal{T}_n\subset\big[0,1\big]\cup\big\{+\infty\big\}$. Then

- the set $\mathcal{T}_n$ satisfies ACC for $n\leqslant 3$ (see [169], [1], [109], [141], [58]),
- we expect that $\mathcal{T}_n$ satisfies ACC for every n (see [105, Conjecture 8.8]),
- it follows from [105, Proposition 8.8] that that $\mathcal{T}_{n-1}\subset\mathcal{T}_n$ and
$$\mathcal{T}_{n-1}\setminus\big\{1,+\infty\big\}\subseteq\partial\mathcal{T}_n,$$
where $\partial\mathcal{T}_n$ is the set of all accumulation points of $\mathcal{T}_n$,
- it follows from [169] and [109] that $\partial\mathcal{T}_2=\mathcal{T}_1\setminus\{1,+\infty\}$, where $\mathcal{T}_1=\{1/n\mid n\in\mathbb{Z}_{\geqslant 0}\}\cup\{0\}$,
- it follows from [145] and [146] that
$$\partial\mathcal{T}_3\cap\left[\frac{1}{2},1\right]=\mathcal{T}_2\cap\left[\frac{1}{2},1\right]=\left\{\frac{1}{2}+\frac{1}{n}\ \middle|\ 3\leqslant n\in\mathbb{Z}_{>0}\right\},$$
which implies that $5/6$ is the largest accumulation point of $\mathcal{T}_3$ (cf. [110]),
- it follows from [124] that $\partial\mathcal{T}_3=\mathcal{T}_2\setminus\{1,+\infty\}$ (cf. [106]),
- it follows from [104] that $41/42$ is is the maximal element of the set $\mathcal{T}_3\cap[0,1)$.

Remark 1.6. If X is smooth, and D is a Weil divisor, then it follows from [131] that

$$\mathrm{lct}\big(X,D\big)=\dim\big(X\big)-\sup\left\{\frac{\dim\big(D_m\big)}{m+1}\ \middle|\ m\in\mathbb{Z}_{\geqslant 0}\right\},$$

where D_m is the m-th jet scheme of the divisor D (see [131]).

Suppose that X is a Fano variety with at most log terminal singularities (see [98]).

Definition 1.7. Global log canonical threshold of the Fano variety X is the number

$$\mathrm{lct}\big(X\big)=\inf\Big\{\mathrm{lct}\big(X,D\big)\ \Big|\ D \text{ is an effective } \mathbb{Q}\text{-divisor on } X \text{ such that } D\ \sim_{\mathbb{Q}}-K_X\Big\}\geqslant 0.$$

Remark 1.8. To define the number $\mathrm{lct}(X)\in\mathbb{R}$, we only need to assume that

$$\Big|-nK_X\Big|\neq\varnothing$$

for some $n\gg 0$. This property is shared by many varieties (toric varieties, weak Fano varieties), but all the currently known applications are related to the case when $-K_X$ is ample.

The number $\mathrm{lct}(X)$ is an algebraic counterpart of the α-invariant introduced in [179]. One has

$$\mathrm{lct}\big(X\big)=\sup\left\{\varepsilon\in\mathbb{Q}\ \middle|\ \begin{array}{l}\text{the log pair } \Big(X,\dfrac{\varepsilon}{n}D\Big) \text{ is log canonical for}\\ \text{every divisor } D\in\big|-nK_X\big| \text{ and all } n\in\mathbb{Z}_{>0}\end{array}\right\}.$$

Recall that every Fano variety X is rationally connected (see [170], [193]). Thus, the group $\mathrm{Pic}(X)$ is torsion free. Then

$$\mathrm{lct}\big(X\big) = \sup\left\{\lambda \in \mathbb{Q}\ \middle|\ \begin{array}{l}\text{the log pair } \Big(X, \lambda D\Big) \text{ is log canonical}\\ \text{for every effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X\end{array}\right\}.$$

Example 1.9. Let X be a smooth hypersurface in $\mathbb{P}^n$ of degree $m < n$. Then

$$\mathrm{lct}\big(X\big) = \frac{1}{n+1-m}$$

as shown in [20] (see Corollary 2.16). In particular, the equality $\mathrm{lct}(\mathbb{P}^n) = 1/(n+1)$ holds.

Example 1.10. Let X be a rational homogeneous space such that $-K_X \sim rD$ and

$$\mathrm{Pic}\big(X\big) = \mathbb{Z}\big[D\big],$$

where D is an ample divisor and $r \in \mathbb{Z}_{>0}$. Then $\mathrm{lct}(X) = 1/r$ (see [78], [79]).

Example 1.11. Let X be a general intersection of hypersurfaces $F_1, \ldots, F_k \subset \mathbb{P}^n$ such that

$$\sum_{i=1}^{k} \deg\big(F_i\big) = n \geqslant 5k+1 \geqslant 11,$$

where $\deg(F_k) \geqslant \ldots \geqslant \deg(F_1) \geqslant 2$ and $\deg(F_k) \geqslant 8$. Then $\mathrm{lct}(X) = 1$ (see [158]).

In general, the number $\mathrm{lct}(X)$ depends on small deformations of the variety X.

Example 1.12. Let X be a smooth hypersurface in $\mathbb{P}(1,1,1,1,3)$ of degree 6. Then

$$\mathrm{lct}\big(X\big) \in \left\{\frac{5}{6}, \frac{43}{50}, \frac{13}{15}, \frac{33}{38}, \frac{7}{8}, \frac{33}{38}, \frac{8}{9}, \frac{9}{10}, \frac{11}{12}, \frac{13}{14}, \frac{15}{16}, \frac{17}{18}, \frac{19}{20}, \frac{21}{22}, \frac{29}{30}, 1\right\}$$

by [157] and [37], and all these values are attained.

Example 1.13. Let X be a smooth hypersurface in $\mathbb{P}(1^{n+1}, n)$ of degree $2n$. The inequalities

$$1 \geqslant \mathrm{lct}\big(X\big) \geqslant \frac{2n-1}{2n}$$

hold (see [37]). But the equality $\mathrm{lct}(X) = 1$ holds if X is general and $n \geqslant 3$.

Example 1.14. Let X be a smooth hypersurface in $\mathbb{P}^n$ of degree $n \geqslant 2$. Then the inequalities

$$1 \geqslant \mathrm{lct}\big(X\big) \geqslant \frac{n-1}{n}$$

hold (see [20]). Then it follows from [157] and [37] that

$$\mathrm{lct}\big(X\big) \geqslant \begin{cases} 1 \text{ if } n \geqslant 6,\\ 22/25 \text{ if } n = 5,\\ 16/21 \text{ if } n = 4,\\ 3/4 \text{ if } n = 3,\end{cases}$$

whenver X is general. But $\mathrm{lct}(X) = 1 - 1/n$ if X contains a cone of dimension $n-2$.

Example 1.15. Let X be a quasismooth hypersurface in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ of degree $\sum_{i=1}^{4} a_i$ such that X has at most terminal singularities (see [102]), where $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. Then

$$-K_X \sim \mathcal{O}_{\mathbb{P}(1,\,a_1,\,a_2,\,a_3,\,a_4)}\big(1\big)\Big|_X,$$

and there are 95 possibilities for the quadruple (a_1, a_2, a_3, a_4) (see [89], [82]). Then

$$1 \geqslant \mathrm{lct}\big(X\big) \geqslant \begin{cases} 16/21 \text{ if } a_1 = a_2 = a_3 = a_4 = 1,\\ 7/9 \text{ if } (a_1, a_2, a_3, a_4) = (1,1,1,2),\\ 4/5 \text{ if } (a_1, a_2, a_3, a_4) = (1,1,2,2),\\ 6/7 \text{ if } (a_1, a_2, a_3, a_4) = (1,1,2,3),\\ 1 \text{ in the remaining cases,}\end{cases}$$

if X is general (see [27], [37], [29], [30]). The global log canonical threshold of the hypersurface

$$w^2 = t^3 + z^9 + y^{18} + x^{18} \subset \mathbb{P}(1,1,2,6,9) \cong \mathrm{Proj}\Big(\mathbb{C}\big[x,y,z,t,w\big]\Big)$$

is equal to 17/18 (see [27]), where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 2$, $\mathrm{wt}(t) = 6$, $\mathrm{wt}(w) = 9$.

Example 1.16. It follows from Lemma 6.1 that

$$\mathrm{lct}\Big(\mathbb{P}\big(a_0, a_1, \ldots, a_n\big)\Big) = \frac{a_0}{\sum_{i=0}^{n} a_i},$$

where $\mathbb{P}(a_0, a_1, \ldots, a_n)$ is well-formed (see [89]), and $a_0 \leqslant a_1 \leqslant \ldots \leqslant a_n$.

Example 1.17. Let X be a smooth hypersurface in $\mathbb{P}(1^{n+1}, d)$ of degree $2d$. Then

$$\mathrm{lct}\big(X\big) = \frac{1}{n+1-d}$$

in the case when the inequalities $2 \leqslant d \leqslant n-1$ hold (see Proposition 20 in [28]).

Example 1.18. Let X be smooth surface del Pezzo. It follows from [31] that

$$\mathrm{lct}\big(X\big) = \begin{cases} 1 \text{ if } K_X^2 = 1 \text{ and } |-K_X| \text{ contains no cuspidal curves,} \\ 5/6 \text{ if } K_X^2 = 1 \text{ and } |-K_X| \text{ contains a cuspidal curve,} \\ 5/6 \text{ if } K_X^2 = 2 \text{ and } |-K_X| \text{ contains no tacnodal curves,} \\ 3/4 \text{ if } K_X^2 = 2 \text{ and } |-K_X| \text{ contains a tacnodal curve,} \\ 3/4 \text{ if } X \text{ is a cubic in } \mathbb{P}^3 \text{ with no Eckardt points,} \\ 2/3 \text{ if either } X \text{ is a cubic in } \mathbb{P}^3 \text{ with an Eckardt point, or } K_X^2 = 4, \\ 1/2 \text{ if } X \cong \mathbb{P}^1 \times \mathbb{P}^1 \text{ or } K_X^2 \in \{5, 6\}, \\ 1/3 \text{ in the remaining cases.} \end{cases}$$

It would be interesting to compute global log canonical thresholds of del Pezzo surfaces with at most canonical singularities that are of Picard rank one, which has been classified in [62].

Example 1.19. Let X be a singular cubic surface in $\mathbb{P}^3$ such that X has at most canonical singularities. The singularities of the surface X are classified in [16]. It follows from [32] that

$$\mathrm{lct}\big(X\big) = \begin{cases} 2/3 \text{ if } \mathrm{Sing}\big(X\big) = \big\{\mathbb{A}_1\big\}, \\ 1/3 \text{ if } \mathrm{Sing}\big(X\big) \supseteq \big\{\mathbb{A}_4\big\}, \\ 1/3 \text{ if } \mathrm{Sing}\big(X\big) = \big\{\mathbb{D}_4\big\}, \\ 1/3 \text{ if } \mathrm{Sing}\big(X\big) \supseteq \big\{\mathbb{A}_2, \mathbb{A}_2\big\}, \\ 1/4 \text{ if } \mathrm{Sing}\big(X\big) \supseteq \big\{\mathbb{A}_5\big\}, \\ 1/4 \text{ if } \mathrm{Sing}\big(X\big) = \big\{\mathbb{D}_5\big\}, \\ 1/6 \text{ if } \mathrm{Sing}\big(X\big) = \big\{\mathbb{E}_6\big\}, \\ 1/2 \text{ in the remaining cases.} \end{cases}$$

It is unknown whether $\mathrm{lct}(X) \in \mathbb{Q}$ or not[2] (cf. Question 1 in [181]).

Conjecture 1.20. There is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ on the variety X such that

$$\mathrm{lct}\big(X\big) = \mathrm{lct}\big(X, D\big) \in \mathbb{Q}.$$

Let $G \subset \mathrm{Aut}(X)$ be an arbitrary subgroup.

Definition 1.21. Global G-invariant log canonical threshold of the Fano variety X is

$$\mathrm{lct}\big(X, G\big) = \sup\left\{ \varepsilon \in \mathbb{Q} \;\middle|\; \begin{array}{l} \text{the log pair } \Big(X, \dfrac{\varepsilon}{n}\mathcal{D}\Big) \text{ has log canonical singularities for every} \\ G\text{-invariant linear system } \mathcal{D} \subset \big|-nK_X\big| \text{ and every } n \in \mathbb{Z}_{>0} \end{array} \right\}.$$

[2]It is even unknown whether $\mathrm{lct}(X) \in \mathbb{Q}$ or not if X is a del Pezzo surfaces with log terminal singularities.

Remark 1.22. To define the threshold $\mathrm{lct}(X,G) \in \mathbb{R} \cup \{+\infty\}$, we only need to assume that
$$\Big| -nK_X \Big| \neq \varnothing$$
for some $n \gg 0$. But all known applications require $-K_X$ to be ample, and G to be compact.

In the case when the Fano variety X is smooth and G is compact, the equality
$$\mathrm{lct}\big(X,G\big) = \alpha_G\big(X\big),$$
holds (see Appendix A), where $\alpha_G(X)$ is the α-invariant introduced in [179]. It is clear that
$$\mathrm{lct}\big(X,G\big) = \sup\left\{\lambda \in \mathbb{Q}\ \left|\ \begin{array}{l}\text{the log pair } \Big(X,\lambda D\Big) \text{ has log canonical singularities}\\ \text{for every } G\text{-invariant effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X\end{array}\right.\right\}$$
in the case when $|G| < +\infty$. Note that $0 \leqslant \mathrm{lct}(X) \leqslant \mathrm{lct}(X,G) \in \mathbb{R} \cup \{+\infty\}$.

Example 1.23. The simple group $\mathrm{PGL}(2,\mathrm{F}_7)$ is a group of automorphisms of the quartic
$$x^3y + y^3z + z^3x = 0 \subset \mathbb{P}^2 \cong \mathrm{Proj}\Big(\mathbb{C}[x,y,z]\Big),$$
which induces an embedding $\mathrm{PGL}(2,\mathrm{F}_7) \subset \mathrm{Aut}(\mathbb{P}^2)$. Then $\mathrm{lct}(\mathbb{P}^2, \mathrm{PGL}(2,\mathrm{F}_7)) = 4/3$ (see [31]).

Example 1.24. Let X be a smooth del Pezzo surface such that $K_X^2 = 5$. Then

- the isomorphism $\mathrm{Aut}(X) \cong \mathrm{S}_5$ holds (see [161]),
- the equalities $\mathrm{lct}(X,\mathrm{S}_5) = \mathrm{lct}(X,\mathrm{A}_5) = 2$ hold (see [31]).

Example 1.25. Let X be the cubic surface in $\mathbb{P}^3$ given by the equation
$$x^3 + y^3 + z^3 + t^3 = 0 \subset \mathbb{P}^3 \cong \mathrm{Proj}\Big(\mathbb{C}\big[x,y,z,t\big]\Big),$$
and let $G = \mathrm{Aut}(X) \cong \mathbb{Z}_3^3 \rtimes \mathrm{S}_4$. Then $\mathrm{lct}(X,G) = 4$ by [31].

The following result was proved in [179], [132], [49] (see Appendix A).

Theorem 1.26. Suppose that X has at most quotient singularities, and the inequality
$$\mathrm{lct}\big(X,G\big) > \frac{\dim\big(X\big)}{\dim\big(X\big)+1}$$
holds. Then X admits an orbifold Kähler–Einstein metric.

Let us show how to apply Theorem 1.26 (cf. Examples 1.13, 1.14, 1.15).

Example 1.27. Let X be a quasismooth hypersurface in $\mathbb{P}(a_0,a_1,a_2,a_3)$ of degree $\sum_{i=0}^{3} a_i - 1$, where $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3$. Then it follows from [49], [81], [13], [2] (cf. [14], [15]) that

- either the surface X is smooth, which implies that
$$\big(a_0,a_1,a_2,a_3\big) \in \Big\{\big(1,1,1,1\big),\ \big(1,1,1,2\big),\ \big(1,1,2,3\big)\Big\},$$
and all possible values of $\mathrm{lct}(X)$ are contained in Example 1.18,
- or $(a_0,a_1,a_2,a_3) = (2,2n+1,2n+1,4n+1)$ and $\mathrm{lct}(X) = 1$, where $n \in \mathbb{Z}_{\geqslant 2}$,
- or we have the following sporadic possibilities:
  - $(a_0,a_1,a_2,a_3) = (2,3,3,5)$ and $\mathrm{lct}(X) \geqslant 33/38$;
  - $(a_0,a_1,a_2,a_3) = (1,2,3,5)$ and $\mathrm{lct}(X) > 2/3$;
  - $(a_0,a_1,a_2,a_3) = (1,3,5,7)$ and $\mathrm{lct}(X) > 2/3$ if $X \subset \mathbb{P}(1,3,5,7)$ is general;
  - $(a_0,a_1,a_2,a_3) = (1,3,5,8)$ and $\mathrm{lct}(X) \geqslant 11/16$ if $X \subset \mathbb{P}(1,3,5,8)$ is general;
  - the inequality $\mathrm{lct}(X) > 1$ holds and
$$\big(a_0,a_1,a_2,a_3\big) \in \left\{\begin{array}{l}\big(2,3,5,9\big), \big(3,3,5,5\big), \big(3,5,7,11\big), \big(3,5,7,14\big), \big(3,5,11,18\big)\\ \big(5,14,17,21\big), \big(5,19,27,31\big), \big(5,19,27,50\big), \big(7,11,27,37\big)\\ \big(7,11,27,44\big), \big(9,15,17,20\big), \big(9,15,23,23\big), \big(11,29,39,49\big)\\ \big(11,49,69,128\big), \big(13,23,35,57\big), \big(13,35,81,128\big)\end{array}\right\}.$$

Example 1.28. Let X be a quasismooth hypersurface in $\mathbb{P}(a_0, \ldots, a_4)$ of degree $\sum_{i=0}^{4} a_i - 1$, where $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. Then it follows from [82] that

- the inequality $\mathrm{lct}(X) > 3/4$ holds for at least 1936 quintuples $(a_0, a_1, a_2, a_3, a_4)$,
- the inequality $\mathrm{lct}(X) \geqslant 1$ holds for at least 1605 quintuples $(a_0, a_1, a_2, a_3, a_4)$.

Example 1.29. Let X be one of the following smooth Fano varieties:

- a Fermat hypersurface in $\mathbb{P}^n$ of degree $n/2 \leqslant d \leqslant n$ (cf. Example 1.25);
- a general complete intersection of three quadrics in $\mathbb{P}^6$ that is given by
$$\sum_{i=0}^{6} x_i^2 = \sum_{i=0}^{6} \lambda_i x_i^2 = \sum_{i=0}^{6} \mu_i x_i^2 = 0 \subseteq \mathbb{P}^6 \cong \mathrm{Proj}\Big(\mathbb{C}\big[x_0, \ldots, x_6\big]\Big),$$
where λ_i and μ_i are complex numbers;
- a smooth complete intersection of two quadrics in $\mathbb{P}^5$ that is given by
$$\sum_{i=0}^{5} x_i^2 = \sum_{i=0}^{5} \zeta^i x_i^2 = 0 \subseteq \mathbb{P}^5 \cong \mathrm{Proj}\Big(\mathbb{C}\big[x_0, \ldots, x_5\big]\Big),$$
where ζ is a primitive sixth root of unity;
- a smooth complete intersection of a quadric and a cubic in $\mathbb{P}^5$ that is given by
$$\sum_{i=0}^{5} x_i^3 = \sum_{i=0}^{5} \zeta^i x_i^2 = 0 \subseteq \mathbb{P}^5 \cong \mathrm{Proj}\Big(\mathbb{C}\big[x_0, \ldots, x_5\big]\Big),$$
where ζ is a nontrivial cube root of unity;
- a hypersurface in $\mathbb{P}(1^{n+1}, q)$ of degree pq that is given by the equation
$$w^p = \sum_{i=0}^{5} x_i^{pq} \subseteq \mathbb{P}\big(1^{n+1}, q\big) \cong \mathrm{Proj}\Big(\mathbb{C}\big[x_0, \ldots, x_n, w\big]\Big),$$
such that $pq - q \leqslant n$, where $\mathrm{wt}(x_0) = \ldots = \mathrm{wt}(x_n) = 1$, $\mathrm{wt}(w) = q \in \mathbb{Z}_{>0}$ and $p \in \mathbb{Z}_{>0}$;

and let $G = \mathrm{Aut}(X)$. Then G is finite, and the inequality $\mathrm{lct}(X, G) \geqslant 1$ holds (see [179], [132]).

Example 1.30. Let X be a blow up of $\mathbb{P}^3$ along a disjoint union of two lines, let G be a maximal compact subgroup in $\mathrm{Aut}(X)$. Then the inequality $\mathrm{lct}(X, G) \geqslant 1$ holds by [132] (cf. Lemma 9.26).

If a variety with quotient singularities admits an orbifold Kähler–Einstein metric, then

- either its canonical divisor is numerically trivial;
- or its canonical divisor is ample (variety of general type);
- or its canonical divisor is antiample (Fano variety).

Remark 1.31. Every variety with at most quotient singularities that has numerically trivial or ample canonical divisor always admits an orbifold Kähler–Einstein metric (see [5], [190], [191]).

There are several known obstructions for the Fano variety X to admit a Kähler–Einstein metric. For example, if the variety X is smooth, then it does not admit a Kähler–Einstein metric if

- either the group $\mathrm{Aut}(X)$ is not reductive (see [122]),
- or the tangent bundle of X is not polystable with respect to $-K_X$ (see [114]),
- or the Futaki character of holomorphic vector fields on X does not vanish (see [68]),
- or the pair $(X, -K_X)$ is not K-semistable (see [182], [53], [54], [162], [163]).

Example 1.32. The following varieties admit no Kähler–Einstein metrics:

- a blow up of $\mathbb{P}^2$ in one or two points (see [122]),
- a smooth Fano threefold $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ (see [176]),
- a smooth Fano fourfold
$$\mathbb{P}\Big(\alpha^*\big(\mathcal{O}_{\mathbb{P}^1}(1)\big) \oplus \beta^*\big(\mathcal{O}_{\mathbb{P}^2}(1)\big)\Big),$$
where $\alpha\colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ and $\beta\colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ are natural projections (see [68]).

Example 1.33. Let X be a smooth Fano threefold such that
$$\mathrm{Pic}\big(X\big) = \mathbb{Z}\big[-K_X\big]$$
and $-K_X^3 = 22$. Then $\mathrm{lct}(X) \leqslant 2/3$ by Lemma 11.3 (see Section 3). But

- the tangent bundle of the threefold X is stable (see [176]),
- the group $\mathrm{Aut}(X)$ is trivial if the threefold X is general (see [167]),
- there is X such that $\mathrm{Aut}(X) = \{1\}$ and X admits no Kähler–Einstein metrics (see [182]);
- there is X, whose group of automorphisms $\mathrm{Aut}(X)$ is not non-reductive (see [142]);
- if $\mathrm{Aut}(X) \cong \mathrm{PSL}(2, \mathbb{C})$, then X has a Kähler–Einstein metric (see [55] and Remark 3.2).

Recently new obstruction for the existence of orbifold Kähler–Einstein metrics on Fano varieties with at most quotient singularities have been found (see [69], [175], [11]).

Example 1.34. Let X be a quasismooth hypersurface in $\mathbb{P}(a_0, \ldots, a_n)$ of degree $d < \sum_{i=0}^{n} a_i$, where $a_0 \leqslant \ldots, \leqslant a_n$ and $\mathbb{P}(a_0, \ldots, a_n)$ is well-formed (see [89]). Then X is a Fano variety. If
$$\sum_{i=0}^{n} a_i > d + na_0,$$
then X admits no orbifold Kähler–Einstein metric (see [69], [175]).

The problem of existence of Kähler–Einstein metrics on smooth toric Fano varieties is completely solved. Namely, the following result holds (see [115], [7], [187], [133]).

Theorem 1.35. If X is smooth and toric, then the following conditions are equivalent:

- the variety X admits a Kähler–Einstein metric;
- the Futaki character of holomorphic vector field of X vanishes;
- the baricenter of the reflexive polytope of X is zero.

It should be pointed out that the assertion of Theorem 1.26 gives only a sufficient condition for the existence of a Kähler–Einstein metric on X (cf. [119], [182]).

Example 1.36. Let X be a general cubic surface in $\mathbb{P}^3$ that has an Eckardt point (see Definition 4.1). Then $\mathrm{Aut}(X) \cong \mathbb{Z}_2$ (see [52]) and
$$\mathrm{lct}\Big(X,\ \mathrm{Aut}\big(X\big)\Big) = \mathrm{lct}\big(X\big) = \frac{2}{3}$$
by [31]. But every smooth del Pezzo surface that has a reductive automorphism groups admits a Kähler–Einstein metric (see [183], [180]).

Example 1.37. Let X be a general hypersurface in $\mathbb{P}(1^5, 3)$ of degree 6. Then $\mathrm{Aut}(X) \cong \mathbb{Z}_2$ (see [123]) and
$$\mathrm{lct}\Big(X,\ \mathrm{Aut}\big(X\big)\Big) = \mathrm{lct}\big(X\big) = \frac{1}{2}$$
by [28]. But X admit a Kähler–Einstein metric (see [3]).

The problem of existence of Kähler–Einstein metrics on singular Fano varities that have quotient singularities is not well studied even for del Pezzo surfaces with canonical singularities.

Example 1.38. Let X be a cubic surface in $\mathbb{P}^3$. Then

- the surface X admits a Kähler–Einstein metric if $\mathrm{Sing}(X) = \varnothing$ (see [180]),
- the surface X does not admit an orbifold Kähler–Einstein metric in the case when it has at least one singular point that is not a singular point of type $\mathbb{A}_1$ or $\mathbb{A}_2$ (see [51]),
- the surface that is given by the equation
$$xyz + xyt + xzt + yzt = 0 \subset \mathbb{P}^3 \cong \mathrm{Proj}\Big(\mathbb{C}\big[x, y, z, t\big]\Big)$$
admits an orbifold Kähler–Einstein metric and has 4 singular points of type $\mathbb{A}_1$ (see [32]),
- the surface that is given by the equation
$$xyz = t^3 \subset \mathbb{P}^3 \cong \mathrm{Proj}\Big(\mathbb{C}\big[x, y, z, t\big]\Big)$$
admits an orbifold Kähler–Einstein metric and has 3 singular points of type $\mathbb{A}_2$ (see [32]).

Example 1.39. Let X be a complete intersection of two quadrics in $\mathbb{P}^4$. Then

- the surface X admits a Kähler–Einstein metric if $\mathrm{Sing}(X)=\varnothing$ (see [180]),
- the surface X does not admit an orbifold Kähler–Einstein metric in the case when it has at least one singular point that is not an ordinary double point (see [88]),
- if the surface $X\subset\mathbb{P}^4$ can be given by the equations
$$\sum_{i=0}^{4}x_i^2=\sum_{i=0}^{4}\lambda_i x_i^2=0\subset\mathbb{P}^4\cong\mathrm{Proj}\Big(\mathbb{C}\big[x_0,\ldots,x_4\big]\Big)$$
and X has at most ordinary double points, then $\mathrm{lct}(X,\mathbb{Z}_2^4)=1$ (see [116]).

Remark 1.40. Let X be a del Pezzo surface with canonical singularities such that $K_X^2\leqslant 2$. Then

- the surface X admits a Kähler–Einstein metric in the following cases:
  - if $K_X^2=2$ and X has at most singular points of type $\mathbb{A}_1$ or $\mathbb{A}_2$ (see [70]);
  - if $K_X^2=1$ and X has at most ordinary double points (see [31]);
- we expect X to admit no Kähler–Einstein metrics if X has relatively bad singularities (cf. [51], [113]).

The numbers $\mathrm{lct}(X)$ and $\mathrm{lct}(X,G)$ play an important role in birational geometry.

Example 1.41. Let V and $\bar{V}$ be varieties with at most terminal and $\mathbb{Q}$-factorial singularities, and let Z be a smooth curve. Suppose that there is a commutative diagram

$$\begin{tikzcd} V \arrow[r, dashed, "\rho"] \arrow[d, "\pi"'] & \bar{V} \arrow[d, "\bar{\pi}"] \\ Z \arrow[r, equal] & Z \end{tikzcd}$$

such that π and $\bar{\pi}$ are flat morphisms, and ρ is a birational map that induces an isomorphism
$$V\setminus X\cong\bar{V}\setminus\bar{X},$$
where X and $\bar{X}$ are scheme fibers of π and $\bar{\pi}$ over a point $O\in Z$, respectively. Suppose that

- the fibers X and $\bar{X}$ are irreducible and reduced,
- the divisors $-K_V$ and $-K_{\bar{V}}$ are π-ample and $\bar{\pi}$-ample, respectively,
- the varieties X and $\bar{X}$ have at most log terminal singularities,

and ρ is not an isomorphism. Then it follows from [136] and [32] that
$$\mathrm{lct}\big(X\big)+\mathrm{lct}\big(\bar{X}\big)\leqslant 1, \tag{1.42}$$
where X and $\bar{X}$ are Fano varieties by the adjunction formula.

In general, the inequality 1.42 is sharp (see [152], [72], [73], [137]).

Example 1.43. Let $\pi\colon V\to Z$ be a surjective flat morphism such that

- the variety V is a smooth threefold,
- the variety Z is a smooth curve,
- the divisor $-K_V$ is π-ample,

let X be a scheme fiber of the morphism π over a point $O\in Z$ such that X is a smooth cubic surface in $\mathbb{P}^3$ that has an Eckardt poin $P\in X$ (cf. Definition 4.1), let $L_1,L_2,L_3\subset X$ be the lines that pass through the point P. Then it follows from [41] that there is a commutative diagram

$$\begin{tikzcd} & U \arrow[r, dashed, "\psi"] \arrow[dl, "\alpha"'] & \bar{U} \arrow[dr, "\beta"] & \\ V \arrow[rrr, dashed, "\rho"] \arrow[dr, "\pi"'] & & & \bar{V} \arrow[dl, "\bar{\pi}"] \\ & Z \arrow[r, equal] & Z & \end{tikzcd}$$

such that α is a blow up of the point P, the map ψ is an antiflip in the proper transforms of the curves L_1,L_2,L_3, and β is a contraction of the proper transform of the fiber X. Then

- the birational map ρ is not an isomorphism,
- the threefold $\bar{V}$ has terminal and $\mathbb{Q}$-factorial singularities,
- the divisor $-K_{\bar{V}}$ is a Cartier $\bar{\pi}$-ample divisor,
- the map ρ induces an isomorphism

$$V \setminus X \cong \bar{V} \setminus \bar{X},$$

  where $\bar{X}$ is a scheme fiber of $\bar{\pi}$ over the point O,
- the surface $\bar{X}$ is a cubic surface with a singular point of type $\mathbb{D}_4$.

The latter assertion implies that $\mathrm{lct}(X) + \mathrm{lct}(\bar{X}) = 1$ (see Examples 1.18 and 1.19).

Global log canonical thresholds can be used to prove that some Fano varieties are non-rational.

Definition 1.44. The variety X is said to be birationally superrigid if the following conditions hold:

- $\mathrm{rk}\,\mathrm{Pic}(X) = 1$;
- the variety X has terminal $\mathbb{Q}$-factorial singularities;
- there is no rational dominant map $\rho\colon X \dashrightarrow Y$ such that
  - general fiber of the map ρ is rationally connected,
  - the inequality $\dim(Y) \geqslant 1$ holds;
- there is no non-biregular birational map $\rho\colon X \dashrightarrow Y$ such that
  - the variety Y has terminal $\mathbb{Q}$-factorial singularities;
  - the equality $\mathrm{rk}\,\mathrm{Pic}(Y) = 1$ holds.

The following result is known as the Noether–Fano inequality (see [40], [96], [22], [159]).

Theorem 1.45. The following conditions are equivalent:

- the variety X is birationally superrigid;
- the following conditions hold:
  - the equality $\mathrm{rk}\,\mathrm{Pic}(X) = 1$ holds;
  - the variety X has terminal $\mathbb{Q}$-factorial singularities;
  - for every linear system $\mathcal{M}$ on the variety X that does not have fixed components, the log pair $(X, \lambda\mathcal{M})$ has canonical singularities, where $K_X + \lambda\mathcal{M} \sim_{\mathbb{Q}} 0$.

Proof. Because one part of the required assertion is well-known (see [40], [22], [159]), we prove only another part of the required assertion. Suppose that

- the variety X is birationally superrigid,
- but there is a linear system $\mathcal{M}$ on the variety X such that $\mathcal{M}$ has no fixed components, the singularities of $(X, \lambda\mathcal{M})$ are not canonical, where $K_X + \lambda\mathcal{M} \sim_{\mathbb{Q}} 0$.

It follows from [75] that there is birational morphism $\pi\colon V \to X$ such that

- the variety V is smooth,
- the proper transform of $\mathcal{M}$ on the variety V has no base points,

and let $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety V. Then

$$K_V + \lambda\mathcal{B} \sim_{\mathbb{Q}} \pi^*\Big(K_X + \lambda\mathcal{M}\Big) + \sum_{i=1}^{r} a_i E_i \sim_{\mathbb{Q}} \sum_{i=1}^{r} a_i E_i,$$

where E_i is an exceptional divisor of π, and $a_i \in \mathbb{Q}$.

It follows from [9] that there is a commutative diagram

$$\begin{tikzcd} & V \arrow[dl, dashed, "\rho"'] \arrow[dr, "\pi"] & \\ U \arrow[rr, "\phi"'] & & X \end{tikzcd}$$

such that ρ is a birational map, the morphism ϕ is birational, the divisor

$$K_U + \lambda\rho(\mathcal{B}) \sim_{\mathbb{Q}} \phi^*\Big(K_X + \lambda\mathcal{M}\Big) + \sum_{i=1}^{r} a_i \rho(E_i) \sim_{\mathbb{Q}} \sum_{i=1}^{r} a_i \rho(E_i)$$

is ϕ-nef, the variety U is $\mathbb{Q}$-factorial, the log pair $(U, \lambda\rho(\mathcal{B})$ has terminal singularities.

The morphism ϕ is not an isomorphism. It follows from [169, 1.1] that

$$a_i > 0 \Longrightarrow \dim\Big(\rho\big(E_i\big)\Big) \leqslant \dim\big(X\big) - 2,$$

but it follows from the construction of the map ρ that there is $k \in \{1, \ldots, r\}$ such that

- the inequality $a_k < 0$ holds,
- the subvariety $\rho(E_k) \subset U$ is a divisor,

because the singularities of the log pair $(X, \lambda\mathcal{M})$ are not canonical.

The divisor $K_U + \lambda\rho(\mathcal{B})$ is not pseudo-effective. Then it follows from [9] that there is a diagram

$$\begin{tikzcd} U \arrow[r, dashed, "\psi"] \arrow[d, "\phi"'] & Y \arrow[d, "\tau"] \\ X & Z \end{tikzcd}$$

such that ψ is a birational map, the morphism τ is a Mori fibred space (see [102]), and the divisor

$$-\Big(K_Y + \lambda\big(\psi \circ \rho\big)\big(\mathcal{B}\big)\Big)$$

is τ-ample. The variety Y has terminal $\mathbb{Q}$-factorial singularities, and $\operatorname{rk}\operatorname{Pic}(Y/Z) = 1$. Then

- the birational map $\psi \circ \rho \circ \pi^{-1}$ is not an isomorphism, because $K_X + \lambda\mathcal{M} \sim_{\mathbb{Q}} 0$,
- general fiber of the morphism τ is rationally connected (see [193]),

which contradicts the assumption that X is birationally superrigid. □

Birationally superrigid Fano varieties are non-rational. In particular, if the variety X is birationally superrigid, then $\dim(X) \neq 2$ (cf. [117], [118], [95]).

Example 1.46. It follows from [97] that smooth quartic hypersurface

$$x^4 + xw^3 + y^4 - 6y^2z^2 + z^4 + t^4 + t^3w = 0 \subset \mathbb{P}^4 \cong \operatorname{Proj}\Big(\mathbb{C}\big[x, y, z, t, w\big]\Big)$$

is smooth and unirational (cf. [120]) and birationally superrigid (cf. [38]).

Example 1.47. The following smooth Fano varieties are birationally superrigid:

- a smooth hypersurface in $\mathbb{P}^n$ of degree n for $4 \leqslant n \leqslant 12$ (see [97], [147], [19], [155], [59]);
- a general hypersurface in $\mathbb{P}^n$ of degree $n \geqslant 4$ (see [151]);
- a smooth hypersurface in $\mathbb{P}(1^{n+1}, n)$ of degree $2n \geqslant 6$ (see [94], [148]).
- a general complete intersection of hypersurfaces $F_1, \ldots, F_k \subset \mathbb{P}^n$ such that

  $$\sum_{i=1}^{k} \deg\big(F_i\big) = n \geqslant 3k + 1 \geqslant 7,$$

  where $\deg(F_k) \geqslant \ldots \geqslant \deg(F_2) \geqslant \deg(F_1) \geqslant 2$ (see [154]);
- a smooth fourfold complete intersection in $\mathbb{P}^6$ of degree 8 containing no planes (see [21]);
- a smooth Fano variety X such that there is a double cover

  $$\tau \colon X \longrightarrow V \subset \mathbb{P}^n$$

  where V is a hypersurface, τ is ramified in a divisor $R \in |\mathcal{O}_{\mathbb{P}^n}(2n - 2\deg(V))|_V|$, and
  - either $\deg(V) = 2$ and $n \geqslant 5$ (see [148]),
  - or V and $R \in |\mathcal{O}_{\mathbb{P}^n}(2n - 2\deg(V))|_V|$ are general and $n \geqslant 5$ (see [153]),
  - or $3 \leqslant \deg(V) \leqslant 4$ and $n \geqslant 8$ (see [24]).
- a sextic hypersurface in $\mathbb{P}^6$ with at most ordinary double points (see [25]).

Example 1.48. Let $\pi \colon X \to \mathbb{P}^3$ be a double cover branched along a surface $S \subset \mathbb{P}^3$ of degree 6 such that the sextic surface S has at most ordinary double points. Then

- the inequality $|\operatorname{Sing}(S)| \leqslant 65$ holds (see [84], [188]),
- for any $65 \geqslant k \in \mathbb{Z}_{>0}$, there exists $S \subset \mathbb{P}^3$ such that $|\operatorname{Sing}(S)| = k$ (see [18], [6]),
- the variety X is birationally superrigid in the case when $\operatorname{rk}\operatorname{Cl}(X) = 1$ (see [97], [34]),
- the equality $\operatorname{rk}\operatorname{Cl}(X) = 1$ holds if $|\operatorname{Sing}(S)| \leqslant 14$ (see [34]),

- suppose that the surface S is a Barth sextic (see [6]) that is given by

$$4\Big(\tau^2x^2-y^2\Big)\Big(\tau^2y^2-z^2\Big)\Big(\tau^2z^2-x^2\Big)=t^2\Big(1+2\tau\Big)\Big(x^2+y^2+z^2-t^2\Big)^2\subset\mathbb{P}^3\cong\mathrm{Proj}\Big(\mathbb{C}\big[x,y,z,t\big]\Big),$$

  where $\tau=(1+\sqrt{5})/2$; then $\mathrm{rk}\,\mathrm{Cl}(X)=14$ (see [56]) and the diagram

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, dashed, "\rho"'] & \mathbb{P}^4 \arrow[d, dashed, "\psi"] \\ X \arrow[r, "\pi"'] & \mathbb{P}^3 \end{tikzcd}$$

  commutes (see [56], [140]), where
  - the variety Y is a determinantal quartic threefold in $\mathbb{P}^4$ such that $|\mathrm{Sing}(Y)|=42$,
  - the map ψ is the projection from a singular point of the quartic Y,
  - the map ρ is a birational map,

  which implies that the threefold X is rational.

The following result is proved in [157].

Theorem 1.49. Let $X_1, X_2, \dots, X_r$ be birationally superrigid Fano varieties such that $\mathrm{lct}(X_1)\geqslant 1, \mathrm{lct}(X_2)\geqslant 1, \dots, \mathrm{lct}(X_r)\geqslant 1$. Then

- the variety $X_1\times\ldots\times X_r$ is non-rational and

$$\mathrm{Bir}\Big(X_1\times\ldots\times X_r\Big)=\mathrm{Aut}\Big(X_1\times\ldots\times X_r\Big),$$

- for every rational dominant map

$$\rho\colon X_1\times\ldots\times X_r \dashrightarrow Y,$$

  whose general fiber is rationally connected, there is a commutative diagram

$$\begin{tikzcd} X_1\times\ldots\times X_r \arrow[d, "\pi"'] \arrow[drr, dashed, "\rho"] & & \\ X_{i_1}\times\ldots\times X_{i_k} \arrow[rr, dashed, "\xi"'] & & Y \end{tikzcd}$$

  for some $\{i_1,\dots,i_k\}\subseteq\{1,\dots,r\}$, where ξ is a birational map, and π is a projection.

Varieties satisfying all hypotheses of Theorem 1.49 exist (see Examples 1.11, 1.13, 1.14, 1.47).

Example 1.50. Let X be a hypersurface that is given by

$$w^2=x^6+y^6+z^6+t^6+x^2y^2zt\subset\mathbb{P}\big(1,1,1,1,3\big)\cong\mathrm{Proj}\Big(\mathbb{C}\big[x,y,z,t,w\big]\Big),$$

where $\mathrm{wt}(x)=\mathrm{wt}(y)=\mathrm{wt}(z)=\mathrm{wt}(t)=1$ and $\mathrm{wt}(w)=3$. Then

- the threefold X is smooth and birationally superrigid (see [94]),
- it follows from [37] that the equality $\mathrm{lct}(X)=1$ holds.

Example 1.51. Let X be a hypersurface that is given by

$$w^2=\sum_{i=0}^{n}x_i^{2n}+\varepsilon\sum_{i=1}^{n-4}\left(\sum_{j=0}^{n}a_{ij}x_i\right)^{2n}\subset\mathbb{P}\big(1^{n+1},n\big)\cong\mathrm{Proj}\Big(\mathbb{C}\big[x_0,x_1,\dots,x_n,w\big]\Big)$$

for some $\varepsilon\in\mathbb{C}\ni a_{ij}$, where $\mathrm{wt}(x_i)=1$ and $\mathrm{wt}(w)=n$. Suppose that any $n+1$ forms among

$$x_0,x_1,\dots,x_n,\ \sum_{i=0}^{n}a_{1i}x_i,\ \sum_{i=0}^{n}a_{2i}x_i,\ \dots,\ \sum_{i=0}^{n}a_{n-4\,i}x_i$$

are linearly independent. Put $\Delta=\max\{|a_{ij}|\}$. Suppose that the inequalities $n\geqslant 8$ and

$$1>|\varepsilon|\Big(2n\Delta+2\Delta\Big)^{2m}$$

hold. Then X is birationally superrigid and $\mathrm{lct}(X) = 1$ (see [148], [160]). The hypersurface

$$w^2 = n^{2n^2} \sum_{i=0}^{n} x_i^{2n} + \sum_{i=1}^{n-4} \left(\sum_{j=0}^{n} (j+1)^i x_i \right)^{2n}$$

is birationally superrigid, and its global log canonical threshold is 1 (see [160]).

Suppose, in addition, that the subgroup $G \subset \mathrm{Aut}(X)$ is finite (cf. [71]).

Definition 1.52. The Fano variety X is G-birationally superrigid if

- the G-invariant subgroup of the group $\mathrm{Cl}(X)$ is isomorphic to $\mathbb{Z}$,
- the variety X has terminal singularities,
- there is no G-equivariant rational dominant map $\rho\colon X \dashrightarrow Y$ such that
  - general fiber of the map ρ is rationally connected,
  - the inequality $\dim(Y) \geqslant 1$ holds,
- there is no G-equivariant non-biregular birational map $\rho\colon X \dashrightarrow Y$ such that
  - the G-invariant subgroup of the group $\mathrm{Cl}(Y)$ is isomorphic to $\mathbb{Z}$,
  - the variety Y has terminal singularities.

Arguing as in the proof of Theorem 1.45, we obtain the following result.

Theorem 1.53. The following conditions are equivalent:

- the variety X is G-birationally superrigid;
- the following conditions hold:
  - the G-invariant subgroup of the group $\mathrm{Cl}(X)$ is isomorphic to $\mathbb{Z}$;
  - the variety X has terminal singularities;
  - for every G-invariant linear system $\mathcal{M}$ on variety X that has no fixed components, the log pair $(X, \lambda\mathcal{M})$ is canonical, where $K_X + \lambda\mathcal{M} \sim_{\mathbb{Q}} 0$.

If X is birationally superrigid, then X is G-birationally superrigid for any $G \subset \mathrm{Aut}(X)$.

Example 1.54. Let X be a smooth surface in $\mathbb{P}(1,1,2,3)$ of degree 6 such that the G-invariant subgroup of the group $\mathrm{Pic}(X)$ is $\mathbb{Z}$. Then X is G-birationally superrigid (see [117], [118],[95]).

The proof of Theorem 1.49 implies the following result (see [31]).

Theorem 1.55. Let X_i be a Fano variety, and let $G_i \subset \mathrm{Aut}(X_i)$ be a finite subgroup such that

- the variety X_i is G_i-birationally superrigid,
- the inequality $\mathrm{lct}(X_i, G_i) \geqslant 1$ holds for any $i = 1, \ldots, r$.

Then the following assertions hold:

- there is no $G_1 \times \ldots \times G_r$-equivariant birational map $\rho\colon X_1 \times \ldots \times X_r \dashrightarrow \mathbb{P}^n$;
- every $G_1 \times \ldots \times G_r$-equivariant birational automorphism of $X_1 \times \ldots \times X_r$ is biregular;
- for every $G_1 \times \ldots \times G_r$-equivariant rational dominant map

  $$\rho\colon X_1 \times \ldots \times X_r \dashrightarrow Y,$$

  whose general fiber is rationally connected, there a commutative diagram

  $$\begin{tikzcd} X_1 \times \ldots \times X_r \arrow[d, "\pi"'] \arrow[dr, dashed, "\rho"] & \\ X_{i_1} \times \ldots \times X_{i_k} \arrow[r, dashed, "\xi"'] & Y \end{tikzcd}$$

  where ξ is a birational map, π is a natural projection, and $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, r\}$.

Varieties satisfying all hypotheses of Theorem 1.55 do exist (see Example 1.25).

Example 1.56. The simple group A_6 is a group of automorphisms of the sextic

$$10x^3y^3 + 9zx^5 + 9zy^5 + 27z^6 = 45x^2y^2z^2 + 135xyz^4 \subset \mathbb{P}^2 \cong \mathrm{Proj}\Big(\mathbb{C}\big[x, y, z\big]\Big),$$

which induces an embedding $\mathrm{A}_6 \subset \mathrm{Aut}(\mathbb{P}^2)$. It follows from [44] that $\mathbb{P}^2$ is A_6-birationally superrigid. But the equality $\mathrm{lct}(\mathbb{P}^2, \mathrm{A}_6) = 2$ holds (see [31]). Thus, there is an induced embedding

$$\mathrm{A}_6 \times \mathrm{A}_6 \cong \Omega \subset \mathrm{Bir}\big(\mathbb{P}^4\big)$$

such that Ω is not conjugate to any subgroup in $\mathrm{Aut}(\mathbb{P}^4)$ by Theorem 1.55.

Example 1.57. Suppose that X be a smooth cubic surface in $\mathbb{P}^3$ that is given by

$$x^2y + xz^2 + zt^2 + tx^2 = 0 \subset \mathbb{P}^3 \cong \mathrm{Proj}\Big(\mathbb{C}\big[x, y, z, t\big]\Big).$$

Then $\mathrm{Aut}(X) \cong \mathrm{S}_5$ (see [52]). Hence, by [31]

$$\mathrm{lct}\big(X, \mathrm{S}_5\big) = \mathrm{lct}\big(X, \mathrm{A}_5\big) = 2,$$

and the surface X is A_5-birationally superrigid (see Example 1.54).

Let us consider Fano varieties that are close to being birationally superrigid.

Definition 1.58. The Fano variety X is birationally rigid[3] if

- the equality $\mathrm{rk}\,\mathrm{Pic}(X) = 1$ holds,
- the variety X has $\mathbb{Q}$-factorial and terminal singularities,
- there is no rational dominant map $\rho\colon X \dashrightarrow Y$ such that
  - a general fiber of the map ρ is rationally connected,
  - the inequality $\dim(Y) \geqslant 1$ holds,
- there is no birational map $\rho\colon X \dashrightarrow Y$ such that
  - the varieties Y and X are not biregular,
  - the variety Y has terminal $\mathbb{Q}$-factorial singularities,
  - the equality $\mathrm{rk}\,\mathrm{Pic}(Y) = 1$ holds.

Arguing as in the proof of Theorem 1.45, we obtain the following result.

Theorem 1.59. The following conditions are equivalent:

- the variety X is birationally rigid;
- the following conditions hold:
  - the equality $\mathrm{rk}\,\mathrm{Pic}(X) = 1$ holds;
  - the variety X has $\mathbb{Q}$-factorial and terminal singularities;
  - for every linear system $\mathcal{M}$ on the Fano variety X that does not have fixed components, there is birational automorphism $\xi \in \mathrm{Bir}(X)$ such that the log pair

$$\Big(X,\ \lambda\,\xi\big(\mathcal{M}\big)\Big)$$

has canonical singularities, where $K_X + \lambda\,\xi(\mathcal{M}) \sim_{\mathbb{Q}} 0$.

Remark 1.60. For every $n \geqslant 5$, there exists a smooth Fano variety X of dimension n such that

$$\mathrm{Pic}\big(X\big) = \mathbb{Z}\big[-K_X\big]$$

and the variety X is not birationally rigid (see [17]).

Birationally rigid Fano varieties are non-rational (see [40], [96], [22], [159]).

Example 1.61. The following varieties are birationally rigid but not birationally superrigid:

- a general complete intersection of a quadric and a cubic in $\mathbb{P}^5$ (see [99]);
- a smooth double cover of a quadric in $\mathbb{P}^4$ branched over a surface of degree 8 (see [94]).

One usually seeks for the birational automorphism in Definition 1.58 in a given set of birational automorphisms. This leads to the following definition.

[3]There are several definitions of birational rigidity and birational superrigidity (see [40], [42], [96], [22], [159]).

Definition 1.62. Suppose that X is birationally rigid. A subset $\Gamma \subset \mathrm{Bir}(X)$ untwists all maximal singularities if for every linear system $\mathcal{M}$ on the variety X that has no fixed components, there is a birational automorphism $\xi \in \Gamma$ such that the log pair

$$\Big(X,\ \lambda\xi(\mathcal{M})\Big)$$

has canonical singularities, where λ is a rational number such that $K_X + \lambda\xi(\mathcal{M}) \sim_{\mathbb{Q}} 0$.

If X is birationally rigid and there is $\Gamma \subset \mathrm{Bir}(X)$ that untwists all maximal singularities, then the group $\mathrm{Bir}(X)$ is generated by Γ and $\mathrm{Aut}(X)$.

Example 1.63. Let X be a sufficiently general hypersurface in $\mathbb{P}^n$ of degree $n \geqslant 5$ that has one ordinary singular point $O = \mathrm{Sing}(X)$ of multiplicity $n-2$. Then the projection

$$\psi\colon X \dashrightarrow \mathbb{P}^{n-1}$$

from the point O induces an involution that untwists all maximal singularities (see [156]).

If X is defined over a perfect field, then Definition 1.58 still makes sense (see [117], [118], [95]).

Definition 1.64. The variety X is universally birationally rigid if for any variety U, the variety

$$X \otimes \mathrm{Spec}\Big(\mathbb{C}\big(U\big)\Big)$$

is birationally rigid over a field of rational functions $\mathbb{C}(U)$ of the variety U.

Example 1.65. Let X be a smooth Fano threefold such that there is a double cover

$$\pi\colon X \longrightarrow Q \subset \mathbb{P}^3,$$

where Q is a quadric threefold, and π is branched in a surface $S \subset Q$ of degree 8. Put

$$\mathcal{C} = \Big\{C \subset X\ \Big|\ C \text{ is a smooth curve such that } -K_X \cdot C = 1\Big\},$$

then $\mathcal{C}$ is a one-dimensional family. For every curve $C \in \mathcal{C}$ there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, dashed, "\psi_C"'] & Q \arrow[d, dashed, "\phi_C"] \\ \mathbb{P}^2 \arrow[r, equal] & \mathbb{P}^2 \end{tikzcd}$$

where ϕ_C is a projection from the line $\pi(C)$. General fiber of the map ψ_C is an elliptic curve, the map ψ_C induces an elliptic fibration with a section and an involution $\tau_C \in \mathrm{Bir}(X)$. Then

$$\psi_C \in \mathrm{Aut}\big(X\big) \iff C \subset S,$$

and S contains no curves in $\mathcal{C}$ if X is general. It follows from [94] that there is an exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \mathrm{Bir}\big(X\big) \longrightarrow \mathrm{Aut}\big(X\big) \longrightarrow 1,$$

where Γ is a free product of subgroups that are generated by non-biregular birational involutions constructed above. Hence the Fano variety X is universally birationally rigid (see [94]).

Example 1.66. Let X be a quartic threefold in $\mathbb{P}^4$ that has at most ordinary double points. Then

- the inequality $|\mathrm{Sing}(X)| \leqslant 45$ holds (see [186]),
- in general, the variety X is not birationally superrigid if $\mathrm{Sing}(X) \neq \varnothing$,
- the variety X is universally birationally rigid if $\mathrm{rk}\,\mathrm{Cl}(X) = 1$ (see [97], [149], [125], [172]),
- the inequality $\mathrm{rk}\,\mathrm{Cl}(X) \leqslant 16$ holds (see [100], [101]),
- the equality $\mathrm{rk}\,\mathrm{Cl}(X) = 1$ holds if $|\mathrm{Sing}(X)| \leqslant 8$ (see [23]),
- the equality $\mathrm{rk}\,\mathrm{Cl}(X) = 1$ holds if the following conditions hold:
  - the inequality $|\mathrm{Sing}(X)| \leqslant 12$ holds;
  - the quartic X contains neither planes or quadric surfaces (see [171]);
- in the case when $|\mathrm{Sing}(X)| = 45$, it follows from [83] that X can be given by the equation

  $$w^4 - w\Big(x^3 + y^3 + z^3 + t^3\Big) + 3xyzt = 0 \subset \mathbb{P}^4 \cong \mathrm{Proj}\Big(\mathbb{C}\big[x, y, z, t, w\big]\Big),$$

  the quartic X is determinantal and rational, and $\mathrm{rk}\,\mathrm{Cl}(X) = 16$ (see [76], [100], [101]).

Birationally superrigid Fano varieties are universally birationally rigid.

Definition 1.67. Suppose that X is universally birationally rigid. A subset $\Gamma \subset \mathrm{Bir}(X)$ universally untwists all maximal singularities if for every variety U the induced subset

$$\Gamma \subset \mathrm{Bir}\big(X\big) \subseteq \mathrm{Bir}\Big(X \otimes \mathrm{Spec}\Big(\mathbb{C}\big(U\big)\Big)\Big)$$

untwists all maximal singularities on $X \otimes \mathrm{Spec}(\mathbb{C}(U))$.

An identity map universally untwists all maximal singularities if X is birationally superrigid.

Remark 1.68. Suppose that X is birationally rigid, and $\dim(X) \neq 1$. Let $\Gamma \subseteq \mathrm{Bir}(X)$ be a subset. It follows from [107] that the following conditions are equivalent:

- the subset Γ universally untwists all maximal singularities;
- the subset Γ untwists all maximal singularities, and $\mathrm{Bir}(X)$ is countable.

Example 1.69. In the assumptions of Example 1.15, suppose that X is general. Then

- the hypersurface X is universally birationally rigid (see [43]),
- there are involutions $\tau_1, \ldots, \tau_k \in \mathrm{Bir}(X)$ such that that the sequence of groups
$$1 \longrightarrow \big\langle \tau_1, \ldots, \tau_k \big\rangle \longrightarrow \mathrm{Bir}\big(X\big) \longrightarrow \mathrm{Aut}\big(X\big) \longrightarrow 1$$
is exact (see [43], [35]), where $\langle \tau_1, \ldots, \tau_k \rangle$ is a subgroup generated by $\tau_1, \ldots, \tau_k$,
- the subgroup $\langle \tau_1, \ldots, \tau_k \rangle$ universally untwists all maximal singularities (see [43]).

All relations between the involutions $\tau_1, \ldots, \tau_k$ are found in [35]. The papers [164], [165], [35], [26] classify all maps $\psi\colon X \dashrightarrow \mathbb{P}^2$ such that the diagram

$$\begin{tikzcd} & V \arrow[dl, "\alpha"'] \arrow[dr, "\omega"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^2 \end{tikzcd}$$

commutes, where α is a birational morphism, and ω is an elliptic fibration. The papers [164], [165], [36] classify all maps $\phi\colon X \dashrightarrow \mathbb{P}^1$ such that the diagram

$$\begin{tikzcd} & V \arrow[dl, "\beta"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\phi"'] & & \mathbb{P}^1 \end{tikzcd}$$

commutes, where β is a birational morphism, and η is a fibration into surfaces of Kodaira dimension zero.

Let $X_1, \ldots, X_r$ be Fano varieties that have at most $\mathbb{Q}$-factorial and terminal singularities, let

$$\pi_i\colon X_1 \times \ldots \times X_{i-1} \times X_i \times X_{i+1} \times \ldots \times X_r \longrightarrow X_1 \times \ldots \times X_{i-1} \times \widehat{X_i} \times X_{i+1} \times \ldots \times X_r$$

be a natural projection, and let $\mathcal{X}_i$ be a scheme general fiber of the projection π_i, which is defined over $\mathbb{C}(X_1 \times \ldots \times X_{i-1} \times \widehat{X_i} \times X_{i+1} \times \ldots \times X_r)$. Suppose that $\mathrm{rk}\,\mathrm{Pic}(X_1) = \ldots = \mathrm{rk}\,\mathrm{Pic}(X_r) = 1$.

Remark 1.70. There are natural embeddings of groups

$$\prod_{i=1}^{r} \mathrm{Bir}\big(X_i\big) \subseteq \Big\langle \mathrm{Bir}\big(\mathcal{X}_1\big), \ldots, \mathrm{Bir}\big(\mathcal{X}_r\big) \Big\rangle \subseteq \mathrm{Bir}\Big(X_1 \times \ldots \times X_r\Big).$$

The following generalization of Theorem 1.49 holds (see [27]).

Theorem 1.71. Suppose that $X_1, X_2, \ldots, X_r$ are universally birationally rigid. Then

- the variety $X_1 \times \ldots \times X_r$ is non-rational and
$$\mathrm{Bir}\Big(X_1 \times \ldots \times X_r\Big) = \Big\langle \mathrm{Bir}\big(\mathcal{X}_1\big), \ldots, \mathrm{Bir}\big(\mathcal{X}_r\big), \mathrm{Aut}\Big(X_1 \times \ldots \times X_r\Big) \Big\rangle,$$

- for every rational dominant map $\rho\colon X_1\times\ldots\times X_r\dashrightarrow Y$, whose general fiber is rationally connected, there is a subset $\{i_1,\ldots,i_k\}\subseteq\{1,\ldots,r\}$ and a commutative diagram

$$\begin{tikzcd}
X_1\times\ldots\times X_r \arrow[r, dashed, "\sigma"] \arrow[d, "\pi"'] & X_1\times\ldots\times X_r \arrow[dr, dashed, "\rho"] & \\
X_{i_1}\times\ldots\times X_{i_k} \arrow[rr, dashed, "\xi"'] & & Y
\end{tikzcd}$$

where ξ and σ are birational maps, and π is a projection,

in the case when the inequalities $\mathrm{lct}(X_1)\geqslant 1,\mathrm{lct}(X_2)\geqslant 1,\ldots,\mathrm{lct}(X_r)\geqslant 1$ hold.

Corollary 1.72. Suppose that there are subgroups $\Gamma_1\subseteq\mathrm{Bir}(X_1),\ldots,\Gamma_r\subseteq\mathrm{Bir}(X_r)$ that universally untwists all maximal singularities, and $\mathrm{lct}(X_1)\geqslant 1,\mathrm{lct}(X_2)\geqslant 1,\ldots,\mathrm{lct}(X_r)\geqslant 1$. Then

$$\mathrm{Bir}\Big(X_1\times\ldots\times X_r\Big)=\Big\langle\prod_{i=1}^{r}\Gamma_i,\ \mathrm{Aut}\Big(X_1\times\ldots\times X_r\Big)\Big\rangle.$$

The following four examples are implied by Examples 1.14, 1.15, 1.47, 1.69 and [123].

Example 1.73. Let X be a general hypersurface in $\mathbb{P}(1,1,4,5,10)$ of degree 20. The sequence

$$1\longrightarrow\prod_{i=1}^{m}\Big(\mathbb{Z}_2*\mathbb{Z}_2\Big)\longrightarrow\mathrm{Bir}\Big(\underbrace{X\times\ldots\times X}_{m\ \text{times}}\Big)\longrightarrow\mathrm{S}_m\longrightarrow 1$$

is exact, where S_m is a permutation group, and $\mathbb{Z}_2*\mathbb{Z}_2$ is the infinite dihedral group.

Example 1.74. Let X be a general hypersurface in $\mathbb{P}(1,1,3,4,5)$ of degree 13. Then

$$\mathrm{Bir}\Big(X\times V\Big)\cong\mathbb{Z}_2*\mathbb{Z}_2*\mathbb{Z}_2,$$

where V is a general hypersurface in $\mathbb{P}^n$ of degree $n\geqslant 6$.

Example 1.75. Let X be a general hypersurface in $\mathbb{P}(1,1,2,3,3)$ of degree 9. Then

$$\mathrm{Bir}\Big(X\times V\Big)\cong\Big\langle a,b,c\ \big|\ a^2=b^2=c^2=\big(abc\big)^2=1\Big\rangle,$$

where V is a general hypersurface in $\mathbb{P}^n$ of degree $n\geqslant 6$.

Example 1.76. Let X be a general hypersurface in $\mathbb{P}(1,1,2,2,3)$ of degree 8. Then

$$\mathrm{Bir}\Big(X\times V\Big)\cong\mathbb{Z}_2*\mathbb{Z}_2*\mathbb{Z}_2*\mathbb{Z}_2*\mathbb{Z}_2,$$

where V is a general hypersurface in $\mathbb{P}^n$ of degree $n\geqslant 6$.

Suppose now that X is a smooth Fano threefold (see [98]). Let

$$\gimel\big(X\big)\in\Big\{1.1,1.2,\ldots,1.17,2.1,\ldots,2.36,3.1,\ldots,3.31,4.1,\ldots,4.13,5.1,\ldots,5.7,5.8\Big\}$$

be the ordinal number of the deformation type of the threefold X in the notation of Table 1.

Remark 1.77. The threefold X lies in 105 deformation families (see [92], [93], [126], [128], [129], [127]).

The main purpose of this paper is to prove the following result.

Theorem 1.78. The following assertions hold:

- $\mathrm{lct}(X)=1/5$ for $\gimel(X)\in\{2.36,3.29\}$;
- $\mathrm{lct}(X)=1/4$ for

$$\gimel\big(X\big)\in\Big\{1.17,2.28,2.30,2.33,2.35,3.23,3.26,3.30,4.12\Big\};$$

- $\mathrm{lct}(X)=1/3$ for

$$\gimel\big(X\big)\in\Big\{1.16,2.29,2.31,2.34,3.9,3.18,\ldots,3.22,3.24,3.25,3.28,3.31,4.4,4.8,\ldots4.11,5.1,5.2\Big\};$$

- $\mathrm{lct}(X)=3/7$ for $\gimel(X)=4.5$;

- $\mathrm{lct}(X)=1/2$ for
$$\gimel(X)\in\left\{\begin{array}{l}1.11, 1.12, 1.13, 1.14, 1.15, 2.1, 2.3, 2.18, 2.25, 2.27, 2.32, 3.4, 3.10, 3.11, 3.12,\\ 3.14, 3.15, 3, 16, 3.17, 3.24, 3.27, 4.1, 4.2, 4.3, 4.6, 4.7, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8\end{array}\right\};$$
- if the threefold X is general in moduli, then
$$\mathrm{lct}\big(X\big)=\begin{cases}1/3 \text{ if } \gimel(X)=2.23,\\ 1/2 \text{ if } \gimel(X)\in\{2.5, 2.8, 2.10, 2.11, 2.14, 2.15, 2.19, 2.24, 2.26, 3.2, 3.5, \ldots, 3.8, 4.13\},\\ 2/3 \text{ if } \gimel(X)=3.3,\\ 3/4 \text{ if } \gimel(X)\in\{2.4, 3.1\},\\ 1 \text{ if } \gimel(X)=1.1.\end{cases}$$

Hence, if the threefold X is general in moduli, then we do not know $\mathrm{lct}(X)$ only when
$$\gimel\big(X\big)\in\Big\{1.2, 1.3, 1.4, 1.5, \ldots, 1.10, 2.2, 2.6, 2.7, 2.9, 2.12, 2.13, 2.16, 2.17, 2.20, 2.21, 2.22, 3.13\Big\},$$
and the generality condition in Theorem 1.78 can not be omitted in many cases.

Example 1.79. Suppose that $\gimel(X)=4.13$. Note that this deformation type was omitted in [126], and it has been discovered only twenty years later (see [127]). There is a birational morphism
$$\alpha\colon X\longrightarrow\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$$
that contracts a surface $E\subset X$ to a curve C such that $C\cdot F_1=C\cdot F_2=1$ and $C\cdot F_3=3$, where
$$F_1\cong F_2\cong F_3\cong\mathbb{P}^1\times\mathbb{P}^1$$
are fibers of three different projections $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\to\mathbb{P}^1$, respectively. Then
$$\mathrm{lct}\big(X\big)=1/2$$
by Theorem 1.78 if X is general. There is a surface $G\in|F_1+F_2|$ such that $C\subset G$. Then
$$-K_X\sim 2\bar{G}+E+\bar{F}_3,$$
where $\bar{F}_3\subset X\supset\bar{G}$ are proper transforms of F_3 and G, respectively. Then $\mathrm{lct}(X)\leqslant 1/2$. But
$$\mathrm{lct}\big(X\big)\leqslant\mathrm{lct}\Big(X,\ 2\bar{G}+E+\bar{F}_3\Big)\leqslant 4/9<1/2$$
in the case when the intersection $F_3\cap C$ consists of a single point.

We hope that the proof of Theorem 1.78 can be used to

- study the slope stability of the threefold X in the sense of [162] and [163],
- study the problem of existence of a Kähler–Einstein metric on the threefold X,
- compute $\mathrm{lct}(X,G)$ for various subgroups $G\subset\mathrm{Aut}(X)$.

Remark 1.80. The stability of the tangent bundle of X was studied in [176]. It is known that

- the tangent bundle of X is unstable with respect to $-K_X$ when
$$\gimel\big(X\big)\in\Big\{2.35, 2.36, 3.29, 3.30, 3.31, 4.11, 4.12\Big\},$$
- the tangent bundle of X is semistable with respect to $-K_X$ when
$$\gimel\big(X\big)\in\Big\{2.33, 2.34, 3.27, 3.28, 4.10, 5.2, \ldots, 5.8\Big\},$$
- the tangent bundle of X is stable with respect to $-K_X$ when
$$\gimel\big(X\big)\not\in\Big\{2.33, 2.34, 2.35, 2.36, 3.27, \ldots, 3.31, 4.10, 4.11, 4.12, 5.2, \ldots, 5.8\Big\}.$$

We organize the paper in the following way:

- in Section 2, we consider auxiliary results that are used in the proof of Theorem 1.78;
- in Section 3, we find the global log canonical threshold of the Mukai–Umemura threefold;
- in Section 4, we prove Theorem 4.2 that is required for Example 5.4 and Lemma 8.2;
- in Section 5, we consider facts on surfaces that are used in the proof of Theorem 1.78;
- in Section 6, we compute global log canonical thresholds of toric Fano varieties;

- in Section 7, we prove Theorem 1.78 for smooth Fano threefolds of index 2, i.e., for

$$\daleth(X) \in \Big\{1.11, 1.12, 1.13, 1.14, 1.15, 2.32, 2.35, 3.27\Big\};$$

- in Section 8, we prove Theorem 1.78 in the case when $\operatorname{rk}\operatorname{Pic}(X) = 2$;
- in Section 9, we prove Theorem 1.78 in the case when $\operatorname{rk}\operatorname{Pic}(X) = 3$;
- in Section 10, we prove Theorem 1.78 in the case when $\operatorname{rk}\operatorname{Pic}(X) \geqslant 4$;
- in Section 11, we find upper bounds for $\operatorname{lct}(X)$ in the case when

$$\daleth(X) \in \Big\{1.8, 1.9, 1.10, 2.9, 2.12, 2.13, 2.16, 2.17, 2.20, 2.21, 2.22, 3.13\Big\};$$

- in Appendix A, written by J.-P. Demailly, the relation between global log canonical thresholds of smooth Fano varieties and the α-invariants of smooth Fano varieties introduced by G. Tian in [179] for the study of the existence of Kähler–Einstein metrics has been studied;
- in Appendix B, we put Table 1 that contains the list of all smooth Fano threefolds together with the known values and bounds for their global log canonical thresholds.

We use a standard notation $D_1 \sim D_2$ (resp., $D_1 \sim_{\mathbb{Q}} D_2$) for the linearly equivalent (resp., $\mathbb{Q}$-linearly equivalent) divisors (resp., $\mathbb{Q}$-divisors). If a divisor (resp., a $\mathbb{Q}$-divisor) D is linearly equivalent to a line bundle $\mathcal{L}$ (resp., is $\mathbb{Q}$-linearly equivalent to a divisor that is linearly equivalent to $\mathcal{L}$), we write $D \sim \mathcal{L}$ (resp., $D \sim_{\mathbb{Q}} \mathcal{L}$). Recall that $\mathbb{Q}$-linear equivalence coincides with numerical equivalence in the case of Fano varieties.

A divisor D on $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \ldots \times \mathbb{P}^{n_m}$ is said to be of multidegree $(a_1, a_2, \ldots, a_m) \in \mathbb{Z}^m$ if $D \sim \sum_{i=1}^{m} a_i H_i$, where

$$H_i \sim \pi_i^*\Big(\mathcal{O}_{\mathbb{P}^{n_i}}(a_i)\Big),$$

is a pull-back of a hyperplane section of $\mathbb{P}^{n_i}$ under the projection $\pi_i\colon \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \ldots \times \mathbb{P}^{n_m} \to \mathbb{P}^{n_i}$ is a natural projection. A curve $C \subset \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \ldots \times \mathbb{P}^{n_m}$ is said to be of multidegree $(a_1, a_2, \ldots, a_m) \in \mathbb{Z}^m$ if

$$\pi_i^*\Big(\mathcal{O}_{\mathbb{P}^{n_i}}(1)\Big) \cdot C = a_i$$

for $i = 1, \ldots, m$.

The projectivisation $\mathbb{P}_Y(\mathcal{E})$ of a vector bundle $\mathcal{E}$ on a variety Y is the variety of hyperplanes in the fibers of $\mathcal{E}$. The symbol $\mathbb{F}_n$ denotes the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$.

We always refer to a smooth Fano threefold X using the ordinal number $\daleth(X)$ introduced in Table 1.

We are very grateful to J.-P. Demailly for writing Appendix A, and to A. Iliev, A. G. Kuznetsov and Yu. G. Prokhorov for useful discussions. The first author would like to express his gratitude to IHES (Bures-sur-Yvette, France) and MPIM (Bonn, Germany) for hospitality.

2. Preliminaries

Let X be a variety with log terminal singularities. Let us consider a $\mathbb{Q}$-divisor

$$B_X = \sum_{i=1}^{r} a_i B_i,$$

where B_i is a prime Weil divisor on the variety X, and a_i is an arbitrary non-negative rational number. Suppose that B_X is a $\mathbb{Q}$-Cartier divisor such that $B_i \neq B_j$ for $i \neq j$.

Let $\pi\colon \bar{X} \to X$ be a birational morphism such that $\bar{X}$ is smooth (see [75]). Put

$$B_{\bar{X}} = \sum_{i=1}^{r} a_i \bar{B}_i,$$

where $\bar{B}_i$ is a proper transform of the divisor B_i on the variety $\bar{X}$. Then

$$K_{\bar{X}} + B_{\bar{X}} \sim_{\mathbb{Q}} \pi^*\Big(K_X + B_X\Big) + \sum_{i=1}^{n} c_i E_i,$$

where $c_i \in \mathbb{Q}$, and E_i is an exceptional divisor of the morphism π. Suppose that

$$\left(\bigcup_{i=1}^{r}\bar{B}_i\right)\bigcup\left(\bigcup_{i=1}^{n}E_i\right)$$

is a divisor with simple normal crossing. Put

$$B^{\bar{X}} = B_{\bar{X}} - \sum_{i=1}^{n} c_i E_i.$$

Definition 2.1. The singularities of (X, B_X) are log canonical (resp., log terminal) if

- the inequality $a_i \leqslant 1$ holds (resp., the inequality $a_i < 1$ holds),
- the inequality $c_j \geqslant -1$ holds (resp., the inequality $c_j > -1$ holds),

for every $i = 1, \ldots, r$ and $j = 1, \ldots, n$.

One can show that Definition 2.1 does not depend on the choice of the morphism π. Put

$$\mathrm{LCS}\Big(X, B_X\Big) = \left(\bigcup_{a_i \geqslant 1} B_i\right)\bigcup\left(\bigcup_{c_i \leqslant -1} \pi\big(E_i\big)\right) \subsetneq X,$$

then $\mathrm{LCS}(X, B_X)$ is called the locus of log canonical singularities of the log pair (X, B_X).

Definition 2.2. A proper irreducible subvariety $Y \subsetneq X$ is said to be a center of log canonical singularities of the log pair (X, B_X) if one of the following conditions is satisfied:

- either the inequality $a_i \geqslant 1$ holds and $Y = B_i$,
- or the inequality $c_i \leqslant -1$ holds and $Y = \pi(E_i)$,

for some choice of the birational morphism $\pi\colon \bar{X} \to X$.

Let $\mathbb{LCS}(X, B_X)$ be the set of all centers of log canonical singularities of (X, B_X). Then

$$Y \in \mathbb{LCS}\Big(X, B_X\Big) \Longrightarrow Y \subseteq \mathrm{LCS}\Big(X, B_X\Big)$$

and $\mathbb{LCS}(X, B_X) = \varnothing \iff \mathrm{LCS}(X, B_X) = \varnothing \iff$ the log pair (X, B_X) is log terminal.

Remark 2.3. Let $\mathcal{H}$ be a linear system on X that has no base points, let H be a sufficiently general divisor in the linear system $\mathcal{H}$, and let $Y \subsetneq X$ be an irreducible subvariety. Put

$$Y\Big|_{H} = \sum_{i=1}^{m} Z_i,$$

where $Z_i \subset H$ is an irreducible subvariety. It follows from Definition 2.2 (cf. Theorem 2.20) that

$$Y \in \mathbb{LCS}\Big(X,\ B_X\Big) \iff \Big\{Z_1, \ldots, Z_m\Big\} \subseteq \mathbb{LCS}\Big(H,\ B_X\Big|_H\Big).$$

Example 2.4. Let $\alpha\colon V \to X$ be a blow up of a smooth point $O \in X$. Then

$$B_V \sim_{\mathbb{Q}} \alpha^*\big(B_X\big) - \mathrm{mult}_O\big(B_X\big)E,$$

where $\mathrm{mult}_O(B_X) \in \mathbb{Q}$, and E is the exceptional divisor of the blow up α. Then

$$\mathrm{mult}_O\big(B_X\big) > 1$$

if the log pair (X, B_X) is not log canonical at the point O. Put

$$B^V = B_V + \Big(\mathrm{mult}_O\big(B_X\big) - \dim\big(X\big) + 1\Big)E,$$

and suppose that $\mathrm{mult}_O(B_X) \geqslant \dim(X) - 1$. Then $O \in \mathbb{LCS}(X, B_X)$ if and only if

- either $E \in \mathbb{LCS}(V, B^V)$, i.e. $\mathrm{mult}_O(B_X) \geqslant \dim(X)$,
- or there is a subvariety $Z \subsetneq E$ such that $Z \in \mathbb{LCS}(V, B^V)$.

The locus $\mathrm{LCS}(X, B_X) \subset X$ can be equipped with a scheme structure (see [132], [169]). Put

$$\mathcal{I}\Big(X, B_X\Big) = \pi_*\Big(\sum_{i=1}^{n}\lceil c_i\rceil E_i - \sum_{i=1}^{r}\lfloor a_i\rfloor \bar{B}_i\Big),$$

and let $\mathcal{L}(X, B_X)$ be a subscheme that corresponds to the ideal sheaf $\mathcal{I}(X, B_X)$.

Definition 2.5. For the log pair (X, B_X), we say that

- the subscheme $\mathcal{L}(X, B_X)$ is the subscheme of log canonical singularities of (X, B_X),
- the ideal sheaf $\mathcal{I}(X, B_X)$ is the multiplier ideal sheaf of (X, B_X).

It follows from the construction of the subscheme $\mathcal{L}(X, B_X)$ that

$$\mathrm{Supp}\Big(\mathcal{L}(X, B_X)\Big) = \mathrm{LCS}\Big(X, B_X\Big) \subset X.$$

The following result is the Nadel–Shokurov vanishing theorem (see [169], [111, Theorem 9.4.8]).

Theorem 2.6. Let H be a nef and big $\mathbb{Q}$-divisor on X such that

$$K_X + B_X + H \sim_{\mathbb{Q}} D$$

for some Cartier divisor D on the variety X. Then for every $i \geqslant 1$

$$H^i\Big(X,\ \mathcal{I}(X, B_X) \otimes D\Big) = 0.$$

Proof. It follows from the Kawamata–Viehweg vanishing theorem (see [105]) that

$$R^i\pi_*\Big(\pi^*(K_X + B_X + H) + \sum_{i=1}^{n}\lceil c_i\rceil E_i - \sum_{i=1}^{r}\lfloor a_i\rfloor \bar{B}_i\Big) = 0$$

for every $i > 0$. It follows from the equality of sheaves

$$\pi_*\Big(\pi^*(K_X + B_X + H) + \sum_{i=1}^{n}\lceil c_i\rceil E_i - \sum_{i=1}^{r}\lfloor a_i\rfloor \bar{B}_i\Big) = \mathcal{I}(X, B_X) \otimes D$$

and from the degeneration of a local-to-global spectral sequence that

$$H^i\Big(X, \mathcal{I}(X, B_X) \otimes D\Big) = H^i\Big(\bar{X}, \pi^*(K_X + B_X + H) + \sum_{i=1}^{n}\lceil c_i\rceil E_i - \sum_{i=1}^{r}\lfloor a_i\rfloor \bar{B}_i\Big),$$

for every $i \geqslant 0$. But for $i > 0$, the cohomology group

$$H^i\Big(\bar{X}, \pi^*(K_X + B_X + H) + \sum_{i=1}^{n}\lceil c_i\rceil E_i - \sum_{i=1}^{r}\lfloor a_i\rfloor \bar{B}_i\Big)$$

is trivial by the Kawamata–Viehweg vanishing theorem (see [105]). □

For every Cartier divisor D on the variety X, let us consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}(X, B_X) \otimes D \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_{\mathcal{L}(X,\, B_X)}(D) \longrightarrow 0,$$

and let us consider the corresponding exact sequence of cohomology groups

$$H^0\Big(\mathcal{O}_X(D)\Big) \longrightarrow H^0\Big(\mathcal{O}_{\mathcal{L}(X,\, B_X)}(D)\Big) \longrightarrow H^1\Big(\mathcal{I}(X, B_X) \otimes D\Big).$$

Theorem 2.7. Suppose that $-(K_X + B_X)$ is nef and big. Then $\mathrm{LCS}(X, B_X)$ is connected.

Proof. Put $D = 0$. Then it follows from Theorem 2.6 that the sequence

$$\mathbb{C} = H^0\Big(\mathcal{O}_X\Big) \longrightarrow H^0\Big(\mathcal{O}_{\mathcal{L}(X,\, B_X)}\Big) \longrightarrow H^1\Big(\mathcal{I}(X, B_X)\Big) = 0$$

is exact. Thus, the locus

$$\mathrm{LCS}\Big(X,\ B_X\Big) = \mathrm{Supp}\Big(\mathcal{L}(X,\ B_X)\Big)$$

is connected. □

Let us consider few elementary applications of Theorem 2.7 (cf. Example 1.18).

Lemma 2.8. Suppose that $\mathrm{LCS}(X, B_X) \neq \varnothing$, where $X \cong \mathbb{P}^n$, and

$$B_X \sim_{\mathbb{Q}} -\lambda K_X$$

for some rational number $0 < \lambda < n/(n+1)$. Then

- the inequality $\dim(\mathrm{LCS}(X, B_X)) \geqslant 1$ holds,
- the subscheme $\mathcal{L}(X, B_X)$ does not contain isolated zero-dimensional components.

Proof. Suppose that there is a point $O \in X$ such that

$$\mathrm{LCS}\Big(X, \lambda B_X\Big) = O \cup \Sigma,$$

where $\Sigma \subset X$ is a possibly empty subset such that $O \notin X$.

Let H be a general line in $X \cong \mathbb{P}^2$. Then the locus

$$\mathrm{LCS}\Big(X, \lambda B_X + H\Big) = O \cup H \cup \Sigma$$

is disconnected. But the divisor $-(K_X+\lambda B_X+H)$ is ample, which contradicts Theorem 2.7. $\square$

Lemma 2.9. Suppose that $\mathrm{LCS}(X, B_X) \neq \varnothing$, where $X \cong \mathbb{P}^3$, and

$$B_X \sim_{\mathbb{Q}} -\lambda K_X$$

for some rational number $0 < \lambda < 1/2$. Then $\mathbb{LCS}(X, B_X)$ contains a surface.

Proof. Suppose that $\mathbb{LCS}(X, B_X)$ contains no surfaces. Let $S \subset \mathbb{P}^3$ be a general plane. The locus

$$\mathrm{LCS}\Big(\mathbb{P}^3,\ B_X + S\Big)$$

is connected by Theorem 2.7. Then $(S, B_X|_S)$ is not log terminal by Remark 2.3. But the locus

$$\mathrm{LCS}\Big(S,\ B_X\Big|_S\Big)$$

consists of finitely many points, which is impossible by Lemma 2.8. $\square$

Lemma 2.10. Suppose that $\mathrm{LCS}(X, B_X) \neq \varnothing$, where X is a smooth quadric threefold in $\mathbb{P}^4$, and

$$B_X \sim_{\mathbb{Q}} -\lambda K_X$$

for some rational number $0 < \lambda < 1/2$. Then $\mathbb{LCS}(X, B_X)$ contains a surface.

Proof. Let $L \subset X$ be a general line, let $P_1 \in L \ni P_2$ be two general points, let H_1 and H_2 be the hyperplane sections of X that are tangent to X at the points P_1 and P_2, respectively. Then

$$\mathrm{LCS}\left(X,\ \lambda B_X + \frac{3}{4}\Big(H_1 + H_2\Big)\right) = \mathrm{LCS}\Big(X,\ \lambda B_X\Big) \cup L$$

is disconnected, which is impossible by Theorem 2.7. $\square$

Remark 2.11. One can prove Lemmas 2.9, 2.10 and 2.29 using the following trick. Suppose that

$$B_X \sim_{\mathbb{Q}} -\lambda K_X$$

for some $\lambda \in \mathbb{Q}$ such that $0 < \lambda < 1/2$, where X is either $\mathbb{P}^3$, or $\mathbb{P}^1 \times \mathbb{P}^2$, or a smooth quadric threefold, and the set $\mathbb{LCS}(X, B_X)$ contains no surfaces. Then

$$\mathrm{LCS}\Big(X,\ B_X\Big) \subseteq \Sigma,$$

where $\Sigma \subset X$ is a (possibly reducible) curve. For a general $\phi \in \mathrm{Aut}(X)$ we have

$$\phi(\Sigma) \cap \Sigma = \varnothing,$$

which implies that $\mathrm{LCS}(X, \phi(B_X)) \cap \mathrm{LCS}(X, B_X) = \varnothing$. But

$$\mathrm{LCS}\Big(X,\ \phi(B_X) + B_X\Big) = \mathrm{LCS}\Big(X,\ \phi(B_X)\Big) \bigcup \mathrm{LCS}\Big(X,\ B_X\Big)$$

whenever ϕ is sufficiently general. The latter contradicts Theorem 2.7 since $\lambda < 1/2$.

Lemma 2.12. Suppose that $\mathrm{LCS}(X, B_X) \neq \varnothing$, where X is a blow up of $\mathbb{P}^3$ in one point, and

$$B_X \sim_{\mathbb{Q}} -\lambda K_X$$

for some rational number $0 < \lambda < 1/2$. Then $\mathbb{LCS}(X, B_X)$ contains a surface.

Proof. Suppose that the set $\mathbb{LCS}(X, B_X)$ contains no surfaces. Let

$$\alpha\colon X \longrightarrow \mathbb{P}^3$$

be the blow up of a point, and let E be the exceptional divisor of α. In the case when

$$\mathrm{LCS}\Big(X,\ \lambda B_X\Big) \not\subseteq E,$$

we can apply Lemma 2.9 to the pair $(\mathbb{P}^3, \alpha(B_X))$ to get a contradiction. Hence $\mathrm{LCS}(X, B_X) \subseteq E$.

Let $H \subset \mathbb{P}^3$ a general hyperplane, and let $H_1 \subset \mathbb{P}^3 \supset H_2$ be general hyperplanes that pass through $\alpha(E)$. Denote by $\bar{H}$, $\bar{H}_1$ and $\bar{H}_2$ the proper transforms of these planes on X. Then

$$\mathrm{LCS}\left(X,\ B_X + \frac{1}{2}\Big(\bar{H}_1 + \bar{H}_2 + 2\bar{H}\Big)\right)$$

is disconnected, which is impossible by Theorem 2.7. □

Lemma 2.13. Suppose that X is a cone in $\mathbb{P}^4$ over a smooth quadric surface, and

$$B_X \sim_{\mathbb{Q}} -\lambda K_X$$

for some rational number $0 < \lambda < 1/3$. Then $\mathbb{LCS}(X, B_X) = \varnothing$.

Proof. Suppose that $\mathbb{LCS}(X, B_X) \neq \varnothing$. Let $S \subset X$ be a general hyperplane section. Then

$$\mathrm{LCS}\Big(S,\ B_X\Big|_S\Big) = \varnothing,$$

because $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathrm{lct}(\mathbb{P}^1 \times \mathbb{P}^1) = 1/2$ (see Example 1.18).

One has $|\mathrm{LCS}(X, B_X)| < +\infty$ by Remark 2.3. Then the locus

$$\mathrm{LCS}\Big(X,\ B_X + S\Big)$$

is disconnected, which contradicts Theorem 2.7. □

The following result is a corollary Theorem 2.6 (see [132, Theorem 4.1]).

Lemma 2.14. Suppose that $-(K_X + B_X)$ is nef and big and $\dim(\mathrm{LCS}(X, B_X)) = 1$. Then

- the locus $\mathrm{LCS}(X, B_X)$ is a connected union of smooth rational curves,
- every two irreducible components of the locus $\mathrm{LCS}(X, B_X)$ meet in at most one point,
- every intersecting irreducible components of the locus $\mathrm{LCS}(X, B_X)$ meet transversally,
- no three irreducible components of the locus $\mathrm{LCS}(X, B_X)$ meet in one point,
- the locus $\mathrm{LCS}(X, B_X)$ does not contain a cycle of smooth rational curves.

Proof. Arguing as in the proof of Theorem 2.7, we see that the locus $\mathrm{LCS}(X, B_X)$ is a connected tree of smooth rational curves with simple normal crossings. □

To consider another application of Theorem 2.7, we need the following result (see [150], [21]).

Lemma 2.15. Suppose that X is a smooth complete intersection $\cap_{i=1}^{k} G_i \subset \mathbb{P}^m$, and

$$B_X \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}^m}(1)\Big|_X,$$

where G_i is a hypersurface. Let $S \subsetneq X$ be an irreducible subvariety such that $\dim(S) \geqslant k$. Then

$$\mathrm{mult}_S\big(B_X\big) \leqslant 1.$$

Proof. We may assume $\dim(S) = k \leqslant (m-1)/2$. Let P be a sufficiently general point in $\mathbb{P}^m$, and let $C \subset \mathbb{P}^m$ be a cone over the subvariety S with vertex in the point P. Then

$$C \cap X = S \cup R,$$

where R is a curve on X. Let us calculate $|R \cap S|$. Let

$$\pi\colon X \longrightarrow \mathbb{P}^{m-1},\ \pi_1\colon G_1 \longrightarrow \mathbb{P}^{m-1},\ \ldots,\ \pi_k\colon G_k \longrightarrow \mathbb{P}^{m-1}$$

be projections from the point P, let $D \subset X$ and $D_i \subset G_i$ be the ramification subvarieties of the projections π and π_i, respectively. Put $C \cap G_i = S \cup R_i$. Then

$$R_i \cap S = D_i \cap S$$

by [155, Lemma 3]. Hence, it follows from $R=\cap_{i=1}^{k}R_i$ and $D=\cap_{i=1}^{k}D_i$ that $R\cap S=D\cap S$.

Let $(z_0,\ldots,z_m)$ be homogeneous coordinates on $\mathbb{P}^m$ such that
$$P=\big(p_0,\ldots,p_m\big),$$
and $G_i\subset\mathbb{P}^m$ is given by the equation $F_i(z_0,\ldots,z_m)=0$. Then $D\subset X$ is cut out by
$$\sum_{i=0}^{m}\frac{\partial F_1\big(z_0,\ldots,z_m\big)}{\partial z_i}p_i=\sum_{i=0}^{m}\frac{\partial F_2\big(z_0,\ldots,z_m\big)}{\partial z_i}p_i=\ldots=\sum_{i=0}^{m}\frac{\partial F_k\big(z_0,\ldots,z_m\big)}{\partial z_i}p_i=0.$$
Let $\mathcal{F}_r$ be a linear system on $\mathbb{P}^m$ that contains divisors
$$\sum_{i=0}^{m}\lambda_i\frac{\partial F_r\big(z_0,\ldots,z_m\big)}{\partial z_i}=0, r=1,\ldots,k.$$
The variety X is smooth. Hence
$$\big|R\cap S\big|=\big|D\cap S\big|=\deg\big(S\big)\prod_{i=1}^{k}\Big(\deg\big(F_i\big)-1\Big),$$
because $\mathcal{F}_1,\ldots,\mathcal{F}_k$ do not base points on X. Therefore, we have the inequality
$$\deg\big(S\big)\prod_{i=1}^{k}\Big(\deg\big(F_i\big)-1\Big)=B_X\cdot R\geqslant\sum_{O\in R\cap S}\mathrm{mult}_S\big(B_X\big)=\mathrm{mult}_S\big(B_X\big)\big|R\cap S\big|,$$
which implies $\mathrm{mult}_S(B_X)\leqslant 1$. □

Using Remark 2.3, Theorem 2.7 and Lemma 2.15, we obtain the following result.

Corollary 2.16. Let X is a smooth complete intersection $\cap_{i=1}^{k}G_i\subset\mathbb{P}^m$ such that
$$m+1-\sum_{i=1}^{k}\deg\big(G_i\big)\geqslant k+1,$$
where G_i is a hypersurface in $\mathbb{P}^m$. Then X is a Fano variety and
$$\mathrm{lct}\big(X\big)=\frac{1}{m+1-\sum_{i=1}^{k}\deg\big(G_i\big)}.$$

Let us consider another simple application of Theorem 2.7 and Lemma 2.15.

Lemma 2.17. Let X be a cubic hypersurface in $\mathbb{P}^4$ such that $|\mathrm{Sing}(X)|<+\infty$. Suppose that
$$B_X\sim_{\mathbb{Q}}-K_X,$$
and there is a positive rational number $\lambda<1/2$ such that $\mathrm{LCS}(X,\lambda B_X)\neq\varnothing$. Then
$$\mathrm{LCS}\Big(X,\ \lambda B_X\Big)=L,$$
where L is a line in $X\subset\mathbb{P}^4$ such that $L\cap\mathrm{Sing}(X)\neq\varnothing$.

Proof. Let S be a general hyperplane section of X. Then
$$S\cup\mathrm{LCS}\Big(X,\ \lambda B_X\Big)\subseteq\mathrm{LCS}\Big(X,\ \lambda B_X+S\Big),$$
which implies that $\dim(\mathrm{LCS}(X,\lambda B_X))\geqslant 1$ by Theorem 2.7. Then
$$\mathrm{LCS}\Big(S,\ \lambda B_X\Big|_S\Big)\neq\varnothing$$
by Remark 2.3. But $|\mathrm{LCS}(S,\ \lambda B_X|_S)|<+\infty$ by Lemma 2.15. There is a point $O\in S$ such that
$$\mathrm{LCS}\Big(S,\ \lambda B_X\Big|_S\Big)=O$$
by Theorem 2.7. Therefore, there is a line $L\subset X$ such that $\mathrm{LCS}(X,\lambda B_X)=L$ by Remark 2.3.

Arguing as in the proof of Lemma 2.15, we see that $L\cap\mathrm{Sing}(X)\neq\varnothing$. □

Similar to Lemma 2.17, one can prove the following result.

Lemma 2.18. Suppose that there is a double cover $\tau\colon X \to \mathbb{P}^3$ branched over an irreducible reduced quartic surface $R \subset \mathbb{P}^3$ that has at most ordinary double points, the equivalence

$$B_X \sim_{\mathbb{Q}} -\lambda K_X$$

holds and $\mathrm{LCS}(X, B_X) \neq \varnothing$, where $\lambda < 1/2$. Then $\mathrm{Sing}(X) \neq \varnothing$ and

$$\mathrm{LCS}\Big(X,\ B_X\Big) = L,$$

where L is an irreducible curve on X such that $-K_X \cdot L = 2$ and $L \cap \mathrm{Sing}(X) \neq \varnothing$.

Proof. We have $-K_X \sim 2H$, where H is a Cartier divisor on X such that

$$H \sim \tau^*\Big(\mathcal{O}_{\mathbb{P}^3}(1)\Big).$$

The variety X is a Fano threefold, and $H^3 = 2$. Then

$$\mathrm{LCS}\Big(X,\ B_X + H\Big)$$

must be connected by Theorem 2.7. Thus, there is a curve

$$C \in \mathbb{LCS}\Big(X,\ B_X\Big),$$

which implies that $\mathrm{mult}_C(B_X) \geqslant 1/\lambda > 2$.

Let S be a general surface in $|H|$. Put $B_S = B_X|_S$. Then

$$-K_S \sim H\Big|_S \sim_{\mathbb{Q}} \frac{1}{\lambda} B_S,$$

but the log pair (S, B_S) is not log canonical in every point of the intersection $S \cap \mathrm{LCS}(X, B_X)$.

The surface H is a smooth surface in $\mathbb{P}(1, 1, 1, 2)$ of degree 4.

Let P be any point in $S \cap \mathrm{LCS}(X, B_X)$. Then there is a birational morphism

$$\rho\colon S \longrightarrow \bar{S}$$

such that $\bar{S}$ is a cubic surface in $\mathbb{P}^3$ and ρ is an isomorphism in a neighborhood of P. Then

$$\Big(\bar{S},\ \rho(B_S)\Big)$$

is not log terminal at the point $\rho(P)$. Thus, we have $\mathrm{LCS}(\bar{S}, \rho(B_S)) \neq \varnothing$. But

$$\frac{1}{\lambda}\rho(B_S) \sim_{\mathbb{Q}} -K_{\bar{S}} \sim \mathcal{O}_{\mathbb{P}^3}(1)\Big|_{\bar{S}},$$

which implies that $\mathrm{LCS}(\bar{S}, \rho(B_S))$ consists of one point by Lemma 2.15 and Theorem 2.7. Then

$$P = S \cap C = S \cap \mathrm{LCS}\Big(X, B_X\Big)$$

if the point P is sufficiently general. Therefore, we see that

$$\mathrm{LCS}\Big(X, B_X\Big) = C,$$

the curve C is irreducible and $-K_X \cdot C = 2$. Then $\tau(C) \subset \mathbb{P}^3$ is a line.

Suppose that $C \cap \mathrm{Sing}(X) = \varnothing$. Let us derive a contradiction.

Suppose that $\tau(C) \subset R$. Take a general point $O \in C$. Let

$$\tau\big(O\big) \in \Pi \subset \mathbb{P}^3$$

be a plane that is tangent to R at the point $\tau(O)$. Arguing as in the proof of Lemma 2.15, we see that $R|_{\Pi}$ is reduced along $\tau(C)$, because $\tau(C) \cap \mathrm{Sing}(R) = \varnothing$. Fix a general line

$$\Gamma \subset \Pi \subset \mathbb{P}^3$$

such that $\tau(O) \in \Gamma$. Let $\bar{\Gamma} \subset X$ be an irreducible curve such that $\tau(\bar{\Gamma}) = \Gamma$. Then

$$\bar{\Gamma} \not\subseteq \mathrm{Supp}\Big(B_X\Big),$$

because Γ spans a dense subset in $\mathbb{P}^3$ when we vary the point $O \in C$ and the line $\Gamma \subset \Pi$. Note that $H \cdot \bar{\Gamma}$ equals either 1 or 2, and $\mathrm{mult}_O(\bar{\Gamma}) = 2$ in the case when $H \cdot \bar{Gamma} = 2$. Hence

$$H \cdot \bar{\Gamma} > 2\lambda H \cdot \bar{\Gamma} = \bar{\Gamma} \cdot B_X \geqslant \mathrm{mult}_O\big(\bar{\Gamma}\big)\mathrm{mult}_C\big(B_V\big) \geqslant H \cdot \bar{\Gamma},$$

which is a contradiction. Thus, we see that $\tau(C)\not\subset R$.

There is an irreducible reduced curve $\bar{C}\subset X$ such that

$$\tau(\bar{C})=\tau(C)\subset\mathbb{P}^3$$

and $\bar{C}\neq C$. Let Y be a general surface in $|H|$ that passes through the curves $\bar{C}$ and C. Then Y is smooth, because $C\cap\mathrm{Sing}(X)=\varnothing$, and

$$\bar{C}\cdot\bar{C}=C\cdot C=-2$$

on the surface Y.

By construction, we have $Y\not\subset\mathrm{Supp}(B_X)$. Put $B_Y=B_X|_Y$. Then

$$B_Y=\mathrm{mult}_{\bar{C}}\big(B_X\big)\bar{C}+\mathrm{mult}_{C}\big(B_X\big)C+\Delta$$

where Δ is an effective $\mathbb{Q}$-divisor on the surface Y such that $\bar{C}\not\subset\mathrm{Supp}(\Delta)\not\supset C$. But

$$B_Y\sim_{\mathbb{Q}}2\lambda\Big(\bar{C}+C\Big),$$

which implies, in particular, that

$$\Big(2\lambda-\mathrm{mult}_C\big(B_X\big)\Big)C\cdot C=\Big(\mathrm{mult}_{\bar{C}}\big(B_X\big)-2\lambda\Big)\bar{C}\cdot C+\Delta\cdot C\geqslant\Big(\mathrm{mult}_{\bar{C}}\big(B_X\big)-2\lambda\Big)\bar{C}\cdot C\geqslant 0,$$

because $\Delta\cdot C\geqslant 0$ and $\bar{C}\cdot C\geqslant 0$. Then $\mathrm{mult}_{\bar{C}}(B_X)\geqslant 2\lambda$, because $C\cdot C<0$. Thus, we have

$$-\Delta\sim_{\mathbb{Q}}\Big(\mathrm{mult}_{\bar{C}}\big(B_X\big)-2\lambda\Big)\bar{C}+\Big(\mathrm{mult}_{C}\big(B_X\big)-2\lambda\Big)C$$

which is impossible, because $\mathrm{mult}_C(B_X)>2\lambda$ and Y is projective. □

One can generalize Theorem 2.7 in the following way (see [169, Lemma 5.7]).

Theorem 2.19. Let $\psi\colon X\to Z$ be a morphism. Then the set

$$\mathrm{LCS}\Big(\bar{X},\ B^{\bar{X}}\Big)$$

is connected in a neighborhood of every fiber of the morphism $\psi\circ\pi\colon X\to Z$ in the case when

- the morphism ψ is surjective and has connected fibers,
- the divisor $-(K_X+B_X)$ is nef and big with respect to ψ.

Let us consider one important application of Theorem 2.19 (see [108, Theorem 5.50]).

Theorem 2.20. Suppose that B_1 is a Cartier divisor, $a_1=1$, and B_1 has at most log terminal singularities. Then the following assertions are equivalent:

- the log pair (X,B_X) is log canonical in a neighborhood of the divisor B_1;
- the singularities of the log pair $(B_1,\sum_{i=2}^r a_iB_i|_{B_1})$ are log canonical.

Proof. Suppose that the singularities of the log pair (X,B_X) are not log canonical in a neighborhood of the divisor $B_1\subset X$. Let us show that $(B_1,\sum_{i=2}^r a_iB_i|_{B_1})$ is not log canonical.

In the case when $a_m>1$ and $B_m\cap B_1\neq\varnothing$ for some $m\geqslant 2$, the log pair

$$\Big(B_1,\ \sum_{i=2}^{r}a_iB_i\Big|_{B_1}\Big)$$

is not log canonical by Definition 2.1. Thus, we may assume that $a_i\leqslant 1$ for every i. Then

$$\Big(X,\ B_1+\sum_{i=2}^{r}\lambda a_iB_i\Big)$$

is not log canonical as well for some rational number $\lambda<1$. Then

$$K_{\bar{X}}+\bar{B}_1+\sum_{i=2}^{r}\lambda a_i\bar{B}_i\sim_{\mathbb{Q}}\ \pi^*\Big(K_X+B_1+\sum_{i=2}^{r}\lambda a_iB_i\Big)+\sum_{i=1}^{n}d_iE_i$$

for some rational numbers $d_1,\ldots,d_n$. It follows from Theorem 2.19 that

$$\bar{B}_1\cap E_k\neq\varnothing$$

and the inequality $d_k \leqslant -1$ holds for some k. But

$$K_{\bar{B}_1} + \sum_{i=2}^{r} \lambda a_i \bar{B}_i\Big|_{B_1} \sim_{\mathbb{Q}} \phi^*\Big(K_{B_1} + \sum_{i=2}^{r} \lambda a_i B_i\Big|_{B_1}\Big) + \sum_{i=1}^{n} d_i E_i\Big|_{B_1},$$

where $\phi\colon \bar{B}_1 \to B_1$ is a birational morphism that is induced by π.

Thus, the log pair $(B_1, \sum_{i=2}^{r} \lambda a_i B_i|_{B_1})$ is not log terminal. Then the log pair

$$\Big(B_1,\ \sum_{i=2}^{r} a_i B_i\Big|_{B_1}\Big)$$

is not log canonical. The rest of the proof is similar (see the proof of [105, Theorem 7.5]). $\square$

The simplest application of Theorem 2.20 is a non-obvious result (see [108, Corollary 5.57]).

Lemma 2.21. Suppose that $\dim(X) = 2$ and $a_1 \leqslant 1$. Then

$$\Big(\sum_{i=2}^{r} a_i B_i\Big) \cdot B_1 > 1$$

whenever (X, B_X) is not log canonical at some point $O \in B_1$ such that $O \notin \mathrm{Sing}(X) \cup \mathrm{Sing}(B_1)$.

Proof. Suppose that (X, B_X) is not log canonical in a point $O \in B_1$. By Theorem 2.20, we have

$$\Big(\sum_{i=2}^{r} a_i B_i\Big) \cdot B_1 \geqslant \mathrm{mult}_O\Big(\sum_{i=2}^{r} a_i B_i\Big|_{B_1}\Big) > 1$$

if $O \notin \mathrm{Sing}(X) \cup \mathrm{Sing}(B_1)$, because $(X, B_1 + \sum_{i=2}^{r} a_i B_i)$ is not log canonical at the point O. $\square$

Let us consider another application of Theorem 2.20 (cf. Lemma 2.30).

Lemma 2.22. Suppose that X is a Fano variety with log terminal singularities. Then

$$\mathrm{lct}\Big(\mathbb{P}^1 \times X\Big) = \min\left(\frac{1}{2},\ \mathrm{lct}\big(X\big)\right).$$

Proof. The inequalities $1/2 \geqslant \mathrm{lct}(V \times U) \leqslant \mathrm{lct}(X)$ are obvious. Suppose that

$$\mathrm{lct}\Big(\mathbb{P}^1 \times X\Big) < \min\left(\frac{1}{2},\ \mathrm{lct}\big(X\big)\right),$$

and let us show that this assumption leads to a contradiction.

There is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_{\mathbb{P}^1 \times X}$ such that the log pair

$$\Big(\mathbb{P}^1 \times X,\ \lambda D\Big)$$

is not log canonical in some point $P \in \mathbb{P}^1 \times X$, where $\lambda < \min(1/2, \mathrm{lct}(X))$.

Let F be a fiber of the projection $\mathbb{P}^1 \times X \to \mathbb{P}^1$ such that $P \in F$. Then

$$D = \mu F + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor on $\mathbb{P}^1 \times X$ such that $F \not\subset \mathrm{Supp}(\Omega)$.

Let L be a general fiber of the projection $\mathbb{P}^1 \times X \to X$. Then

$$2 = D \cdot L = \mu + \Omega \cdot L \geqslant \mu,$$

which implies that the log pair $(\mathbb{P}^1 \times X, F + \lambda\Omega)$ is not log canonical at the point P. Then

$$\Big(F,\ \lambda\Omega\Big|_F\Big)$$

is not log canonical at the point P by Theorem 2.20. But

$$\Omega\Big|_F \sim_{\mathbb{Q}} D\Big|_F \sim_{\mathbb{Q}} -K_F,$$

which is impossible, because $X \cong F$ and $\lambda < \mathrm{lct}(X)$. $\square$

Let P be a point in X. Let us consider an effective divisor

$$\Delta = \sum_{i=1}^{r} \varepsilon_i B_i \sim_{\mathbb{Q}} B_X,$$

where ε_i is a non-negative rational number. Suppose that

- the divisor Δ is a $\mathbb{Q}$-Cartier divisor,
- the equivalence $\Delta \sim_{\mathbb{Q}} B_X$ holds,
- the log pair (X, Δ) is log canonical in the point $P \in X$.

Remark 2.23. Suppose that (X, B_X) is not log canonical in the point $P \in X$. Put

$$\alpha = \min \left\{ \frac{a_i}{\varepsilon_i} \;\Big|\; \varepsilon_i \neq 0 \right\},$$

where α is well defined, because there is $\varepsilon_i \neq 0$. Then $\alpha < 1$, the log pair

$$\left(X,\ \sum_{i=1}^{r} \frac{a_i - \alpha\varepsilon_i}{1-\alpha} B_i \right)$$

is not log canonical in the point $P \in X$, the equivalence

$$\sum_{i=1}^{r} \frac{a_i - \alpha\varepsilon_i}{1-\alpha} B_i \sim_{\mathbb{Q}} B_X \sim_{\mathbb{Q}} \Delta$$

holds, and at least one irreducible component of the divisor $\mathrm{Supp}(\Delta)$ is not contained in

$$\mathrm{Supp}\left(\sum_{i=1}^{r} \frac{a_i - \alpha\varepsilon_i}{1-\alpha} B_i \right).$$

The assertion of Remark 2.23 is obvious. Nevertheless it is very useful.

Lemma 2.24. Suppose that $X \cong C_1 \times C_2$, where C_1 and C_2 are smooth curves, suppose that

$$B_X \sim_{\mathbb{Q}} \lambda E + \mu F$$

where $E \cong C_1$ and $F \cong C_2$ are curves on the surface X such that

$$E \cdot E = F \cdot F = 0$$

and $E \cdot F = 1$, and λ and μ are non-negative rational numbers. Then

- the pair (X, B_X) is log terminal if $\lambda < 1$ and $\mu < 1$,
- the pair (X, B_X) is log canonical if $\lambda \leqslant 1$ and $\mu \leqslant 1$.

Proof. Suppose that $\lambda, \mu < 1$, but (X, B_X) is not log terminal at some point $P \in X$. Then

$$\mathrm{mult}_P\big(B_X\big) \geqslant 1,$$

and we may assume that $E \not\subset \mathrm{Supp}(B_X)$ or $F \not\subset \mathrm{Supp}(B_X)$ by Remark 2.23. But

$$E \cdot B_X = \mu,\ \ F \cdot B_X = \lambda,$$

which immediately leads to a contradiction, because $\mathrm{mult}_P(B_X) \geqslant 1$. □

Let $[B_X]$ be a class of $\mathbb{Q}$-rational equivalence of the divisor B_X. Put

$$\mathrm{lct}\Big(X, \big[B_X\big]\Big) = \inf \Big\{ \mathrm{lct}\big(X, D\big) \;\Big|\; D \text{ is an effective } \mathbb{Q}\text{-divisor on } X \text{ such that } D \sim_{\mathbb{Q}} B_X \Big\} \geqslant 0,$$

and put $\mathrm{lct}(X, [B_X]) = +\infty$ if $B_X = 0$. Note that B_X is an effective by assumption.

Remark 2.25. The equality $\mathrm{lct}(X, [-K_X]) = \mathrm{lct}(X)$ holds (see Definition 1.7).

Arguing as in the proof of Lemma 2.22, we obtain the following result.

Lemma 2.26. Suppose that there is a surjective morphism with connected fibers

$$\phi\colon X \longrightarrow Z$$

such that $\dim(Z) = 1$. Let F be a fiber of ϕ that has log terminal singularities. Then either

$$\mathrm{lct}_F\Big(X, B_X\Big) \geqslant \mathrm{lct}\Big(F, \Big[B_X\big|_F\Big]\Big),$$

or there is a positive rational number $\varepsilon < \mathrm{lct}(F, [B_X|_F])$ such that $F \subseteq \mathrm{LCS}(X, \varepsilon B_X)$.

Proof. Suppose that $\mathrm{lct}_F(X, B_X) < \mathrm{lct}(F, [B_X|_F])$. Then there is a rational number

$$\varepsilon < \mathrm{lct}\Big(F, \Big[B_X\big|_F\Big]\Big)$$

such that the log pair $(X, \varepsilon B_X)$ is not log canonical at some point $P \in F$. Put

$$B_X = \mu F + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor on X such that $F \not\subset \mathrm{Supp}(\Omega)$.

We may assume that $\varepsilon\mu \leqslant 1$. Then $(X, F + \varepsilon\Omega)$ is not canonical at the point P. Then

$$\Big(F,\ \varepsilon\Omega\Big|_F\Big)$$

is not log canonical at P by Theorem 2.20. But $\Omega|_F \sim_{\mathbb{Q}} B_X|_F$, which is a contradiction. $\square$

Let us show how to apply Lemma 2.26.

Lemma 2.27. Let $Q \subset \mathbb{P}^4$ be a cone over a smooth quadric surface, and let $\alpha\colon X \to Q$ be a blow up along a smooth conic $C \subset Q \setminus \mathrm{Sing}(Q)$. Then $\mathrm{lct}(X) = 1/3$.

Proof. Let H be a general hyperplane section of $Q \subset \mathbb{P}^4$ that contains C, and let $\bar{H}$ be a proper transform of the surface H on the threefold X. Then

$$-K_X \sim 3\bar{H} + 2E,$$

where E is the exceptional divisor of α. In particular, the inequality $\mathrm{lct}(X) \leqslant 1/3$ holds.

We suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/3$.

There is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ Q \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^1 \end{tikzcd}$$

where β is a morphism given by the linear system $|\bar{H}|$, and ψ is a projection from the two-dimensional linear subspace that contains the conic C.

Suppose that $\mathbb{LCS}(X, \lambda D)$ contains a surface $M \subset X$. Then

$$D = \mu M + \Omega,$$

where $\mu \geqslant 1/\lambda$, and Ω is an effective $\mathbb{Q}$-divisor such that $M \not\subset \mathrm{Supp}(\Omega)$.

Let F be a general fiber of β. Then $F \cong \mathbb{P}^1 \times \mathbb{P}^1$ and

$$D\Big|_F = \mu M\Big|_F + \Omega\Big|_F \sim_{\mathbb{Q}} -K_F,$$

which immediately implies that M is a fiber of the morphism β. But

$$\alpha\big(D\big) = \mu\alpha\big(M\big) + \alpha(\Omega) \sim_{\mathbb{Q}} -K_Q \sim 3\alpha\big(M\big),$$

which is impossible, because $\mu \geqslant 1/\lambda > 3$. Thus, the set $\mathbb{LCS}(X, \lambda D)$ contains no surfaces.

There is a fiber S of the morphism β such that

$$S \neq S \cap \mathrm{LCS}\Big(X,\ \lambda D\Big) \neq \varnothing,$$

which implies that S is singular by Lemma 2.26, because $\mathrm{lct}(\mathbb{P}^1 \times \mathbb{P}^1) = 1/2$.

Thus, the surface S is an irreducible quadric cone in $\mathbb{P}^3$. Then

$$\mathrm{LCS}\Big(X,\ \lambda D\Big)\subseteq S$$

by Theorem 2.7. We may assume that either $S\not\subseteq \mathrm{Supp}(D)$ or $E\not\subseteq \mathrm{Supp}(D)$ by Remark 2.23, because

$$\Big(X,\ S+\frac{2}{3}E\Big)$$

has log canonical singularities, and the equivalence $3S+2E\sim_{\mathbb{Q}} D$ holds.

Put $\Gamma=E\cap S$. The curve Γ is an irreducible conic in S. Then

$$\mathrm{LCS}\Big(X,\ \lambda D\Big)\subseteq \Gamma$$

by Lemma 2.13. Intersecting D with a general ruling of the cone $S\subset\mathbb{P}^3$, and intersecting D with a general fiber of the projection $E\to C$, we see that

$$\Gamma\not\subseteq \mathrm{LCS}\Big(X,\lambda D\Big),$$

which implies that $\mathrm{LCS}(X,\lambda D)$ consists of a single point $O\in\Gamma$ by Theorem 2.7.

Let R be a general (not passing through O) surface in $|\alpha^*(H)|$. Then

$$\mathrm{LCS}\left(X,\ \lambda D+\frac{1}{2}\Big(\bar{H}+2R\Big)\right)=R\cup O,$$

which is impossible by Theorem 2.7, since $-K_X\sim \bar{H}+2R\sim_{\mathbb{Q}} D$ and $\lambda<1/3$. □

The following generalization of Lemma 2.26 is proved in [79].

Theorem 2.28. Suppose that there is a surjective morphism with connected fibers

$$\phi\colon X\longrightarrow Z$$

such that ϕ is smooth in a neighborhood of a fiber F of the morphism ϕ. Then either

$$\mathrm{lct}_F\Big(X,B_X\Big)\geqslant \mathrm{lct}\Big(F,\ \Big[B_X\big|_F\Big]\Big),$$

or the equality $\mathrm{lct}_O(X,B_X)=\mathrm{lct}_Q(X,B_X)$ holds for every two points $O\in F\ni Q$.

Let us consider two elementary applications of Theorem 2.28.

Lemma 2.29. Suppose that $\mathrm{LCS}(X,B_X)\neq\varnothing$, where $X\cong\mathbb{P}^1\times\mathbb{P}^2$ and

$$B_X\sim_{\mathbb{Q}}-\lambda K_X$$

for some rational number $0<\lambda<1/2$. Then $\mathbb{LCS}(X,B_X)$ contains a surface.

Proof. Suppose that $\mathbb{LCS}(X,B_X)$ contains no surfaces. By Theorems 2.7 and 2.28, we have

$$\mathrm{LCS}\Big(X,\ B_X\Big)=F,$$

where F is a fiber of the natural projection $\pi_2\colon X\to\mathbb{P}^2$. Let S be a general surface in

$$\Big|\pi_1^*\Big(\mathcal{O}_{\mathbb{P}^2}(1)\Big)\Big|,$$

let M_1 and M_2 be general fibers of the natural projection $\pi_1\colon X\to\mathbb{P}^1$. Then the locus

$$\mathrm{LCS}\left(X,\ \lambda D+\frac{1}{2}\Big(M_1+M_2+3S\Big)\right)=F\cup S$$

is disconnected, which is impossible by Theorem 2.7. □

Lemma 2.30. Let V and U be smooth Fano varieties. Then

$$\mathrm{lct}\Big(V\times U\Big)=\min\Big(\mathrm{lct}\big(V\big),\ \mathrm{lct}\big(U\big)\Big).$$

Proof. The inequalities $\mathrm{lct}(U) \geqslant \mathrm{lct}(V \times U) \leqslant \mathrm{lct}(U)$ are obvious. Suppose that

$$\mathrm{lct}\Big(V \times U\Big) < \min\Big(\mathrm{lct}(V),\ \mathrm{lct}(U)\Big),$$

and let us show that this assumption leads to a contradiction.

There is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_{V\times U}$ such that the log pair

$$\Big(V \times U,\ \lambda D\Big)$$

is not log canonical in some point $P \in V \times U$, where $\lambda < \min(\mathrm{lct}(V), \mathrm{lct}(U))$.

Let us identify V with a fiber of the projection $V \times U \to U$ that contains the point P. Then

$$\mathrm{lct}_O\Big(V \times U,\ D\Big) = \mathrm{lct}_Q\Big(V \times U,\ D\Big)$$

for every points $O \in V \ni Q$ by Theorem 2.28, because the inequalities

$$\mathrm{lct}(V) > \lambda > \mathrm{lct}_V\Big(V \times U,\ D\Big) \geqslant \mathrm{lct}\Big(V, \Big[D\Big|_V\Big]\Big) = \mathrm{lct}\Big(V, \big[-K_V\big]\Big) = \mathrm{lct}(V)$$

are inconsistent. So, the log pair $(V \times U, \lambda D)$ is not log canonical in every point of $V \subset V \times U$.

Let us identify U with a general fiber of the projection $V \times U \to V$. Then

$$D\Big|_U \sim_{\mathbb{Q}} -K_U,$$

and $(U, \lambda D|_U)$ is not log canonical in $U \cap V$ by Remark 2.3 (applied $\dim(V)$ times here), which contradicts the inequality $\lambda < \mathrm{lct}(U)$. □

We believe that the assertion of Lemma 2.30 holds for log terminal varieties (cf. Lemma 2.22).

3. The Mukai–Umemura threefold

The main purpose of this section is to compute the global log canonical threshold of one remarkable smooth Fano threefold (cf. [55]) to illustrate the proof of Theorem 1.78.

Lemma 3.1. Let X be the smooth Fano threefold such that[4]

$$\mathrm{Pic}(X) = \mathbb{Z}\big[-K_X\big],$$

the equality $-K_X^3 = 22$ holds, and $\mathrm{Aut}(X) \cong \mathrm{PSL}(2, \mathbb{C})$. Then $\mathrm{lct}(X) = 1/2$.

Proof. Let $U \subset \mathbb{C}[x, y]$ be a subspace of forms of degree 12. Consider $U \cong \mathbb{C}^{13}$ as the affine part of

$$\mathbb{P}\Big(U \oplus \mathbb{C}\Big) \cong \mathbb{P}^{13},$$

and let us identify $\mathbb{P}(U)$ with the hyperplane at infinity.

The natural action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathbb{C}[x, y]$ induces an action on $\mathbb{P}(U \oplus \mathbb{C})$. Put

$$\phi = xy\Big(x^{10} - 11x^5y^5 - y^{10}\Big) \in U$$

and consider the closure $\overline{\mathrm{SL}(2, \mathbb{C}) \cdot [\phi + 1]} \subset \mathbb{P}(U \oplus \mathbb{C})$. It follows from [130] that

$$X \cong \overline{\mathrm{SL}\big(2, \mathbb{C}\big) \cdot \big[\phi + 1\big]},$$

and the embedding $X \subset \mathbb{P}(U \oplus \mathbb{C}) \cong \mathbb{P}^{13}$ is induced by $|-K_X|$.

The action of $\mathrm{SL}(2, \mathbb{C})$ on X has the following orbits (see [98, Theorem 5.2.13]):

- the three-dimensional orbit $\Sigma_3 = \mathrm{SL}(2, \mathbb{C}) \cdot [\phi + 1]$;
- the two-dimensional orbit $\Sigma_2 = \mathrm{SL}(2, \mathbb{C}) \cdot [xy^{11}]$;
- the one-dimensional orbit $\Sigma_1 = \mathrm{SL}(2, \mathbb{C}) \cdot [y^{12}]$.

The orbit Σ_3 is open, the orbit $\Sigma_1 \cong \mathbb{P}^1$ is closed, and

$$\overline{\Sigma}_2 = \Sigma_1 \cup \Sigma_2,$$

so that the orbit Σ_2 is neither open nor closed. One has

$$X \cap \mathbb{P}(U) = \Sigma_1 \cup \Sigma_2$$

and $X = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. Put $R = X \cap \mathbb{P}(U)$. It follows from [130] that

[4] The threefold X satisfying these assumptions is unique (see [130] and [142]).

- the surface R is swept out by lines on $X \subset \mathbb{P}^{13}$,
- the surface R contains all lines on $X \subset \mathbb{P}^{13}$,
- for any lines $L_1 \subset R \supset L_2$ such that $L_1 \neq L_2$, one has $L_1 \cap L_2 = \varnothing$,
- the surface R is singular along the orbit $\Sigma_1 \cong \mathbb{P}^1$,
- the normalization of the surface R is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$,
- for every point $P \in \Sigma_1$, the surface R is locally isomorphic to
$$x^2 = y^3 \subset \mathbb{C}^3 \cong \mathrm{Spec}\Big(\mathbb{C}\big[x, y, z\big]\Big),$$
which implies that $\mathrm{lct}(X, R) = 5/6$.

The structure of the surface R can be seen as follows. We see that
$$\Sigma_2 = \Big\{\Big[\big(ax + by\big)\big(cx + dy\big)^{11}\Big] \ \Big|\ ad - bc = 1\Big\} \subset \mathbb{P}\big(U\big),$$
which implies that there is a birational morphism $\nu\colon \mathbb{P}^1 \times \mathbb{P}^1 \to R$ that is defined by
$$\nu\colon \big[a : b\big] \times \big[c : d\big] \mapsto \Big[\big(ax + by\big)\big(cx + dy\big)^{11}\Big] \in R,$$
so that ν is a normalization of the surface R.

Let V_5 be a smooth Fano threefold such that
$$-K_{V_5} \sim 2H$$
and $H^3 = 5$, where H is a Cartier divisor on V_5. Then $|H|$ induces an embedding $X \subset \mathbb{P}^6$.

Let $L \cong \mathbb{P}^1$ be a line on X. Then $\mathcal{N}_{L/X} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Let
$$\alpha_L\colon U_L \longrightarrow X$$
be a blow up of the line L, and let E_L be the exceptional divisor of α_L. Then
$$E_L \cong \mathbb{F}_3,$$
and it follows from Theorem 4.3.3 in [98] (see [45], [178]) that there is a commutative diagram
$$\begin{tikzcd} U_L \arrow[r, dashed, "\rho_L"] \arrow[d, "\alpha_L"'] & W_L \arrow[d, "\beta_L"] \\ X \arrow[r, dashed, "\psi_L"'] & V_5 \end{tikzcd}$$
where ρ_L is a flop in the exceptional section of $E \cong \mathbb{F}_3$, the morphism β_L contracts a surface $D_L \subset W_L$ to a smooth rational curve of degree 5, and ψ_L is a double projection from the line L.

Let $\bar{D}_L \subset X$ be the proper transform of the surface D_L. Then
$$\mathrm{mult}_L\big(\bar{D}_L\big) = 3$$
and $\bar{D}_L \sim -K_X$. It follows from [63] that $X \setminus \bar{D}_L \cong \mathbb{C}^3$ (cf. [138], [139], [65]).

It follows from [64] that there is an open subset $\breve{D}_L \subset \bar{D}_L$ that is given by
$$\mu_0 x^4 + \Big(\mu_1 yz + \mu_2 z^3\Big)x^3 + \Big(\mu_3 y^3 + \mu_4 y^2 z^2 + \mu_5 yz^4\Big)x^2 + \Big(\mu_6 y^4 z + \mu_7 y^3 z^3\Big)x + \mu_8 y^6 + \mu_9 y^5 z^2 = 0$$
in $\mathbb{C}^3 \cong \mathrm{Spec}(\mathbb{C}[x, y, z])$, where $L \cap \Sigma_1 \in \breve{D}_L$ is given by the equations $x = y = z = 0$, and
$$\mu_0 = -2^8 5^2,\ \mu_1 = 2^9 3^3 5,\ \mu_2 = -2^6 3^4 5,\ \mu_3 = -2^8 3^3 7,\ \mu_4 = -2^4 3^4 127,$$
$$\mu_5 = 2^9 3^5,\ \mu_6 = 2^2 3^6 89,\ \mu_7 = -2^8 3^6,\ \mu_8 = -3^6 5^3,\ \mu_9 = 2^5 3^7.$$

Put $O_L = \Sigma_1 \cap L$. Then $\mathrm{mult}_{O_L}(\bar{D}_L) = 4$, and it follows from Proposition 8.14 in [105] that
$$\mathrm{LCS}\left(X,\ \frac{1}{2}\bar{D}_L\right) = O_L$$
and $\mathrm{lct}(X, \bar{D}_L) = 1/2$. Thus, we see that $\mathrm{lct}(X) \leqslant 1/2$.

Suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor
$$D \sim_{\mathbb{Q}} -K_X$$
such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$.

By Remark 2.23, we may assume that $R \not\subset \mathrm{Supp}(D)$, because $\mathrm{lct}(X, R) = 5/6$.

Let C be a line in X such that $C \not\subset \mathrm{Supp}(D)$. Then

$$1 = D \cdot C \geqslant \mathrm{mult}_{O_C}\big(D\big)\mathrm{mult}_{O_C}\big(C\big) = \mathrm{mult}_{O_C}\big(D\big),$$

which implies that $O_C \notin \mathrm{LCS}(X, \lambda D)$. In particular, we see that $\Sigma_1 \notin \mathrm{LCS}(X, \lambda D)$.

Let Γ be an irreducible curve in $\mathrm{Supp}(D)$ such that $O_C \in \Gamma$. Then

$$\mathrm{mult}_{\Gamma}\left(\frac{1}{2}\bar{D}_C + \lambda D\right) = \frac{\mathrm{mult}_{\Gamma}\big(\bar{D}_C\big)}{2} + \lambda\mathrm{mult}_{\Gamma}\big(D\big) \leqslant \frac{\mathrm{mult}_{\Gamma}\big(\bar{D}_C\big)}{2} + \lambda\mathrm{mult}_{O_C}\big(D\big) < 1,$$

because $\lambda < 1/2$ and $\mathrm{Sing}(\bar{D}_C) = C$, because $\bar{D}_C \neq R$. Thus, we see that

$$\Gamma \not\subseteq \mathrm{LCS}\left(X,\ \frac{1}{2}\bar{D}_C + \lambda D\right) \supseteq \mathrm{LCS}\Big(X, \lambda D\Big) \cup O_C,$$

which is impossible by Theorem 2.7, because $O_C \notin \mathrm{LCS}(X, \lambda D)$ and $\lambda < 1/2$. □

The threefold satisfying all hypotheses of Lemma 3.1 is called the Mukai–Mumemura threefold.

Remark 3.2. Let X be the Mukai–Mumemura threefold. Then it follows from [55] that

$$\mathrm{lct}\Big(X,\ \mathrm{SO}\big(3\big)\Big) = \frac{5}{6}.$$

Remark 3.3. Let X be a smooth Fano threefold such that $\mathrm{Pic}(X) = \mathbb{Z}[-K_X]$. Then it follows from the papers [143], [91], [74] that the following conditions are equivalent:

- $-K_X^3 = 22$ and the threefold X is the Mukai–Mumemura threefold;
- $-K_X^3 \geqslant 16$ and for any curve $\mathbb{P}^1 \cong L \subset X$ such that $-K_X \cdot L = 1$, we have
$$\mathcal{N}_{L/X} \cong \mathcal{O}_{\mathbb{P}^1}\big(1\big) \oplus \mathcal{O}_{\mathbb{P}^1}\big(-2\big),$$
- $-K_X^3 \geqslant 6$ and for any two curves $L_1 \subset X \supset L_2$ such that
$$-K_X \cdot L_1 = -K_X \cdot L_2 = 1,$$
either $L_1 = L_2$, or $L_1 \cap L_2 = \varnothing$.

Remark 3.4. Let $Z \subset \mathbb{P}^2$ be a plane quartic curve. Then Z is given by an equation

$$\zeta\big(x, y, z\big) = 0 \subset \mathbb{P}^2 \cong \mathrm{Proj}\Big(\mathbb{C}[x, y, z]\Big),$$

where $\zeta(x, y, z)$ is a form of degree 4. A polar hexagon of the curve $Z \subset \mathbb{P}^2$ is the union

$$\Gamma = \bigcup_{i=1}^{6}\Big(\xi_i\big(x, y, z\big) = 0\Big) \subset \mathbb{P}^2$$

such that $\zeta(x, y, z) = \sum_{i=1}^{6} \xi_i^4(x, y, z)$, where $\xi_i(x, y, z)$ is a non-zero linear form. Put

$$X_\zeta = \overline{\Big\{\Gamma \in \mathrm{Hilb}_6\big(\mathbb{P}^2\big)\ \Big|\ \Gamma \text{ is polar to } Z \subset \mathbb{P}^2\Big\}},$$

where we identify the polar hexagon Γ with a point in $\mathrm{Hilb}_6(\mathbb{P}^2)$. Then it follows from [167] that the variety X_ζ is a smooth Fano threefold such that

$$\mathrm{Pic}\big(X_\zeta\big) = \mathbb{Z}\big[-K_{X_\zeta}\big]$$

and the equality $-K_{X_\zeta}^3 = 22$ holds in the case when the homogeneous form $\zeta(x, y, z)$ is sufficiently general[5]. It follows from [130] that X_ζ is the Mukai–Mumemura threefold if

$$\zeta\big(x, y, z\big) = \Big(x^2 + y^2 + z^2\Big)^2.$$

[5]Varieties X_ζ that are smooth compactifications of $\mathbb{C}^3$ were studied in [138], [139], [63], [144], [64], [65].

4. Cubic surfaces

Let X be a cubic surface in $\mathbb{P}^3$ that has at most one ordinary double point.

Definition 4.1. A point $O \in X$ is said to be an Eckardt point if $O \notin \mathrm{Sing}(X)$ and

$$O = L_1 \cap L_2 \cap L_3,$$

where L_1, L_2, L_3 are different lines on the surface $X \subset \mathbb{P}^3$.

General cubic surfaces have no Eckardt points. It follows from Example 1.18 and 1.19 that

$$\mathrm{lct}\big(X\big) = \begin{cases} 3/4 \text{ when } X \text{ has no Eckardt points and } \mathrm{Sing}\big(X\big) = \varnothing \\ 2/3 \text{ when } X \text{ has an Eckardt point or } \mathrm{Sing}\big(X\big) \neq \varnothing, \end{cases}$$

Let D be an effective $\mathbb{Q}$-divisor on X such that $D \sim_{\mathbb{Q}} -K_X$, and let ω be a positive rational number such that $\omega < 3/4$. In this section we prove the following result (cf. [20], [33], [31]).

Theorem 4.2. Suppose that $(X, \omega D)$ is not log canonical. Then

$$\mathrm{LCS}\Big(X,\ \omega D\Big) = O,$$

where $O \in X$ is either a singular point or an Eckardt point.

Suppose that $(X, \omega D)$ is not log canonical. Let P be a point in $\mathrm{LCS}(X, \omega D)$. Suppose that

- neither $P = \mathrm{Sing}(X)$,
- nor P is an Eckardt point.

Lemma 4.3. One has $\mathrm{LCS}(X, \omega D) = P$.

Proof. Suppose that $\mathrm{LCS}(X, \omega D) \neq P$. Then there is a curve $C \subset X$ such that

$$P \in C \subseteq \mathrm{LCS}\Big(X, \omega D\Big)$$

by Theorem 2.7. Then there is an effective $\mathbb{Q}$-divisor Ω on X such that $C \not\subset \mathrm{Supp}(\Omega)$ and

$$D = \mu C + \Omega,$$

where $\mu \geqslant 1/\omega$. Let H be a general hyperplane section of X. Then

$$3 = H \cdot D = \mu H \cdot C + H \cdot \Omega \geqslant \mu \mathrm{deg}\big(C\big),$$

which implies that either $\deg(C) = 1$, or $\deg(C) = 2$.

Suppose that $\deg(C) = 1$. Let Z be a general conic on X such that $-K_X \sim C + Z$. Then

$$2 = Z \cdot D = \mu Z \cdot C + Z \cdot \Omega \geqslant \mu Z \cdot C = \begin{cases} 2\mu \text{ if } C \cap \mathrm{Sing}\big(X\big) = \varnothing \\ 3\mu/2 \text{ if } C \cap \mathrm{Sing}\big(X\big) \neq \varnothing, \end{cases}$$

which implies that $\mu \leqslant 4/3$. But $\mu \geqslant 1/\omega > 4/3$, which gives a contradiction.

We see that $\deg(C) = 2$. Let L be a line on X such that $-K_X \sim C + L$. Then

$$D = \mu C + \lambda L + \Upsilon,$$

where $\lambda \in \mathbb{Q}$ such that $\lambda \geqslant 0$, and Υ is an effective $\mathbb{Q}$-divisor such that $C \not\subset \mathrm{Supp}(\Upsilon) \not\supset L$. Then

$$1 = L \cdot D = \mu L \cdot C + \lambda L \cdot L + L \cdot \Upsilon \geqslant \mu L \cdot C + \lambda L \cdot L = \begin{cases} 2\mu - \lambda \text{ if } C \cap \mathrm{Sing}\big(X\big) = \varnothing \\ 3\mu/2 - \lambda/2 \text{ if } C \cap \mathrm{Sing}\big(X\big) \neq \varnothing, \end{cases}$$

which implies that $\mu \leqslant 7/6 < 4/3$, because $\lambda \leqslant 4/3$ (see the case when $\deg(C) = 1$). But $\mu > 4/3$, which gives a contradiction. ◻

Let $\pi \colon U \to X$ be a blow up of P, and let E be the π-exceptional curve. Then

$$\bar{D} \sim_{\mathbb{Q}} \pi^*\big(D\big) + \mathrm{mult}_P\big(D\big)E,$$

where $\mathrm{mult}_P(D) \geqslant 1/\omega$ and $\bar{D}$ is a proper transform of D on the surface U. The log pair

$$\Big(U,\ \omega\bar{D} + \big(\omega\mathrm{mult}_P\big(D\big) - 1\big)E\Big)$$

is not log canonical at some point $Q \in E$. Then either $\mathrm{mult}_P(D) \geqslant 2/\omega$, or

$$\mathrm{mult}_Q\big(\bar{D}\big) + \mathrm{mult}_P\big(D\big) \geqslant 2/\omega > 8/3, \tag{4.4}$$

because the divisor $\omega\bar{D} + (\omega\mathrm{mult}_P(D) - 1)E$ is effective.

Let T be the unique hyperplane section of X that is singular at P. We may assume that

$$\mathrm{Supp}\big(T\big) \not\subseteq \mathrm{Supp}\big(D\big)$$

by Remark 2.23, because $(X, \omega T)$ is log canonical. The following cases are possible:

- the curve T is irreducible;
- the curve T is a union of a line and an irreducible conic;
- the curve T consists of 3 lines.

Hence T is reduced. Note that $\mathrm{mult}_P(T) = 2$ since P is not an Eckardt point. We exclude these cases one by one.

Lemma 4.5. The curve T is reducible.

Proof. Suppose that T is irreducible. Then there is a commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "\pi"'] \arrow[dr, "\psi"] & \\ X \arrow[rr, dashed, "\rho"'] & & \mathbb{P}^2 \end{tikzcd}$$

where ψ is a double cover branched over a quartic curve, and ρ is the projection from $P \in X$.

Let $\bar{T}$ be the proper transform of T on the surface U. Suppose that $Q \in \bar{T}$. Then

$$3-2\mathrm{mult}_P\big(D\big) = \bar{T}\cdot\bar{D} \geqslant \mathrm{mult}_Q\big(\bar{T}\big)\mathrm{mult}_Q\big(\bar{D}\big) > \mathrm{mult}_Q\big(\bar{T}\big)\Big(8/3-\mathrm{mult}_P\big(D\big)\Big) \geqslant 8/3-\mathrm{mult}_P\big(D\big),$$

which implies that $\mathrm{mult}_P(D) \leqslant 1/3$. But $\mathrm{mult}_P(D) > 4/3$. Thus, we see that $Q \notin \bar{T}$.

Let $\tau \in \mathrm{Aut}(U)$ be an involution[6] induced by ψ. It follows from [118] that

$$\tau^*\Big(\pi^*\big(-K_X\big)\Big) \sim \pi^*\big(-2K_X\big) - 3E,$$

and $\tau(\bar{T}) = E$. Put $\breve{Q} = \pi\circ\tau(Q)$. Then $\breve{Q} \neq P$, because $Q \notin \bar{T}$.

Let H be the hyperplane section of X that is singular at $\breve{Q}$. Then $T \neq H$, because $P \neq \breve{Q}$ and T is smooth outside of the point P. Hence $P \notin H$, because otherwise

$$3 = H\cdot T \geqslant \mathrm{mult}_P\big(H\big)\mathrm{mult}_P\big(T\big) + \mathrm{mult}_{\breve{Q}}\big(H\big)\mathrm{mult}_{\breve{Q}}\big(T\big) \geqslant 4.$$

Let $\bar{H}$ be the proper transform of H on the surface U. Put $\bar{R} = \tau(\bar{H})$ and $R = \pi(\bar{R})$. Then

$$\bar{R} \sim \pi^*\big(-2K_X\big) - 3E,$$

ant the curve $\bar{R}$ must be singular at the point Q.

Suppose that R is irreducible. Taking into account the possible singularities of $\bar{R}$, we see that

$$\Big(X,\ \frac{3}{8}R\Big)$$

is log canonical. Thus, we may assume that $R \not\subseteq \mathrm{Supp}(D)$ by Remark 2.23. Then

$$6 - 3\mathrm{mult}_P\big(D\big) = \bar{R}\cdot\bar{D} \geqslant \mathrm{mult}_Q(\bar{R})\mathrm{mult}_Q\big(\bar{D}\big) > 2\Big(8/3 - \mathrm{mult}_P\big(D\big)\Big),$$

which implies that $\mathrm{mult}_P(D) < 2/3$. But $\mathrm{mult}_P(D) > 4/3$. The curve R must be reducible.

The curves R and H are reducible. So, there is a line $L \subset X$ such that $P \notin L \ni \breve{Q}$.

Let $\bar{L}$ be the proper transform of L on the surface U. Put $\bar{Z} = \tau(\bar{L})$. Then $\bar{L}\cdot E = 0$ and

$$\bar{L}\cdot\bar{T} = \bar{L}\cdot\pi^*(-K_X) = 1,$$

which implies that $\bar{Z}\cdot E = 1$ and $\bar{Z}\cdot\pi^*\big(-K_X\big) = 2$. We have $Q \in \bar{Z}$. Then

$$2 - \mathrm{mult}_P\big(D\big) = \bar{Z}\cdot\bar{D} \geqslant \mathrm{mult}_Q\big(\bar{D}\big) > 8/3 - \mathrm{mult}_P\big(D\big) > 2 - \mathrm{mult}_P\big(D\big)$$

in the case when $\bar{Z} \not\subseteq \mathrm{Supp}(\bar{D})$. Hence, we see that $\bar{Z} \subseteq \mathrm{Supp}(\bar{D})$.

[6]The involution τ induces an involution in $\mathrm{Bir}(X)$ that is called a Geiser involution.

Put $Z=\pi(\bar{Z})$. Then Z is a conic such that $P\in Z$ and

$$-K_X\sim L+Z,$$

which means that $L\cup Z$ is cut out by the plane in $\mathbb{P}^3$ that passes through Z. Put

$$D=\varepsilon Z+\Upsilon,$$

where $\varepsilon\in\mathbb{Q}$ such that $\varepsilon\geqslant 0$, and Υ is an effective $\mathbb{Q}$-divisor such that $Z\not\subset\mathrm{Supp}(\Upsilon)$.

We may assume that $L\not\subseteq\mathrm{Supp}(\Upsilon)$ by Remark 2.23. Then

$$1=L\cdot D=\varepsilon Z\cdot L+L\cdot\Upsilon\geqslant\varepsilon Z\cdot L=\begin{cases}2\varepsilon \text{ if } Z\cap\mathrm{Sing}\big(X\big)=\varnothing,\\ 3\varepsilon/2 \text{ if } Z\cap\mathrm{Sing}\big(X\big)\neq\varnothing,\end{cases}$$

which implies that $\varepsilon\leqslant 2/3$.

Let $\bar{\Upsilon}$ be the proper transform of Υ on the surface U. Then the log pair

$$\Big(U,\ \varepsilon\omega\bar{Z}+\omega\bar{\Upsilon}+\big(\omega\mathrm{mult}_P\big(D\big)-1\big)E\Big)$$

is not log canonical at $Q\in\bar{Z}$. Then

$$\omega\bar{\Upsilon}\cdot\bar{Z}+\Big(\omega\mathrm{mult}_P\big(D\big)-1\Big)=\Big(\omega\bar{\Upsilon}+\big(\omega\mathrm{mult}_P\big(D\big)-1\big)E\Big)\cdot\bar{Z}>1$$

by Lemma 2.21, because $\varepsilon\leqslant 2/3$. In particular, we see that

$$8/3-\mathrm{mult}_P\big(D\big)<\bar{Z}\cdot\bar{\Upsilon}=2-\mathrm{mult}_P\big(D\big)-\varepsilon\bar{Z}\cdot\bar{Z}=\begin{cases}2-\mathrm{mult}_P\big(D\big)+\varepsilon \text{ if } Z\cap\mathrm{Sing}\big(X\big)=\varnothing,\\ 2-\mathrm{mult}_P\big(D\big)+\varepsilon/2 \text{ if } Z\cap\mathrm{Sing}\big(X\big)\neq\varnothing,\end{cases}$$

which implies that $\varepsilon>2/3$. But $\varepsilon\leqslant 2/3$. $\square$

Therefore, there is a line $L_1\subset X$ such that $P\in L_1$. Put

$$D=m_1L_1+\Omega,$$

where $m_1\in\mathbb{Q}$ such that Ω is an effective $\mathbb{Q}$-divisor such that $L_1\not\subset\!\!\in\mathrm{Supp}(\Omega)$. Then

$$4/3<1/\omega<\Omega\cdot L_1=1-m_1L_1\cdot L_1=\begin{cases}1+m_1 \text{ if } L_1\cap\mathrm{Sing}\big(X\big)=\varnothing,\\ 1+m_1/2 \text{ if } L_1\cap\mathrm{Sing}\big(X\big)\neq\varnothing.\end{cases}$$

Corollary 4.6. The following inequality holds:

$$m_1>\begin{cases}1/3 \text{ if } L_1\cap\mathrm{Sing}\big(X\big)=\varnothing,\\ 2/3 \text{ if } L_1\cap\mathrm{Sing}\big(X\big)\neq\varnothing.\end{cases}$$

Remark 4.7. Suppose that X is singular. Put $O=\mathrm{Sing}(X)$. It follows from [16] that

$$O=\Gamma_1\cap\Gamma_2\cap\Gamma_3\cap\Gamma_4\cap\Gamma_5\cap\Gamma_6,$$

where $\Gamma_1,\ldots,\Gamma_6$ are different lines on the surface $X\subset\mathbb{P}^3$. The equivalence

$$-2K_X\sim\sum_{i=1}^{6}\Gamma_i$$

holds. Suppose that $L_1=\Gamma_1$. Let $\Pi_2,\ldots,\Pi_6\subset\mathbb{P}^3$ be planes such that

$$L_1\subset\Pi_i\supset\Gamma_i,$$

and let $\Lambda_2,\ldots,\Lambda_6$ be lines on the surface X such that

$$L_1\cup\Gamma_i\cup\Lambda_i=\Pi_i\cap X\subset X\subset\mathbb{P}^3,$$

which implies that $-K_X\sim L_1+\Gamma_i+\Lambda_i$. Then

$$-5K_X\sim 4L_1+\sum_{i=2}^{6}\Lambda_i+\Big(L_1+\sum_{i=2}^{6}\Gamma_i\Big)\sim 4L_1+\sum_{i=2}^{6}\Lambda_i-2K_X,$$

which implies that $-3K_X \sim 4L_1 + \sum_{i=2}^{6}\Lambda_i$. But the log pair

$$\left(X,\ L_1 + \frac{\sum_{i=2}^{6}\Lambda_i}{3}\right)$$

is log canonical at the point P. Thus, we may assume that

$$\mathrm{Supp}\left(\sum_{i=2}^{6}\Lambda_i\right) \not\subseteq \mathrm{Supp}\big(D\big)$$

thanks to Remark 2.23, because $L_1 \subseteq \mathrm{Supp}(D)$. Then there is Λ_k such that

$$1 = D\cdot\Lambda_k = \Big(m_1L_1 + \Omega\Big)\cdot\Lambda_k = m_1 + \Omega\cdot\Lambda_k \geqslant m_1,$$

because $O \notin \Lambda_k$. Thus, we may assume that $m_1 \leqslant 1$ if $L_1 \cap \mathrm{Sing}(X) \neq \varnothing$.

Arguing as in the proof of Lemma 2.15, we see that $m_1 \leqslant 1$ if $L_1 \cap \mathrm{Sing}(X) = \varnothing$.

Lemma 4.8. There is a line $L_2 \subset X$ such that $L_1 \neq L_2$ and $P \in L_2$.

Proof. Suppose that there is no line $L_2 \subset X$ such that $L_1 \neq L_2$ and $P \in L_2$. Then

$$T = L_1 + C,$$

where C is an irreducible conic on the surface $X \subset \mathbb{P}^3$ such that $P \in C$.

It follows from Remark 2.23 that we may assume that $C \not\subseteq \mathrm{Supp}(\Omega)$, because $m_1 \neq 0$.

Let $\bar{L}_1$ and $\bar{C}$ be the proper transforms of L_1 and C on the surface U, respectively. Then

$$\bar{D} \sim_{\mathbb{Q}} m_1\bar{L}_1 + \bar{\Omega} \sim_{\mathbb{Q}} \pi^*\Big(m_1L_1 + \Omega\Big) - \Big(m_1 + \mathrm{mult}_P\big(\Omega\big)\Big)E \sim_{\mathbb{Q}} \pi^*\big(D\big) - \mathrm{mult}_P\big(D\big)E,$$

where $\bar{\Omega}$ is the proper transform of the divisor Ω on the surface U. We have

$$0 \leqslant \bar{C}\cdot\bar{\Omega} = 2 - \mathrm{mult}_P\big(D\big) + m_1\bar{C}\cdot\bar{L} < 2/3 - m_1\bar{C}\cdot\bar{L}_1 = \begin{cases} 2/3 - m_1, \text{ if } L_1\cap\mathrm{Sing}\big(X\big) = \varnothing,\\ 2/3 - m_1/2, \text{ if } L_1\cap\mathrm{Sing}\big(X\big) \neq \varnothing,\end{cases}$$

which implies that $m_1 < 2/3$ if $L_1 \cap \mathrm{Sing}(X) = \varnothing$. It follows from inequality 4.4 that

$$\mathrm{mult}_Q\big(\bar{\Omega}\big) > 8/3 - \mathrm{mult}_P\big(\Omega\big) - m_1\Big(1 + \mathrm{mult}_Q\big(\bar{L}_1\big)\Big).$$

Suppose that $Q \in \bar{L}_1$. Then it follows from Lemma 2.21 that

$$8/3 < \bar{L}_1\cdot\Big(\bar{\Omega} + \big(\mathrm{mult}_P\big(\Omega\big) + m_1\big)E\Big) = 1 - m_1\bar{L}_1\cdot\bar{L}_1 = \begin{cases} 1 + 2m_1, \text{ if } L_1\cap\mathrm{Sing}\big(X\big) = \varnothing,\\ 1 + 3m_1/2, \text{ if } L_1\cap\mathrm{Sing}\big(X\big) \neq \varnothing,\end{cases}$$

which is impossible, because $m_1 \leqslant 1$ if $L_1 \cap \mathrm{Sing}(X) \neq \varnothing$ by Remark 4.7.

We see that $Q \notin \bar{L}_1$. Suppose that $Q \in \bar{C}$. Then

$$2 - \mathrm{mult}_P\big(\Omega\big) - m_1 - m_1\bar{C}\cdot\bar{L}1 = \bar{C}\cdot\bar{\Omega} > 8/3 - \mathrm{mult}_P\big(\Omega\big) - m_1,$$

which is impossible, because $m_1\bar{C}\cdot\bar{L}_1 \geqslant 0$. Hence, we see that $Q \notin \bar{C}$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\zeta"] \arrow[d, "\pi"'] & W \arrow[d, "\psi"] \\ X \arrow[r, dashed, "\rho"'] & \mathbb{P}^2 \end{tikzcd}$$

where ζ is a birational morphism that contracts the curve $\bar{L}_1$, the morphism ψ is a double cover branched over a quartic curve, and ρ is a linear projection from the point $P \in X$.

Let τ be the birational involution of U induced by ψ. Then

- the involution τ is biregular $\iff$ $L_1 \cap \mathrm{Sing}(X) = \varnothing$,
- the involution τ acts biregularly on $U \setminus \bar{L}_1$ if $L_1 \cap \mathrm{Sing}(X) \neq \varnothing$,
- it follows from the construction of τ that $\tau(E) = \bar{C}$,

- if $L_1\cap\mathrm{Sing}(X)=\varnothing$, then
$$\tau^*\big(\bar{L}_1\big)\sim\bar{L}_1,\ \tau^*\big(E\big)\sim\bar{C},\ \tau^*\Big(\pi^*\big(-K_X\big)\Big)\sim\pi^*\big(-2K_X\big)-3E-\bar{L}_1.$$

Let H be the hyperplane section of X that is singular at $\pi\circ\tau(Q)\in C$. Then $P\not\in H$, because C is smooth. Let $\bar{H}$ be the proper transform of H on the surface U. Then
$$\bar{L}_1\not\subseteq\mathrm{Supp}\big(\bar{H}\big)\not\supseteq\bar{C},$$
and we can put $\bar{R}=\tau(\bar{H})$ and $R=\pi(\bar{R})$. Then $\bar{R}$ is singular at the point Q, and
$$\bar{R}\sim\pi^*\big(-2K_X\big)-3E-\bar{L}_1,$$
because R does not pass through a singular point of X if $\mathrm{Sing}(X)\ne\varnothing$.

Suppose that R is irreducible. Then $R+L_1\sim-2K_X$, but the log pair
$$\Big(X,\ \frac{3}{8}\Big(R+L_1\Big)\Big)$$
is log canonical. Thus, we may assume that $R\not\subseteq\mathrm{Supp}(D)$ by Remark 2.23. Then
$$5-2\Big(m_1+\mathrm{mult}_P\big(\Omega\big)\Big)+m_1\Big(1+\bar{L}_1\cdot\bar{L}_1\Big)=\bar{R}\cdot\bar{\Omega}\geqslant 2\mathrm{mult}_Q\big(\bar{\Omega}\big)>2\Big(8/3-m_1-\mathrm{mult}_P\big(\Omega\big)\Big),$$
which implies that $m_1<0$. The curve R must be reducible.

There is a line $L\subset X$ such that $P\not\in L$ and $\pi\circ\tau(Q)\in L$. Then
$$L\cap L_1=\varnothing,$$
because $\pi\circ\tau(Q)\in C$ and $(C+L_1)\cdot L=T\cdot L=1$. Thus, there is unique conic $Z\subset X$ such that $-K_X\sim L+Z$ and $P\in Z$. Then Z is irreducible and $P=Z\cap L_1$, because $(L+Z)\cdot L_1=1$.

Let $\bar{L}$ and $\bar{Z}$ be the proper transform of L and Z on the surface U, respectively. Then
$$\bar{L}\cdot\bar{C}=\bar{Z}\cdot E=1,\ \bar{L}_1\cdot\bar{Z}=\bar{L}\cdot E=\bar{L}\cdot\bar{L}_1=0,\ \bar{Z}\cdot\bar{Z}=1-\bar{L}\cdot\bar{Z},\ \bar{L}\cdot\bar{Z}=\begin{cases}2\text{ if }L\cap\mathrm{Sing}\big(X\big)=\varnothing,\\ 3/2\text{ if }L\cap\mathrm{Sing}\big(X\big)\ne\varnothing.\end{cases}$$

We have $\tau(\bar{Z})=\bar{L}$. Then $Q\in\bar{Z}$. Suppose that $\bar{Z}\not\subseteq\mathrm{Supp}(\bar{\Omega})$. Then
$$2-m_1-\mathrm{mult}_P\big(\Omega\big)=\bar{Z}\cdot\bar{\Omega}>8/3-m_1-\mathrm{mult}_P\big(\Omega\big),$$
which is impossible. Thus, we see that $\bar{Z}\subseteq\mathrm{Supp}(\bar{\Omega})$. But the log pair
$$\Big(X,\ \omega(L+Z)\Big)$$
is log canonical at the point P. Hence, we may assume that $\bar{L}\not\subseteq\mathrm{Supp}(\bar{\Omega})$ by Remark 2.23. Put
$$D=\varepsilon Z+m_1L_1+\Upsilon,$$
where Υ is an effective $\mathbb{Q}$-divisor such that $Z\not\subset\mathrm{Supp}(\Upsilon)\not\supset L_1$. Then
$$1=L\cdot D=\varepsilon L\cdot Z+m_1L\cdot L_1+L\cdot\Upsilon=\varepsilon L\cdot Z+L\cdot\Upsilon\geqslant\varepsilon L\cdot Z=\begin{cases}2\varepsilon\text{ if }L\cap\mathrm{Sing}\big(X\big)=\varnothing,\\ 3\varepsilon/2\text{ if }L\cap\mathrm{Sing}\big(X\big)\ne\varnothing,\end{cases}$$
which implies that $\varepsilon\leqslant 2/3$. But $\bar{Z}\cap\bar{L}_1=\varnothing$. Then it follows from Lemma 2.21 that
$$2-\mathrm{mult}_P\big(D\big)-\varepsilon\bar{Z}\cdot\bar{Z}=\bar{Z}\cdot\bar{\Upsilon}>8/3-\mathrm{mult}_P\big(D\big),$$
where $\bar{\Upsilon}$ is a proper transform of Υ on the surface U. We deduce that $\varepsilon>2/3$. But $\varepsilon\leqslant 2/3$. □

Therefore, we see that $T=L_1+L_2+L_3$, where L_3 is a line such that $P\not\in L_3$. Put
$$D=m_1L_1+m_2L_2+\Delta,$$
where Δ is an effective $\mathbb{Q}$-divisor such that $L_2\not\subseteq\mathrm{Supp}(\Delta)\not\supseteq L_2$.

The inequalities $m_1>1/3$ and $m_2>1/3$ hold by Corollary 4.6. We may assume that $L_3\not\subseteq\mathrm{Supp}\big(\Delta\big)$ by Remark 2.23. If the singular point of X (provided that there exists one) is contained in either L_1 or L_2, we may assume without loss of generality that it is contained in L_1. Then $L_3\cdot L_2=1$ and $L_3\cdot L_1=1/2$ in the case when $L_1\cap\mathrm{Sing}(X)\ne\varnothing$, and
$$L_3\cdot L_2=L_3\cdot L_1=1$$
in the case when $L_1\cap\mathrm{Sing}(X)=\varnothing$. Then $1-m_1L_1\cdot L_3-m_2=L_3\cdot\Delta\geqslant 0$.

Let $\bar{L}_1$ and $\bar{L}_3$ be the proper transforms of L_1 and L_2 on the surface U, respectively. Then

$$m_1\bar{L}_1 + m_2\bar{L}_2 + \bar{\Delta} \sim_{\mathbb{Q}} \pi^*\Big(m_1L_1 + m_2L_2 + \Delta\Big) - \Big(m_1 + m_2 + \mathrm{mult}_P\big(\Delta\big)\Big)E,$$

where $\bar{\Delta}$ is the proper transform of Δ on the surface U. The inequality 4.4 implies that

$$\mathrm{mult}_Q\big(\bar{\Delta}\big) > 8/3 - \mathrm{mult}_P\big(\Delta\big) - m_1\Big(1 + \mathrm{mult}_Q\big(\bar{L}_1\big)\Big) - m_1\Big(1 + \mathrm{mult}_Q\big(\bar{L}_2\big)\Big). \tag{4.9}$$

Lemma 4.10. The curve $\bar{L}_2$ does not contain the point Q.

Proof. Suppose that $Q \in \bar{L}_2$. Then

$$1 - \mathrm{mult}_P\big(\Delta\big) - m_1 + m_2 = \bar{L}_2 \cdot \bar{\Delta} > 8/3 - \mathrm{mult}_P\big(\Delta\big) - m_1 - m_2$$

by Lemma 2.21. Thus, we have $m_2 > 5/6$. It follows from Lemma 2.21 that

$$1 - m_2 - m_1L_1 \cdot L_1 = \Delta \cdot L_1 > 4/3 - m_2,$$

but $L_1 \cdot L_1 = -1$ if $L_1 \cap \mathrm{Sing}(X) = \varnothing$, and $L_1 \cdot L_1 = -1/2$ if $L_1 \cap \mathrm{Sing}(X) \neq \varnothing$. Then

$$m_1 > \begin{cases} 1/3 \text{ if } L_1 \cap \mathrm{Sing}\big(X\big) = \varnothing, \\ 2/3 \text{ if } L_1 \cap \mathrm{Sing}\big(X\big) \neq \varnothing, \end{cases}$$

by Corollary 4.6, which is impossible because $m_2 > 5/6$ and

$$1 > m_1L_1 \cdot L_3 + m_2.$$

□

Lemma 4.11. The curve $\bar{L}_1$ does not contain the point Q.

Proof. Suppose that $Q \in \bar{L}_1$. Arguing as in the proof of Lemma 4.10, we see that

$$L_1 \cap \mathrm{Sing}\big(X\big) \neq \varnothing,$$

which implies that $\bar{L}_1 \cdot \bar{L}_1 = -1/2$. Then $m_1 > 10/9$, because

$$1 + 3m_1/2 = \bar{L}_2 \cdot \Big(\bar{\Delta} + \big(\mathrm{mult}_P\big(\Delta\big) - m_1 - m_2\big)E\Big) > 8/3$$

by Lemma 2.21. But $m_1 \leqslant 1$ by Remark 4.7. □

Therefore, we see that $\bar{L}_1 \not\ni Q \notin \bar{L}_2$. There is a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\zeta"] \arrow[d, "\pi"'] & W \arrow[d, "\psi"] \\ X \arrow[r, dashed, "\rho"'] & \mathbb{P}^2 \end{tikzcd}$$

where ζ is a birational morphism that contracts the curves $\bar{L}_1$ and $\bar{L}_2$, the morphism ψ is a double cover branched over a quartic curve, and ρ is the projection from the point P.

Let τ be the birational involution of U induced by ψ. Then

- the involution τ is biregular $\iff$ $L_1 \cap \mathrm{Sing}(X) = \varnothing$,
- the involution τ acts biregularly on $U \setminus \bar{L}_1$ if $L_1 \cap \mathrm{Sing}(X) \neq \varnothing$,
- the equality $\tau(\bar{L}_2) = \bar{L}_2$ holds,
- if $L_1 \cap \mathrm{Sing}(X) = \varnothing$, then $\tau(\bar{L}_1) = \bar{L}_1$ and

$$\tau^*\Big(\pi^*\big(-K_X\big)\Big) \sim \pi^*\big(-2K_X\big) - 3E - \bar{L}_1 - \bar{L}_2.$$

Let $\bar{L}_3$ be a proper transform of L_3 on the surface U. Then $\tau(E) = \bar{L}_3$ and

$$L_1 \cup L_2 \not\ni \pi \circ \tau\big(Q\big) \in L_3.$$

Lemma 4.12. The line L_3 is the only line on X that passes through the point $\pi \circ \tau(Q)$.

Proof. Suppose that there is a line $L \subset X$ such that $L \neq L_3$ and $\pi \circ \tau(Q) \in L$. Then

$$L \cap L_1 = L \cap L_2 = \varnothing,$$

because $\pi \circ \tau(Q) \in L_3$ and $(L_1 + L_2 + L_3) \cdot L = 1$. Thus, there is unique conic $Z \subset X$ such that $-K_X \sim L + Z$ and $P \in Z$. Then Z is irreducible, because $P \notin L$ and P is not an Eckardt point.

Let $\bar{L}$ and $\bar{Z}$ be the proper transform of L and Z on the surface U, respectively. Then

$$\bar{L} \cdot \bar{L}_3 = \bar{Z} \cdot E = 1,\ \bar{Z} \cdot \bar{Z} = 1 - \bar{L} \cdot \bar{Z},\ \bar{L} \cdot \bar{Z} = \begin{cases} 2 \text{ if } L \cap \mathrm{Sing}\big(X\big) = \varnothing, \\ 3/2 \text{ if } L \cap \mathrm{Sing}\big(X\big) \neq \varnothing, \end{cases}$$

and $\bar{L}_1 \cdot \bar{Z} = \bar{L}_2 \cdot \bar{Z} = \bar{L} \cdot E = \bar{L} \cdot \bar{L}_1 = \bar{L} \cdot \bar{L}_2 = 0$.

We have $\tau(\bar{Z}) = \bar{L}$. Then $Q \in \bar{Z}$, which implies that $\bar{Z} \subseteq \mathrm{Supp}(\bar{\Delta})$, because

$$2 - \mathrm{mult}_P\big(\Delta\big) - m_1 - m_2 = \bar{Z} \cdot \bar{\Omega} > 8/3 - \mathrm{mult}_P\big(\Delta\big) - m_1 - m_2$$

in the case when $\bar{Z} \not\subseteq \mathrm{Supp}(\bar{\Delta})$. On the other hand, the log pair

$$\Big(X,\ \omega(L + Z)\Big)$$

is log canonical at the point P. Hence, we may assume that $\bar{L} \not\subseteq \mathrm{Supp}(\bar{\Delta})$ by Remark 2.23. Put

$$D = \varepsilon Z + m_1 L_1 + m_2 L_2 + \Upsilon,$$

where Υ is an effective $\mathbb{Q}$-divisor such that $Z \not\subseteq \mathrm{Supp}(\Upsilon)$. Then

$$1 = L \cdot D = \varepsilon L \cdot Z + m_1 L \cdot L_1 + L \cdot \Upsilon = \varepsilon L \cdot Z + L \cdot \Upsilon \geqslant \varepsilon L \cdot Z = \begin{cases} 2\varepsilon \text{ if } L \cap \mathrm{Sing}\big(X\big) = \varnothing, \\ 3\varepsilon/2 \text{ if } L \cap \mathrm{Sing}\big(X\big) \neq \varnothing, \end{cases}$$

which implies that $\varepsilon \leqslant 2/3$. But $\bar{Z} \cap \bar{L}_1 = \varnothing$. Then it follows from Lemma 2.21 that

$$2 - \mathrm{mult}_P\big(D\big) - \varepsilon \bar{Z} \cdot \bar{Z} = \bar{Z} \cdot \bar{\Upsilon} > 8/3 - \mathrm{mult}_P\big(D\big),$$

where $\bar{\Upsilon}$ is a proper transform of Υ on the surface U. We deduce that $\varepsilon > 2/3$. But $\varepsilon \leqslant 2/3$. ☐

Therefore, there is an unique irreducible conic $C \subset X$ such that

$$-K_X \sim L_3 + C$$

and $\pi \circ \tau(Q) \in C$. Then $C + L_3$ is a hyperplane section of X that is singular at $\pi \circ \tau(Q)$.

Let $\bar{C}$ be the proper transform of C on the surface U. Put $\bar{Z} = \tau(\bar{C})$ and $Z = \pi(\bar{Z})$.

Lemma 4.13. One has $L_1 \cap \mathrm{Sing}(X) \neq \varnothing$.

Proof. Suppose that $L_1 \cap \mathrm{Sing}(X) = \varnothing$. Then

$$C \cap L_1 = C \cap L_2 = \varnothing,$$

because $(L_1 + L_2 + L_3) \cdot C = L_3 \cdot C = 2$. One can easily check that

$$\bar{Z} \sim \pi^*\big(-2K_X\big) - 4E - \bar{L}_1 - \bar{L}_2,$$

and Z is singular at P. Then $-2K_X \sim Z + L_1 + L_2$. But the log pair

$$\left(U,\ \frac{1}{2}\Big(Z + L_1 + L_2\Big)\right)$$

is log canonical at P. Thus, we may assume that $Z \not\subseteq \mathrm{Supp}(D)$ by Remark 2.23.

We have $Q \in \bar{Z}$ and $\bar{Z} \cdot E = 2$. Then it follows from the inequality 4.4 that

$$4 - 2\mathrm{mult}_P\big(D\big) = \bar{Z} \cdot \bar{D} \geqslant \mathrm{mult}_Q\big(\bar{D}\big) > 8/3 - \mathrm{mult}_P\big(D\big),$$

which implies that $\mathrm{mult}_P(D) < 4/3$. But $\mathrm{mult}_P(D) > 4/3$. ☐

Thus, we see that $L_1 \cap L_3 = \mathrm{Sing}(X) \neq \varnothing$. Then $L_1 \cap L_2 \in C$, which implies that

$$\bar{Z} \sim \pi^*\big(-2K_X\big) - 4E - 2\bar{L}_1 - \bar{L}_2,$$

and Z is smooth rational cubic curve. Then $-2K_X \sim Z + 2L_1 + L_2$. But the log pair

$$\left(U,\ \frac{1}{2}\Big(Z + 2L_1 + L_2\Big)\right)$$

is log canonical at P. Thus, we may assume that $Z \not\subseteq \mathrm{Supp}(D)$ by Remark 2.23.

We have $Q \in \bar{Z}$ and $\bar{Z} \cdot E = \bar{L}_1 = 1$. Then it follows from the inequality 4.4 that

$$3 - \mathrm{mult}_P(\Delta) - 2m_1 - m_2 = \bar{Z} \cdot \bar{\Delta} \geqslant \mathrm{mult}_Q(\bar{\Delta}) > 8/3 - \mathrm{mult}_P(\Delta) - m_1 - m_2,$$

which implies that $m_1 < 1/3$. But $m_1 > 2/3$ by Corollary 4.6.

The obtained contradiction completes the proof Theorem 4.2.

5. Del Pezzo surfaces

Let X be a del Pezzo surface that has at most canonical singularities, let O be a point of the surface X, and let B_X be an effective $\mathbb{Q}$-divisor on the surface X. Suppose that

- the point O is either smooth or an ordinary double point of X,
- the surface X is smooth outside the point $O \in X$.

Lemma 5.1. Suppose that $\mathrm{Sing}\,(X) = O$, $K_X^2 = 2$ and the equivalence

$$B_X \sim_{\mathbb{Q}} -\mu K_X$$

holds, where $0 < \mu < 2/3$. Then $\mathbb{LCS}(X, \mu B_X) = \varnothing$.

Proof. Suppose that $\mathbb{LCS}(X, \mu B_X) \neq \varnothing$. There is a curve $\mathbb{P}^1 \cong L \subset X$ such that

$$\mathrm{LCS}\Big(X, \mu B_X\Big) \not\subseteq L$$

the equality $L \cdot L = -1$ holds, and $L \cap \mathrm{Sing}(X) = \varnothing$. Thus, there is a birational morphism $\pi\colon X \to S$ that contracts the curve L. Then

$$\mathbb{LCS}\Big(S,\ \mu\pi\big(B_X\big)\Big) \neq \varnothing$$

due to the choice of the curve $L \subset X$. But $-K_S \sim_{\mathbb{Q}} \pi(B_X)$, and $S \subset \mathbb{P}^3$ is a cubic surface that has at most one ordinary double point, which is impossible (see Examples 1.19 and 1.18). $\square$

Lemma 5.2. Suppose that $\mathrm{Sing}(X) = \varnothing$, $K_X^2 = 5$, the equivalence

$$B_X \sim_{\mathbb{Q}} -\mu K_X$$

holds, where $\mu \in \mathbb{Q}$ is such that $0 < \mu < 2/3$. Assume that $\mathbb{LCS}(X, B_X) \neq \varnothing$. Then

- either the set $\mathbb{LCS}(X, B_X)$ contains a curve,
- or there are a curve $\mathbb{P}^1 \cong L \subset X$ and a point $P \in L$ such that $L \cdot L = -1$ and

$$\mathrm{LCS}\Big(X,\ B_X\Big) = P.$$

Proof. Suppose that $\mathbb{LCS}(X, B_X)$ contains no curves. Then it follows from Theorem 2.7 that

$$\mathrm{LCS}\Big(X,\ B_X\Big) = P,$$

where $P \in X$ is a point. We may assume that P does not lie on any curve $\mathbb{P}^1 \cong L \subset X$ such that the equality $L \cdot L = -1$ holds. Then there is a birational morphism $\phi\colon X \longrightarrow \mathbb{P}^2$ such that ϕ is an isomorphism in a neighborhood of the point P. Note that

$$\phi\big(P\big) \in \mathrm{LCS}\Big(\mathbb{P}^2,\ \phi\big(B_X\big)\Big) \neq \varnothing,$$

the set $\mathbb{LCS}(\mathbb{P}^2, \phi(B_X))$ contains no curves, and

$$\phi\big(B_X\big) \sim_{\mathbb{Q}} -\mu K_{\mathbb{P}^2}.$$

Since $\mu < 2/3$, the latter is impossible by Lemma 2.8. $\square$

Example 5.3. Suppose that $O = \mathrm{Sing}(X)$ and $K_X^2 = 5$. Let $\alpha\colon V \to X$ be a blow up of O, and let E be the exceptional divisor of α. Then there is a birational morphism $\omega\colon V \to \mathbb{P}^2$ such that

- the morphism ω contracts the curves E_1, E_2, E_3, E_4,
- the curve $\omega(E)$ is a line in $\mathbb{P}^2$ that contains $\omega(E_1)$, $\omega(E_2)$, $\omega(E_3)$, but $\omega(E) \not\ni \omega(E_4)$.

Let Z be a line in $\mathbb{P}^2$ such that $\omega(E_1) \in Z \ni \omega(E_4)$. Then

$$2E + \bar{Z} + 2E_1 + E_2 + E_3 \sim -K_V,$$

where $\bar{Z} \subset V$ is a proper transform of Z. One has

$$\mathrm{lct}\Big(X,\ \alpha\big(\bar{Z} + 2\alpha(E_1) + \alpha(E_2) + \alpha(E_3)\big)\Big) = \frac{1}{2},$$

which implies $\mathrm{lct}(X) \leqslant 1/2$. Suppose that $-K_X \sim_{\mathbb{Q}} 2B_X$, but (X, B_X) is not log canonical. Then

$$K_V + B_V + mE \sim_{\mathbb{Q}} \alpha^*\Big(K_X + B_X\Big),$$

for some $m \geqslant 0$, and B_V is the proper transform of B_X on the surface V. Then

$$\Big(V,\ B_V + mE\Big)$$

is not log canonical at some point $P \in V$. There is a birational morphism $\pi\colon V \to U$ such that

- the surface U is a smooth del Pezzo surface with $K_U^2 = 6$,
- the morphism π is an isomorphism in a neighborhood of $P \in X$,

which implies that $(U, \pi(B_V) + m\pi(E))$ is not log canonical at $\pi(P)$. But

$$\pi\big(B_V\big) + m\pi\big(E\big) \sim_{\mathbb{Q}} -\frac{1}{2}K_U,$$

which is impossible, because $\mathrm{lct}(U) = 1/2$ (see Example 1.18). We see that $\mathrm{lct}(X) = 1/2$.

Example 5.4. Suppose that $K_X^2 = 4$. Arguing as in Example 5.3, we see that the equality

$$\mathrm{lct}\big(X\big) = \begin{cases} 1/2 \text{ when } O = \mathrm{Sing}\big(X\big), \\ 2/3 \text{ when } \mathrm{Sing}\big(X\big) = \varnothing, \end{cases}$$

holds (cf. [189]). Take $\lambda \leqslant 1$. Suppose that

$$B_X \sim_{\mathbb{Q}} -K_X,$$

and $(X, \lambda B_X)$ is not log canonical at some point $P \in X \setminus O$. There is a commutative diagram

$$\begin{tikzcd} & V \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "\psi"'] & & U \end{tikzcd}$$

where U is a cubic surface in $\mathbb{P}^3$ that has canonical singularities, the morphism α is a blow up of the point P, the morphism β is birational, and ψ is a projection from the point $P \in X$. Then

$$K_V + \lambda B_V + \Big(\lambda \mathrm{mult}_P\big(B_X\big) - 1\Big)E \sim_{\mathbb{Q}} \alpha^*\Big(K_X + \lambda B_X\Big),$$

where E is the exceptional divisor of α, and B_V is the proper transform of B_X. Note that

$$\Big(V,\ \lambda B_V + \Big(\lambda \mathrm{mult}_P\big(B_X\big) - 1\Big)E\Big)$$

is not log canonical at some point $Q \in E$ and $\mathrm{mult}_P(B_X) > 1/\lambda$. Then the log pair

$$\Big(V,\ \lambda B_V + \Big(\lambda \mathrm{mult}_P\big(B_X\big) - \lambda\Big)E\Big)$$

is not log canonical at the point $Q \in E$ as well. But the equivalences

$$B_V + \Big(\mathrm{mult}_P\big(B_X\big) - 1\Big)E \sim_{\mathbb{Q}} -K_V + \alpha^*\Big(K_X + B_X\Big) \sim_{\mathbb{Q}} -K_V,$$

hold. Suppose that P is not contained in any line on the surface X. Then

- the morphism $\beta\colon V \to U$ is an isomorphism,
- the cubic surface U is smooth outside of the point $\psi(O)$,
- the point $\psi(O)$ is at most ordinary double point of the surface U,

which implies that $\lambda > 2/3$ (see Example 1.19).

Suppose that $\lambda = 3/4$. Then the point

$$\psi(Q) \in U \subset \mathbb{P}^3$$

must be an Eckardt point (see Definition 4.1) of the surface U by Theorem 4.2. But

$$\beta(E) \subset U \subset \mathbb{P}^3$$

is a line. So, there are two conics $C_1 \neq C_2$ contained in X such that $P = C_1 \cap C_2$ and

$$C_1 + C_2 \sim -K_X.$$

Lemma 5.5. Suppose that $O = \mathrm{Sing}(X)$ and $K_X^2 = 6$ such that there is a diagram

$$\begin{tikzcd} & V \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X & & \mathbb{P}^2 \end{tikzcd}$$

where β is a blow up of three points $P_1, P_2, P_3 \in \mathbb{P}^2$ lying on a line $L \subset \mathbb{P}^2$, and α is a birational morphism that contracts an irreducible curve $\bar{L}$ to the point O such that $\beta(\bar{L}) = L$. Then

$$\mathrm{LCS}\Big(X,\ \lambda B_X\Big) = O$$

in the case when $\mathrm{LCS}(X, \lambda B_X) \neq \varnothing$, $B_X \sim_{\mathbb{Q}} -K_X$ and $\lambda < 1/2$.

Proof. Suppose that $B_X \sim_{\mathbb{Q}} -K_X$ and

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda B_X\Big) \neq O,$$

let $M \subset \mathbb{P}^2$ be a general line, and let $\bar{M} \subset V$ be its proper transform. Then

$$-K_X \sim 2\alpha\big(\bar{M}\big)$$

and $O \in \alpha(\bar{M})$. Thus, the set $\mathbb{LCS}(X, \lambda B_X)$ contains a curve, because otherwise the locus

$$\mathrm{LCS}\Big(X,\ \lambda B_X + \alpha\big(\bar{M}\big)\Big)$$

would be disconnected, which is impossible by Theorem 2.7.

Let $C \subset X$ be an irreducible curve such that $C \subseteq \mathrm{LCS}(X, \lambda B_X)$. Then

$$B_X = \varepsilon C + \Omega,$$

where $\varepsilon > 2$, and Ω is an effective $\mathbb{Q}$-divisor such that $C \not\subset \mathrm{Supp}(\Omega)$.

Let $\Gamma_i \subset X$ be a proper transform of a general line in $\mathbb{P}^2$ that passes through P_i. Then

$$O \notin \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$$

and $-K_X \cdot \Gamma_1 = -K_X \cdot \Gamma_2 = -K_X \cdot \Gamma_3 = 2$. But

$$-K_X \sim_{\mathbb{Q}} \Gamma_1 + \Gamma_2 + \Gamma_3,$$

which implies that there is $m \in \{1, 2, 3\}$ such that $C \cdot \Gamma_m \neq 0$. Then

$$2 = B_X \cdot \Gamma_m = \Big(\varepsilon C + \Omega\Big) \cdot \Gamma_m \geqslant \varepsilon C \cdot \Gamma_m \geqslant \varepsilon > 2,$$

because $\Gamma_m \not\subset \mathrm{Supp}(B_X)$. The obtained contradiction completes the proof. $\square$

Remark 5.6. Suppose that $O = \mathrm{Sing}(X)$ and $K_X^2 = 6$. Let $\alpha\colon V \to X$ be a blow up of the point $O \in X$, and let E be the exceptional divisor of α. Then

$$K_V + B_V + mE \sim_{\mathbb{Q}} \alpha^*\Big(K_X + B_X\Big)$$

for some $m \geqslant 0$, and B_V is the proper transform of B_X on V. Note that $\mathrm{lct}(X) \leqslant 1/3$. Suppose that $\mathrm{lct}(X) < 1/3$, i. e. there exists an effective $\mathbb{Q}$-divisor $B_X \sim_{\mathbb{Q}} -K_X$, such that the log pair $(X, 1/3B_X)$ is not log canonical. Then the log pair

$$\Big(V, \frac{1}{3}\big(B_V + mE\big)\Big)$$

is not log canonical at some point $P \in V$. There is a birational morphism $\pi\colon V \longrightarrow U$ such that U is either $\mathbb{F}_1$ or $\mathbb{P}^1 \times \mathbb{P}^1$ and π is an isomorphism in a neighborhood of $P \in X$. Then the log pair

$$\Big(U, \frac{1}{3}\big(\pi(B_V) + m\pi(E)\big)\Big)$$

is not log canonical at the point $\pi(P)$. But we know that

$$-K_U \sim_{\mathbb{Q}} \pi\big(B_V\big) + m\pi\big(E\big),$$

so the latter contradicts Example 1.18. Hence $\mathrm{lct}(X) = 1/3$.

Lemma 5.7. Suppose that $X \cong \mathbb{P}(1,1,2)$ and $B_X \sim_{\mathbb{Q}} -K_X$, but there is a point $P \in X$ such that

$$O \neq P \in \mathrm{LCS}\Big(X,\ \lambda B_X\Big)$$

for some $\lambda < 1/2$. Take $L \in |\mathcal{O}_{\mathbb{P}(1,1,2)}(1)|$ such that $P \in L$. Then $L \subseteq \mathrm{LCS}(X, \lambda B_X)$.

Proof. Suppose that there is a curve $\Gamma \in \mathrm{LCS}(X, \lambda B_X)$ such that $P \in \Gamma \neq L$. Then

$$B_X = \mu\Gamma + \Omega,$$

where $\mu > 2$, and Ω is an effective $\mathbb{Q}$-divisor such that $\Gamma \not\subset \mathrm{Supp}(\Omega)$. Hence

$$\mu\Gamma + \Omega \sim_{\mathbb{Q}} 4L$$

and $\Gamma \sim mL$, where $m \in \mathbb{Z}_{>0}$. But $m \geqslant 2$, because $P \in \Gamma \neq L$, which is a contradiction.

Suppose that $L \not\subseteq \mathrm{LCS}(X, \lambda B_X)$. Then it follows from Theorem 2.7 that

$$\mathrm{LCS}\Big(X, \lambda B_X\Big) = P,$$

because we proved that $\mathbb{LCS}(X, \lambda B_X)$ contains no curves passing through $P \in X$.

Let $C \in |\mathcal{O}_{\mathbb{P}(1,1,2)}(1)|$ be a general curve. Then

$$\mathrm{LCS}\Big(X, \lambda B_X + C\Big) = P \cup C,$$

which is impossible by Theorem 2.7. □

Lemma 5.8. Suppose that $X \cong \mathbb{F}_1$. Then there are $0 \leqslant \mu \in \mathbb{Q} \ni \lambda \geqslant 0$ such that

$$B_X \sim_{\mathbb{Q}} \mu C + \lambda L,$$

where C and L are irreducible curves on the surface X such that

$$C \cdot C = -1,\ C \cdot L = 1$$

and $L \cdot L = 0$. Suppose that $\mu < 1$ and $\lambda < 1$. Then $\mathbb{LCS}(X, B_X) = \varnothing$.

Proof. The set $\mathbb{LCS}(X, B_X)$ contains no curves, because L and C generate the cone of effective divisors of the surface X. Suppose that $\mathbb{LCS}(X, B_X)$ contains a point $O \in X$. Then

$$K_X + B_X + \Big(\big(1-\mu\big)C + \big(2-\lambda\big)L\Big) \sim_{\mathbb{Q}} -\big(L + C\big),$$

because $-K_X \sim_{\mathbb{Q}} 2C + 3L$. But it follows from Theorem 2.6 that the map

$$0 = H^0\Big(\mathcal{O}_X\big(-L - C\big)\Big) \longrightarrow H^0\Big(\mathcal{O}_{\mathcal{L}(X,\, B_X)}\Big) \neq 0$$

is surjective, because the divisor $(1-\mu)C + (2-\lambda)L$ is ample. □

Lemma 5.9. Suppose that $\mathrm{Sing}(X) = \varnothing$ and $K_X^2 = 7$. Then

$$L_1 \cdot L_1 = L_2 \cdot L_2 = L_3 \cdot L_3 = -1,\ L_1 \cdot L_2 = L_2 \cdot L_3 = 1,\ L_1 \cdot L_3 = 0$$

where L_1, L_2, L_3 are exceptional curves on X. Suppose that $\mathrm{LCS}(X, B_X) \neq \varnothing$ and

$$B_X \sim_{\mathbb{Q}} -\mu K_X,$$

where $\mu < 1/2$. Then $\mathrm{LCS}(X, B_X) = L_2$.

Proof. Let P be a point in $\mathrm{LCS}(X, B_X)$. Then $P \in L_2$, because $\mathrm{lct}(\mathbb{P}^1 \times \mathbb{P}^1) = 1/2$, and there is a birational morphism $X \to \mathbb{P}^1 \times \mathbb{P}^1$ that contracts only the curve L_2.

Suppose that $\mathrm{LCS}(X, B_X) \neq L_2$. Then $\mathrm{LCS}(X, B_X) = P$ by Theorem 2.7.

We may assume that $P \notin L_3$. There is a birational morphism $\phi\colon X \to \mathbb{P}^2$ that contracts the curves L_1 and L_3. Let C_1 and C_3 be proper transforms on X of sufficiently general lines in $\mathbb{P}^2$ that pass through the points $\phi(L_1)$ and $\phi(L_3)$, respectively. Then

$$-K_X \sim C_1 + 2C_3 + L_3,$$

and $C_1 \not\ni P \notin C_2$. Therefore, we see that

$$C_2 \cup P \subseteq \mathrm{LCS}\left(X,\ \lambda D + \frac{1}{2}\Big(C_1 + 2C_2 + L_3\Big)\right) \subseteq C_2 \cup P \cup L_3,$$

which is impossible by Theorem 2.7, because $P \notin L_3$. □

Lemma 5.10. Suppose that $O = \mathrm{Sing}(X)$, the equality $K_X^2 = 7$ holds, the equivalence

$$B_X \sim_{\mathbb{Q}} C + \frac{4}{3}L$$

holds, where $L \cong \mathbb{P}^1 \cong C$ are curves on the surface X such that

$$L \cdot L = -1/2,\ C \cdot C = -1,\ C \cdot L = 1,$$

but the log pair (X, B_X) is not log canonical at some point $P \in C$. Then $P \in L$.

Proof. Let S be a quadratic cone in $\mathbb{P}^3$. Then $S \cong \mathbb{P}(1,1,2)$ and there is a birational morphism

$$\phi\colon X \longrightarrow S \subset \mathbb{P}^3$$

that contracts the curve C to a smooth point $Q \in S$. Then $Q \in \phi(L) \in |\mathcal{O}_{\mathbb{P}(1,1,2)}(1)|$.

Suppose that $P \notin L$. Then it follows from Remark 2.23 that to complete the proof we may assume that either $C \not\subset \mathrm{Supp}(B_X)$ or $L \not\subset \mathrm{Supp}(B_X)$, because the log pair

$$\left(X,\ C + \frac{4}{3}L\right)$$

is log canonical in the point $P \in X$. Suppose that $C \not\subset \mathrm{Supp}(B_X)$. Then

$$\frac{1}{3} = B_X \cdot C \geqslant \mathrm{mult}_P\big(B_X\big) > 1,$$

which is impossible. Therefore, we see that $C \subset \mathrm{Supp}(B_X)$. Then $L \not\subset \mathrm{Supp}(B_X)$. Put

$$B_X = \varepsilon C + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor such that $C \not\subset \mathrm{Supp}(\Omega)$. Then

$$\frac{1}{3} = B_X \cdot L = \varepsilon + \Omega \cdot L \geqslant \varepsilon,$$

which implies that $\varepsilon \leqslant 1/3$. Then

$$1 < \Omega \cdot C = 1/3 + \varepsilon \leqslant 2/3$$

by Lemma 2.21, which is a contradiction. □

6. Toric varieties

The purpose of this section is to prove Lemma 6.1 (cf. [7], [174], [39]).

Let $N = \mathbb{Z}^n$ be a lattice of rank n, and let $M = \mathrm{Hom}(N, \mathbb{Z})$ be the dual lattice. Put $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Let X be a toric variety defined by a complete fan $\Sigma \subset N_{\mathbb{R}}$, let

$$\Delta_1 = \big\{v_1, \ldots, v_m\big\}$$

be a set of generators of one-dimensional cones of the fan Σ. Put

$$\Delta = \Big\{w \in M\ \Big|\ \big\langle w, v_i\big\rangle \geqslant -1 \text{ for all } i = 1, \ldots, m\Big\}.$$

Put $T = (\mathbb{C}^*)^n \subset \mathrm{Aut}(X)$. Let $\mathcal{N}$ be the normalizer of T in $\mathrm{Aut}(X)$ and $\mathcal{W} = \mathcal{N}/T$.

Lemma 6.1. Let $G \subset \mathcal{W}$ be a subgroup. Suppose that X is $\mathbb{Q}$-factorial. Then

$$\mathrm{lct}\Big(X,\ G\Big) = \frac{1}{1+\max\Big\{\langle w, v\rangle \mid w \in \Delta^G,\ v\in \Delta_1\Big\}},$$

where Δ^G is the set of the points in Δ that are fixed by the group G.

Proof. Put $\mu = 1+\max\{\langle w, v\rangle \mid w \in \Delta^G,\ v\in \Delta_1\}$. Then $\mu \in \mathbb{Q}$ is the largest number such that

$$-K_X \sim_{\mathbb{Q}} \lambda R + H,$$

where R is an integral $T \rtimes G$-invariant effective divisor, and H is an effective $\mathbb{Q}$-divisor. Hence

$$\mathrm{lct}\big(X, G\big) \leqslant \frac{1}{\mu}.$$

Suppose that $\mathrm{lct}(X, G) < 1/\mu$. Then there is an effective G-invariant $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some rational $\lambda < 1/\mu$.

There exists a family $\{D_t \mid t \in \mathbb{C}\}$ of G-invariant effective $\mathbb{Q}$-divisors such that

- the equivalence $D_t \sim_{\mathbb{Q}} D$ holds for every $t \in \mathbb{C}$,
- the equality $D_1 = D$ holds,
- for every $t \neq 0$ there is $\phi_t \in \mathrm{Aut}(X)$ such that

$$D_t = \phi_t\big(D\big) \cong D,$$

- the divisor D_0 is T-invariant,

which implies that $(X, \lambda D_0)$ is not log canonical (see [49]).

On the other hand, the divisor D_0 does not have components with multiplicity greater then μ, which implies that $(X, \lambda D_0)$ is log canonical (see [61], [121]), which is a contradiction. ☐

Corollary 6.2. Let $X = \mathbb{P}_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^n}(-a_k))$, $a_i \geqslant 0$ for $i = 1, \ldots, k$. Then

$$\mathrm{lct}(X) = \frac{1}{1+\max\Big\{k, n+\sum_{i=1}^k a_i\Big\}}.$$

Proof. Note that X is a toric variety, and Δ_1 consists of the following vectors:

$$\begin{gathered}
(\overbrace{1,0,\ldots,0}^{k},\overbrace{0,0,\ldots,0}^{n}),\\
\ldots\\
(0,\ldots,0,1,0,0,\ldots,0),\\
(-1,\ldots,-1,0,0,\ldots,0),\\
(0,0,\ldots,0,1,0,\ldots,0),\\
\ldots\\
(0,0,\ldots,0,0,\ldots,0,1),\\
(-a_1,\ldots,-a_k,-1,\ldots,-1),
\end{gathered}$$

which implies the required assertion by Lemma 6.1. ☐

Applying Corollary 6.2, we obtain the following result.

Corollary 6.3. In the notation of section 1 one has

$$\mathrm{lct}\big(X\big) = \begin{cases} 1/4 \text{ whenever } \daleth(X) \in \{2.33, 2.35\},\\ 1/5 \text{ whenever } \daleth(X) = 2.36. \end{cases}$$

On the other hand, straightforward computations using Lemma 6.1 imply the following result.

Corollary 6.4. In the notation of section 1 one has

$$\mathrm{lct}\big(X\big) = \begin{cases} 1/3 \text{ whenever } \daleth(X) \in \{3.25, 3.31, 4.9, 4.11, 5.2\},\\ 1/4 \text{ whenever } \daleth(X) \in \{3.26, 3.30, 4.12\},\\ 1/5 \text{ whenever } \daleth(X) = 3.29. \end{cases}$$

Another application of Lemma 6.1 is an immediate proof of the following result, obtained in [77].

Example 6.5. Let X be a blow up of $\mathbb{P}^n$, $n \geqslant 2$, at $k+1$ points that are not contained in a linear subspace of dimension $k-1$. Assume that $k \leqslant n$. Then the points are in general position (i. e. no $r+1$ of them are contained in a linear subspace of dimension $r-1$), so that X is toric, and

$$\operatorname{lct}\big(X\big) = \frac{1}{n+1}$$

by Lemma 6.1. Note that X is a Fano variety only if $k=1$, or $n=2$ and $k \leqslant 3$ (cf. [77, Chapter 5]).

Remark 6.6. Let X be a blow up of $\mathbb{P}^n$, $n \geqslant 2$, at $n+1$ points that are not contained in a linear subspace of dimension $n-1$. Then X is not Fano whether $n \geqslant 3$, but the points are still in general position, so that X is toric, and

$$\operatorname{lct}\big(X\big) = \frac{1}{n}$$

by Lemma 6.1. On the other hand, there is a natural action of the symmetric group S_{n+1} on X such that

$$\operatorname{lct}(X, S_{n+1}) = 1.$$

7. Del Pezzo threefolds

We use the assumptions and notation of Theorem 1.78. Suppose that $-K_X \sim 2H$, where H is a Cartier divisor that is indivisible in $\operatorname{Pic}(X)$.

The purpose of this section is to prove the following result.

Theorem 7.1. *The equality $\operatorname{lct}(X) = 1/2$ holds, unless $\daleth(X) = 2.35$ when $\operatorname{lct}(X) = 1/4$.*

It follows from Theorems 3.1.14 and 3.3.1 in [98] that

$$\daleth(X) \in \Big\{1.11, 1.12, 1.13, 1.14, 1.15, 1.17, 2.32, 2.35, 3.27\Big\},$$

and by [20] and [28] (see also Lemma 2.18) one has $\operatorname{lct}(X) = 1/2$ if $\daleth(X) \in \{1.12, 1.13\}$.

It follows from Lemma 2.30 that $\operatorname{lct}(X) = 1/2$ when $\daleth(X) = 3.27$.

Lemma 7.2. *Suppose that $\daleth(X) = 2.35$. Then $\operatorname{lct}(X) = 1/4$.*

Proof. There is a birational morphism $\pi\colon X \to \mathbb{P}^3$ that contracts a surface $E \cong \mathbb{P}^2$ to a point $P \in \mathbb{P}^3$. Hence $\operatorname{lct}(X) \leqslant 1/4$, because

$$-K_X \sim 2E + 4T,$$

where T is the proper transform of a plane in $\mathbb{P}^3$ that passes through $P \in \mathbb{P}^3$.

Suppose that $\operatorname{lct}(X) < 1/4$. Then there is an effective $\mathbb{Q}$-divisor D such that

$$D \sim_{\mathbb{Q}} \frac{1}{2}E + T,$$

but the log pair $(X, \lambda D)$ is not log canonical for some rational number $\lambda < 1$.

Let R be a proper transform on X of a general plane in $\mathbb{P}^3$. Then

$$\operatorname{LCS}\Big(X,\ \lambda D + R\Big)$$

must be connected by Theorem 2.7, because $-(K_X + \lambda D + R)$ is ample. Thus, we see that

- the subscheme $\mathcal{L}(X, \lambda D)$ is not zero-dimensional,
- the locus $\operatorname{LCS}(X, \lambda D)$ is not contained in E,

which implies that $(\mathbb{P}^3, \lambda\pi(D))$ is not log canonical. But $\operatorname{lct}(\mathbb{P}^3) = 1/4$ and $\pi(D) \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}^3}(1)$, which is a contradiction. □

Remark 7.3. Actually, the assertion of Lemma 7.2 is contained in Corollary 6.3, but we still prefer to give a detailed proof that may have further applications.

We are left with the cases

$$\gimel(X) \in \Big\{1.11, 1.14, 1.15, 2.32\Big\},$$

while the inequality $\mathrm{lct}(X) \leqslant 1/2$ is obvious, because $|H|$ is not empty.

Lemma 7.4. Suppose that $\gimel(X) = 2.32$. Then $\mathrm{lct}(X) = 1/2$.

Proof. We may suppose that $\mathrm{lct}(X) < 1/2$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} H$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1$.

The threefold X is a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bi-degree $(1, 1)$. There are two natural $\mathbb{P}^1$-bundles $\pi_1 \colon X \to \mathbb{P}^2$ and $\pi_2 \colon X \to \mathbb{P}^2$, and applying Theorem 2.28 to π_1 and π_2 we immediately obtain a contradiction. $\square$

Remark 7.5. Suppose that $\mathrm{Pic}(X) = \mathbb{Z}[H]$, and there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} H$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1$. Put

$$D = \varepsilon S + \Omega \sim_{\mathbb{Q}} H,$$

where S is an irreducible surface and Ω is an effective $\mathbb{Q}$-divisor such that

$$\mathrm{Supp}(\Omega) \not\supset S.$$

Then $\varepsilon \leqslant 1$, because $\mathrm{Pic}(X) = \mathbb{Z}[H]$, which implies that the set $\mathbb{LCS}(X, \lambda D)$ contains no surfaces. Moreover, for any choice of $H \in |H|$ the locus

$$\mathrm{LCS}\Big(X,\ \lambda D + H\Big)$$

is connected by Theorem 2.7. Let H be a general surface in $|H|$. Since $\mathrm{LCS}(X, \lambda D + H)$ is connected, one obtains that the locus $\mathrm{LCS}(X, \lambda D + H)$ has no isolated zero-dimensional components outside the base locus of the linear system $|H|$. Note that $|H|$ has no base points at all, unless $\gimel(X) = 1.11$ when $\mathrm{Bs}|H|$ consists of a single point O. Note that in the latter case $O \notin \mathrm{LCS}(X, \lambda D)$, since X is covered by the curves of anticanonical degree 2 passing through O. Hence the locus $\mathrm{LCS}(X, \lambda D)$ never has isolated zero-dimensional components; in particular, it contains an (irreducible) curve C. Put $D|_H = \bar{D}$. Then

$$-K_H \sim H\big|_H \sim_{\mathbb{Q}} \bar{D},$$

but $(H, \lambda\bar{D})$ is not log canonical in every point of the intersection $H \cap C$. The locus $\mathrm{LCS}(H, \lambda\bar{D})$ is connected by Theorem 2.7. But the scheme $\mathcal{L}(H, \lambda\bar{D})$ is zero-dimensional. We see that

$$H \cdot C = \big|H \cap C\big| = 1,$$

and the locus $\mathrm{LCS}(X, \lambda D)$ contains no curves besides the irreducible curve C.

Lemma 7.6. Suppose that $\gimel(X) = 1.15$. Then $\mathrm{lct}(X) = 1/2$.

Proof. We may suppose that $\mathrm{lct}(X) < 1/2$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} H$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1$.

The linear system $|H|$ induces an embedding $X \subset \mathbb{P}^6$. Thus, it follows from Remark 7.5 that the locus $\mathrm{LCS}(X, \lambda D)$ consists of a single line $C \subset X \subset \mathbb{P}^6$.

It follows from [98, Proposition 3.4.1] (see [92] and [67]) that there is a commutative diagram

$$\begin{tikzcd} & V \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "\psi"'] & & Q \end{tikzcd}$$

where Q is a quadric in $\mathbb{P}^4$, the morphism α is a blow up of C, the morphism β is a blow up of a smooth rational cubic curve $Z \subset Q$, and the map ψ is a projection from the line C.

Let S be the exceptional divisor of β, and let L be a fiber of the morphism β over a general point of the curve Z. Put $\bar{S} = \alpha(S)$ and $\bar{L} = \alpha(L)$. Then $\bar{S} \sim H$, the curve $\bar{L}$ is a line, and the surface $\bar{S}$ is singular along C. Moreover, the singularity of $\bar{S}$ at a general point of C is locally isomorphic to $T \times \mathbb{A}^1$, where T is a germ of a nodal curve. In particular, the pair $(X, \bar{S})$ is log canonical.

We may assume that $\mathrm{Supp}(D) \not\supset \bar{S}$ by Remark 2.23. Then

$$1 = \bar{L} \cdot D \geqslant \mathrm{mult}_C\big(D\big) > 1/\lambda > 1,$$

which is a contradiction. $\square$

Lemma 7.7. Let $\daleth(X) = 1.14$. Then $\mathrm{lct}(X) = 1/2$.

Proof. We may suppose that $\mathrm{lct}(X) < 1/2$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} H$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1$.

The linear system $|H|$ induces an embedding $X \subset \mathbb{P}^5$ such that X is a complete intersection of two quadrics. Then the locus $\mathrm{LCS}(X, \lambda D)$ consists of a single line $C \subset X \subset \mathbb{P}^5$ by Remark 7.5.

It follows from [98, Proposition 3.4.1] that there is a commutative diagram

$$\begin{tikzcd} & V \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^3 \end{tikzcd}$$

where ψ is a projection from C, the morphism α is a blow up of the line C, and β is a blow up of a smooth curve $Z \subset \mathbb{P}^3$ of degree 5 and genus 2.

Let S be the exceptional divisor of β, and let L be a fiber of the morphism β over a general point of the curve Z. Put $\bar{S} = \alpha(S)$ and $\bar{L} = \alpha(L)$. Then $\bar{S} \sim 2H$, the curve $\bar{L}$ is a line, and $\mathrm{mult}_C(\bar{S}) = 3$. But the log pair $(X, 1/2\bar{S})$ is log canonical, which implies that we may assume that $\mathrm{Supp}(D) \not\supset \bar{S}$ by Remark 2.23. Then

$$1 = \bar{L} \cdot D \geqslant \mathrm{mult}_C\big(D\big) > 1/\lambda > 1,$$

which is a contradiction. $\square$

Remark 7.8. Let $V \subset \mathbb{P}^5$ be a complete intersection of two quadric hypersurfaces that has isolated singularities, and let B_V be an effective $\mathbb{Q}$-divisor on V such that $B_V \sim_{\mathbb{Q}} -K_V$ and

$$\mathrm{LCS}\Big(V, \mu B_V\Big) \neq \varnothing,$$

where $\mu < 1/2$. Arguing as in the proof of Lemma 7.7, we see that

$$\mathrm{LCS}\Big(V, \mu B_V\Big) \subseteq L,$$

where $L \subset V$ is a line such that $L \cap \mathrm{Sing}\,(V) \neq \varnothing$.

Lemma 7.9. Suppose that $\daleth(X) = 1.11$. Then $\mathrm{lct}(X) = 1/2$.

Proof. We may suppose that $\mathrm{lct}(X) < 1/2$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} H$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1$.

Recall that the threefold X can be given by an equation

$$w^2 = t^3 + t^2 f_2\big(x,y,z\big) + t f_4\big(x,y,z\big) + f_6\big(x,y,z\big) \subset \mathbb{P}\big(1,1,1,2,3\big) \cong \mathrm{Proj}\Big(\mathbb{C}\big[x,y,z,t,w\big]\Big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = \mathrm{wt}(z) = 1$, $\mathrm{wt}(t) = 2$, $\mathrm{wt}(w) = 3$, and f_i is a polynomial of degree i.

The locus $\mathrm{LCS}(X, \lambda D)$ consists of a single curve $C \subset X$ such that $H \cdot C = 1$ by Remark 7.5.

Let $\psi\colon X \dashrightarrow \mathbb{P}^2$ be the natural projection. Then ψ is not defined in a point O that is cut out by $x = y = z = 0$. The curve C does not contain the point O, because otherwise we get

$$1 = \Gamma \cdot D \geqslant \mathrm{mult}_O\big(D\big)\mathrm{mult}_O\big(\Gamma\big) \geqslant \mathrm{mult}_C\big(D\big) > 1/\lambda > 1,$$

where Γ is a general fiber of the projection ψ. Thus, we see that $\psi(C) \subset \mathbb{P}^2$ is a line.

Let S be the unique surface in $|H|$ such that $C \subset S$. Let L be a sufficiently general fiber of the rational map ψ that intersects the curve C. Then $L \subset \mathrm{Supp}\,(D)$, since otherwise

$$1 = D \cdot L \geqslant \mathrm{mult}_C(D) > 1/\lambda > 1.$$

We may assume that $D = S$ by Remark 2.23. Then S has a cuspidal singularity along C.

We may assume that the surface S is cut out on X by the equation $x=0$, and we may assume that the curve C is given by $w=t=x=0$. Then S is given by

$$w^2 = t^3 + t^2 f_2\big(0,y,z\big) + t f_4\big(0,y,z\big) \subset \mathbb{P}\big(1,1,2,3\big) \cong \mathrm{Proj}\Big(\mathbb{C}\big[y,z,t,w\big]\Big),$$

and $f_6(x,y,z) = x f_5(x,y,z)$, where $f_5(x,y,z)$ is a homogeneous polynomial of degree 5.

Since the surface S is singular along the curve C, one has

$$f_4\big(x,y,z\big) = x f_3\big(x,y,z\big),$$

where $f_3(x,y,z)$ is a homogeneous polynomial of degree 3. Then every point of the set

$$x = f_5\big(x,y,z\big) = t = w = 0 \subset \mathbb{P}\big(1,1,1,2,3\big)$$

must be singular on X, which is a contradiction, because X is smooth. $\square$

The assertion of Theorem 7.1 is completely proved.

8. Fano threefolds with $\rho = 2$

We use the assumptions and notation introduced in section 1.

Lemma 8.1. Suppose that $\daleth(X) = 2.1$ or $\daleth(X) = 2.3$. Then $\mathrm{lct}(X) = 1/2$.

Proof. There is a birational morphism $\alpha\colon X \to V$ that contracts a surface $E \subset X$ to a smooth elliptic curve $C \subset V$, where V is one of the following Fano threefolds:

- smooth hypersurface in $\mathbb{P}(1,1,1,2,3)$ of degree 6;
- smooth hypersurface in $\mathbb{P}(1,1,1,1,2)$ of degree 4.

The curve C is contained in a surface $H \subset V$ such that

$$\mathrm{Pic}\big(V\big) = \mathbb{Z}\big[H\big]$$

and $-K_X \sim 2H$. Then C is a complete intersection of two surfaces in $|H|$, and

$$-K_X \sim 2\bar{H} + E,$$

where E is the exceptional divisor of the birational morphism α, and $\bar{H}$ is a proper transform of the surface H on the threefold X. In particular, the inequality $\mathrm{lct}(X) \leqslant 1/2$ holds.

We suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$. Then

$$\varnothing \neq \mathrm{LCS}\Big(X, \lambda D\Big) \subseteq E,$$

since $\mathrm{lct}(V) = 1/2$ by Theorem 7.1 and $\alpha(D) \sim_{\mathbb{Q}} 2H \sim -K_V$.

Put $k = H \cdot C$. Then $k = H^3 \in \{1,2\}$. Note that

$$\mathcal{N}_{C/V} \cong \mathcal{O}_C\Big(H\big|_C\Big) \oplus \mathcal{O}_C\Big(H\big|_C\Big),$$

which implies that $E \cong C \times \mathbb{P}^1$. Let $Z \cong C$ and $L \cong \mathbb{P}^1$ be curves on E such that

$$Z \cdot Z = L \cdot L = 0$$

and $Z \cdot L = 1$. Then $\alpha^*(H)|_E \sim kL$, and since

$$-2Z \sim K_E \sim \Big(K_X + E\Big)\Big|_E \sim \Big(2E - 2\alpha^*(H)\Big)\Big|_E \sim -2kL + 2E\Big|_E,$$

we see that $E|_E \sim -Z + kL$. Put

$$D = \mu E + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor on X such that $E \not\subset \mathrm{Supp}(\Omega)$.

The pair $(X, E + \lambda\Omega)$ is not log canonical in the neighborhood of E. Hence, the log pair

$$\Big(E,\ \lambda\Omega\Big|_E\Big)$$

is also not log canonical by Theorem 2.20. But

$$\Omega\Big|_E \sim_{\mathbb{Q}} \Big(-K_X - \mu E\Big)\Big|_E \sim_{\mathbb{Q}} \Big(2\alpha^*(H) - \big(1+\mu\big)E\Big)\Big|_E \sim_{\mathbb{Q}} \big(1+\mu\big)Z + k\big(1-\mu\big)L,$$

and $0 \leqslant \lambda k(1-\mu) \leqslant 1$, which contradicts Lemma 2.24. $\square$

Lemma 8.2. Suppose that $\gimel(X) = 2.4$ and X is general. Then $\mathrm{lct}(X) = 3/4$.

Proof. There is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathbb{P}^3 \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^1 \end{tikzcd}$$

where ψ is a rational map, α is a blow up of a smooth curve $C \subset \mathbb{P}^3$ such that

$$C = H_1 \cdot H_2$$

for some $H_1, H_2 \in |\mathcal{O}_{\mathbb{P}^3}(3)|$, and β is a fibration into cubic surfaces.

Let $\mathcal{P}$ be a pencil in $|\mathcal{O}_{\mathbb{P}^3}(3)|$ generated by H_1 and H_2. Then ψ is given by $\mathcal{P}$.

We assume that X satisfies the following generality conditions:

- every surface in $\mathcal{P}$ has at most one ordinary double point;
- the curve C contains no Eckardt points[7] (see Definition 4.1) of any surface in $\mathcal{P}$.

Let E be the exceptional divisor of the birational morphism α. Then

$$\frac{4}{3}\bar{H}_1 + \frac{1}{3}E \sim_{\mathbb{Q}} \frac{4}{3}\bar{H}_2 + \frac{1}{3}E \sim_{\mathbb{Q}} -K_X,$$

where $\bar{H}_i$ is a proper transform of H_i on the threefold X. We see that $\mathrm{lct}(X) \leqslant 3/4$.

We suppose that $\mathrm{lct}(X) < 3/4$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 3/4$.

Suppose that the set $\mathbb{LCS}(X, \lambda D)$ contains a (irreducible) surface $S \subset X$. Then

$$D = \varepsilon S + \Delta,$$

where $\varepsilon \geqslant 1/\lambda$, and Δ is an effective $\mathbb{Q}$-divisor such that $S \not\subset \mathrm{Supp}(\Delta)$. Then

$$\Big(\bar{H}_1,\ D\Big|_{\bar{H}_1}\Big)$$

is not log canonical by Remark 2.3 if $S \cap \bar{H}_1 \neq \varnothing$. But

$$D\Big|_{\bar{H}_1} \sim_{\mathbb{Q}} -K_{\bar{H}_1}.$$

We can choose $\bar{H}_1$ to be a smooth cubic surface in $\mathbb{P}^3$. Thus, it follows from Theorem 4.2 that

$$S \cap \bar{H}_1 = \varnothing,$$

which implies that $S \sim \bar{H}_1$. Thus, we see that $\alpha(S)$ is a cubic surface in $\mathcal{P}$. Then

$$\varepsilon\alpha\big(S\big) + \alpha\big(\Delta\big) \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}^3}\big(4\big),$$

which is impossible, because $\varepsilon \geqslant 1/\lambda > 4/3$.

Let F be a fiber of β such that $F \cap \mathrm{LCS}(X, \lambda D) \neq \varnothing$. Put

$$D = \mu F + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor such that $F \not\subset \mathrm{Supp}(\Omega)$. Then the log pair $(F, \lambda\Omega|_F)$ is not log canonical by Theorem 2.20, because $\lambda\mu < 1$. It follows from Theorem 4.2 that

$$\mathrm{LCS}\Big(F,\ \lambda\Omega\Big|_F\Big) = O,$$

where either O is an Eckardt point of the surface F, or $O = \mathrm{Sing}(F)$. By Theorem 2.7

$$\mathrm{LCS}\Big(X,\ \lambda D\Big) = \mathrm{LCS}\Big(X,\ \lambda\mu F + \lambda\Omega D\Big) = O,$$

because it follows from Theorem 2.20 that $(X, F + \lambda\Omega D)$ is not log canonical at O and is log canonical in a punctured neighborhood of O. But $O \notin E$ by our generality assumptions. Then

$$\alpha\big(O\big) \subset \mathrm{LCS}\Big(\mathbb{P}^3,\ \lambda\alpha\big(D\big)\Big) \subseteq \alpha(O) \cup C,$$

[7] Note that C does not contain singular points of the surfaces in $\mathcal{P}$ since C is a complete intersection of two surfaces from $\mathcal{P}$.

where $\alpha(O) \notin C$. But $\lambda < 3/4$, which contradicts Lemma 2.8. $\square$

Lemma 8.3. Suppose that $\daleth(X) \in \{2.5, 2.10, 2.14\}$ and X is general. Then $\mathrm{lct}(X) = 1/2$.

Proof. There is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ V \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^1 \end{tikzcd}$$

where V is a smooth Fano threefold such that $-K_V \sim 2H$ for some $H \in \mathrm{Pic}(V)$ and

$$\daleth(V) \in \big\{1.13,\ 1.14,\ 1.15\big\},$$

the morphism α is a blow up of a smooth curve $C \subset V$ such that

$$C = H_1 \cdot H_2$$

for some $H_1, H_2 \in |H|$, $H_1 \neq H_2$, the morphism β is a del Pezzo fibration, and ψ is a linear projection.

Let E be the exceptional divisor of the birational morphism α. Then

$$2\bar{H}_1 + E \sim 2\bar{H}_2 + E \sim -K_X,$$

where $\bar{H}_i$ is a proper transform of H_i on the threefold X. We see that $\mathrm{lct}(X) \leqslant 1/2$.

We suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$. Then

$$\varnothing \neq \mathrm{LCS}\Big(X, \lambda D\Big) \subseteq E,$$

because $\alpha(D) \sim_{\mathbb{Q}} -K_V$ and $\mathrm{lct}(V) = 1/2$ by Theorem 7.1.

We assume that the threefold X satisfies the following generality condition: every fiber of the del Pezzo fibration β has at most one singular point that is an ordinary double point.

Let F be a fiber of β such that $F \cap \mathrm{LCS}(X, \lambda D) \neq \varnothing$. Put

$$D = \mu F + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor on X such that $F \not\subset \mathrm{Supp}(\Omega)$. Then

$$\alpha\big(D\big) = \mu\alpha\big(F\big) + \alpha\big(\Omega\big) \sim_{\mathbb{Q}} 2\alpha\big(F\big) \sim_{\mathbb{Q}} -K_V,$$

which implies that $\mu \leqslant 2$. Then $(F, \lambda\Omega|_F)$ is also not log canonical by Theorem 2.20. But

$$\Omega\Big|_F \sim_{\mathbb{Q}} -K_F,$$

which implies that $\mathrm{lct}(F) \leqslant \lambda < 1/2$. But F has at most one ordinary double point and

$$K_F^2 = H^3 \leqslant 5,$$

which implies that $\mathrm{lct}(F) \geqslant 1/2$ (see Examples 1.18, 1.19, 5.3 and 5.4), which is a contradiction. $\square$

Lemma 8.4. Suppose that $\daleth(X) = 2.8$ and X is general. Then $\mathrm{lct}(X) = 1/2$.

Proof. Let $O \in \mathbb{P}^3$ be a point, and let $\alpha\colon V_7 \to \mathbb{P}^3$ be a blow up of the point O. Then

$$V_7 \cong \mathbb{P}\Big(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}\big(1\big)\Big),$$

and there is a $\mathbb{P}^1$-bundle $\pi\colon V_7 \to \mathbb{P}^2$. Let E be the exceptional divisor of α. Then E is a section of π.

There is a quartic surface $R\subset\mathbb{P}^3$ such that $\mathrm{Sing}(R)=O$, the point O is an isolated double point of the surface R, and there is a commutative diagram

$$\begin{tikzcd}
 & & X \arrow[dll, bend right, "\beta"'] \arrow[dl, "\eta"'] \arrow[dd, "\phi"] \\
V_2 \arrow[dr, "\omega"'] & V_7 \arrow[d, "\alpha"'] \arrow[dr, "\pi"] & \\
 & \mathbb{P}^3 \arrow[r, dashed, "\psi"] & \mathbb{P}^2
\end{tikzcd}$$

where ω is a double cover branched in R, the morphism η is a double cover branched in the proper transform of R, the morphism β is a birational morphism that contracts a surface $\bar{E}$ such that $\eta(\bar{E})=E$ to the singular point of V_2 and

$$\omega\Big(\mathrm{Sing}\big(V_2\big)\Big)=O,$$

the map ψ is a projection from the point O, and ϕ is a conic bundle.

We assume that X satisfies the following mild generality condition: the point O is an *ordinary* double point of the surface R. Then $\bar{E}\cong\mathbb{P}^1\times\mathbb{P}^1$.

Let $\bar{H}$ be the proper transform on X of the general plane in $\mathbb{P}^3$ that passes through O. Then

$$-K_X\sim 2\bar{H}+\bar{E},$$

which implies that $\mathrm{lct}(X)\leqslant 1/2$.

We suppose that $\mathrm{lct}(X)<1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the log pair $(X,\lambda D)$ is not log canonical for some positive rational number $\lambda<1/2$.

It follows from Lemma 2.18 that $\mathrm{LCS}(X,D)\cap\bar{E}\neq\varnothing$. Put

$$D=\mu\bar{E}+\Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor on X such that $\bar{E}\not\subset\mathrm{Supp}(\Omega)$. Then

$$2=D\cdot\Gamma=\Big(\mu\bar{E}+\Omega\Big)\cdot\Gamma=2\mu+\Omega\cdot\Gamma\geqslant 2\mu,$$

where Γ is a general fiber of the conic bundle ϕ. Hence $\mu\leqslant 1$. Thus, the log pair

$$\Big(\bar{E},\ \lambda\Omega\Big\vert_{\bar{E}}\Big)$$

is not log canonical by Theorem 2.20, because $\mathrm{LCS}(X,D)\cap\bar{E}\neq\varnothing$. But

$$\Omega\Big\vert_{\bar{E}}\sim_{\mathbb{Q}}-\frac{1+\mu}{2}K_{\bar{E}},$$

which is impossible by Lemma 2.24. $\square$

Lemma 8.5. Suppose that $\daleth(X)=2.11$ and X is general. Then $\mathrm{lct}(X)=1/2$.

Proof. Let V be a cubic hypersurface in $\mathbb{P}^4$. Then there is a commutative diagram

$$\begin{tikzcd}
 & V \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^2
\end{tikzcd}$$

such that α contracts a surface $E\subset X$ to a line $L\subset V$, the map ψ is a linear projection from the line L, the morphism β is a conic bundle.

We assume that X satisfies the following generality condition: the normal bundle $\mathcal{N}_{L/V}$ to the line L on the variety V is isomorphic to $\mathcal{O}_L\oplus\mathcal{O}_L$.

Let H be a hyperplane section of V such that $L\subset H$. Then

$$-K_X\sim 2\bar{H}+E,$$

where $\bar{H}\subset X$ is the proper transform of the surface H. In particular, $\mathrm{lct}(X)\leqslant 1/2$.

We suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$. Then

$$\varnothing \neq \mathrm{LCS}\Big(X, \lambda D\Big) \subseteq E,$$

since $\mathrm{lct}(V) = 1/2$ and $\alpha(D) \sim_{\mathbb{Q}} -K_V$. Note that $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ by the generality condition.

Let $F \subset E$ be a fiber of the induced projection $E \to L$, let $Z \subset E$ be a section of this projection such that $Z \cdot Z = 0$. Then $\alpha^*(H)|_E \sim F$ and $E|_E \sim -Z$, because

$$-2Z - 2F \sim K_E \sim \Big(K_X + E\Big)\Big|_E \sim 2\Big(E - \alpha^*(H)\Big)\Big|_E \sim -2F + 2E\Big|_E.$$

Put $D = \mu E + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor on X such that $E \not\subset \mathrm{Supp}(\Omega)$. Then

$$2 = D \cdot \Gamma = \mu E \cdot \Gamma + \Omega \cdot \Gamma \geqslant \mu E \cdot \Gamma = 2\mu,$$

where Γ is a general fiber of the conic bundle β. Thus, we see that $\mu \leqslant 1$.

The log pair $(E, \lambda\Omega|_E)$ is not log canonical by Theorem 2.20. But

$$\Omega\Big|_E \sim_{\mathbb{Q}} \Big(-K_X - \mu E\Big)\Big|_E \sim_{\mathbb{Q}} \big(1 + \mu\big)Z + 2F,$$

which contradicts Lemma 2.24, because $\mu \leqslant 1$ and $\lambda < 1/2$. □

Lemma 8.6. Suppose that $\gimel(X) = 2.15$ and X is general. Then $\mathrm{lct}(X) = 1/2$.

Proof. There is a birational morphism $\alpha\colon X \to \mathbb{P}^3$ that contracts a surface $E \subset X$ to a smooth curve $C \subset \mathbb{P}^3$ that is complete intersection of an irreducible quadric $Q \subset \mathbb{P}^3$ and a cubic $F \subset \mathbb{P}^3$.

We assume that X satisfies the following generality condition: the quadric Q is smooth.

Let $\bar{Q}$ be a proper transform of Q on the threefold X. Then there is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathbb{P}^3 & & V \arrow[ll, dashed, "\gamma"] \end{tikzcd}$$

where V is a cubic in $\mathbb{P}^4$ that has one ordinary double point $P \in V$, the morphism β contracts the surface $\bar{Q}$ to the point P, and γ is a linear projection from the point P.

Let E be the exceptional divisor of α. Then

$$-K_X \sim 2\bar{Q} + E$$

and $\beta(E) \subset V$ is a surface that contains all lines on V that pass through P. We see that $\mathrm{lct}(X) \leqslant 1/2$.

We suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$.

It follows from Lemma 2.17 that either

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq \bar{Q},$$

or the set $\mathrm{LCS}(X, \lambda D)$ contains a fiber of the natural projection $E \to C$. In both cases

$$\mathrm{LCS}\Big(X,\ \lambda D\Big) \cap \bar{Q} \neq \varnothing.$$

We have $\bar{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Put

$$D = \mu\bar{Q} + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor on X such that $\bar{Q} \not\subset \mathrm{Supp}(\Omega)$. Then

$$\alpha\big(D\big) \sim_{\mathbb{Q}} \mu Q + \alpha\big(\Omega\big) \sim_{\mathbb{Q}} -K_{\mathbb{P}^3},$$

which gives $\mu \leqslant 2$. The log pair $(\bar{Q}, \lambda\Omega|_{\bar{Q}})$ is not log canonical by Theorem 2.20. But

$$\Omega\Big|_{\bar{Q}} \sim_{\mathbb{Q}} -\frac{1+\mu}{2}K_{\bar{Q}},$$

which implies that $\mu > 1$ by Lemma 2.24.

It follows from Remark 2.23 that we may assume that $E \not\subset \mathrm{Supp}(D)$. Then

$$1 = D \cdot F = \mu\bar{Q} \cdot F + \Omega \cdot F = \mu + \Omega \cdot F \geqslant \mu,$$

where F is a general fiber the natural projection $E \to C$. But $\mu > 1$, which is a contradiction. $\square$

Lemma 8.7. Suppose that $\gimel(X) = 2.18$. Then $\mathrm{lct}(X) = 1/2$.

Proof. There is a smooth divisor $B \subset \mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(2, 2)$ such that the diagram

$$\begin{tikzcd}
& X \arrow[dl, "\phi_1"'] \arrow[d, "\pi"] \arrow[dr, "\phi_2"] & \\
\mathbb{P}^1 & \mathbb{P}^1 \times \mathbb{P}^2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & \mathbb{P}^2
\end{tikzcd}$$

commutes, where π is a double cover that is branched in B, the morphisms π_1 and π_2 are natural projections, the morphism ϕ_1 is a quadric fibration, and ϕ_2 is a conic bundle.

Let H_1 be a general fiber of π_1, and let H_2 be a general surface in $|\pi_2^*(\mathcal{O}_{\mathbb{P}^2}(1))|$. Then $B \sim 2H_1 + 2H_2$.

Let $\bar{H}_1$ be a general fiber of ϕ_1, and let $\bar{H}_2$ be a general surface in $|\phi_2^*(\mathcal{O}_{\mathbb{P}^2}(1))|$. Then

$$-K_X \sim \bar{H}_1 + 2\bar{H}_2,$$

which implies that $\mathrm{lct}(X) \leqslant 1/2$.

We suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$.

Applying Lemma 2.26 to the fibration ϕ_1 we see that

$$\varnothing \ne \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq Q,$$

where Q is a singular fiber of ϕ_1. Moreover, applying Theorem 2.28 to the fibration ϕ_2, we see that

$$\varnothing \ne \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq Q \cap R,$$

where $R \subset X$ be an irreducible surface that is swept out by singular fibers of ϕ_2. In particular, the set $\mathbb{LCS}(X, \lambda D)$ contains no surfaces.

Suppose that $\mathrm{LCS}(X, \lambda D)$ is zero-dimensional. Then

$$\mathrm{LCS}\left(X,\ \lambda D + \frac{1}{2}\Big(\bar{H}_1 + 2\bar{H}_2\Big)\right) = \mathrm{LCS}(X,\ \lambda D) \cup \bar{H}_2,$$

which is impossible by Theorem 2.7.

We see that the set $\mathbb{LCS}(X, \lambda D)$ contains a curve $\Gamma \subset Q \cap R$. Put

$$D = \mu Q + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor such that $Q \not\subset \mathrm{Supp}(\Omega)$. Then

$$\Big(Q,\ \lambda\Omega\Big|_{Q}\Big)$$

is not log canonical along Γ by Theorem 2.20. But

$$\Omega\Big|_{Q} \sim_{\mathbb{Q}} \Big(-K_X - \mu Q\Big)\Big|_{Q} \sim_{\mathbb{Q}} -K_Q,$$

which implies that Γ is a ruling of the cone $Q \subset \mathbb{P}^3$ by Lemma 5.7. Then $\phi_2(\Gamma) \subset \mathbb{P}^2$ is a line, and

$$\phi_2\big(\Gamma\big) \subseteq \phi_2\big(R\big).$$

But $\phi_2(R) \subset \mathbb{P}^2$ is a curve of degree 4. Thus, we see that

$$\phi_2\big(R\big) = \phi_2\big(\Gamma\big) \cup Z,$$

where $Z \subset \mathbb{P}^2$ is a reduced cubic curve. Then ϕ_2 induces a double cover of

$$\phi_2\big(\Gamma\big) \setminus \Big(\phi_2\big(\Gamma\big) \cap Z\Big)$$

that must be unramified (see [192]). But the quartic curve $\phi_2(R)$ has at most ordinary double points (see [192], [166]). Then

$$\big|\phi_2(\Gamma) \cap Z\big| = 3,$$

which is impossible, because $\phi_2(\Gamma) \cong \mathbb{P}^1$. $\square$

Lemma 8.8. Suppose that $\daleth(X) = 2.19$ and X is general. Then $\mathrm{lct}(X) = 1/2$.

Proof. It follows from [98, Proposition 3.4.1] that there is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ V \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^3 \end{tikzcd}$$

where V is a complete intersection of two quadric fourfolds in $\mathbb{P}^5$, the morphism α is a blow up of a line $L \subset V$, the morphism β is a blow up of a smooth curve $C \subset \mathbb{P}^3$ of degree 5 and genus 2, and the map ψ is a linear projection from the line L.

Let E and R be the exceptional divisors of α and β, respectively. Then

- the surface $\beta(E) \subset \mathbb{P}^3$ is an irreducible quadric,
- the surface $\alpha(R) \subset V$ is swept out by lines that intersect the line L.

We assume that X satisfies the following generality condition: the surface $\beta(E)$ is smooth.

Let H be any hyperplane section of $V \subset \mathbb{P}^5$ such that $L \subset H$. Then

$$2\bar{H} + E \sim R + 2E \sim -K_X,$$

where $\bar{H}$ is a proper transform of H on the threefold X. We see that $\mathrm{lct}(X) \leqslant 1/2$.

We suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$. Then

$$\varnothing \neq \mathrm{LCS}\Big(X, \lambda D\Big) \subseteq E \cong \mathbb{P}^1 \times \mathbb{P}^1,$$

because $\alpha(D) \sim_{\mathbb{Q}} -K_V$ and $\mathrm{lct}(V) = 1/2$ by Theorem 7.1.

Let F be a fiber of the projection $E \to L$, and let Z be a section of this projection such that $Z \cdot Z = 0$. Then $\alpha^*(H)|_E \sim F$ and $E|_E \sim -Z$, because

$$-2Z - 2F \sim K_E \sim \Big(K_X + E\Big)\Big|_E \sim 2\Big(E - \alpha^*(H)\Big)\Big|_E \sim 2E\Big|_E - 2F.$$

By Remark 2.23, we may assume that either $E \not\subset \mathrm{Supp}(D)$, or $R \not\subset \mathrm{Supp}(D)$, because the log pair

$$\Big(X, \lambda\big(R + 2E\big)\Big)$$

is log canonical and $-K_X \sim R + 2E$. Put

$$D = \mu E + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor on X such that $E \not\subset \mathrm{Supp}(\Omega)$.

Suppose that $\mu \leqslant 1$. Then $(X, E + \lambda\Omega)$ is not log canonical, which implies that

$$\Big(E,\ \lambda\Omega\Big|_E\Big)$$

is also not log canonical by Theorem 2.20. But

$$\Omega\Big|_E \sim_{\mathbb{Q}} \Big(-K_X - \mu E\Big)\Big|_E \sim_{\mathbb{Q}} \big(1+\mu\big)Z + 2F,$$

which contradicts Lemma 2.24, because $\mu \leqslant 1$ and $\lambda < 1/2$.

Thus, we see that $\mu > 1$. Then we may assume that $R \not\subset \mathrm{Supp}(D)$.

Let Γ be a general fiber of the projection $R \to C$. Then $\Gamma \not\subset \mathrm{Supp}(D)$ and

$$1 = -K_X \cdot \Gamma = \mu E \cdot \Gamma + \Omega \cdot \Gamma = \mu + \Omega \cdot \Gamma \geqslant \mu,$$

which is a contradiction. $\square$

Lemma 8.9. Suppose that $\daleth(X) = 2.23$ and X is general. Then $\mathrm{lct}(X) = 1/3$.

Proof. There is a birational morphism $\alpha\colon X\to Q$ such that $Q\subset\mathbb{P}^4$ is a smooth quadric threefold, and α contracts a surface $E\subset X$ to a smooth curve $C\subset Q$ that is a complete intersection of a hyperplane section $H\subset Q$ and a divisor $F\in|\mathcal{O}_Q(2)|$.

We assume that X satisfies the following generality condition: the quadric surface H is smooth.

Let $\bar{H}$ be a proper transform of H on the threefold X. Then there is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ Q & & V \arrow[ll, dashed, "\gamma"] \end{tikzcd}$$

where V is a complete intersection of two quadrics in $\mathbb{P}^5$ such that V has one ordinary double point $P\in V$, the morphism β contracts $\bar{H}$ to the point P, and γ is a projection from P.

Let E be the exceptional divisor of α. Then

$$-K_X\sim 3\bar{H}+2E$$

and $\beta(E)\subset V$ is a surface that contains all lines that pass through P. In particular, $\mathrm{lct}(X)\leqslant 1/3$.

We suppose that $\mathrm{lct}(X)<1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the log pair $(X,\lambda D)$ is not log canonical for some positive rational number $\lambda<1/3$.

It follows from Remark 7.8 that either

$$\varnothing\neq\mathrm{LCS}\Big(X,\ \lambda D\Big)\subseteq\bar{H},$$

or the set $\mathrm{LCS}(X,\lambda D)$ contains a fiber of the natural projection $E\to C$. In both cases

$$\mathrm{LCS}\Big(X,\ \lambda D\Big)\cap\bar{H}\neq\varnothing.$$

We have $\bar{H}\cong\mathbb{P}^1\times\mathbb{P}^1$. There is a non-negative rational number μ such that

$$D=\mu\bar{H}+\Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor on X such that $\bar{H}\not\subset\mathrm{Supp}(\Omega)$. Then

$$\alpha\big(D\big)\sim_{\mathbb{Q}}\mu H+\alpha\big(\Omega\big)\sim_{\mathbb{Q}}-K_Q,$$

which gives $\mu\leqslant 3$. The log pair $(\bar{H},\lambda\Omega|_{\bar{H}})$ is not log canonical by Theorem 2.20. But

$$\Omega\Big|_{\bar{H}}\sim_{\mathbb{Q}}-\frac{1+\mu}{2}K_{\bar{H}},$$

which implies that $\mu>1$ by Lemma 2.24.

It follows from Remark 2.23 that we may assume that $E\not\subset\mathrm{Supp}(D)$, because the log pair

$$\Big(X,\lambda\big(3\bar{H}+2E\big)\Big)$$

is log canonical. Let F be a general fiber the natural projection $E\to C$. Then

$$1=D\cdot F=\mu\bar{H}\cdot F+\Omega\cdot F=\mu+\Omega\cdot F\geqslant\mu,$$

which is a contradiction, because $\mu>1$. □

Lemma 8.10. Suppose that $\daleth(X)=2.24$ and X is general. Then $\mathrm{lct}(X)=1/2$.

Proof. The threefold X is a divisor on $\mathbb{P}^2\times\mathbb{P}^2$ of bidegree $(1,2)$. Let H_i be a surface in

$$\Big|\pi_i^*\Big(\mathcal{O}_{\mathbb{P}^2}(1)\Big)\Big|,$$

where $\pi_i\colon X\to\mathbb{P}^2$ is a projection of X onto the i-th factor of $\mathbb{P}^2\times\mathbb{P}^2$, $i\in\{1,2\}$. Then

$$-K_X\sim 2H_1+H_2,$$

which implies that $\mathrm{lct}(X)\leqslant 1/2$. Note that π_1 is a conic bundle, and π_2 is a $\mathbb{P}^1$-bundle.

Let $\Delta\subset\mathbb{P}^2$ be the degeneration curve of the conic bundle π_1. Then Δ is a cubic curve.

We suppose that X satisfies the following generality condition: the curve Δ is irreducible.

Assume that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$.

Suppose that the set $\mathbb{LCS}(X, \lambda D)$ contains a surface $S \subset X$. Then

$$D = \mu S + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor such that $S \not\subset \mathrm{Supp}(\Omega)$, and $\mu > 1/\lambda$. Let F_i be a general fiber of π_i, $i \in \{1, 2\}$. Then

$$2 = D \cdot F_i = \mu S \cdot F_i + \Omega \cdot F_i \geqslant \mu S \cdot F_i,$$

but either $S \cdot F_1 \geqslant 1$ or $S \cdot F_2 \geqslant 1$. Thus, we see that $\mu \leqslant 2$, which is a contradiction.

By Theorem 2.28 and Theorem 2.7, there is a fiber Γ_2 of the $\mathbb{P}^1$-bundle π_2 such that

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) = \Gamma_2,$$

because the set $\mathbb{LCS}(X, \lambda D)$ contains no surfaces.

Applying Theorem 2.28 to the conic bundle π_1, we see that

$$\pi_1\big(\Gamma_2\big) \subset \Delta,$$

which is impossible, because $\Delta \subset \mathbb{P}^2$ is an irreducible cubic curve and $\pi_1(\Gamma_2) \subset \mathbb{P}^2$ is a line. $\square$

Lemma 8.11. Suppose that $\daleth(X) = 2.25$. Then $\mathrm{lct}(X) = 1/2$.

Proof. Recall that X is a blow up $\alpha\colon X \to \mathbb{P}^3$ along a normal elliptic curve C of degree 4.

Let $Q \subset \mathbb{P}^3$ be a general quadric containing C, and let $\bar{Q} \subset X$ be a proper transform of Q. Then

$$-K_X \sim 2\bar{Q} + E,$$

where E is the exceptional divisor of α. In particular, $\mathrm{lct}(X) \leqslant 1/2$.

We suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$.

Note that the linear system $|\bar{Q}|$ defines a quadric fibration

$$\phi\colon X \longrightarrow \mathbb{P}^1,$$

such that every fiber of ϕ is irreducible. Then the log pair $(X, \lambda D)$ is log canonical along every nonsingular fiber $\tilde{Q}$ of the fibration ϕ by Theorem 2.28 since $\mathrm{lct}(\tilde{Q}) = 1/2$ (see Example 1.18).

The locus $\mathrm{LCS}(X, \lambda D)$ does not contain any fiber of ϕ, because $\alpha(D) \sim_{\mathbb{Q}} 2Q$ and every fiber of ϕ is irreducible. Therefore, we see that $\dim(\mathrm{LCS}(X, \lambda D)) \leqslant 1$.

Let Z be an element in $\mathbb{LCS}(X, \lambda D)$. There is a singular fiber $\bar{Q}_1$ of the fibration ϕ such that $Z \subset \bar{Q}_1$. Note that ϕ has 4 singular fibers and each of them is a proper transform of a quadric cone in $\mathbb{P}^3$ with vertex outside C.

Let $\bar{Q}_2$ be a singular fiber of ϕ such that $\bar{Q}_1 \neq \bar{Q}_2$, let $\bar{H}$ be a proper transform of a general plane in $\mathbb{P}^3$ that is tangent to the cone $\alpha(\bar{Q}_2) \subset \mathbb{P}^3$ along one of its rulings $L \subset \alpha(\bar{Q}_2)$, and let $\bar{R}$ be a proper transform of a very general plane in $\mathbb{P}^3$. Put

$$\Delta = \lambda D + \frac{1}{2}\Big(\big(1+\varepsilon\big)\bar{Q}_2 + \big(2-\varepsilon\big)\bar{H} + 3\varepsilon R\Big)$$

for some positive rational number $\varepsilon < 1 - 2\lambda$. Then

$$\Delta \sim_{\mathbb{Q}} -\Big(\lambda + \frac{1}{2}\big(1+\varepsilon\big)\Big)K_X \sim_{\mathbb{Q}} -\frac{1+\varepsilon+2\lambda}{2}K_X,$$

which implies that $-(K_X + \Delta)$ is ample.

Let $\bar{L}$ be a proper transform on X of the line L. Then

$$Z \cup \bar{L} \subset \mathrm{LCS}\big(X, \Delta\big) \subset \bar{Q}_1 \cup \bar{Q}_2,$$

which is impossible by Theorem 2.7, because $-(K_X + \Delta)$ is ample. $\square$

Lemma 8.12. Suppose that $\daleth(X) = 2.26$ and X is general. Then $\mathrm{lct}(X) = 1/2$.

Proof. Let V be a smooth Fano threefold such that $-K_V \sim 2H$ and

$$\mathrm{Pic}\big(V\big) = \mathbb{Z}\big[H\big],$$

where H is a Cartier divisor such that $H^3 = 5$ (i. e. $\gimel(V) = 1.15$). Then $|H|$ induces an embedding $X \subset \mathbb{P}^6$.

It follows from [98, Proposition 3.4.1] (see also [67]) that there is a line

$$L \subset X \subset \mathbb{P}^6$$

such that there is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ V \arrow[rr, dashed, "\psi"'] & & Q \end{tikzcd}$$

where Q is a smooth quadric hypersurface in $\mathbb{P}^4$, the morphism α is a blow up of the line $L \subset V$, the morphism β is a blow up of a twisted cubic curve $\mathbb{P}^1 \cong C \subset Q$, and ψ is a projection from the line L.

Let S be the exceptional divisor of the morphism β. Put $\bar{S} = \alpha(S)$. Then $\bar{S} \sim H$, and $\bar{S}$ is singular along the line L. Let E be the exceptional divisor of the morphism α. Then

$$\beta\big(E\big) \sim \mathcal{O}_{\mathbb{P}^4}\big(1\big)\Big|_{Q},$$

which implies that $\beta(E)$ is an irreducible quadric surface.

We suppose that X satisfies the following generality condition: the surface $\beta(E)$ is smooth.

The equivalence $-K_X \sim 2S + 3E$ holds. Moreover, the log pair

$$\Bigg(X,\ \frac{1}{3}\Big(2S + 3E\Big)\Bigg)$$

is log canonical but not log terminal. Thus, we see that $\mathrm{lct}(X) \leqslant 1/3$.

We suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/3$. Then

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq E,$$

because $\alpha(D) \sim_{\mathbb{Q}} -K_V$ and $\mathrm{lct}(V) = 1/2$ by Theorem 7.1.

Note that $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ by our generality condition. Let F be a fiber of the projection $E \to L$, and let Z be a section of this projection such that $Z \cdot Z = 0$. Then $\alpha^*(H)|_E \sim F$ and $E|_E \sim -Z$, because

$$-2Z - 2F \sim K_E \sim \Big(K_X + E\Big)\Big|_E \sim 2\Big(E - \alpha^*\big(H\big)\Big)\Big|_E \sim 2E\Big|_E - 2F.$$

By Remark 2.23, we may assume that either $E \not\subset \mathrm{Supp}(D)$, or $S \not\subset \mathrm{Supp}(D)$. Put

$$D = \mu E + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor on X such that $E \not\subset \mathrm{Supp}(\Omega)$.

Suppose that $\mu \leqslant 2$. Then $(X, E + \lambda\Omega)$ is not log canonical, which implies that

$$\Big(E,\ \lambda\Omega\Big|_E\Big)$$

is also not log canonical by Theorem 2.20. But

$$\Omega\Big|_E \sim_{\mathbb{Q}} \Big(-K_X - \mu E\Big)\Big|_E \sim_{\mathbb{Q}} \big(1 + \mu\big)Z + 2F,$$

which contradicts Lemma 2.24, because $\mu \leqslant 2$ and $\lambda < 1/3$.

Thus, we see that $\mu > 2$. Then we may assume that $S \not\subset \mathrm{Supp}(D)$.

Let Γ be a general fiber of the projection $S \to C$. Then $\Gamma \not\subset \mathrm{Supp}(D)$ and

$$1 = -K_X \cdot \Gamma = \mu E \cdot \Gamma + \Omega \cdot \Gamma = \mu + \Omega \cdot \Gamma \geqslant \mu,$$

which is a contradiction. $\square$

Lemma 8.13. Suppose that $\gimel(X) = 2.27$. Then $\mathrm{lct}(X) = 1/2$.

Proof. There is a morphism $\alpha\colon X\to\mathbb{P}^3$ contracting a surface E to a twisted cubic curve $C\subset\mathbb{P}^3$, and $X\cong\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a stable rank two vector bundle on $\mathbb{P}^2$ with $c_1(\mathcal{E})=0$ and $c_1(\mathcal{E})=2$ such that the sequence

$$0\longrightarrow\mathcal{O}_{\mathbb{P}_2}\big(-1\big)\oplus\mathcal{O}_{\mathbb{P}_2}\big(-1\big)\longrightarrow\mathcal{O}_{\mathbb{P}_2}\oplus\mathcal{O}_{\mathbb{P}_2}\oplus\mathcal{O}_{\mathbb{P}_2}\oplus\mathcal{O}_{\mathbb{P}_2}\longrightarrow\mathcal{E}\otimes\mathcal{O}_{\mathbb{P}_2}\big(1\big)\longrightarrow 0$$

is exact (see [50] and [177, Application 1]).

Let $Q\subset\mathbb{P}^3$ be a general quadric containing C, and let $\bar{Q}\subset X$ be a proper transform of Q. Then

$$-K_X\sim 2\bar{Q}+E,$$

where E is the exceptional divisor of α. In particular, we see that $\mathrm{lct}(X)\leqslant 1/2$.

We suppose that $\mathrm{lct}(X)<1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the log pair $(X,\lambda D)$ is not log canonical for some positive rational number $\lambda<1/2$.

Suppose that the set $\mathbb{LCS}(X,\lambda D)$ contains a surface $S\subset X$. Put

$$D=\mu F+\Omega,$$

where $\mu\geqslant 1/\lambda$ and Ω is an effective $\mathbb{Q}$-divisor such that $F\not\subset\mathrm{Supp}(\Omega)$.

Let $\phi\colon X\to\mathbb{P}^2$ be the natural $\mathbb{P}^1$-bundle. Then

$$2=D\cdot\Gamma=\mu F\cdot\Gamma+\Omega\cdot\Gamma=\mu F\cdot\Gamma+\Omega\cdot F\geqslant\mu F\cdot\Gamma,$$

where Γ is a general fiber of ϕ. Thus, we see that F is swept out by the fibers of ϕ. Then

$$\alpha\big(F\big)\sim\mathcal{O}_{\mathbb{P}^3}\big(d\big)$$

and $d\geqslant 2$. But $\alpha(D)\sim_{\mathbb{Q}}\mu\alpha(F)+\alpha(\Omega)\sim_{\mathbb{Q}}\mathcal{O}_{\mathbb{P}^3}(4)$, which is a contradiction.

We wee that the locus $\mathrm{LCS}(X,\lambda D)$ contains no surfaces. Applying Theorem 2.28 to $(X,\lambda D)$ and ϕ, we see that

$$L\subseteq\mathrm{LCS}\Big(X,\ \lambda D\Big),$$

where L is a fiber of ϕ. Note that $\alpha(L)$ is a secant line of the curve $C\subset\mathbb{P}^3$. One has

$$\alpha\big(L\big)\subseteq\mathrm{LCS}\Big(\mathbb{P}^3,\lambda\alpha\big(D\big)\Big)\subseteq\alpha\Big(\mathrm{LCS}\big(X,\ \lambda D\big)\Big)\cup C,$$

which is impossible by Lemma 2.9. □

Lemma 8.14. Suppose that $\daleth(X)=2.28$. Then $\mathrm{lct}(X)=1/4$.

Proof. There is a blow up $\alpha\colon X\to\mathbb{P}^3$ along a plane cubic curve $C\subset\mathbb{P}^3$. One has

$$-K_X\sim 4G+3E,$$

where E is the exceptional divisor of α and G is a proper transform of the plane in $\mathbb{P}^3$ that contains the curve C. In particular, we see that the inequality $\mathrm{lct}(X)\leqslant 1/4$ holds.

We suppose that $\mathrm{lct}(X)<1/4$. Then there exists an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the log pair $(X,\lambda D)$ is not log canonical for some rational number $\lambda<1/4$. One has

$$\varnothing\neq\mathrm{LCS}\Big(X,\ \lambda D\Big)\subseteq E,$$

since $\mathrm{lct}(\mathbb{P}^4)=1/4$. Computing the intersections with a strict transform of a general line in $\mathbb{P}^3$ intersecting the curve C, one obtains that $\mathrm{LCS}(X,\lambda D)$ does not contain the divisor E. Moreover, any curve $\Gamma\in\mathbb{LCS}(X,\lambda D)$ must be a fiber of the natural projection

$$\psi\colon E\longrightarrow C$$

by Lemma 2.14. Therefore, we see that either the locus $\mathrm{LCS}(X,\lambda D)$ consists of a single point, or the locus $\mathrm{LCS}(X,\lambda D)$ consists of a single fiber of the projection ψ by Theorem 2.7.

Let R be a sufficiently general cone in $\mathbb{P}^3$ over the curve C, and let H be a sufficiently general plane in $\mathbb{P}^3$ that passes through the point $\mathrm{Sing}\,(R)$. Then

$$\mathrm{LCS}\left(X,\ \lambda D+\frac{3}{4}\Big(\bar{R}+\bar{H}\Big)\right)=\mathrm{LCS}\Big(X,\ \lambda D\Big)\bigcup\mathrm{Sing}\,\big(\bar{R}\big),$$

where $\bar{R}$ and $\bar{H}$ are proper transforms of R and H on the threefold X. But the divisor

$$-\Big(K_X+\lambda D+\frac{3}{4}\Big(\bar{R}+\bar{H}\Big)\Big)\sim_{\mathbb{Q}}(\lambda-1/4)K_X$$

is ample, which contradicts Theorem 2.7. $\square$

Lemma 8.15. Suppose that $\gimel(X)=2.29$. Then $\mathrm{lct}(X)=1/3$.

Proof. There is a birational morphism $\alpha\colon X\to Q$ such that Q is a smooth quadric hypersurface, and α is a blow up along a smooth conic $C\subset Q$.

Let H be a general hyperplane section of $Q\subset\mathbb{P}^4$ that contains C, and let $\bar{H}$ be a proper transform of the surface H on the threefold X. Then

$$-K_X\sim 3\bar{H}+2E,$$

where E is the exceptional divisor of α. In particular, the inequality $\mathrm{lct}(X)\leqslant 1/3$ holds.

We suppose that $\mathrm{lct}(X)<1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the log pair $(X,\lambda D)$ is not log canonical for some rational $\lambda<1/3$. Then

$$\varnothing\neq\mathrm{LCS}\Big(X,\lambda D\Big)\subseteq E,$$

since $\mathrm{lct}(Q)=1/3$ (see Example 1.9) and $\alpha(D)\sim_{\mathbb{Q}}-K_Q$.

The linear system $|\bar{H}|$ has no base points and defines a morphism $\beta\colon X\to\mathbb{P}^1$, whose general fiber is a smooth quadric surface. Then the log pair $(X,\lambda D)$ is log canonical along the smooth fibers of β by Theorem 2.28 (see Example 1.18).

It follows from Theorem 2.7 that there is a singular fiber $S\sim\bar{H}$ of the morphism β such that

$$\varnothing\neq\mathrm{LCS}\Big(X,\lambda D\Big)\subseteq E\cap S,$$

and $\alpha(S)\subset\mathbb{P}^3$ is a quadratic cone. Put $\Gamma=E\cap S$. Then Γ is an irreducible conic, the log pair

$$\Big(X,\ S+\frac{2}{3}E\Big)$$

has log canonical singularities, and $3S+2E\sim_{\mathbb{Q}}D$. Therefore, it follows from Remark 2.23 that to complete the proof we may assume that either $S\not\subseteq\mathrm{Supp}(D)$ or $E\not\subseteq\mathrm{Supp}(D)$.

Intersecting the divisor D with a strict transform of a general ruling of the cone $\alpha(S)\subset\mathbb{P}^3$ and with a general fiber of the projection $E\to C$, we see that

$$\Gamma\not\subseteq\mathrm{LCS}\Big(X,\lambda D\Big),$$

which implies that $\mathrm{LCS}(X,\lambda D)$ consists of a single point $O\in\Gamma$ by Theorem 2.7.

Let R be a general (not passing through O) surface in $|\alpha^*(H)|$. Then

$$\mathrm{LCS}\left(X,\ \lambda D+\frac{1}{2}\Big(\bar{H}+2R\Big)\right)=R\cup O,$$

which is impossible by Theorem 2.7, since $-K_X\sim\bar{H}+2R\sim_{\mathbb{Q}}D$ and $\lambda<1/3$. $\square$

Lemma 8.16. Suppose that $\gimel(X)=2.30$. Then $\mathrm{lct}(X)=1/4$.

Proof. There is a commutative diagram

$$\begin{tikzcd}
 & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
\mathbb{P}^3 & & Q \arrow[ll, dashed, "\gamma"]
\end{tikzcd}$$

where Q is a smooth quadric threefold in $\mathbb{P}^4$, the morphism α is a blow up of a smooth conic $C\subset\mathbb{P}^3$, the morphism β is a blow up of a point, and γ is a projection from a point.

Let G be a proper transform on the variety X of the unique plane in $\mathbb{P}^3$ that contains the conic C. Then the surface G is contracted by the morphism β, and

$$-K_X\sim 4G+3E,$$

where E is the exceptional divisor of the blow up α. Thus, we see that $\mathrm{lct}(X)\leqslant 1/4$.

We suppose that $\mathrm{lct}(X) < 1/4$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some rational $\lambda < 1/4$. Then

$$\varnothing \neq \mathrm{LCS}\Big(X, \lambda D\Big) \subseteq E \cap G,$$

because $\mathrm{lct}(\mathbb{P}^4) = 1/4$ and $\mathrm{lct}(Q) = 1/3$.

We may assume that either $G \not\subset \mathrm{Supp}(X)$ or $E \not\subset \mathrm{Supp}(X)$ by Remark 2.23.

Intersecting D with lines in $G \cong \mathbb{P}^2$ and with fibers of the projection $E \to C$, we see that

$$\mathrm{LCS}\Big(X, \lambda D\Big) \subsetneq E \cap G$$

which implies that there is a point $O \in E \cap G$ such that $\mathrm{LCS}(X, \lambda D) = O$ by Theorem 2.7.

Let R be a general surface in $|\alpha^*(H)|$ and F a general surface in $|\alpha^*(2H) - E|$. Then

$$\mathrm{LCS}\left(X,\ \lambda D + \frac{1}{2}\Big(F + 2R\Big)\right) = R \cup O,$$

which is impossible by Theorem 2.7 since $-K_X \sim F + 2R \sim_{\mathbb{Q}} D$ and $\lambda < 1/4$. $\square$

Lemma 8.17. Suppose that $\daleth(X) = 2.31$. Then $\mathrm{lct}(X) = 1/3$.

Proof. There is a birational morphism $\alpha\colon X \to Q$ such that Q is a smooth quadric hypersurface, and α is a blow up of the quadric Q along a line $L \subset Q$.

Let H be a sufficiently general hyperplane section of the quadric $Q \subset \mathbb{P}^4$ that passes through the line L, and let $\bar{H}$ be a proper transform of the surface H on the threefold X. Then

$$-K_X \sim 3\bar{H} + 2E,$$

where E is the exceptional divisor of α. In particular, $\mathrm{lct}(X) \leqslant 1/3$.

We suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some rational $\lambda < 1/3$. Then

$$\varnothing \neq \mathrm{LCS}\Big(X, \lambda D\Big) \subseteq E,$$

since $\mathrm{lct}(Q) = 1/3$ and $\alpha(D) \sim_{\mathbb{Q}} -K_Q$.

The linear system $|\bar{H}|$ defines a $\mathbb{P}^1$-bundle $\phi\colon X \to \mathbb{P}^2$ such that the induced morphism $E \cong \mathbb{F}_1 \to \mathbb{P}^2$ contracts an irreducible curve $Z \subset E$. One has

$$\mathrm{LCS}\Big(X, \lambda D\Big) = Z \subset E$$

by Theorem 2.28. Put

$$D = \mu E + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor on X such that $E \not\subset \mathrm{Supp}(\Omega)$. Then

$$2 = D \cdot F = \mu E \cdot F + \Omega \cdot F = \mu + \Omega \cdot F \geqslant \mu,$$

where F is a general fiber of ϕ. Note that the log pair

$$\Big(X,\ E + \lambda\Omega\Big)$$

is not log canonical, because $\lambda < 1/3$. Then $(E, \lambda\Omega|_E)$ is not log canonical by Theorem 2.20.

Let C be a fiber of the natural projection $E \to L$. Then

$$\Omega\Big|_E \sim_{\mathbb{Q}} 3C + \big(1 + \mu\big)Z,$$

which implies that $(E, \lambda\Omega|_E)$ is log canonical by Lemma 5.8, which is a contradiction. $\square$

We use the assumptions and notation introduced in section 1.

Lemma 9.1. Suppose that $\daleth(X) = 3.1$ and X is general. Then $\mathrm{lct}(X) = 3/4$.

Proof. There is a double cover
$$\omega\colon X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$
branched over a divisor of tridegree $(2,2,2)$. The projection
$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$
onto the i-th factor induces a morphism $\pi_i\colon X \to \mathbb{P}^1$, whose fibers are del Pezzo surfaces of degree 4.

Let R_1 be a singular fiber of the fibration π_1, let Q be a singular point of the surface R_1, and let R_2 and R_3 be fibers of π_2 and π_3 such that
$$R_2 \ni Q \in R_3,$$
respectively. Then $\mathrm{mult}_Q(R_1 + R_2 + R_3) = 4$, which implies that the log pair
$$\Big(X,\ \frac{3}{4}\Big(R_1 + R_2 + R_3\Big)\Big)$$
is not log terminal at Q. But $-K_X \sim R_1 + R_2 + R_3$. Thus, we see that $\mathrm{lct}(X) \leqslant 3/4$.

We suppose that the threefold X satisfies the following generality condition: for an arbitrary point $O \in X$, there is $k \in \{1,2,3\}$ such that

- the fiber F_k of the fibration π_k that contains O is smooth at the point O,
- the singularities of the surface F_k consist of at most one ordinary double point,
- for every smooth curve $\Gamma \subset F_k$ such that $-K_{F_k} \cdot \Gamma = 1$, we have $O \notin \Gamma$,
- for every smooth curves $\Delta_1 \subset S_k \supset \Delta_2$ such that
$$-K_{F_k} \cdot \Delta_1 = -K_{F_k} \cdot \Delta_2 = 2$$
and $\Delta_1 + \Delta_2 \sim -K_{F_k}$, we have $O \neq \Delta_1 \cap \Delta_2$.

We suppose that $\mathrm{lct}(X) < 3/4$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical at some point $P \in X$ for some rational number $\lambda < 3/4$.

Let S_i be a fiber of π_i such that $P \in S_i$. Without loss of generality, we may assume that

- the surface S_1 is smooth at the point P,
- the singularities of the surface S_1 consist of at most one ordinary double point,
- for every smooth curve $L \subset S_1$ such that $-K_{S_1} \cdot L = 1$, we have $P \notin L$,
- for every smooth curves $C_1 \subset S_1 \supset C_2$ such that
$$-K_{S_1} \cdot C_1 = -K_{S_1} \cdot C_2 = 2$$
and $C_1 + C_2 \sim -K_{S_1}$, we have $P \neq C_1 \cap C_2$.

The surface S_1 is a del Pezzo surface of degree 4. One has
$$D = \mu S_1 + \Omega,$$
where Ω is an effective $\mathbb{Q}$-divisor on X such that $S_1 \not\subset \mathrm{Supp}(\Omega)$.

Let $\phi\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ be a natural conic bundle induced by the linear system
$$\Big|S_2 + S_3\Big|,$$
and let Γ be a general fiber of the conic bundle ϕ. Then
$$2 = D \cdot \Gamma = \mu S_1 \cdot \Gamma + \Omega \cdot \Gamma = 2\mu + \Omega \cdot \Gamma \geqslant 2\mu,$$
which implies that $\mu \leqslant 1$. Then $(X, S_1 + \lambda\Omega)$ is not canonical at the point P. Hence
$$\Big(S_1,\ \lambda\Omega\Big|_{S_1}\Big)$$
is not log canonical at the point P by Theorem 2.20. But
$$\Omega\Big|_{S_1} \sim_{\mathbb{Q}} D\Big|_{S_1} \sim_{\mathbb{Q}} -K_{S_k},$$

which is impossible (see Example 5.4 and mind the generality assumption). $\square$

Lemma 9.2. Suppose that $\gimel(X) = 3.2$ and X is general. Then $\mathrm{lct}(X) = 1/2$.

Proof. The threefold X is a primitive Fano threefold (see [128, Definition 1.3]). Put

$$U = \mathbb{P}\Big(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\big(-1,-1\big) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\big(-1,-1\big)\Big),$$

let $\pi\colon U \to \mathbb{P}^1 \times \mathbb{P}^1$ be a natural projection, and let L be a tautological line bundle on U. Then

$$X \in \Big|2L + \pi^*\Big(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\big(2,3\big)\Big)\Big|.$$

Let us show that $\mathrm{lct}(X) \leqslant 1/2$. Let E_1 and E_2 be surfaces in X such that

$$\pi\big(E_1\big) \subset \mathbb{P}^1 \times \mathbb{P}^1 \supset \pi\big(E_2\big)$$

are divisors on $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(1,0)$ and $(0,1)$, respectively. Then

$$-K_X \sim L\Big|_X + 2E_1 + E_2,$$

which implies that $\mathrm{lct}(X) \leqslant 1/2$.

We suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical at some point $P \in X$ for some rational number $\lambda < 1/2$.

It follows from [86, Proposition 3.8] that there is a commutative diagram

$$\begin{tikzcd}
U_1 \arrow[r, "\gamma_1"] \arrow[dd, "\psi_1"'] & V & U_2 \arrow[l, "\gamma_2"'] \arrow[dd, "\psi_2"] \\
 & X \arrow[ul, "\beta_1"] \arrow[u, "\alpha"'] \arrow[ur, "\beta_2"'] \arrow[dl, "\phi_1"'] \arrow[d, "\omega"'] \arrow[dr, "\phi_2"] & \\
\mathbb{P}^1 & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & \mathbb{P}^1
\end{tikzcd}$$

where V is a Fano threefold that has one ordinary double point $O \in V$ such that

$$\mathrm{Pic}\big(V\big) = \mathbb{Z}\big[-K_V\big]$$

and $-K_V^3 = 16$, the morphism α contracts a unique surface

$$\mathbb{P}^1 \times \mathbb{P}^1 \cong S \subset X$$

such that $S \sim L|_X$ to the point $O \in V$, the morphism β_i contracts S to a smooth rational curve, the morphism γ_i contracts the curve $\beta_i(S)$ to the point $O \in V$ so that the rational map

$$\gamma_2 \circ \gamma_1^{-1}\colon U_1 \dashrightarrow U_2$$

is a flop in $\beta_1(S) \cong \mathbb{P}^1$, the morphism ψ_2 is a quadric fibration, and the morphisms ψ_1, ϕ_1, ϕ_2 are fibrations whose fibers are del Pezzo surfaces of degree 4, 3 and 6, respectively. The morphisms π_1 and π_2 are natural projections, and $\omega = \pi|_X$. Then

$$\mathrm{Cl}\big(V\big) = \mathbb{Z}\big[\alpha\big(E_1\big)\big] \oplus \mathbb{Z}\big[\alpha\big(E_2\big)\big],$$

and ω is a conic bundle. The curve $\beta_1(S)$ is a section of ψ_1, and $\beta_2(S)$ is a 2-section of ψ_2.

We assume that the threefold X satisfies the following mild generality condition: every singular fiber of the del Pezzo fibration ϕ_2 has at most $\mathbb{A}_1$ singularities.

Applying Lemma 2.26 to the fibration ϕ_1, we see that

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq S_1$$

where S_1 is a singular fiber of the del Pezzo fibration ϕ_1, because the global log canonical threshold of a smooth del Pezzo surface of degree 6 is equal to $1/2$ by Example 1.18.

Applying Lemma 2.26 to ϕ_2, we obtain a contradiction by Example 1.38. $\square$

Lemma 9.3. Suppose that $\gimel(X) = 3.3$ and X is general. Then $\mathrm{lct}(X) = 2/3$.

Proof. The threefold X is a divisor on $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^2$ of tridegree $(1,1,2)$. In particular,

$$-K_X\sim\pi_1^*\Big(\mathcal{O}_{\mathbb{P}^1}(1)\Big)+\pi_2^*\Big(\mathcal{O}_{\mathbb{P}^1}(1)\Big)+\phi^*\Big(\mathcal{O}_{\mathbb{P}^2}(1)\Big),$$

where $\pi_1\colon X\to\mathbb{P}^1$ and $\pi_1\colon X\to\mathbb{P}^1$ are fibrations into del Pezzo surfaces of degree 4 that are induced by the projections of the variety $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^2$ onto its first and second factor, respectively, and $\phi\colon X\to\mathbb{P}^2$ is conic bundle that is induced by the projection $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^2\to\mathbb{P}^2$.

Let $\alpha_2\colon X\to\mathbb{P}^1\times\mathbb{P}^2$ be a birational morphism induced by the linear system

$$\Big|\pi_2^*\Big(\mathcal{O}_{\mathbb{P}^1}(1)\Big)+\phi^*\Big(\mathcal{O}_{\mathbb{P}^2}(1)\Big)\Big|,$$

let $H_i\in|\pi_i^*(\mathcal{O}_{\mathbb{P}^1}(1))|$ and $R\in|\phi^*(\mathcal{O}_{\mathbb{P}^2}(1))|$ be general surfaces. Then

$$H_1\sim H_2+2R-E_2,$$

where E_2 is the exceptional divisor of the birational morphism α_2. Hence

$$-K_X\sim H_1+H_2+R\sim_{\mathbb{Q}}\frac{3}{2}H_1+\frac{1}{2}H_2+\frac{1}{2}E_2,$$

which implies that $\mathrm{lct}(X)\leqslant 2/3$.

We suppose that the threefold X satisfies the following generality conditions: for an arbitrary point $O\in X$, there is $k\in\{1,2\}$ such that

- the fiber F_k of the fibration π_k that contains the point O is smooth at the point O,
- the singularities of the surface F_k consist of at most one ordinary double point,
- for every smooth curve $\Gamma\subset F_k$ such that $-K_{F_k}\cdot\Gamma=1$, we have $O\notin\Gamma$ if $\mathrm{Sing}(F_k)\neq\varnothing$.

We suppose that $\mathrm{lct}(X)<2/3$. Then there exists an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the log pair $(X,\lambda D)$ is not log canonical at some point $P\in X$ for some rational number $\lambda<2/3$.

Let S_i be a fiber of π_i such that $P\in S_i$. Then we may assume that

- the surface S_1 is smooth at the point P,
- the singularities of the surface S_1 consist of at most one ordinary double point,
- for every smooth curve $L\subset S_1$ such that $-K_{S_1}\cdot L=1$, we have $P\notin L$ if $\mathrm{Sing}(S_1)\neq\varnothing$.

Put $D=\mu S_1+\Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $S_1\not\subset\mathrm{Supp}(\Omega)$. Then

$$\Big(H_2,\ \lambda\mu S_1\Big|_{H_2}+\lambda\Omega\Big|_{H_2}\Big)$$

is log canonical because $\mathrm{lct}(H_2)=2/3$. Thus, we see that $\mu\leqslant 1/\lambda$. Hence

$$\Big(S_1,\ \lambda\Omega\Big|_{S_1}\Big)$$

is not log canonical at the point P by Theorem 2.20. But

$$\Omega\Big|_{S_1}\sim_{\mathbb{Q}}-K_{S_1},$$

which is impossible (see Example 5.4). ☐

Lemma 9.4. Suppose that $\daleth(X)=3.4$. Then $\mathrm{lct}(X)=1/2$.

Proof. Let O be a point in $\mathbb{P}^2$. Then there is a commutative diagram

$$\begin{tikzcd}
 & X \arrow[d, "\alpha"] \arrow[ddl, bend right, "\eta_1"'] \arrow[dr, "\eta_2"] \arrow[drr, bend left, "\phi"] & & \\
 & V \arrow[dl, "\gamma_1"'] \arrow[d, "\omega"] \arrow[dr, "\gamma_2"] & \mathbb{F}_1 \arrow[r, "\upsilon"'] \arrow[d, "\beta"] & \mathbb{P}^1 \\
\mathbb{P}^1 & \mathbb{P}^1\times\mathbb{P}^2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & \mathbb{P}^2 &
\end{tikzcd}$$

such that π_i and υ are natural projections, ω is a double cover branched over a divisor $B\subset\mathbb{P}^1\times\mathbb{P}^2$ of bi-degree $(2,2)$, the morphism γ_1 is a fibration into quadrics, γ_2 and η_2 are conic bundles, the morphism β is a blow up of the point O, the morphism α is a blow up of a smooth curve that is a fiber of γ_2 over the point O, the morphism η_1 is a fibration into del Pezzo surfaces of degree 6, and ϕ is a fibration into del Pezzo surfaces of degree 4.

Let H be a general fiber of η_1, and let S be a general fiber of ϕ. Then

$$-K_X \sim H + 2S + E,$$

where E is the exceptional divisor of α. Thus, we see that $\mathrm{lct}(X) \leqslant 1/2$.

We suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$. Then

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq E,$$

because $\alpha(D) \sim_{\mathbb{Q}} -K_V$ and $\mathrm{lct}(V) = 1/2$ by Lemma 8.7.

Let Γ be a fiber of η_2 such that $\Gamma \cap \mathrm{LCS}(X, \lambda D) \neq \varnothing$. Then

$$\Gamma \subseteq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq E,$$

by Theorem 2.28. Hence $(H, \lambda D|_H)$ is not log canonical at the points $H \cap \Gamma$. But

$$D\Big|_H \sim_{\mathbb{Q}} -K_X\Big|_H \sim -K_H$$

and $\mathrm{lct}(H) = 1/2$, because H is a del Pezzo surface of degree 6, which is a contradiction. □

Lemma 9.5. Suppose that $\daleth(X) = 3.5$ and X is general. Then $\mathrm{lct}(X) = 1/2$.

Proof. There is a birational morphism $\alpha \colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ that contracts a surface $E \subset X$ to a curve $C \subset \mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(5, 2)$. Let $\pi_1 \colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ and $\pi_2 \colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ be natural projections. There is

$$Q \in \Big|\pi_2^*\Big(\mathcal{O}_{\mathbb{P}^1}(2)\Big)\Big|$$

such that $C \subset Q$. Let H_1 be a general fiber of π_1, let H_2 be a surface in $|\pi_2^*(\mathcal{O}_{\mathbb{P}^1}(1))|$. We have

$$-K_X \sim 2\bar{H}_1 + \bar{H}_2 + \bar{Q},$$

where $\bar{H}_1, \bar{H}_2, \bar{Q} \subset X$ are proper transforms of H_1, H_2, Q, respectively. In particular, $\mathrm{lct}(X) \leqslant 1/2$.

We suppose that X satisfies the following generality condition: every fiber F of $\pi_1 \circ \alpha$ is singular at most at one ordinary double point.

Assume that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$.

Let S be an irreducible surface on the threefold X. Put

$$D = \mu S + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor such that $S \not\subset \mathrm{Supp}(\Omega)$. Then

$$\left(\bar{H}_1,\ \frac{1}{2}\Big(\mu S + \Omega\Big)\Big|_{\bar{H}_1}\right)$$

is log canonical (see Example 1.18). Thus, either $\mu \leqslant 2$, or S is a fiber of $\pi_1 \circ \alpha$.

Let $\Gamma \cong \mathbb{P}^1$ be a general fiber of the conic bundle $\pi_2 \circ \alpha$. Then

$$2 = D \cdot \Gamma = \mu S \cdot \Gamma + \Omega \cdot \Gamma \geqslant \mu S \cdot \Gamma,$$

which implies that $\mu \leqslant 2$ in the case when S is a fiber of $\pi_1 \circ \alpha$.

We see that the set $\mathbb{LCS}(X, \lambda D)$ contains no surfaces. Now, applying Lemma 2.26 to $\pi_1 \circ \alpha$, we obtain a contradiction with Example 5.4. □

Lemma 9.6. Suppose that $\daleth(X) = 3.6$ and X is general. Then $\mathrm{lct}(X) = 1/2$.

Proof. Let $\varepsilon \colon V \to \mathbb{P}^3$ be a blow up of a line $L \subset \mathbb{P}^3$. Then

$$V \cong \mathbb{P}\Big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\Big)$$

and there is a natural $\mathbb{P}^2$-bundle $\eta\colon V \to \mathbb{P}^1$. There is a smooth elliptic curve $C \subset \mathbb{P}^3$ of degree 4 such that $L \cap C = \varnothing$ and there is a commutative diagram

$$\begin{tikzcd}
Y \arrow[d, "\varepsilon"'] & X \arrow[l, "\gamma"'] \arrow[r, "\phi"] \arrow[d, "\beta"] & \mathbb{P}^1 \\
\mathbb{P}^3 & V \arrow[l, "\delta"] \arrow[ur, "\eta"'] &
\end{tikzcd}$$

where δ is a blow up of C, the morphism β is a blow up of the proper transform of the line L, the morphism γ is a blow up of the proper transform of the curve C, and ϕ is a del Pezzo fibration.

We suppose that X satisfies the following generality condition: for every fiber F of ϕ, the surface F has at most one singular point that is an ordinary double point of the surface F.

Let E and G be the exceptional surfaces of β and γ, respectively, let $H \subset \mathbb{P}^3$ be a general plane that passes through L, and let $Q \subset \mathbb{P}^3$ a quadric surface that passes through C. Then

$$-K_X \sim 2\bar{H} + \bar{Q} + E,$$

where $\bar{H} \subset X \supset \bar{Q}$ are proper transforms of H and Q, respectively. We have $\mathrm{lct}(X) \leqslant 1/2$.

We suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$.

It follows from Lemma 8.11 that $\mathrm{lct}(V) = 1/2$. Therefore, we see that

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq G.$$

Note that every fiber of the fibration ϕ is a del Pezzo surface of degree 5 that has at most one ordinary double point. Thus, applying Lemma 2.26 to ϕ, we obtain a contradiction with Example 5.3. $\square$

Lemma 9.7. Suppose that $\daleth(X) = 3.7$ and X is general. Then $\mathrm{lct}(X) = 1/2$.

Proof. Let W be a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bi-degree $(1,1)$. Then $-K_W \sim 2H$, where H is a Cartier divisor on W. There is a commutative diagram

$$\begin{tikzcd}
 & X \arrow[dl, "\beta"'] \arrow[dd, "\alpha"] \arrow[dr, "\gamma"] \arrow[ddd, bend left=60, "\omega"] & \\
\mathbb{P}^1 \times \mathbb{P}^2 \arrow[d, "\phi"'] & & \mathbb{P}^1 \times \mathbb{P}^2 \arrow[d, "\psi"] \\
\mathbb{P}^2 & W \arrow[l, "\xi"'] \arrow[r, "\zeta"] \arrow[d, dashed, "\rho"'] & \mathbb{P}^2 \\
 & \mathbb{P}^1 &
\end{tikzcd}$$

where ϕ and ψ are natural projections, α is a blow up of a smooth curve $C \subset W$ such that

$$C = H_1 \cap H_2,$$

where $H_1 \neq H_2$ are surfaces in $|H|$, the map ρ is induced by the pencil generated by H_1 and H_2, the morphism ω is a del Pezzo fibration of degree 6, the morphisms ζ and ξ are $\mathbb{P}^1$-bundles, while β and γ contract surfaces $\bar{M}_1 \subset X \supset \bar{M}_2$ such that $\phi\circ\beta(\bar{M}_1) = \xi(C)$ and $\psi\circ\gamma(\bar{M}_2) = \zeta(C)$.

Note that $\mathrm{lct}(X) \leqslant 1/2$, because

$$-K_X \sim 2\bar{H}_1 + E,$$

where $\bar{H}_1 \subset X$ is the proper transform of H_1, and E is the exceptional surface of α.

We suppose that X satisfies the following generality condition: all singular fibers of the fibration ω satisfy the hypotheses of Lemma 5.5.

Assume that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$. Then

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq E,$$

because $\mathrm{lct}(W)=1/2$ by Theorem 7.1. Applying Lemma 2.26, we see that

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq E \cap F,$$

where F is a singular fiber of ω. Note that F is a del Pezzo surface of degree 6. Put

$$D = \mu F + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor such that $F \not\subset \mathrm{Supp}(\Omega)$. Then

$$\Omega\Big|_{F} \sim_{\mathbb{Q}} D\Big|_{F} \sim_{\mathbb{Q}} -K_F,$$

and the surface F is smooth along the curve $E \cap F$. But the log pair $(F, \lambda\Omega|_F)$ is not log canonical at some point $P \in E \cap F$ by Theorem 2.20, which is impossible by Lemma 5.5. $\square$

Remark 9.8. Let us use the notation and assumptions of the proof of Lemma 9.7. Then

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq E \cap F,$$

where F is a singular fiber of the fibration ω. Applying Theorem 2.28 to ϕ and ψ, we see that

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq E \cap F \cap \bar{M}_1 \cap \bar{M}_2,$$

by Lemma 2.29. Regardless to how singular F is, if the threefold X is sufficiently general, then

$$E \cap F \cap \bar{M}_1 \cap \bar{M}_2 = \varnothing,$$

which implies that an alternative generality condition can be used in Lemma 9.7.

Lemma 9.9. Suppose that $\daleth(X) = 3.8$ and X is general. Then $\mathrm{lct}(X) = 1/2$.

Proof. Let $\pi_1\colon \mathbb{F}_1 \times \mathbb{P}^2 \to \mathbb{F}_1$ and $\pi_2\colon \mathbb{F}_1 \times \mathbb{P}^2 \to \mathbb{P}^2$ be natural projections. Then

$$X \in \Bigg|\big(\alpha \circ \pi_1\big)^*\Big(\mathcal{O}_{\mathbb{P}^2}(1)\Big) \otimes \pi_2^*\Big(\mathcal{O}_{\mathbb{P}^2}(2)\Big)\Bigg|,$$

where $\alpha\colon \mathbb{F}_1 \to \mathbb{P}^2$ is a blow up of a point. Let H be a surface in $|\pi_2^*(\mathcal{O}_{\mathbb{P}^2}(1))|$. Then

$$-K_X \sim E + 2L + H,$$

where $E \subset X \supset L$ are irreducible surfaces such that $\pi_1(E) \subset \mathbb{F}_1$ is the exceptional curve of α, and $\pi_1(L) \subset \mathbb{F}_1$ is a fiber of the natural projection $\mathbb{F}_1 \to \mathbb{P}^2$. We have $\mathrm{lct}(X) \leqslant 1/2$.

The projection π_1 induces a fibration $\phi\colon X \to \mathbb{P}^1$ into del Pezzo surfaces of degree 5.

We suppose that X satisfies the following generality condition: for every fiber F of ϕ, the surface F has at most one singular point that is an ordinary double point of the surface F.

Assume that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$.

Applying Lemma 2.26 to the morphism ϕ, we obtain a contradiction with Example 5.3. $\square$

Lemma 9.10. Suppose that $\daleth(X) = 3.9$. Then $\mathrm{lct}(X) = 1/3$.

Proof. Let $O_1 \in V_1 \cong V_2 \ni O_2$ be singular points of $V_1 \cong V_2 \cong \mathbb{P}(1,1,1,2)$, respectively, let

$$O_1 \notin S_1 \in \Big|\mathcal{O}_{\mathbb{P}(1,1,1,2)}(2)\Big|$$

be a smooth surface, and let $C_1 \subset S_1 \cong \mathbb{P}^2$ be a smooth quartic curve. Then the diagram

$$\begin{tikzcd}
 & & X \arrow[dll, "\beta_1"'] \arrow[drr, "\beta_2"] & & \\
U_1 \arrow[d, "\alpha_1"] \arrow[ddrr, bend left, "\gamma_1"] & & & & U_2 \arrow[d, "\alpha_2"'] \arrow[ddll, bend right, "\gamma_2"'] \\
V_1 \arrow[drr, dashed, "\psi_1"'] & & & & V_2 \arrow[dll, dashed, "\psi_2"] \\
 & & \mathbb{P}^2 & &
\end{tikzcd}$$

commutes, where ψ_i is a natural projection, α_i is a blow up of the point O_i with weights $(1,1,1)$, the morphism γ_i is a $\mathbb{P}^1$-bundle, and β_i is a birational morphism that contracts a surface

$$\mathbb{P}^1 \times C_1 \cong G_i \subset X$$

to a smooth curve $C_1 \cong C_i \subset U_i$.

Let $E_i \subset X$ be the proper transform of the exceptional divisor of α_i. Then

$$S_1 = \alpha_1 \circ \beta_1\big(E_2\big) \subset V_1 \cong \mathbb{P}\big(1,1,1,2\big) \cong V_2 \supset \alpha_2 \circ \beta_2\big(E_1\big)$$

are surfaces in $|\mathcal{O}_{\mathbb{P}(1,1,1,2)}(2)|$ that contain the curves C_1 and C_2, respectively. On the other hand,

$$\alpha_1 \circ \beta_1\big(G_2\big) \subset V_1 \cong \mathbb{P}\big(1,1,1,2\big) \cong V_2 \supset \alpha_2 \circ \beta_2\big(G_1\big)$$

are surfaces in $|\mathcal{O}_{\mathbb{P}(1,1,1,2)}(4)|$ that contain $O_1 \cup C_1$ and $O_2 \cup C_2$, respectively.

Let $\bar{H} \subset X$ be the proper transform of a general surface in $|\mathcal{O}_{\mathbb{P}(1,1,1,2)}(1)|$. Then

$$-K_X \sim 3\bar{H} + E_2 + E_1,$$

which gives $\mathrm{lct}(X) \leqslant 1/3$.

Suppose that $\mathrm{lct}(X) < 1/3$. Then there is an effective $\mathbb{Q}$-divisor

$$D \sim_{\mathbb{Q}} -K_X \sim_{\mathbb{Q}} \frac{5}{2}\Big(G_1 + G_2\Big) - 5\Big(E_1 + E_2\Big)$$

such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/3$. Put

$$D = \mu_1 E_1 + \mu_2 E_2 + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor on X such that

$$E_1 \not\subseteq \mathrm{Supp}\big(\Omega\big) \not\supseteq E_2.$$

Let Γ be a general fiber of the conic bundle $\gamma_1 \circ \beta_1$. Then

$$2 = \Gamma \cdot D = \Gamma \cdot \Big(\mu_1 E_1 + \mu_2 E_2 + \Omega\Big) = \mu_1 + \mu_2 + \Gamma \cdot \Omega \geqslant \mu_1 + \mu_2,$$

and without loss of generality we may assume that $\mu_1 \leqslant \mu_2$. Then $\mu_1 \leqslant 1$.

Suppose that there is a surface $S \in \mathbb{LCS}(X, \lambda D)$. Then $S \neq E_1$ and $S \neq G_1$, because $\alpha_2 \circ \beta_2(G_1) \in |\mathcal{O}_{\mathbb{P}(1,1,1,2)}(4)|$ and $\alpha_2 \circ \beta_2(D) \in |\mathcal{O}_{\mathbb{P}(1,1,1,2)}(5)|$. Hence $S \cap E_1 \neq \varnothing$. But

$$-\frac{1}{3}K_{E_1} \sim_{\mathbb{Q}} D\Big|_{E_1} = -\frac{2\mu_1}{3}K_{E_1} + \Omega\Big|_{E_1}$$

and $E_1 \cong \mathbb{P}^2$, which is impossible by Theorem 2.20, because $\lambda < 1/3 = \mathrm{lct}(\mathbb{P}^2)$.

We see that the set $\mathbb{LCS}(X, \lambda D)$ contains no surfaces. Let $P \in \mathrm{LCS}(X, \lambda D)$ be a point. Suppose that $P \notin G_1$. Let Z be a fiber of γ_1 such that $\beta_1(P) \in Z$. Then

$$Z \subseteq \mathrm{LCS}\Big(U_1,\ \lambda\beta_1(D)\Big)$$

by Theorem 2.28. Put $\bar{E}_1 = \beta_1(E_1)$. Then we have

$$Z \cap \bar{E}_1 \in \mathrm{LCS}\Big(\bar{E}_1,\ \lambda\Omega\Big|_{\bar{E}_1}\Big)$$

by Theorem 2.20, which is impossible by Lemma 2.8, because $\mu_1 \leqslant 1$. Hence $\mathrm{LCS}(X, \lambda D) \subsetneq G_1$.

Suppose that $\mathrm{LCS}(X, \lambda D) \subseteq G_1 \cap G_2$. Then

$$\Big|\mathrm{LCS}\Big(X,\ \lambda D\Big)\Big| = 1$$

by Lemma 2.14 and Theorem 2.7. One has

$$\mathrm{LCS}\Big(X,\ \lambda D\Big) \cup \bar{H} \subseteq \mathrm{LCS}\left(X,\ \lambda D + \frac{1}{3}\Big(E_2 + E_2\Big) + \bar{H}\right) \subset \mathrm{LCS}\Big(X,\ \lambda D\Big) \cup \bar{H} \cup E_1 \cup E_1,$$

which contradicts Theorem 2.7, because $\bar{H}$ is a general surface in $|(\beta_1 \circ \gamma_1)^*(\mathcal{O}_{\mathbb{P}^2}(1))|$ and

$$\lambda D + \frac{1}{3}\Big(E_2 + E_2\Big) + \bar{H} \sim_{\mathbb{Q}} \big(\lambda - 1/3\big)K_X.$$

Thus, we see that $G_1 \supsetneq \mathbb{LCS}(X, \lambda D) \not\subseteq G_1 \cap G_2$. Then

$$\varnothing \neq \mathrm{LCS}\Big(U_2,\ \lambda\beta_2(D)\Big) \subsetneq \beta_2(G_1),$$

and it follows from Theorems 2.7 and 2.28 that there is a fibre L of the fibration γ_2 such that

$$\mathrm{LCS}\Big(U_2,\ \lambda\beta_2(D)\Big) = L.$$

Let B be a general surface in $|\alpha_2^*(\mathcal{O}_{\mathbb{P}(1,1,1,2)}(2))|$. Then $\beta_2(D)|_B \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}^2}(5)$ and $B \cong \mathbb{P}^2$. But

$$\mathrm{LCS}\Big(B,\ \lambda\beta_2(D)\Big|_B\Big) = L \cap B$$

and $|L \cap B| = 1$, which is impossible by Lemma 2.8, because $\lambda < 1/3$. $\square$

Lemma 9.11. Suppose that $\gimel(X) = 3.10$. Then $\mathrm{lct}(X) = 1/2$.

Proof. Let $Q \subset \mathbb{P}^4$ be a smooth quadric hypersurface. Let $C_1 \subset Q \supset C_2$ be disjoint (irreducible) conics. Then there is a commutative diagram

$$\begin{tikzcd}
 & & X \arrow[dll, "\phi_1"'] \arrow[dl, "\beta_2"] \arrow[dr, "\beta_1"'] \arrow[drr, "\phi_2"] & & \\
\mathbb{P}^1 & Y_1 \arrow[l, "\psi_1"] \arrow[r, "\alpha_1"'] & Q & Y_2 \arrow[l, "\alpha_2"] \arrow[r, "\psi_2"'] & \mathbb{P}^1
\end{tikzcd}$$

where the morphism α_i is a blow up along the conic C_i, the morphism β_i is a blow up along the proper transform of the conic C_i, the morphism ψ_i is a natural fibration into quadric surfaces, and ϕ_i is fibration, whose general fiber is isomorphic to a smooth del Pezzo surfaces of degree 6.

Let E_i be the exceptional divisor of the morphism β_i, and let H_i be a sufficiently general hyperplane section of the quadric Q that passes through the conic C_i. Then

$$-K_X \sim \bar{H}_1 + 2\bar{H}_2 + E_2,$$

where $\bar{H}_i \subset X$ is the proper transform of the surface H_i. We see that $\mathrm{lct}(X) \leqslant 1/2$.

We suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$.

Using Example 1.18 and applying Lemma 2.28, we see that

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq S_1 \cap S_2,$$

where S_i is a singular fiber of ϕ_i. Hence, the set $\mathbb{LCS}(X, \lambda D)$ contains no surfaces.

It follows from Theorem 2.7 that either $\mathrm{LCS}(X, \lambda D)$ is a point in $E_1 \cup E_2$, or

$$\mathrm{LCS}\Big(X,\ \lambda D\Big) \cap \Big(X \setminus \big(E_1 \cup E_2\big)\Big) \neq \varnothing,$$

which implies that we may assume that $\mathrm{LCS}(X, \lambda D)$ is a point in E_1 by Lemma 2.10.

Since β_2 is an isomorphism on $X \setminus E_2$, we see that

$$P \in \mathrm{LCS}\Big(Y_1,\ \lambda\beta_2(D)\Big) \subset P \cup \beta_2(E_2)$$

for some point $P \in E_1$. Then $\mathrm{LCS}(Y_1, \lambda\beta_2(D)) = P$ by Theorem 2.7, because $P \notin \beta_2(E_2)$.

Let H be a general hyperplane section of the quadric Q. Then

$$-K_{Y_1} \sim \tilde{H}_1 + 2\tilde{H} \sim_{\mathbb{Q}} \beta_2(D),$$

where $\tilde{H} \subset Y_1 \supset \tilde{H}_1$ are proper transforms of H and H_1, respectively. But

$$\mathrm{LCS}\left(Y_1,\ \lambda\beta_2(D) + \frac{1}{2}\Big(\tilde{H}_1 + 2\tilde{H}\Big)\right) = P \cup \tilde{H},$$

which is impossible by Theorem 2.7, because $\lambda < 1/2$. $\square$

Lemma 9.12. Suppose that $\gimel(X) = 3.11$. Then $\mathrm{lct}(X) = 1/2$.

Proof. Let $O \in \mathbb{P}^3$ be a point, let $\delta\colon V_7 \to \mathbb{P}^3$ be a blow up of the point O, and let E be the exceptional divisor of δ. Then

$$V_7 \cong \mathbb{P}\Big(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)\Big),$$

there is a natural $\mathbb{P}^1$-bundle $\eta\colon V_7 \to \mathbb{P}^2$, and E is a section of η. There is a linearly normal elliptic curve $O \in C|subset\mathbb{P}^3$ such that the diagram

$$\begin{tikzcd}
U \arrow[rr, "\gamma"] \arrow[ddr, bend right, "\omega"'] & & \mathbb{P}^3 & & V_7 \arrow[ll, "\delta"'] \arrow[d, "\eta"] \\
& & X \arrow[ull, "\alpha"] \arrow[urr, "\beta"'] \arrow[dl, "\phi"'] \arrow[dr, "\upsilon"] & & \mathbb{P}^2 \\
& \mathbb{P}^1 & & \mathbb{P}^1 \times \mathbb{P}^2 \arrow[ll, "\pi_1"] \arrow[ur, bend left, "\pi_2"'] &
\end{tikzcd}$$

commutes, where π_1 and π_2 are natural projections, the morphism γ contracts a surface

$$C \times \mathbb{P}^1 \cong G \subset U$$

to the curve C, the morphism α is a blow up of the fiber of the morphism γ over the point $O \in \mathbb{P}^3$, the morphism β is a blow up of the proper transform of C, the morphism ω is a fibration into quadric surfaces, the morphism ϕ is a fibration into del Pezzo surfaces of degree 7, and υ contracts a surface

$$C \times \mathbb{P}^1 \cong F \subset X$$

to an elliptic curve $Z \subset \mathbb{P}^1 \times \mathbb{P}^2$ such that $-K_{\mathbb{P}^1\times\mathbb{P}^2} \cdot Z = 13$ and $Z \cong C$.

Let H_1 be a general fiber of ϕ, and let H_2 be a general surface in $|(\eta \circ \beta)^*(\mathcal{O}_{\mathbb{P}^2}(1))|$. Then

$$-K_X \sim H_1 + 2H_2,$$

which implies that $\mathrm{lct}(X) \leqslant 1/2$.

We suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$. Note that

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq \bar{E},$$

where $\bar{E}$ is the exceptional divisor of α, because $\mathrm{lct}(U) = 1/2$ by Lemma 8.11.

Let $\Gamma \cong \mathbb{P}^2$ be the general fiber of $\pi_2 \circ \upsilon$. Then

$$2 = -K_X \cdot \Gamma = D \cdot \Gamma = 2\bar{E} \cdot \Gamma,$$

which implies that $\bar{E} \not\subset \mathrm{LCS}(X, \lambda D)$. Applying Lemma 2.26 to the log pair

$$\Big(V_7,\ \lambda\beta(D)\Big)$$

we see that $\mathrm{LCS}(X, \lambda D) \subseteq \bar{E} \cap G$. Applying Lemma 2.29 to the log pair

$$\Big(\mathbb{P}^1 \times \mathbb{P}^2,\ \lambda\upsilon(D)\Big)$$

we see that $\mathrm{LCS}(X, \lambda D) = \bar{E} \cap F \cap G$, where $|\bar{E} \cap F \cap G| = 1$. Hence

$$\mathrm{LCS}\Big(X,\ \lambda D + H_2\Big) = \mathrm{LCS}\Big(X,\ \lambda D\Big) \cup H_2,$$

and $H_2 \cap \mathrm{LCS}(X, \lambda D) = \varnothing$. But the divisor

$$-\Big(K_X + \lambda D + H_2\Big) = \Big(\lambda - \frac{1}{2}\Big) K_X + \frac{1}{2} H_1$$

is ample, which is impossible by Theorem 2.7. $\square$

Lemma 9.13. Suppose that $\daleth(X) = 3.12$. Then $\mathrm{lct}(X) = 1/2$.

Proof. Let $\varepsilon\colon V \to \mathbb{P}^3$ be a blow up of a line $L \subset \mathbb{P}^3$. There is a natural $\mathbb{P}^2$-bundle $\eta\colon V \to \mathbb{P}^1$, there is a smooth rational cubic curve $C \subset \mathbb{P}^3$ such that $L \cap C = \varnothing$, and the diagram

$$\begin{tikzcd}
\mathbb{P}^1 & & \mathbb{P}^1 \times \mathbb{P}^2 \arrow[ll, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
& X \arrow[ul, "\phi"] \arrow[ur, "\omega"'] \arrow[dl, "\beta"] \arrow[dr, "\gamma"] & & \mathbb{P}^2 \\
V \arrow[uu, "\eta"] \arrow[dr, "\varepsilon"'] & & Y \arrow[ur, "\psi"'] \arrow[dl, "\alpha"] & \\
& \mathbb{P}^3 & &
\end{tikzcd}$$

commutes, where α and β are blow ups of the curve C and its proper transform, respectively, the morphism γ is a blow up of the proper transform of the line L, the morphism ψ is a $\mathbb{P}^1$-bundle, the morphism ω is a birational contraction of a surface $F \subset X$ to a curve such that

$$C \cup L \subset \alpha \circ \gamma\big(F\big) \subset \mathbb{P}^3,$$

and $\alpha \circ \gamma(F)$ consists of secant lines of $C \subset \mathbb{P}^3$ that intersect L, the morphism ϕ is a fibration into del Pezzo surfaces of degree 6, the morphisms π_1 and π_2 are natural projections.

Let E and G be exceptional divisors of β and γ, respectively, let $Q \subset \mathbb{P}^3$ be a general quadric surface that passes through C, let $H \subset \mathbb{P}^3$ be a general plane that passes through L. Then

$$-K_X \sim \bar{Q} + 2\bar{H} + G,$$

where $\bar{Q} \subset X \supset \bar{H}$ are proper transforms of $Q \subset \mathbb{P}^3 \supset H$, respectively. In particular, $\mathrm{lct}(X) \leqslant 1/2$.

We suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$. Note that

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subset G,$$

since $\mathrm{lct}(Y) = 1/2$ by Lemma 8.13. Applying Theorem 2.28 to ϕ we see that

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subset G \cap S_\phi,$$

where S_ϕ is a singular fiber of the del Pezzo fibration ϕ (see Example 1.18). Then we see that

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subset G \cap S_\phi \cap F,$$

by applying Theorem 2.28 to the log pair $(\mathbb{P}^1 \times \mathbb{P}^2, \lambda\omega(D))$ and to the $\mathbb{P}^1$-bundle π_2.

Let $Z_1 \cong \mathbb{P}^1$ be a section of the natural projection

$$\mathbb{P}^1 \times \mathbb{P}^1 \cong G \longrightarrow L \cong \mathbb{P}^1$$

such that $Z_1 \cdot Z_1 = 0$, and let Z_2 a fiber of this projection. Then

$$F\Big|_G \sim Z_1 + 3Z_2$$

and $S_\phi|_G \sim Z_1$. The curve $F \cap G$ is irreducible. Thus, we see that

$$\Big|G \cap F \cap S_\phi\Big| < +\infty,$$

which implies that the set $\mathrm{LCS}(X, \lambda D)$ consists of a single point $P \in G$ by Theorem 2.7.

The log pair $(V, \lambda\beta(D))$ is not log canonical. Since β is an isomorphism on $X \setminus E$, we see that

$$\beta\big(P\big) \in \mathrm{LCS}\Big(V,\ \lambda\beta\big(D\big)\Big) \subseteq \beta\big(P\big) \cup \beta\big(E\big),$$

which implies that $\mathrm{LCS}(V, \lambda\beta(D)) = \beta(P)$ by Theorem 2.7. Let $H \subset \mathbb{P}^3$ be a general plane. Then

$$\mathrm{LCS}\left(V,\ \lambda\beta\big(D\big) + \frac{1}{2}\Big(\tilde{H}_1 + 3\tilde{H}\Big)\right) = \beta\big(P\big) \cup \tilde{H},$$

where $\tilde{H} \subset V \supset \tilde{H}_1$ are proper transforms of $H \subset \mathbb{P}^3 \supset H_1$, respectively. But

$$-K_V \sim \tilde{H}_1 + 3\tilde{H} \sim_{\mathbb{Q}} \beta\big(D\big),$$

which contradicts Theorem 2.7, because $\lambda < 1/2$. $\square$

Lemma 9.14. Suppose that $\daleth(X) = 3.14$. Then $\mathrm{lct}(X) = 1/2$.

Proof. Let $P \in \mathbb{P}^3$ be a point, and let $\alpha\colon V_7 \to \mathbb{P}^3$ be a blow up of the point P. Then there is a natural $\mathbb{P}^1$-bundle $\pi\colon V_7 \to \mathbb{P}^2$.

Let $\zeta\colon Z \to \mathbb{P}(1,1,1,2)$ be a blow up of the singular point of $\mathbb{P}(1,1,1,2)$. Then

$$Z \cong \mathbb{P}\Big(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)\Big),$$

and there is a natural $\mathbb{P}^1$-bundle $\phi\colon Z \to \mathbb{P}^2$.

There is a plane $\Pi \subset \mathbb{P}^3$ and a smooth cubic curve $C \subset \Pi$ such that $P \notin \Pi$ and the diagram

$$\begin{tikzcd}
 & & X \arrow[dl, "\gamma"'] \arrow[dr, "\beta"] \arrow[rr, "\omega"] & & Z \arrow[dd, "\phi"] \arrow[dddr, "\zeta"] & \\
 & Y \arrow[dr, "\varepsilon"] \arrow[ddl, "\eta"'] & & V_7 \arrow[dl, "\alpha"'] \arrow[dr, "\pi"] & & \\
 & & \mathbb{P}^3 \arrow[rr, dashed, "\xi"'] & & \mathbb{P}^2 & \\
U \arrow[rr, hook] & & \mathbb{P}(1,1,1,1,2) \arrow[u, dashed, "\nu"'] \arrow[rrr, dashed, "\upsilon"'] & & & \mathbb{P}(1,1,1,2) \arrow[ul, dashed, "\psi"]
\end{tikzcd}$$

commutes (see [176, Example 3.6]), where we have the following notation:

- the morphism ε is a blow up of the curve C;
- the threefold U is a cubic hypersurface in $\mathbb{P}(1,1,1,1,2)$;
- the rational map ξ is a projection from the point P;
- the morphism γ is a blow up of the point that dominates P;
- the morphism β is a blow up of the proper transform of the curve C;
- the morphism η contracts the proper transform of Π to the point $\mathrm{Sing}(U)$,
- the morphism ω contracts a surface $R \subset X$ to a curve such that

  $$\beta \circ \alpha\big(R\big) \subset \mathbb{P}^3$$

  is a cone over the curve C whose vertex is the point P;
- the rational maps ψ and ν are natural projections;
- the rational map υ is a linear projection from a point.

Let E and G be exceptional divisors of γ and β, respectively, and let $\bar{H} \subset X$ be a proper transform of a general plane in $\mathbb{P}^3$ that passes through the point P. Then

$$-K_X \sim \bar{\Pi} + 3\bar{H} + G,$$

where $\bar{\Pi} \subset X$ is the proper transform of the plane Π. Thus, we see that $\mathrm{lct}(X) \leqslant 1/3$.

We suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/3$.

Let $\bar{L} \subset X$ be a proper transform of a general line in $\mathbb{P}^3$ that intersects the curve C. Then

$$D \cdot \bar{L} = \bar{\Pi} \cdot \bar{L} + 3\bar{H} \cdot \bar{L} + G \cdot \bar{L} = 3\bar{H} \cdot \bar{L} = 3,$$

which implies that $\mathbb{LCS}(X, \lambda D)$ contains no surfaces except possibly $\bar{\Pi}$ and E.

Let Γ be a general fiber of $\pi \circ \beta$. Then

$$D \cdot \Gamma = \bar{\Pi} \cdot \Gamma + 3\bar{H} \cdot \Gamma + G \cdot \Gamma = \bar{\Pi} \cdot \Gamma + G \cdot \Gamma = 2,$$

which implies that $\mathbb{LCS}(X, \lambda D)$ does not contain $\bar{\Pi}$ and E. Thus, by Lemma 2.9, we have

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subsetneq E \cup G.$$

Suppose that $\mathrm{LCS}(X, \lambda D) \subseteq E$. Then

$$\varnothing \neq \mathrm{LCS}\Big(V_7,\ \lambda\beta(D)\Big) \subseteq \beta(E),$$

which contradicts Theorem 2.28, because $\beta(E)$ is a section of π. We see that $\mathrm{LCS}(X, \lambda D) \subsetneq G$.

Applying Theorem 2.28 to $(Z, \lambda\omega(D))$ and ϕ and Theorem 2.7 to $(X, \lambda D)$, we see that

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq F,$$

where F is a fiber of the natural projection $G \to \beta(G)$. Then

$$\varnothing \neq \mathrm{LCS}\Big(Y,\ \lambda\gamma(D)\Big) \subseteq \gamma(F),$$

where $\gamma(F)$ is a fiber of the blow up ε over a point in the curve C.

Let $S \subset \mathbb{P}^3$ be a general cone over the curve C, and let $O \in C$ be an inflection point such that

$$\varepsilon \circ \gamma\big(F\big) \neq O.$$

Let $L \subset S$ be a line that passes through the point O, and let $H \subset \mathbb{P}^3$ be a plane that is tangent to the cone S along the line L. Since O is an inflection point of the curve C, the equality

$$\mathrm{mult}_L\Big(S \cdot H\Big) = 3$$

holds. Let $\breve{S}$, $\breve{H}$ and $\breve{L}$ be the proper transforms of S, H and L on the threefold Y. Then

$$\mathrm{LCS}\left(Y,\ \lambda\gamma(D) + \frac{2}{3}\Big(\breve{S} + \breve{H}\Big)\right) = \mathrm{LCS}\Big(Y,\ \lambda\gamma(D)\Big) \cup \breve{L}$$

due to generality in the choice of S. But $-K_Y \sim \breve{S} + \breve{H}$, which is impossible by Theorem 2.7. $\square$

Lemma 9.15. Suppose that $\daleth(X) = 3.15$. Then $\mathrm{lct}(X) = 1/2$.

Proof. Let $Q \subset \mathbb{P}^4$ be a smooth quadric hypersurface, let $C \subset Q$ be a smooth conic, and let $\varepsilon\colon V \to Q$ be a blow up of the conic $C \subset Q$. Then there is a natural morphism $\eta\colon V \to \mathbb{P}^1$ induced by the projection $Q \dashrightarrow \mathbb{P}^1$ from the two-dimensional linear subspace in $\mathbb{P}^4$ that contains the conic $C \subset Q$. Then a general fiber of η is a smooth quadric surface in $\mathbb{P}^3$.

Take a line $L \subset Q$ such that $L \cap C = \varnothing$; then there is a commutative diagram

$$\begin{tikzcd}
\mathbb{P}^1 & & \mathbb{P}^1 \times \mathbb{P}^2 \arrow[ll, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
& X \arrow[ul, "\phi"'] \arrow[ur, "\omega"] \arrow[dl, "\beta"] \arrow[dr, "\gamma"'] & & \mathbb{P}^2 \\
V \arrow[uu, "\eta"] \arrow[dr, "\varepsilon"'] & & Y \arrow[ur, "\psi"'] \arrow[dl, "\alpha"] & \\
& Q & &
\end{tikzcd}$$

where α and β are blow ups of the line $L \subset Q$ and its proper transform, respectively, the morphism γ is a blow up of the proper transform of the conic C, the morphism ψ is a $\mathbb{P}^1$-bundle, the morphism ω is a birational contraction of a surface $F \subset X$ to a curve such that

$$C \cup L \subset \alpha \circ \gamma\big(F\big) \subset Q,$$

and $\alpha \circ \gamma(F)$ consists of all lines in $Q \subset \mathbb{P}^4$ that intersect L and C, the morphism ϕ is a fibration into del Pezzo surfaces of degree 7, the morphisms π_1 and π_2 are natural projections.

Let E_1 and E_2 be exceptional surfaces of β and γ, respectively, let $H_1, H_2 \subset Q$ be general hyperplane sections that pass through the curves L and C, respectively. We have

$$-K_X \sim \bar{H}_1 + 2\bar{H}_2 + E_2 \sim \bar{H}_2 + 2\bar{H}_1 + E_1,$$

where $\bar{H}_1 \subset X \supset \bar{H}_2$ are proper transforms of $H_1 \subset Q \supset H_2$, respectively. In particular, $\mathrm{lct}(X) \leqslant 1/2$.

We suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$.

Let S be an irreducible surface on the threefold X. Put

$$D = \mu S + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor such that $S \not\subset \mathrm{Supp}(\Omega)$. Then

$$\mathrm{LCS}\left(\bar{H}_2,\ \frac{1}{2}\Big(\mu S + \Omega\Big)\Big|_{\bar{H}_2}\right) \subset E_1 \cap \bar{H}_2$$

by Lemma 5.9. Thus, if $\mu \leqslant 2$, then either $S = E_1$, or S is a fiber of ϕ.

Let $\Gamma \cong \mathbb{P}^1$ be a general fiber of the conic bundle $\psi \circ \gamma$. Then

$$2 = D \cdot \Gamma = \mu S \cdot \Gamma + \Omega \cdot \Gamma \geqslant \mu S \cdot \Gamma,$$

which implies that $\mu \leqslant 2$ in the case when either $S = E_1$, or S is a fiber of ϕ.

Therefore, we see that $\mathbb{LCS}(X, \lambda D)$ does not contain surfaces.

Applying Theorem 2.28 to the log pair $(Y, \lambda\gamma(D))$ and ψ, we see that

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subsetneq E_2 \cup \bar{L},$$

where $\mathbb{P}^1 \cong \bar{L} \subset X$ is a curve such that $\gamma(\bar{L})$ is a fiber of the conic bundle ψ.

Suppose that $\bar{L} \not\subset E_1$ and $\bar{L} \subset \mathrm{LCS}(X, \lambda D)$. Then

$$\alpha \circ \gamma\big(\bar{L}\big) \subseteq \mathrm{LCS}\Big(Q,\ \lambda\alpha \circ \gamma\big(D\big)\Big) \subseteq \alpha \circ \gamma\big(\bar{L}\big) \cup C \cup L,$$

which is impossible by Lemma 2.10. Hence by Theorem 2.7 we see that

- either $\mathrm{LCS}(X, \lambda D) \subsetneq E_2$,
- or $\mathrm{LCS}(X, \lambda D) \subseteq \bar{L}$ and $\bar{L} \subset E_1$.

We may assume that $\bar{L} \subset E_1$. Note that $E_1 \cong \mathbb{F}_1$. One has $\bar{L} \cdot \bar{L} = -1$ on the surface E_1.

Applying Lemma 2.29 to the log pair $(\mathbb{P}^1 \times \mathbb{P}^2, \lambda\omega(D))$, we see that $\mathrm{LCS}(X, \lambda D) \subset F$, because

$$\omega\big(D\big) \sim_{\mathbb{Q}} -K_{\mathbb{P}^1 \times \mathbb{P}^2}$$

and $\lambda < 1/2$. Applying Lemma 2.26 to the log pair $(V, \lambda\beta(D))$ and the fibration η, we see that

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subsetneq E_1 \cup S_\phi,$$

where S_ϕ is a singular fiber of ϕ, because $\mathrm{lct}(\mathbb{P}^1 \times \mathbb{P}^1) = 1/2$ (see Example 1.18).

We have $F \cap \bar{L} = \varnothing$ and $|F \cap \bar{S_\phi} \cap E_2| < +\infty$. Thus, there is point $P \in E_2$ such that

$$\mathrm{LCS}\Big(X,\ \lambda D\Big) = P \in E_2$$

by Theorem 2.7. But $\beta(E_1) \cap \beta(P) = \varnothing$. Thus, it follows from Theorem 2.7 that

$$\mathrm{LCS}\Big(V,\ \lambda\beta\big(D\big)\Big) = \beta\big(P\big).$$

Let $\tilde{H}_1 \subset V \supset \tilde{H}_2$ be the proper transforms of $H_1 \subset Q \supset H_2$, respectively. Then

$$-K_V \sim \tilde{H}_2 + 2\tilde{H}_1 \sim_{\mathbb{Q}} \beta\big(D\big),$$

but it follows from the generality of H_1 and H_2 that

$$\mathrm{LCS}\left(V,\ \lambda\beta\big(D\big) + \frac{1}{2}\Big(\tilde{H}_2 + 2\tilde{H}_1\Big)\right) = \beta\big(P\big) \cup \tilde{H}_1,$$

which is impossible by Theorem 2.7, because $\lambda < 1/2$. $\square$

Lemma 9.16. Suppose that $\daleth(X) = 3.16$. Then $\mathrm{lct}(X) = 1/2$.

Proof. Let $\mathbb{P}^1 \cong C \subset \mathbb{P}^3$ be a twisted cubic curve, let $O \in C$ be a point. There is a commutative diagram

$$\begin{tikzcd}
\mathbb{P}(\mathcal{E}) \arrow[r, equal] & U \arrow[rr, "\gamma"] \arrow[ddr, bend right, "\omega"'] & & \mathbb{P}^3 & & V_7 \arrow[ll, "\delta"'] \arrow[r, equal] \arrow[d, "\eta"] & \mathbb{P}\Big(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)\Big) \\
& & & X \arrow[ull, "\alpha"] \arrow[urr, "\beta"'] \arrow[dr, "\upsilon"] & & \mathbb{P}^2 & \\
& & \mathbb{P}^2 & & W \arrow[ll, "\pi_1"] \arrow[ur, bend right, "\pi_2"'] & &
\end{tikzcd}$$

where $\mathcal{E}$ is a stable rank two vector bundle on $\mathbb{P}^2$ (see the proof of Lemma 8.13), and we have the following notation:

- the morphism δ is a blow up of the point O;
- the morphism γ contracts a surface $G \subset U$ to the curve $C \subset \mathbb{P}^3$;
- the morphism α contracts a surface $E \cong \mathbb{F}_1$ to the fiber of γ over the point $O \in \mathbb{P}^3$;
- the morphism β is a blow up of the proper transform of the curve C;
- the variety W is a smooth divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bi-degree $(1, 1)$;
- the morphisms π_1 and π_2 are natural projections;
- the morphisms ω and η are natural $\mathbb{P}^1$-bundles;
- the morphism υ contracts a surface $F \subset X$ to a curve
$$\mathbb{P}^1 \cong Z \subset W$$
such that $\omega \circ \alpha(E) = \pi_1(Z)$ and $\eta \circ \beta(G) = \pi_2(Z)$.

Take general surfaces $H_1 \in |(\omega \circ \alpha)^*(\mathcal{O}_{\mathbb{P}^2}(1))|$ and $H_2 \in |(\eta \circ \beta)^*(\mathcal{O}_{\mathbb{P}^2}(1))|$. Then

$$-K_X \sim H_1 + 2H_2,$$

which implies that $\mathrm{lct}(X) \leqslant 1/2$.

We suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$. Note that

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq E \cap F,$$

because $\mathrm{lct}(U) = 1/2$ by Lemma 8.11 and $\mathrm{lct}(W) = 1/2$ by Theorem 7.1.

Applying Lemma 2.12 to the log pair $(V_7, \lambda\beta(D))$, we see that

$$\mathrm{LCS}\Big(X, \lambda D\Big) = E \cap F \cap G,$$

where $|E \cap F \cap G| = 1$. Then

$$\mathrm{LCS}\Big(X,\ \lambda D + H_2\Big) = \mathrm{LCS}\Big(X,\ \lambda D\Big) \cup H_2,$$

where $H_2 \cap \mathrm{LCS}(X, \lambda D) = \varnothing$. But the divisor

$$-\Big(K_X + \lambda D + H_2\Big) \sim_{\mathbb{Q}} \Big(\lambda - \frac{1}{2}\Big) K_X + \frac{1}{2} H_1$$

is ample, which is impossible by Theorem 2.7. □

Lemma 9.17. Suppose that $\daleth(X) = 3.17$. Then $\mathrm{lct}(X) = 1/2$.

Proof. The threefold X is a divisor in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of tri-degree $(1, 1, 1)$. Take general surfaces

$$H_1 \in \Big|\pi_1^*\Big(\mathcal{O}_{\mathbb{P}^1}(1)\Big)\Big|,\ H_2 \in \Big|\pi_2^*\Big(\mathcal{O}_{\mathbb{P}^1}(1)\Big)\Big|,\ H_3 \in \Big|\pi_3^*\Big(\mathcal{O}_{\mathbb{P}^2}(1)\Big)\Big|,$$

where π_i is a natural projection of the threefold X onto the i-th factor of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$. Then

$$-K_X \sim H_1 + H_2 + 2H_3,$$

which implies that $\mathrm{lct}(X) \leqslant 1/2$. There is a commutative diagram

$$
\begin{tikzcd}
\mathbb{P}^1 & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[l, "\upsilon_1"'] \arrow[r, "\upsilon_2"] & \mathbb{P}^1 \\
 & X \arrow[ul, "\pi_1"] \arrow[u, "\zeta"'] \arrow[ur, "\pi_2"'] \arrow[dl, "\alpha_1"'] \arrow[d, "\pi_3"] \arrow[dr, "\alpha_2"] & \\
\mathbb{P}^1 \times \mathbb{P}^2 \arrow[uu, "\eta_1"] \arrow[r, "\omega_1"'] & \mathbb{P}^2 & \mathbb{P}^1 \times \mathbb{P}^2 \arrow[uu, "\eta_2"'] \arrow[l, "\omega_2"]
\end{tikzcd}
$$

where ω_i, η_i and υ_i are natural projections, ζ is a $\mathbb{P}^1$-bundle, and α_i is a birational morphism that contracts a surface $E_i \subset X$ to a smooth curve $C_i \subset \mathbb{P}^1 \times \mathbb{P}^2$ such that $\omega_1(C_1) = \omega_2(C_2)$ is a (irreducible) conic.

Note that $E_2 \sim H_1 + H_3 - H_2$ and $E_1 \sim H_2 + H_3 - H_1$.

We suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$.

Suppose that the set $\mathbb{LCS}(X, \lambda D)$ contains a (irreducible) surface $S \subset X$. Put

$$D = \mu S + \Omega,$$

where $\mu \geqslant 1/\lambda$ and Ω is an effective $\mathbb{Q}$-divisor such that $S \not\subset \mathrm{Supp}(\Omega)$. Then

$$2 = D \cdot \Gamma = \mu S \cdot \Gamma + \Omega \cdot \Gamma \geqslant \mu S \cdot \Gamma,$$

where $\Gamma \cong \mathbb{P}^1$ is a general fiber of ζ. Hence $S \cdot \Gamma = 0$, which implies that $E_2 \neq S \neq E_1$. One also has

$$2 = D \cdot \Delta = \mu S \cdot \Delta + \Omega \cdot \Delta \geqslant \mu S \cdot \Delta,$$

where $\Delta \cong \mathbb{P}^1$ is a general fiber of the conic bundle π_2. Hence $S \cdot \Delta = 0$, which implies that

$$S \in \Big|\pi_3^*\Big(\mathcal{O}_{\mathbb{P}^2}(m)\Big)\Big|$$

for some $m \in \mathbb{Z}_{>0}$, because $E_2 \neq S \neq E_1$ and S is an irreducible surface. Then

$$0 = S \cdot \Gamma = m \neq 0,$$

which is a contradiction. Thus, we see that the set $\mathbb{LCS}(X, \lambda D)$ contains no surfaces.

Applying Theorem 2.28 to ζ and using Theorem 2.7, we see that

$$\mathrm{LCS}\Big(X,\ \lambda D\Big) = F \cong \mathbb{P}^1,$$

where F is a fiber of the $\mathbb{P}^1$-bundle ζ. Applying Theorem 2.28 to the conic bundle π_3, we see that every fiber of the conic bundle π_3 that intersects F must be reducible. This means that

$$\pi_3\big(F\big) \subset \omega_1\big(C_1\big) = \omega_2\big(C_2\big) \subset \mathbb{P}^2,$$

which is impossible, because $\pi_3(F)$ is a line, and $\omega_1(C_1) = \omega_2(C_2)$ is an irreducible conic. □

Lemma 9.18. Suppose that $\daleth(X) = 3.18$. Then $\mathrm{lct}(X) = 1/3$.

Proof. Let $Q \subset \mathbb{P}^4$ be a smooth quadric hypersurface, $C \subset Q$ an irreducible conic, and $O \in C$ a point. Then there is a commutative diagram

$$
\begin{tikzcd}
Y \arrow[rrr, bend left, "\eta"] \arrow[dddr, bend right, "\tau"'] & & & \\
 & X \arrow[ul, "\sigma"'] \arrow[r, "\beta"] \arrow[d, "\gamma"'] & V \arrow[r, "\omega"] \arrow[d, "\alpha"] & \mathbb{P}^1 \\
 & U \arrow[r, "\zeta"'] \arrow[d, "\upsilon"'] & Q \arrow[ur, dashed, "\phi"'] \arrow[dl, dashed, "\psi"] & \\
 & \mathbb{P}^3 \arrow[uurr, dashed, bend right, "\xi"'] & &
\end{tikzcd}
$$

where ζ is a blow up of the point O, the morphisms α and γ are blow ups of the conic C and its proper transform, respectively, β is a blow up of the fiber of the morphism α over the point O, the map ψ is a projection from O, the map ϕ is induced by the projection from the two-dimensional linear subspace that contains the conic C, the morphism τ is a blow up of the line $\psi(C)$, the morphism υ is a blow up of an irreducible conic $Z \subset \mathbb{P}^3$ such that

$$\psi\big(C\big) \cap Z \neq \varnothing,$$

and Z and $\psi(C)$ are not contained in one plane, the morphism σ is a blow up of the proper transform of the conic Z, the map ξ is a projection from $\psi(C)$, the morphism η is a $\mathbb{P}^1$-bundle, and ω is a fibration into quadric surfaces.

Let $\bar{H}$ be a general fiber of $\omega \circ \beta$. Then $\bar{H}$ is a del Pezzo surface such that $K_{\bar{H}}^2 = 7$, and

$$-K_X \sim 3\bar{H} + 2E + G,$$

where G and E are the exceptional divisors of β and γ, respectively. In particular, $\mathrm{lct}(X) \leqslant 1/3$.

We suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/3$. Note that

$$\varnothing \neq \mathrm{LCS}\Big(X, \lambda D\Big) \subseteq G,$$

since $\mathrm{lct}(V) = 1/3$ by Lemma 8.15 and $\beta(D) \sim_{\mathbb{Q}} -K_V$.

Applying Lemma 2.26 to the del Pezzo fibration $\omega \circ \beta$ and using Theorem 2.7, we see that there is a unique singular fiber S of the fibration $\omega \circ \beta$ such that

$$\varnothing \neq \mathrm{LCS}\Big(X, \lambda D\Big) \subseteq G \cap S,$$

because the equality $\mathrm{lct}(\bar{H}) = 1/3$ holds (see Example 1.18).

Let $P \in G \cap S$ be an arbitrary point of the locus $\mathrm{LCS}(X, \lambda D)$. Put

$$D = \mu S + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor such that $S \not\subset \mathrm{Supp}(\Omega)$. Then

$$P \in \mathrm{LCS}\Big(S,\ \lambda\Omega\Big|_{S}\Big)$$

by Theorem 2.20.

We can identify the surface $\beta(S)$ with an irreducible quadric cone in $\mathbb{P}^3$. Note that $G \cap S$ is an exceptional curve on S, so that there is a unique ruling of the cone $\beta(S)$ intersecting the curve $\beta(G)$. Let $L \subset S$ be a proper transform of this ruling. Then $L \cap G \neq \varnothing$ (moreover, $|L \cap G \cap S| = 1$), while $L \cap E = \varnothing$. Hence $P = L \cap G$ by Lemma 5.10. We see that $\mathrm{LCS}(X, \lambda D) = P$. One has

$$\bar{H} \cup P \subseteq \mathrm{LCS}\left(X,\ \lambda D + \bar{H} + \frac{2}{3}E\right) \subseteq \bar{H} \cup P \cup E,$$

because $\bar{H}$ is a general fiber of the fibration $\omega \circ \beta$. Therefore, the locus

$$\varnothing \neq \mathrm{LCS}\left(X,\ \lambda D + \bar{H} + \frac{2}{3}E\right) \subset X$$

must be disconnected, because $P \notin \bar{H}$ and $P \notin E$. But

$$-\left(K_X + \lambda D + \bar{H} + \frac{2}{3}E\right) \sim_{\mathbb{Q}} \bar{H} + \frac{2}{3}\Big(E + G\Big) + \big(\lambda - 1/3\big)K_X$$

is an ample divisor, which is impossible by Theorem 2.7. $\square$

The proof of Lemma 9.18 implies the following corollary.

Corollary 9.19. Suppose that $\beth(X) = 4.4$ or $\beth(X) = 5.1$. Then $\mathrm{lct}(X) = 1/3$.

Lemma 9.20. Suppose that $\beth(X) = 3.19$. Then $\mathrm{lct}(X) = 1/3$.

Proof. Let $Q \subset \mathbb{P}^4$ be a smooth quadric, and let $L \subset \mathbb{P}^4$ be a line such that

$$L \cap Q = P_1 \cup P_2,$$

where P_1 and P_2 are different points. Let $\eta\colon Q \dashrightarrow \mathbb{P}^2$ be the projection from L. The diagram

$$\begin{tikzcd}
 & & X \arrow[ddll, "\gamma_1"'] \arrow[dl, "\beta_2"'] \arrow[dr, "\beta_1"] \arrow[ddrr, "\gamma_2"] & & \\
 & U_1 \arrow[r, "\alpha_1"] \arrow[d, "\delta_1"'] & Q \arrow[dl, dashed, "\xi_1"] \arrow[dr, dashed, "\xi_2"'] \arrow[dd, dashed, "\eta"] & U_2 \arrow[l, "\alpha_2"'] \arrow[d, "\delta_2"] & \\
\mathbb{P}\Big(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)\Big) \arrow[r, "\pi_2"] \arrow[drr, "\omega_2"'] & \mathbb{P}^3 \arrow[dr, dashed, "\zeta_2"] & & \mathbb{P}^3 \arrow[dl, dashed, "\zeta_1"'] & \mathbb{P}\Big(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)\Big) \arrow[l, "\pi_1"'] \arrow[dll, "\omega_1"] \\
 & & \mathbb{P}^2 & &
\end{tikzcd}$$

commutes, where α_i is a blow up of the point P_i, the morphism β_i contracts a surface

$$\mathbb{P}^2 \cong E_i \subset X$$

to the point that dominates $P_i \in Q$, the map ξ_i is a projection from P_i, the map ζ_i is a projection from the image of P_i, the morphism δ_i is a contraction of a surface

$$\mathbb{F}_2 \cong G_i \subset U_i$$

to a conic $C_i \subset \mathbb{P}^3$, the morphism π_i is a blow up of the image of P_i, the morphism γ_i contracts the proper transform of G_i to the proper transform of C_i, and ω_i is a natural projection.

The map $\gamma_1 \circ \gamma_2^{-1}$ is an elementary transformation of a conic bundle (see [166]), and

$$\delta_1 \circ \beta_2\big(E_1\big) \subset \mathbb{P}^3 \supset \delta_2 \circ \beta_1\big(E_2\big)$$

are planes that contain the conics C_1 and C_2, respectively.

Let H be a general hyperplane section of Q such that $P_1 \in H \ni P_2$. Then

$$-K_X \sim 3\bar{H} + E_1 + E_2,$$

where $\bar{H}$ is the proper transform of H on the threefold X. We see that $\mathrm{lct}(X) \leqslant 1/3$.

We suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/3$. Note that

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq E_1 \cup E_2,$$

because $\mathrm{lct}(Q) = 1/3$. By Theorem 2.7, we may assume that

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq E_1.$$

Let $\bar{G}_2 \subset X$ be a proper transform of G_2. Then $\bar{G}_2 \cap E_1 = \varnothing$, because $\alpha_2(G_2) \subset Q$ is a quadric cone whose vertex is the point P_2, and the line L is not contained in Q. Hence

$$\varnothing \neq \mathrm{LCS}\Big(\mathbb{P}\Big(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)\Big),\ \lambda\gamma_2\big(D\big)\Big) \subseteq \gamma_2\big(E_1\big),$$

where $\gamma_2(E_1)$ is a section of ω_1. Applying Theorem 2.28 to ω_1, we obtain a contradiction. $\square$

Lemma 9.21. Suppose that $\daleth(X) = 3.20$. Then $\mathrm{lct}(X) = 1/3$.

Proof. Let $Q \subset \mathbb{P}^4$ be a smooth quadric threefold, and let

$$W \subset \mathbb{P}^2 \times \mathbb{P}^2$$

be a smooth divisor of bi-degree $(1,1)$. Let $L_1 \subset Q \supset L_2$ be lines such that $L_1 \cap L_2 = \varnothing$; then there exists a commutative diagram

$$\begin{tikzcd}
 & & W \arrow[dddll, "\upsilon_1"'] \arrow[dddrr, "\upsilon_2"] & & \\
 & & X \arrow[u, "\omega"'] \arrow[dl, "\beta_2"'] \arrow[dr, "\beta_1"] & & \\
 & V_1 \arrow[dl, "\pi_1"] \arrow[dr, "\alpha_1"'] & & V_2 \arrow[dl, "\alpha_2"] \arrow[dr, "\pi_2"'] & \\
\mathbb{P}^2 & & Q \arrow[ll, dashed, "\psi_1"] \arrow[rr, dashed, "\psi_2"'] & & \mathbb{P}^2
\end{tikzcd}$$

where the morphisms α_i and β_i are blow ups of the line L_i and its proper transform, respectively, the morphism ω is a blow up of a smooth curve $C \subset W$ of bi-degree $(1,1)$, the morphisms υ_i and π_i are natural $\mathbb{P}^1$-bundles, and the map ψ_i is a linear projection from the line L_i.

Let $\bar{H}$ be the exceptional divisor of ω, and let E_i be the exceptional divisor of β_i. Then

$$-K_X \sim 3\bar{H} + 2E_1 + 2E_2,$$

because $\alpha_2 \circ \beta_1(\bar{H}) \subset Q$ is a hyperplane section that contains L_1 and L_2. Hence $\mathrm{lct}(X) \leqslant 1/3$.

We suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/3$. Note that

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq E_1 \cap E_2 \cap \bar{H} = \varnothing,$$

because $\mathrm{lct}(V_1) = \mathrm{lct}(V_2) = 1/3$ by Lemma 8.17 and $\mathrm{lct}(W) = 1/2$ by Theorem 7.1, which gives a contradiction. $\square$

Lemma 9.22. Suppose that $\gimel(X) = 3.21$. Then $\mathrm{lct}(X) = 1/3$.

Proof. Let $\pi_1\colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ and $\pi_2\colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ be natural projections. There is a morphism

$$\alpha\colon X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^2$$

that contracts a surface E to a curve C such that $\pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1)) \cdot C = 2$ and $\pi_2^*(\mathcal{O}_{\mathbb{P}^2}(1)) \cdot C = 1$.

The curve $\pi_2(C) \subset \mathbb{P}^2$ is a line. Therefore, there is a unique surface

$$H_2 \in \Big|\pi_2^*\Big(\mathcal{O}_{\mathbb{P}^2}(1)\Big)\Big|$$

such that $C \subset H_2$. Let H_1 be a fiber of the $\mathbb{P}^2$-bundle π_1. Then

$$-K_X \sim 2\bar{H}_1 + 3\bar{H}_2 + 2E,$$

where $\bar{H}_i \subset X$ is a proper transform of the surface H_i. In particular, $\mathrm{lct}(X) \leqslant 1/3$.

We suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some rational $\lambda < 1/3$. Note that

$$\mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq E,$$

because $\mathrm{lct}(\mathbb{P}^1 \times \mathbb{P}^2) = 1/3$ by Lemma 2.22. There is a commutative diagram

$$\begin{tikzcd}
 & & V & & \\
U_1 \arrow[urr, "\delta_1"] \arrow[d, "\omega_1"'] & & X \arrow[u, "\gamma"'] \arrow[ll, "\beta_1"'] \arrow[rr, "\beta_2"] \arrow[d, "\alpha"'] & & U_2 \arrow[ull, "\delta_2"'] \arrow[d, "\omega_2"] \\
\mathbb{P}^1 & & \mathbb{P}^1 \times \mathbb{P}^2 \arrow[ll, "\pi_1"'] \arrow[rr, "\pi_2"] & & \mathbb{P}^2
\end{tikzcd}$$

where V is a Fano threefold of index 2 with one ordinary double point $O \in V$ such that $-K_V^3 = 40$, the birational morphism β_i is a contraction of the surface $\bar{H}_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ to a smooth rational curve, the morphism δ_i contracts the curve $\beta_i(\bar{H}_2)$ to the point $O \in V$ such that the rational map

$$\delta_2 \circ \delta_1^{-1} \colon U_1 \dashrightarrow U_2$$

is a standard flop in $\beta_1(\bar{H}_2) \cong \mathbb{P}^1$, the morphism ω_1 is a fibration whose general fiber is $\mathbb{P}^1 \times \mathbb{P}^1$, the morphism ω_2 is a $\mathbb{P}^1$-bundle, and γ is a birational morphism such that $\gamma(\bar{H}_2) = O \in V$.

The variety V is a section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by a linear subspace of codimension 3. One has

$$-K_V \sim 2\Big(\gamma(\bar{H}_1) + \gamma(E)\Big),$$

and the divisor $\gamma(\bar{H}_1) + \gamma(E)$ is very ample. There is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & V \arrow[r, hook, "\zeta"'] & \mathbb{P}^6 \arrow[d, dashed, "\xi"] \\
\mathbb{P}^1 \times \mathbb{P}^2 \arrow[rr, hook, "\eta"'] & & \mathbb{P}^5
\end{tikzcd}$$

such that the embedding ζ is given by the linear system $|\gamma(\bar{H}_1) + \gamma(E)|$, the map ξ is a linear projection from the point O, the embedding η is given by the linear system $|H_1 + H_2|$.

It follows from [85, Theorem 3.6] (see [86, Theorem 3.13]) that $U_2 \cong \mathbb{P}\big(\mathcal{E}\big)$, where where $\mathcal{E}$ is a stable rank two vector bundle on $\mathbb{P}^2$ such that the sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}_2}\big(1\big) \longrightarrow \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}_2}\big(1\big) \longrightarrow 0$$

is exact, where $\mathcal{I}$ is an ideal sheaf of two general points on $\mathbb{P}^2$. One has $c_1(\mathcal{E}) = -1$ and $c_1(\mathcal{E}) = 2$, and $\mathcal{E}$ is a Hulsbergen bundle (see [80]). It follows from [85, Theorem 3.5] that

$$U_1 \subset \mathbb{P}\Big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}\big(1\big) \oplus \mathcal{O}_{\mathbb{P}^1}\big(1\big) \oplus \mathcal{O}_{\mathbb{P}^1}\big(1\big)\Big),$$

and $U_1 \in |2T - F|$, where T is a tautological bundle on $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, and F is a fiber of the projection $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$.

Either $\bar{H}_1$ is a smooth del Pezzo surface such that $K_{\bar{H}_1}^2 = 7$, or

$$\big|H_1 \cap C\big| = 1,$$

because $H_1 \cdot C = 2$. Applying Lemma 2.26 to the morphism $\omega_1 \circ \beta_1$ and the surface $\bar{H}_1$, we see that

- either $|H_1 \cap C| = 1$,
- or $H_1 \cap \mathrm{LCS}(X, \lambda D) = \varnothing$,

because $\mathrm{lct}(\bar{H}_1) = 1/3$ if $\bar{H}_1$ is smooth. So, there is a fiber L of the projection $E \to C$ such that

$$\mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq L$$

by Theorem 2.7. Put $\bar{C} = \bar{H}_2 \cap E$ and $P = L \cap \bar{C}$. Applying Theorem 2.28 to ω_2 and

$$\Big(U_2,\ \lambda\beta_2\big(D\big)\Big),$$

we see that either $\mathrm{LCS}(X, \lambda D) = P$ or $\mathrm{LCS}(X, \lambda D) = L$ by Theorem 2.7.

Suppose that $\mathrm{LCS}(X, \lambda D) = L$. Then

$$\mathrm{LCS}\Big(V,\ \lambda\gamma\big(D\big)\Big) = \gamma\big(L\big) \subset V \subset \mathbb{P}^6,$$

where $\gamma(L) \subset V \subset \mathbb{P}^6$ is a line, because $-K_V \cdot \gamma(L) = 2$ and $-K_V \sim_{\mathbb{Q}} \gamma(D)$. We have $\mathrm{Sing}(V) = O \in \gamma(L)$.

Let $S \subset V$ be a general hyperplane section of $V \subset \mathbb{P}^6$ such that $\gamma(L) \subset S$. Then

- the surface S is a del Pezzo surface such that $K_S^2 = 5$,
- the point O is an ordinary double point of the surface S,
- the surface S is smooth outside of the point $O \in \gamma(L)$,
- the equivalence $K_S \sim \mathcal{O}_{\mathbb{P}^6}(1)|_S$ holds,

which implies that S contains finitely many lines that intersect the line $\gamma(L)$.

Let $H\subset V$ be a general hyperplane section of $V\subset\mathbb{P}^6$. Put $Q=\gamma(L)\cap H$. Then

$$\mathrm{LCS}\Big(H,\ \lambda\gamma(D)\Big|_H\Big)=Q,$$

by Remark 2.3, which contradicts Lemma 5.2, because $\lambda<1/3$.

Thus, we see that $\mathrm{LCS}(X,\lambda D)=P\in\bar{C}$. Let F_1 be a general fiber of π_1. Then

$$F_1\cap C=P_1\cup P_2\not\ni\alpha\big(P\big),$$

where $P_1\neq P_2$ are two points of the curve C. One has

$$P_1\cup P_2\subset H_2\cap F_1,$$

because $C\subset H_2$. Let Z be a general line in $F_1\cong\mathbb{P}^2$ such that $P_1\in Z$. Then there is a surface

$$F_2\in\Big|\pi_2^*\Big(\mathcal{O}_{\mathbb{P}^2}(1)\Big)\Big|$$

such that $Z\subset F_2$. Let $\bar{F}_1\subset X\supset\bar{F}_2$ be the proper transforms of F_1 and F_2, respectively. Then

$$P\not\in\bar{F}_1\cup\bar{F}_2.$$

Let $\bar{Z}\subset X$ be the proper transform of the curve Z. Then $-K_X\cdot\bar{Z}=2$ and

$$\bar{Z}\subset\bar{F}_1\cap\bar{F}_2,$$

but $\bar{Z}\cap\bar{H}_2=\varnothing$. Thus, the curve $\gamma(\bar{Z})$ is a line on $V\subset\mathbb{P}^6$ such that $\mathrm{Sing}(V)=O\not\in\gamma(\bar{Z})$.

Let T be a general hyperplane section of the threefold $V\subset\mathbb{P}^6$ such that $\gamma(\bar{Z})\subset T$. Then

$$\bar{T}\sim2\bar{H}_2+\bar{H}_1+E\sim2\bar{H}_2+\bar{F}_1+E\sim2\bar{F}_2+\bar{F}_1-E,$$

where $\bar{T}$ is the proper transform of the surface T on the threefold X. Hence

$$\bar{F}_1+\bar{F}_2+\bar{T}\sim3\bar{F}_2+2\bar{F}_1-E\sim2\bar{H}_2+2\bar{H}_1+2E\sim-K_X,$$

and applying Theorem 2.7, we see that the locus

$$P\cup\bar{Z}=\mathrm{LCS}\left(X,\ \lambda D+\frac{2}{3}\Big(\bar{F}_1+\bar{F}_2+\bar{T}\Big)\right)$$

must be connected. But $P\not\in\bar{Z}$, which is a contradiction. □

Lemma 9.23. Suppose that $\beth(X)=3.22$. Then $\mathrm{lct}(X)=1/3$.

Proof. Let $\pi_1\colon\mathbb{P}^1\times\mathbb{P}^2\to\mathbb{P}^1$ and $\pi_2\colon\mathbb{P}^1\times\mathbb{P}^2\to\mathbb{P}^2$ be natural projections. There is a morphism

$$\alpha\colon X\longrightarrow\mathbb{P}^1\times\mathbb{P}^2$$

that contracts a surface E to a curve C contained in a fiber H_1 of π_1 such that $\pi_2(C)$ is a conic.

We have $E\cong\mathbb{F}_2$. Let H_2 be a general surface in $|\pi_2^*(\mathcal{O}_{\mathbb{P}^2}(1))|$. The equivalence

$$-K_X\sim2\bar{H}_1+3\bar{H}_2+E$$

holds, where $\bar{H}_i\subset X$ is a proper transform of the surface H_i. Hence $\mathrm{lct}(X)\leqslant1/3$.

We suppose that $\mathrm{lct}(X)<1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the log pair $(X,\lambda D)$ is not log canonical for some rational $\lambda<1/3$. Note that

$$\mathrm{LCS}\Big(X,\lambda D\Big)\subseteq E,$$

since $\mathrm{lct}(\mathbb{P}^1\times\mathbb{P}^2)=1/3$ by Lemma 2.22.

Let Q be the unique surface in $|\pi_2^*(\mathcal{O}_{\mathbb{P}^2}(2))|$ such that $C\subset Q$, and let $\bar{Q}\subset X$ be the proper transform of the surface Q. Then $\bar{Q}\cap\bar{H}_1=\varnothing$, and there is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\beta"] \arrow[d, "\alpha"] & \mathbb{P}\Big(\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(2)\Big) \arrow[d, "\phi"] \arrow[dr, "\gamma"] & \\
\mathbb{P}^1\times\mathbb{P}^2 \arrow[r, "\pi_2"] & \mathbb{P}_2 & \mathbb{P}\big(1,1,1,2\big) \arrow[l, dashed, "\psi"]
\end{tikzcd}$$

such that β is a contraction of $\bar{Q}$ to a curve, γ is a contraction of the surface $\beta(\bar{H}_1)$ to a point, the morphism ϕ is a natural $\mathbb{P}^1$-bundle, and the map ψ is a natural projection. One has

$$\gamma\circ\beta\big(D\big)\sim_{\mathbb{Q}}\frac{5\gamma\circ\beta\big(E\big)}{2}\sim_{\mathbb{Q}}-K_{\mathbb{P}(1,1,1,2)}\sim_{\mathbb{Q}}\mathcal{O}_{\mathbb{P}(1,1,1,2)}\big(5\big),$$

which implies that $E\not\subseteq \mathrm{LCS}(X,\lambda D)$, because $\lambda<1/3$.

Applying Theorem 2.28 to ϕ, we see that there is a fiber F of the projection $E\to C$ such that

$$\varnothing\neq\mathrm{LCS}\Big(X,\ \lambda D\Big)\subseteq\Big(E\cap\bar{Q}\Big)\cup F,$$

including the possibility that $\mathrm{LCS}(X,\lambda D)\subset E\cap\bar{Q}$.

Suppose that $\mathrm{LCS}(X,\lambda D)\subset E\cap\bar{Q}$. Let $M\subset\mathbb{P}^1\times\mathbb{P}^2$ be a general surface in $|H_1+H_2|$, and let $\bar{M}\subset X$ be the proper transform of the surface M. Then

$$\bar{M}\cap\bar{H}_1=L,$$

where L is a line on $\bar{H}_1\cong\mathbb{P}^2$. Let R be the unique surface in $|\pi_2^*(\mathcal{O}_{\mathbb{P}^2}(1))|$ such that $\alpha(L)\subset R$, and let $\bar{R}$ be a proper transform of the surface R on the threefold X. Then

$$\mathrm{LCS}\Big(X,\lambda D\Big)\cup L\subseteq\mathrm{LCS}\left(X,\lambda D+\frac{2}{3}\Big(\bar{M}+\bar{H}_1+\bar{R}+\bar{H}_2\Big)\right)\subseteq\mathrm{LCS}\Big(X,\lambda D\Big)\cup L\cup\bar{H}_1,$$

but $L\cap E\cap\bar{Q}=\bar{Q}\cap\bar{H}_1=\varnothing$ and $-K_X\sim\bar{M}+\bar{H}_1+\bar{R}+\bar{H}_2$, which contradicts Theorem 2.7.

Therefore, we see that $F\subseteq\mathrm{LCS}(X,\lambda D)$. Put $\breve{F}=\gamma\circ\beta(F)$ and $\breve{D}=\gamma\circ\beta(D)$. Then

$$\breve{F}\subseteq\mathrm{LCS}\Big(\mathbb{P}\big(1,1,1,2\big),\ \lambda\breve{D}\Big)\subseteq\breve{C}\cup\breve{F},$$

where $\breve{C}=\gamma\circ\beta(\bar{Q})\subset\mathbb{P}(1,1,1,2)$ is a curve such that $\psi(\breve{C})=\pi_2(C)$.

Let S be a general surface in $|\mathcal{O}_{\mathbb{P}(1,1,1,2)}(2)|$. Then $S\cong\mathbb{P}^2$ and

$$\breve{F}\cap S\subseteq\mathrm{LCS}\Big(S,\ \lambda\breve{D}\Big|_S\Big)\subseteq\Big(\breve{C}\cup\breve{F}\Big)\cap S;$$

but $3D|_S\sim_{\mathbb{Q}}-5K_S$, which is impossible by Lemma 2.8. ☐

Lemma 9.24. Suppose that $\daleth(X)=3.23$. Then $\mathrm{lct}(X)=1/4$.

Proof. Let $O\in\mathbb{P}^3$ be a point, let $C\subset\mathbb{P}^3$ be a conic such that $O\in C$, let $\Pi\subset\mathbb{P}^3$ be a unique plane such that $C\subset\Pi$, and let $Q\subset\mathbb{P}^4$ be a smooth quadric threefold. Then the diagram

$$\begin{tikzcd}
 & & X \arrow[ddll, "\delta"'] \arrow[d, "\phi"] \arrow[ddrr, "\eta"] & & \\
 & & U \arrow[dl, "\omega"'] \arrow[dr, "\upsilon"] & & \\
V_7 \arrow[r, "\pi"] \arrow[drr, "\alpha"'] & \mathbb{P}^2 & & Q \arrow[ll, dashed, "\zeta"] \arrow[dl, dashed, "\xi"] & Y \arrow[l, "\gamma"'] \arrow[dll, "\beta"] \\
 & & \mathbb{P}^3 \arrow[ul, dashed, "\psi"] & &
\end{tikzcd}$$

commutes, where we have the following notation:

- the morphism α is a blow up of the point O with an exceptional divisor E;
- the morphism π is a natural $\mathbb{P}^1$-bundle;
- the morphisms β and δ are blow ups of C and its proper transform, respectively;
- the morphism γ contracts the proper transform of the plane Π to a point;
- the morphism ϕ contracts the proper transform of the plane Π to a curve;
- the morphism η contracts the proper transform of E to a curve $L\subset Y$ such that
$$\gamma\big(\Pi\big)\in\gamma\big(L\big)\subset Q\subset\mathbb{P}^4$$
and $\gamma(L)$ is a line in $\mathbb{P}^4$;

- the morphism ω is a natural $\mathbb{P}^1$-bundle;
- the morphism υ is a blow up of the line $\gamma(L)$;
- the maps ψ, ξ and ζ are projections from O, $\gamma(\Pi)$ and $\gamma(L)$, respectively.

Note that E is a section of π.

Let $\bar{\Pi} \subset X$ be a proper transform of $\Pi \subset \mathbb{P}^3$. Then $\mathrm{lct}(X) \leqslant 1/4$, because

$$-K_X \sim 4\bar{\Pi} + 2\bar{E} + 3G,$$

where $\bar{E}$ and G are exceptional surfaces of η and δ, respectively.

We suppose that $\mathrm{lct}(X) < 1/4$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/4$. Note that

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq \bar{E} \cap \bar{\Pi} \cap G,$$

because $\mathrm{lct}(V_7) = 1/4$ by Lemma 9.26, $\mathrm{lct}(Y) = 1/4$ by Lemma 8.16 and $\mathrm{lct}(U) = 1/3$ by Lemma 8.17.

Let $R \subset \mathbb{P}^3$ be a general cone over C whose vertex is $P \in \mathbb{P}^3$, let $H_1 \subset \mathbb{P}^3$ be a general plane such that $O \in H_1 \ni P$, and let $H_2 \subset \mathbb{P}^3$ be a general plane such that $P \in H_2$. Then

$$\bar{R} \sim \big(\alpha \circ \delta\big)^*\big(R\big) - \bar{E} - G,\ \bar{H}_1 \sim \big(\alpha \circ \delta\big)^*\big(H_1\big) - \bar{E},\ \bar{H}_2 \sim \big(\alpha \circ \delta\big)^*\big(H_2\big),$$

where $\bar{R}$, $\bar{H}_1$, $\bar{H}_2$ are proper transforms of R, H_1, H_2 on the threefold X, respectively. One has

$$-K_X \sim \bar{Q} + \bar{H}_1 + \bar{H}_2,$$

but it follows from the generality of R, H_1, H_2 that the locus

$$\mathrm{LCS}\left(X,\ \lambda D + \frac{3}{4}\Big(\bar{Q} + \bar{H}_1 + \bar{H}_2\Big)\right) = \mathrm{LCS}\Big(X,\ \lambda D\Big) \cup P,$$

is disconnected, which is impossible by Theorem 2.7. □

Lemma 9.25. Suppose that $\gimel(X) = 3.24$. Then $\mathrm{lct}(X) = 1/3$.

Proof. Let W is a divisor of bi-degree $(1,1)$ on $\mathbb{P}^2 \times \mathbb{P}^2$. There is a commutative diagram

$$\begin{tikzcd}
& X \arrow[r, "\alpha"] \arrow[dl, "\zeta"'] \arrow[d, "\pi"] & W \arrow[d, "\omega_1"] \\
\mathbb{P}^1 & \mathbb{F}_1 \arrow[l, "\xi"] \arrow[r, "\gamma"] & \mathbb{P}^2
\end{tikzcd}$$

where ω_1 is a natural $\mathbb{P}^1$-bundle, the morphism α contracts a smooth surface

$$E \cong \mathbb{P}^1 \times \mathbb{P}^1$$

to a fiber L of ω_1, γ is a blow up of the point $\omega_1(L)$, the morphism ξ is a $\mathbb{P}^1$-bundle, and ζ is a $\mathbb{F}_1$-bundle.

Let $\omega_2 \colon X \to \mathbb{P}^2$ be a natural $\mathbb{P}^1$-bundle that is different from ω_1. Then there is a surface

$$G \in \Big|\omega_2^*\Big(\mathcal{O}_{\mathbb{P}^2}(1)\Big)\Big|$$

such that $L \subset G$, because $\omega_2(L)$ is a line. Let $\bar{G} \subset X$ be a proper transform of G. Then

$$-K_X \sim 2F + 2\bar{G} + 3E,$$

where E is the exceptional divisor of α, and F is a fiber of ζ. We see that $\mathrm{lct}(X) \leqslant 1/3$.

We suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/3$. Note that

$$\varnothing \neq \mathrm{LCS}\Big(X, \lambda D\Big) \subseteq E,$$

since $\mathrm{lct}(W) = 1/2$ by Theorem 7.1. We may assume that $F \cap \mathrm{LCS}(X, \lambda D) \neq \varnothing$. Then

$$\mathbb{F}_1 \cong F \subseteq \mathrm{LCS}\Big(X, \lambda D\Big) \subseteq E \cong \mathbb{P}^1 \times \mathbb{P}^1$$

by Lemma 2.26, because $\mathrm{lct}(F) = 1/3$ (see Example 1.18), which is a contradiction. □

Lemma 9.26. Suppose that $\daleth(X) = 3.25$. Then $\mathrm{lct}(X) = 1/3$.

Proof. Let $L_1 \subset \mathbb{P}^3 \supset L_2$ be lines such that $L_1 \cap L_2 = \varnothing$. Then there is a commutative diagram

$$\begin{tikzcd}
 & \mathbb{P}^3 & \\
V_1 \arrow[ur, "\alpha_1"] \arrow[d, "\omega_1"'] & X \arrow[l, "\beta_2"'] \arrow[r, "\beta_1"] \arrow[d, "\omega"'] & V_2 \arrow[ul, "\alpha_2"'] \arrow[d, "\omega_2"] \\
\mathbb{P}^1 & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[l, "\gamma_1"] \arrow[r, "\gamma_2"'] & \mathbb{P}^1
\end{tikzcd}$$

where the morphisms α_i and β_i are blow ups of the line L_i and its proper transform, respectively, the morphism ω_i is a natural $\mathbb{P}^2$-bundle, the morphisms ω and γ_i are $\mathbb{P}^1$-bundles. Note that

$$V_1 \cong V_2 \cong \mathbb{P}\Big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\Big).$$

Let H_1 and H_2 be proper transforms on X of planes in $\mathbb{P}^3$ such that $L_i \subset \alpha(H_i)$. Then

$$-K_X \sim 2H_1 + 2H_2 + E_1 + E_2 \sim 3H_1 + H_2 + 2E_1 \sim H_1 + 3H_2 + 2E_2,$$

where E_i is an exceptional divisors of β_i. Hence $\mathrm{lct}(X) \leqslant 1/3$.

We suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/3$.

Applying Lemma 2.26 to the $\mathbb{F}_1$-fibrations $\omega_2 \circ \beta_1$ and $\omega_1 \circ \beta_2$, we obtain a contradiction, because the equality $\mathrm{lct}(\mathbb{F}_1) = 1/3$ holds (see Example 1.18). $\square$

Remark 9.27. Actually, the result of Lemma 9.26 is contained in Corollary 6.4, but we still prefer to give a detailed proof that may have further applications.

Lemma 9.28. Suppose that $\daleth(X) = 3.30$. Then $\mathrm{lct}(X) = 1/4$.

Proof. Let $O \in \mathbb{P}^3$ be a point, and let $\gamma\colon V_7 \to \mathbb{P}^3$ be a blow up of the point O. Then there is a $\mathbb{P}^1$-bundle $\pi\colon V_7 \to \mathbb{P}^1$. Take a line $O \in L \subset \mathbb{P}^3$; then the diagram

$$\begin{tikzcd}
 & & \mathbb{F}_1 \arrow[dll, "\upsilon"'] \arrow[drr, "\tau"] & & \\
\mathbb{P}^1 & & X \arrow[u, "\zeta"'] \arrow[dl, "\beta"'] \arrow[dr, "\xi"] & & \mathbb{P}^2 \\
 & V \arrow[ul, "\eta"] \arrow[dr, "\alpha"'] & & V_7 \arrow[ur, "\pi"'] \arrow[dl, "\gamma"] & \\
 & & \mathbb{P}^3 \arrow[uull, dashed, bend left, "\phi"] \arrow[uurr, dashed, bend right, "\psi"'] & &
\end{tikzcd}$$

commutes, where α and ξ are blow ups of the line L and its proper transforms, respectively, the morphism η is is a natural $\mathbb{P}^2$-bundle, the morphism β is a blow up of the curve

$$\mathbb{P}^1 \cong C \subset V \cong \mathbb{P}\Big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\Big)$$

such that $\beta(C) = O$, the morphisms ζ and υ are $\mathbb{P}^1$-bundles, the maps ϕ and ψ are linear projections from L and O, respectively, and τ is a blow up of the point $\psi(L) \in \mathbb{P}^2$.

Let T be the proper transform on X of a general plane in $\mathbb{P}^3$ that passes through $L \subset \mathbb{P}^3$, and let G be the exceptional divisor of the blow up β. Then

$$-K_X \sim 4T + 3E + 2G,$$

where E is the proper transform on X of the exceptional divisor of α. In particular, $\mathrm{lct}(X) < 1/4$.

We suppose that $\mathrm{lct}(X) < 1/4$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some rational number $\lambda < 1/4$. Note that

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq G,$$

because $\mathrm{lct}(V)=1/4$ by Lemma 7.2. However, every fiber of the morphism $\eta\circ\beta$ is isomorphic to $\mathbb{F}_1$, which is impossible by Lemma 2.26, because $\mathrm{lct}(\mathbb{F}_1)=1/3$ by Example 1.18. $\square$

The proof of Lemma 9.28 implies the following corollary (cf. [176, Example 3.3]).

Corollary 9.29. Suppose that $\daleth(X)=4.12$. Then $\mathrm{lct}(X)=1/4$.

Remark 9.30. Actually, the results of Lemma 9.28 and Corollary 9.29 are contained in Corollary 6.4, but we still prefer to give a detailed proof that may have further applications.

10. Fano threefolds with $\rho\geqslant 4$

We use the assumptions and notation introduced in section 1.

Lemma 10.1. Suppose that $\daleth(X)=4.1$. Then $\mathrm{lct}(X)=1/2$.

Proof. The threefold X is a divisor on $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ of multidegree $(1,1,1,1)$. Let

$$\Big[\big(x_1:y_1\big),\ \big(x_2:y_2\big),\ \big(x_3:y_3\big),\ \big(x_4:y_4\big)\Big]$$

be coordinates on $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$. Then X is given by the equation

$$F\big(x_1,y_1,x_2,y_2,x_3,y_3,x_4,y_4\big)=0,$$

where F is a of multidegree $(1,1,1,1)$.

Let $\pi_1\colon X\to\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ be a projection given by

$$\Big[\big(x_1:y_1\big),\ \big(x_2:y_2\big),\ \big(x_3:y_3\big),\ \big(x_4:y_4\big)\Big]\mapsto\Big[\big(x_2:y_2\big),\ \big(x_3:y_3\big),\ \big(x_4:y_4\big)\Big]\in\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1,$$

and let π_2, π_3 and $\pi_4\colon X\to\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ be projections defined in a similar way. Put

$$F=x_1G\big(x_2,y_2,x_3,y_3,x_4,y_4\big)+y_1H\big(x_2,y_2,x_3,y_3,x_4,y_4\big),$$

where $G(x_2,y_2,x_3,y_3,x_4,y_4)$ and $H(x_2,y_2,x_3,y_3,x_4,y_4)$ are multi-linear forms that do not depend on x_1 and y_1. Then π_1 is a blow up of a curve $C_1\subset\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ given by the equations

$$G\big(x_2,y_2,x_3,y_3,x_4,y_4\big)=H\big(x_2,y_2,x_3,y_3,x_4,y_4\big)=0,$$

which define a surface $E_1\subset\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ that is contracted by π_1. The equations

$$x_1=H\big(x_2,y_2,x_3,y_3,x_4,y_4\big)=0$$

define a divisor $H_1\subset X$ such that $-K_X\sim 2H_1+E_1$, which implies that $\mathrm{lct}(X)\leqslant 1/2$.

We suppose that $\mathrm{lct}(X)<1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the log pair $(X,\lambda D)$ is not log canonical for some positive rational number $\lambda<1/2$.

Let E_2, E_3, E_4 be surfaces in X defined in a way similar to E_1. Then

$$\varnothing\ne\mathrm{LCS}\Big(X,\ \lambda D\Big)\subseteq E_1\cap E_2\cap E_3\cap E_4,$$

because $\mathrm{lct}(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1)=1/2$ by Lemma 2.22. But $E_i\subset\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ is given by

$$\frac{\partial F\big(x_1,y_1,x_2,y_2,x_3,y_3,x_4,y_4\big)}{\partial x_i}=\frac{\partial F\big(x_1,y_1,x_2,y_2,x_3,y_3,x_4,y_4\big)}{\partial y_i}=0,$$

which implies that the intersection $E_1\cap E_2\cap E_3\cap E_4$ is given by the equations

$$\frac{\partial F}{\partial x_1}=\frac{\partial F}{\partial y_1}=\frac{\partial F}{\partial x_2}=\frac{\partial F}{\partial y_2}=\frac{\partial F}{\partial x_3}=\frac{\partial F}{\partial y_3}=\frac{\partial F}{\partial x_4}=\frac{\partial F}{\partial y_4}=0.$$

Hence $E_1\cap E_2\cap E_3\cap E_4=\mathrm{Sing}(X)=\varnothing$, and $\mathrm{LCS}(X,\ \lambda D)=\varnothing$. $\square$

Lemma 10.2. Suppose that $\daleth(X)=4.2$. Then $\mathrm{lct}(X)=1/2$.

Proof. Let $Q_1 \subset \mathbb{P}^4 \supset Q_2$ be quadric cones, whose vertices are $O_1 \in \mathbb{P}^4 \ni O_2$, respectively. Let

$$O_1 \notin S_1 \subset Q_1 \subset \mathbb{P}^4$$

be a hyperplane section of Q_1, and let $C_1 \subset |-K_{S_1}|$ be a smooth elliptic curve. Then the diagram

$$\begin{tikzcd}
 & & X \arrow[dl, "\beta_1"'] \arrow[dr, "\beta_2"] \arrow[dddll, bend right, "\zeta_1"'] \arrow[dddrr, bend left, "\zeta_2"] & & \\
 & U_1 \arrow[d, "\alpha_1"] \arrow[ddl, "\eta_1"'] \arrow[ddr, bend left, "\gamma_1"] & & U_2 \arrow[d, "\alpha_2"'] \arrow[ddr, "\eta_2"] \arrow[ddl, bend right, "\gamma_2"'] & \\
 & Q_1 \arrow[dr, dashed, "\psi_1"'] & & Q_2 \arrow[dl, dashed, "\psi_2"] & \\
\mathbb{P}^1 & & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[ll, "\pi_1"] \arrow[rr, "\pi_2"'] & & \mathbb{P}^1
\end{tikzcd}$$

commutes, where $\pi_1 \neq \pi_2$ are natural projections, the map ψ_i is a projection from $O_i \in Q_i \subset \mathbb{P}^4$, the morphism α_i is a blow up of the vertex O_i, the morphism β_i contracts a surface

$$\mathbb{P}^1 \times C_1 \cong G_i \subset X$$

to a curve $C_1 \cong C_i \subset U_i$, the morphism η_i is an $\mathbb{F}_1$-bundle, γ_i is a $\mathbb{P}^1$-bundle, and ζ_i is a fibration into del Pezzo surfaces of degree 6 that has 4 singular fibers.

Let $E_i \subset X$ be the proper transform of the exceptional divisor of α_i. Then

$$S_1 = \alpha_1 \circ \beta_1\big(E_2\big) \subset Q_1 \subset \mathbb{P}^4 \supset Q_2 \supset \alpha_2 \circ \beta_2\big(E_1\big)$$

are hyperplane sections that contain C_1 and C_2, respectively. It is also easy to see that

$$\alpha_1 \circ \beta_1\big(G_2\big) \subset Q_1 \subset \mathbb{P}^4 \supset Q_2 \supset \alpha_2 \circ \beta_2\big(G_1\big)$$

are the cones in $\mathbb{P}^4$ over the curves C_1 and C_2, respectively.

Let $\bar{H} \subset X$ be the proper transform of a hyperplane section of $Q_1 \subset \mathbb{P}^4$ that contains O_1. Then

$$-K_X \sim 2\bar{H} + E_2 + E_1,$$

which gives $\mathrm{lct}(X) \leqslant 1/2$. Suppose that $\mathrm{lct}(X) < 1/2$. Then there is an effective $\mathbb{Q}$-divisor

$$D \sim_{\mathbb{Q}} -K_X \sim E_1 + E_2 + G_1 + G_2$$

such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$. Put

$$D = \mu_1 E_1 + \mu_2 E_2 + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor on X such that

$$E_1 \not\subseteq \mathrm{Supp}\big(\Omega\big) \not\supseteq E_2.$$

Let Γ be a general fiber of the conic bundle $\gamma_1 \circ \beta_1$. Then

$$2 = \Gamma \cdot D = \Gamma \cdot \Big(\mu_1 E_1 + \mu_2 E_2 + \Omega\Big) = \mu_1 + \mu_2 + \Gamma \cdot \Omega \geqslant \mu_1 + \mu_2,$$

and without loss of generality we may assume that $\mu_1 \leqslant \mu_2$. Then $\mu_1 \leqslant 1$.

Suppose that there is a surface $S \in \mathbb{LCS}(X, \lambda D)$. Then $S \neq E_1$. Moreover, we have $S \neq G_1$, because $\alpha_2 \circ \beta_2(G_1)$ is a quadric surface and $\lambda < 1/2$. Hence $S \cap E_1 \neq \varnothing$. But

$$-\frac{1}{2}K_{E_1} \sim_{\mathbb{Q}} D\Big|_{E_1} \sim_{\mathbb{Q}} -\frac{\mu_1}{2}K_{E_1} + \Omega\Big|_{E_1},$$

and $E_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$, which is impossible by Theorem 2.20 and Lemma 2.24.

We see that the set $\mathbb{LCS}(X, \lambda D)$ contains no surfaces. Let $P \in \mathrm{LCS}(X, \lambda D)$ be a point.

Suppose that $P \notin G_1$. Let Z be a fiber of γ_1 such that $\beta_1(P) \in Z$. Then

$$Z \subseteq \mathrm{LCS}\Big(U_1,\ \lambda\beta_1\big(D\big)\Big)$$

by Theorem 2.28. Put $\bar{E}_1 = \beta_1(E_1)$. Then we have

$$Z \cap \bar{E}_1 \in \mathrm{LCS}\Big(\bar{E}_1,\ \lambda\Omega\Big|_{\bar{E}_1}\Big)$$

by Theorem 2.20, which is impossible by Lemma 2.24, because $\mu_1 \leqslant 1$.

Thus, we see that $P \in G_1$. Let $F_1 \subset X \supset F_2$ be fibers of ζ_1 and ζ_2 passing through the point P. Then either F_1 or F_2 is smooth, because $\alpha_1(P) \in C_1$. But

$$\mathrm{lct}\big(F_i\big) = 1/2$$

in the case when F_i is smooth (see Example 1.18), which contradicts Lemma 2.26. □

Lemma 10.3. Suppose that $\beth(X) = 4.3$. Then $\mathrm{lct}(X) = 1/2$.

Proof. Let $F_1 \cong F_2 \cong F_3 \cong \mathbb{P}^1 \times \mathbb{P}^1$ be fibers of three different projections

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1,$$

respectively. There is a contraction $\alpha\colon X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of a surface $E \subset X$ to a curve

$$C \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

such that $C \cdot F_1 = C \cdot F_2 = 1$ and $C \cdot F_3 = 2$. There is a smooth surface

$$\mathbb{P}^1 \times \mathbb{P}^1 \cong G \in \big|F_1 + F_2\big|$$

such that $C \subset G$. In particular, we see that

$$-K_X \sim 2\bar{G} + E + \bar{F}_3,$$

where $\bar{F}_3$ and $\bar{G}$ are proper transforms of F_3 and G, respectively. Hence $\mathrm{lct}(X) \leqslant 1/2$.

We suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$. Note that

$$\varnothing \neq \mathrm{LCS}\Big(X, \lambda D\Big) \subseteq E,$$

because $\mathrm{lct}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = 1/2$ and $\alpha(D) \sim_{\mathbb{Q}} -K_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}$.

There is a smooth surface $H \in |3F_1 + F_3\big|$ such that $C = G \cap H$. Let $\bar{H}$ be a proper transform of the surface H on the threefold X. Then $\bar{H} \cap \bar{G} = \varnothing$ and there is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\phi"'] & & X \arrow[ll, "\gamma"'] \arrow[rr, "\beta"] \arrow[d, "\alpha"] & & V \arrow[d, "\pi"] \\
\mathbb{P}^1 \times \mathbb{P}^1 & & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[ll, "\zeta"'] \arrow[rr, "\xi"] & & \mathbb{P}^1 \times \mathbb{P}^1
\end{tikzcd}$$

such that β and γ are contractions of the surfaces $\bar{G}$ and $\bar{H}$ to smooth curves, the morphisms π and ϕ are $\mathbb{P}^1$-bundles, the morphisms ζ and ξ are projections that are given by the linear systems $|F_1 + F_2|$ and $|F_1 + F_3|$, respectively.

It follows from $\bar{H} \cap \bar{G} = \varnothing$ that

- either the log pair $(V, \lambda\beta(D))$ is not log canonical,
- or the log pair $(U, \lambda\gamma(D))$ is not log canonical.

Applying Theorem 2.28 to the log pairs $(V, \lambda\beta(D))$ or $(U, \lambda\gamma(D))$ (and the fibrations π or ϕ, respectively) and using Theorem 2.7, we see that

$$\mathrm{LCS}\Big(X,\ \lambda D\Big) = \Gamma,$$

where Γ is a fiber of the natural projection $E \to C$.

We may assume that $\alpha(\Gamma) \in F_3$. Let $\bar{F}_3 \subset X$ be the proper transform of the surface F_3. Put

$$D = \mu\bar{F}_3 + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor on X such that $\bar{F}_3 \not\subset \mathrm{Supp}(\Omega)$. Then

$$\mu F_3 + \alpha\big(\Omega\big) \sim_{\mathbb{Q}} 2\Big(F_1 + F_2 + F_3\Big),$$

which gives $\mu \leqslant 2$. Hence the log pair $(\bar{F}_3, \lambda\Omega|_{\bar{F}_3})$ is not log canonical along $\Gamma \subset \bar{F}_3$ by Theorem 2.20. But

$$\Omega\Big|_{\bar{F}_3} \sim_{\mathbb{Q}} -K_{\bar{F}_3},$$

and $\bar{F}_3$ is a del Pezzo surface such that $K^2_{\bar{F}_3} = 6$ and

- either $\bar{F}_3$ is smooth and $|C \cap F_3| = 2$;
- or $\bar{F}_3$ has one ordinary double point and $|C \cap F_3| = 1$.

We have $\mathrm{lct}(\bar{F}_3) \leqslant \lambda$. Then $\bar{F}_3$ is singular by Example 1.18. It follows from Lemma 5.5 that

$$\mathrm{LCS}\Big(\bar{F}_3,\ \lambda\Omega|_{\bar{F}_3}\Big) = \mathrm{Sing}\big(\bar{F}_3\big),$$

but the log pair $(\bar{F}_3, \lambda\Omega|_{\bar{F}_3})$ is not log canonical along the whole curve $\Gamma \subset \bar{F}_3$, which is a contradiction. $\square$

Lemma 10.4. Suppose that $\daleth(X) = 4.5$. Then $\mathrm{lct}(X) = 3/7$.

Proof. Let $Q \subset \mathbb{P}^4$ be a quadric cone, let $V \subset \mathbb{P}^6$ be a a section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by a linear subspace of dimension 6 such that V has one ordinary double point. Then the diagram

$$\begin{tikzcd}
 & & X \arrow[rr, "\theta"] \arrow[dr, "\eta"] \arrow[dd, "\gamma"] \arrow[dddd, "\mu"'] & & Y \arrow[rr, "\iota"] \arrow[drr, "\tau"] & & Q \arrow[d, dashed, "\xi"] & \\
V \arrow[urrrrrr, dashed, bend left, "\phi"] & & & \mathbb{P}^1 \times \mathbb{F}_1 \arrow[rrr, "\sigma"] \arrow[ddd, "\nu"] \arrow[dr, bend right, "\xi"'] & & & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dr, "\upsilon_2"] \arrow[ddddllllll, bend left=60, "\upsilon_1"] & \\
 & & & & & \mathbb{F}_1 \arrow[rr, "\chi"] \arrow[ddrr, bend left, "\zeta"] & & \mathbb{P}^1 \\
U_1 \arrow[uu, "\delta_1"] \arrow[dr, bend right, "\omega_1"'] & & U \arrow[ll, "\beta_1"'] \arrow[rrr, "\beta_2"] \arrow[ull, "\beta"] \arrow[dr, "\alpha"] & & & U_2 \arrow[drr, "\omega_2"] \arrow[uulllll, bend right, "\delta_2"'] & & \\
 & \mathbb{P}^1 & & \mathbb{P}^1 \times \mathbb{P}^2 \arrow[ll, "\pi_1"'] \arrow[rrrr, "\pi_2"] & & & & \mathbb{P}^2 \arrow[uu, dashed, "\psi"']
\end{tikzcd}$$

commutes (cf. [66, Lemma 2.6]), where we have the following notation:

- the morphisms π_i, υ_i, ξ and χ are natural projections;
- the morphism α contracts a surface $\mathbb{F}_3 \cong E \subset U$ to a curve C such that

$$\pi_1^*\Big(\mathcal{O}_{\mathbb{P}^1}(1)\Big) \cdot C = 2,\ \pi_2^*\Big(\mathcal{O}_{\mathbb{P}^2}(1)\Big) \cdot C = 1;$$

- the morphism β contracts a surface $\mathbb{P}^1 \times \mathbb{P}^1 \cong \bar{H}_2 \subset U$ to the singular point of V;
- the morphism β_i contracts the surface $\bar{H}_2$ to a smooth rational curve;
- the morphism δ_i contracts the curve $\beta_i(\bar{H}_2)$ to the singular point of V so that the map

$$\delta_2 \circ \delta_1^{-1} \colon U_1 \dashrightarrow U_2$$

  is a standard flop in the curve $\beta_1(\bar{H}_2) \cong \mathbb{P}^1$;
- the morphism ω_1 is a fibration whose general fiber is $\mathbb{P}^1 \times \mathbb{P}^1$;
- the morphisms ω_2, π_2, ξ, σ, τ are $\mathbb{P}^1$-bundles;
- the morphism ζ is a blow up of a point $O \in \mathbb{P}^2$ such that $O \notin \pi_2(C)$;
- the map ψ is a linear projection from the point $O \in \mathbb{P}^2$;
- the morphism ν contracts a surface $G \cong \mathbb{P}^1 \times \mathbb{P}^1$ to a curve L such that $\pi_2(L) = O$;
- the morphism γ contracts a surface $\breve{G}$ to a curve $\bar{L}$ such that

$$\alpha\big(\bar{L}\big) = L \subset \mathbb{P}^1 \times \mathbb{P}^2$$

  and the curve $\beta(\bar{L})$ is a line in $V \subset \mathbb{P}^6$ such that $\beta(\bar{L}) \cap \mathrm{Sing}(V) = \varnothing$;
- the morphism η contracts a surface $\breve{E}$ to a curve such that $\nu \circ \eta(\breve{E}) = C \subset \mathbb{P}^1 \times \mathbb{P}^2$;

- the morphism θ contracts a surface $\breve{R} \subset X$ to a curve such that $\breve{R} \neq \breve{E}$ and
$$\tau \circ \theta(\breve{R}) = \sigma \circ \eta(\breve{E}) \subset \mathbb{P}^1 \times \mathbb{P}^1;$$
- the morphism μ is a fibration into del Pezzo surfaces of degree 6;
- the morphism ι contracts the surface $\theta(\breve{H}_2)$ to the singular point of the quadric Q;
- the map ϕ is a linear projection from the line $\beta(\bar{L}) \subset V \subset \mathbb{P}^6$.

The curve $\pi_2(C) \subset \mathbb{P}^2$ is a line. Then $\alpha(\bar{H}_2) \in |\pi_2^*(\mathcal{O}_{\mathbb{P}^2}(1))|$ and $C \subset \alpha(\bar{H}_2)$.

The morphism π_1 induces a double cover $C \to \mathbb{P}^1$ branched in two points $Q_1 \in C \ni Q_2$. Let
$$T_i \in \Big|\pi_1^*\Big(\mathcal{O}_{\mathbb{P}^1}(1)\Big)\Big|$$
be the unique surface such that $Q_i \in T_i$. Let $\bar{T}_i \subset U$ be the proper transform of T_i. Then
- the surface $\bar{T}_i$ has one ordinary double point,
- the surface $\bar{T}_i$ is tangent to the surface E along the curve $E \cap \bar{T}_i$,
- the surface $\bar{T}_i$ is a del Pezzo surface such that $K^2_{\bar{T}_i} = 7$.

Let $Z_i \subset \mathbb{P}^2$ be the unique line such that $O \in Z \ni \pi_2 \circ \alpha(Q_i)$. Then there is a unique surface
$$\bar{R}_i \in \Big|(\pi_2 \circ \alpha)^*\Big(\mathcal{O}_{\mathbb{P}^2}(1)\Big)\Big|$$
such that $Z_i \subset \pi_2 \circ \alpha(\bar{R}_i)$. One has $\bar{L} \subset \bar{R}_i$ and
$$-K_U \sim 2\bar{H}_2 + \bar{R}_i + 2\bar{T}_i + E.$$
Let Γ_i be a fiber of the projection $E \to C$ over the point Q_i. Then $\Gamma_i = E \cap \bar{T}_i$ and
$$\Gamma_i \subset \mathrm{LCS}\left(U,\ \frac{3}{7}\Big(2\bar{H}_2 + \bar{R}_i + 2\bar{T}_i + E\Big)\right).$$
Let $\breve{R}_i$ and $\breve{T}_i$ be the proper transforms of $\bar{R}_i$ and $\bar{T}_i$ on the threefold X, respectively. Then
$$-K_X \sim 2\breve{H}_2 + \breve{R}_i + 2\breve{T}_i + \breve{E},$$
because $\bar{L} \subset \bar{R}_i$. Let $\breve{\Gamma}_i \subset X$ be the proper transform of the curve Γ_i. Then the log pair
$$\left(X,\ \frac{3}{7}\Big(2\breve{H}_2 + \breve{R}_i + 2\breve{T}_i + \breve{E}\Big)\right)$$
is log canonical but not log terminal. Thus, we see that $\mathrm{lct}(X) \leqslant 3/7$.

We suppose that $\mathrm{lct}(X) < 3/7$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some rational $\lambda < 3/7$.

The surfaces $\breve{T}_1$ and $\breve{T}_2$ are the only singular fibers of the fibration $\mu\colon X \to \mathbb{P}^1$. Then
$$\breve{T}_i \not\subseteq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subsetneq \breve{T}_1 \cup \breve{T}_2,$$
by Lemma 2.26, because $D \cdot Z = \breve{T}_1 = 2$, where Z is a general fiber of $\pi_2 \circ \alpha \circ \gamma$.

We may assume that $\mathrm{LCS}(X, \lambda D) \subseteq \breve{T}_1$ by Theorem 2.7.

Applying Theorem 2.28 to the log pair $(\mathbb{P}^1 \times \mathbb{F}_1, \lambda\eta(D))$, we see that
$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \neq \breve{T}_1 \cap \breve{G},$$
because $G = \eta(\breve{G})$ is a section of the $\mathbb{P}^1$-bundle σ.

Applying Theorem 2.28 to the log pair $(\mathbb{P}^1 \times \mathbb{P}^2, \lambda\alpha \circ \gamma(D))$, we see that
$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq \breve{T}_1 \cap \breve{E} = \breve{\Gamma}_1$$
by Theorem 2.7, because $\breve{G} \cap \breve{E} = \varnothing$ and T_1 is a section of π_2.

Applying Theorem 2.28 to the log pairs $(Y, \lambda\theta(D))$ and $(U_2, \lambda\beta_2 \circ \gamma(D))$ (and the fibrations τ and ω_2) we see that
$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) = \breve{\Gamma}_1,$$
because $\breve{R} \cap \breve{H}_2 = \varnothing$. Put $\bar{D} = \gamma(D)$. Then $\mathrm{LCS}(U, \lambda\bar{D}) = \Gamma_1$. Put
$$\bar{D} = \varepsilon\bar{H}_2 + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor such that $\bar{H}_2 \not\subseteq \mathrm{Supp}(\Omega)$. Then

$$\Omega\Big|_{\bar{H}_2} \sim_{\mathbb{Q}} -\frac{(1+\varepsilon)}{2} K_{\bar{H}_2},$$

and the log pair $(\bar{H}_2, \lambda\Omega|_{\bar{H}_2})$ is not log canonical by Theorem 2.20. The latter implies that

$$\frac{3}{7} \cdot \frac{1+\varepsilon}{2} > \lambda \frac{1+\varepsilon}{2} > 1/2$$

by Lemma 2.24, and hence $\varepsilon > 4/3$.

We may assume that either $E \not\subseteq \mathrm{Supp}(\bar{D})$ or $\bar{T}_1 \not\subseteq \mathrm{Supp}(\bar{D})$ by Remark 2.23.

Suppose that $E \not\subseteq \mathrm{Supp}(\bar{D})$. Let Z be a general fiber of the projection $E \to C$. Then

$$1 = -K_U \cdot Z = \bar{D} \cdot Z = \varepsilon + \Omega \cdot Z \geqslant \varepsilon,$$

which is a contradiction, because $\varepsilon > 4/3$. Thus, we see that $\bar{T}_1 \not\subseteq \mathrm{Supp}(\bar{D})$.

Let $\bar{\Delta} \subset \bar{T}_1$ be a proper transform of a general line in $T_1 \cong \mathbb{P}^2$ that passes through Q_1. Then

$$2 = -K_U \cdot \bar{\Delta} = \bar{D} \cdot \bar{\Delta} \geqslant \mathrm{mult}_{\Gamma_1}\big(\bar{D}\big) \geqslant 1/\lambda > 7/3,$$

because $\bar{\Delta} \not\subseteq \mathrm{Supp}(\bar{D})$ and $\bar{\Delta} \cap \Gamma_1 \neq \varnothing$. The obtained contradiction completes the proof. $\square$

Lemma 10.5. Suppose that $\daleth(X) = 4.6$. Then $\mathrm{lct}(X) = 1/2$.

Proof. There is a birational morphism $\alpha\colon X \to \mathbb{P}^3$ that blows up three disjoint lines L_1, L_2, L_3.

Let H_i be the proper transform on X of a general plane in $\mathbb{P}^3$ such that $L_i \subset \alpha(H_i)$. Then

$$-K_X \sim 2H_1 + E_1 + H_2 + H_3 \sim 2H_2 + E_2 + H_1 + H_3 \sim 2H_3 + E_3 + H_1 + H_2,$$

where E_i is the exceptional divisor of α such that $\alpha(E_i) = L_i$. In particular, we see that $\mathrm{lct}(X) \leqslant 1/2$.

We suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$.

The surface H_i is a smooth del Pezzo surface such that $K_{H_i}^2 = 7$, the linear system $|H_i|$ has no base points and induces a smooth morphism $\phi_i\colon X \to \mathbb{P}^1$, whose fibers are isomorphic to H_i.

Suppose that $|\mathrm{LCS}(X, \lambda D)| < +\infty$. We may assume that $\mathrm{LCS}(X, \lambda D) \not\subseteq E_1$. Then the set

$$\mathrm{LCS}\left(X,\ \lambda D + H_1 + \frac{1}{2}E_1\right)$$

is disconnected, which is impossible by Theorem 2.7, because $H_2 + H_3 + (\lambda - 1/2)K_X$ is ample.

We may assume that $H_1 \cap \mathrm{LCS}(X, \lambda D) \neq \varnothing$. Then

$$\varnothing \neq H_1 \cap \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq \mathrm{LCS}\Big(H_1,\ \lambda D\Big|_{H_1}\Big)$$

by Remark 2.3. Put $C_2 = E_2|_{H_1}$ and $C_3 = E_3|_{H_1}$. Then

$$C_2 \cdot C_2 = C_3 \cdot C_3 = -1,$$

and there is a unique curve $\mathbb{P}^1 \cong C \subset H_1$ such that $C \cdot C_2 = C \cdot C_3 = 1$ and $C \cdot C = -1$. Note that

$$\mathrm{LCS}\Big(H_1,\ \lambda D\Big|_{H_1}\Big) = C$$

by Lemma 5.9.

There is a unique smooth quadric $Q \subset \mathbb{P}^3$ that contains L_1, L_2, L_3. Note that

$$\bar{Q} \cap H_1 = C,$$

where $\bar{Q} \subset X$ is a proper transform of the surface Q.

There is a morphism $\sigma\colon X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ contracting $\bar{Q}$ to a curve of tri-degree $(1,1,1)$. Since $\bar{Q} \cap H_1 = C$, one obtains (see Remark 2.3) that

$$\mathrm{LCS}\Big(X,\ \lambda D\Big) \supset \bar{Q},$$

and hence $\mathrm{LCS}(X, \lambda D) = \bar{Q}$, because $\mathrm{lct}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = 1/2$. Put

$$D = \mu\bar{Q} + \Omega,$$

where $\mu \geqslant 1/\lambda > 2$, and Ω is an effective $\mathbb{Q}$-divisor such that $\bar{Q} \not\subset \mathrm{Supp}(\Omega)$. Then

$$\alpha\big(D\big) = \mu Q + \alpha\big(\Omega\big),$$

which is impossible, because $\alpha(D) \sim_{\mathbb{Q}} 2Q \sim -K_{\mathbb{P}^3}$ and $\mu > 2$. $\square$

Lemma 10.6. Suppose that $\daleth(X) = 4.7$. Then $\mathrm{lct}(X) = 1/2$.

Proof. There is a birational morphism $\alpha\colon X \to W$ such that

- the variety W is a smooth divisor of bi-degree $(1,1)$ on $\mathbb{P}^2 \times \mathbb{P}^2$;
- the morphism α contracts two (irreducible) surfaces $E_1 \neq E_2$ to two disjoint curves L_1 and L_2;
- the curves L_i are fibers of one natural $\mathbb{P}^1$-bundle $W \to \mathbb{P}^2$.

There is a surface $H \subset W$ such that $-K_X \sim 2H$ and $L_1 \subset H \supset L_2$. Then

$$-K_X \sim 2\bar{H} + E_1 + E_2,$$

where $\bar{H}$ is a proper transform of H on the threefold X. In particular, $\mathrm{lct}(X) \leqslant 1/2$.

We suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$. Then

$$\varnothing \neq \mathrm{LCS}\Big(X, \lambda D\Big) \subseteq E_1 \cup E_2,$$

since $\mathrm{lct}(W) = 1/2$ by Theorem 7.1 and $\alpha(D) \sim_{\mathbb{Q}} -K_W$.

We may assume that $\mathrm{LCS}(X, \lambda D) \cap E_1 \neq \varnothing$. Let $\beta\colon X \to Y$ be a contraction of E_2. Then

$$\mathbb{LCS}\Big(Y,\ \lambda\beta\big(D\big)\Big) \neq \varnothing$$

and $\beta(D) \sim_{\mathbb{Q}} -K_Y$, which contradicts Lemma 9.25. $\square$

Lemma 10.7. Suppose that $\daleth(X) = 4.8$. Then $\mathrm{lct}(X) = 1/3$.

Proof. There is blow up $\alpha\colon X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of a curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ such that $C \subset F_1$ and

$$C \cdot F_2 = C \cdot F_3 = 1,$$

where F_i is a fiber of the projection of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ to its i-th factor. There is a surface

$$\mathbb{P}^1 \times \mathbb{P}^1 \cong G \in \big|F_2 + F_3\big|$$

such that $C \subset G$. Let E be the exceptional divisor of α. Then

$$-K_X \sim 2\bar{F}_1 + 2\bar{G} + 3E,$$

where $\bar{F}_1$ and $\bar{G}$ are proper transforms of F_1 and G, respectively. In particular, $\mathrm{lct}(X) \leqslant 1/3$.

We suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/3$. Note that

$$\varnothing \neq \mathrm{LCS}\Big(X, \lambda D\Big) \subseteq E,$$

because $\mathrm{lct}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = 1/2$ and $\alpha(D) \sim_{\mathbb{Q}} -K_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}$.

Let Q be a quadric cone in $\mathbb{P}^4$. Then there is a commutative diagram

$$\begin{tikzcd}
& & X \arrow[dll, "\alpha"'] \arrow[d, "\beta"] \arrow[drr, "\gamma"] & & \\
\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dr, "\phi"'] & & V \arrow[dl, "\pi"'] \arrow[dr, "\delta"] & & U \arrow[dl, "\xi"] \\
& \mathbb{P}^1 \times \mathbb{P}^1 & & Q \arrow[ll, dashed, "\psi"] &
\end{tikzcd}$$

where we have the following notations:

- V is a variety with $\daleth(V) = 3.31$;
- the morphism β is a contraction of the surface $\bar{G}$ to a curve;
- the morphism γ is a contraction of $\bar{F}_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$ to an ordinary double point;

- the morphism δ is a blow up of the vertex of the quadric cone $Q \subset \mathbb{P}^4$;
- the morphism ξ is a blow up of a smooth conic in Q;
- the map ψ is a projection from the vertex of the cone Q;
- the morphism ϕ is a projection that is given by $|F_2+F_3|$, i. e. the projection of $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ onto the product of the last two factors;
- the morphism π is a natural $\mathbb{P}^1$-bundle.

It follows from Corollary 6.4 that $\mathrm{lct}(V) = 1/3$. On the other hand, $\mathrm{lct}(U) = 1/3$ by Lemma 2.27. Hence

$$\varnothing \neq \mathrm{LCS}\Big(X,\ \lambda D\Big) \subseteq E \cap \bar{G} \cap \bar{F}_1 = \varnothing,$$

which is a contradiction. $\square$

Lemma 10.8. Suppose that $\daleth(X) = 4.9$. Then $\mathrm{lct}(X) = 1/3$.

Proof. There exists a point $O \in \mathbb{P}^3$, and there exist lines $L_1 \subset \mathbb{P}^3 \supset L_2$ such that $L_1 \cap L_2 = \varnothing$, the line L_1 passes through the point O, and there is a commutative diagram

$$\begin{tikzcd}
\mathbb{F}_1 \arrow[rrr, "\phi"] \arrow[ddddd, "\chi"'] & & & \mathbb{P}^2 & & \\
& Z \arrow[ul, "\xi"] \arrow[ddddd, "\nu"'] & & V_7 \arrow[u, "\pi"'] \arrow[ddddd, "\sigma"] & & \mathbb{P}^2\times\mathbb{P}^1 \arrow[ull, bend right, "\iota_1"'] \arrow[dddddd, "\iota_2"] \\
& & U_1 \arrow[ul, "\zeta_1"'] \arrow[ur, "\varepsilon_1"] \arrow[dd, "\delta_1"'] & X \arrow[l, "\upsilon_2"'] \arrow[r, "\upsilon_1"] \arrow[d, "\tau"] \arrow[ull, bend right, "\rho"'] & U_2 \arrow[ul, "\varepsilon_2"'] \arrow[ur, "\zeta_2"'] \arrow[dd, "\delta_2"] & \\
& & & Y \arrow[dl, "\beta_2"'] \arrow[dr, "\beta_1"] \arrow[dddll, "\eta"] & & \\
& & V_1 \arrow[dll, "\omega_1"'] \arrow[ddr, "\alpha_1"] & & V_2 \arrow[ddl, "\alpha_2"'] \arrow[dddr, bend left, "\omega_2"] & \\
\mathbb{P}^1 & & & & & \\
& \mathbb{P}^1\times\mathbb{P}^1 \arrow[ul, "\gamma_1"] \arrow[drrrr, bend right, "\gamma_2"'] & & \mathbb{P}^3 & & \\
& & & & & \mathbb{P}^1
\end{tikzcd}$$

that uses the following notation:

- the morphism σ is a blow up of the point O;
- the morphism π is a natural $\mathbb{P}^1$-bundle;
- the morphism α_i is a blow up of the line L_i;
- the morphisms β_i, υ_i and ε_i are blow ups of the proper transforms of the line L_i;
- the morphisms τ and δ_1 are blow ups of curves that are the preimages of the point O;
- the morphism δ_2 is a blow up of the point that dominates the point O;
- the morphisms ω_1 and ω_2 are natural $\mathbb{P}^2$-bundles, where

$$V_1 \cong V_2 \cong \mathbb{P}\Big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\Big);$$

- the morphisms ι_1, ι_2, γ_1 and γ_2 are natural projections;
- the morphism ϕ is a blow up of the point $\pi(\bar{L}_1)$, where $\bar{L}_1 \subset V_7$ is a proper transform of L_1;

- the morphisms ρ and ζ_i contract the proper transforms of the plane $\Pi \subset \mathbb{P}^3$ such that
$$L_2 \subset \Pi \ni O;$$
- the morphisms η, ν, ξ, χ, γ_1, γ_2 and ι_2 are natural $\mathbb{P}^1$-bundles.

Let $H_i \subset X$ be the proper transform of a general plane in $\mathbb{P}^3$ that contains L_i. Then
$$-K_X \sim 3H_1 + H_2 + 2E_1 + G,$$
where E_i and G be the exceptional divisor of υ_i and τ, respectively. Thus, we have $\mathrm{lct}(X) \leqslant 1/3$.

We suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/3$. Note that
$$\varnothing \ne \mathrm{LCS}\Big(X, \lambda D\Big) \subseteq G,$$
since $\mathrm{lct}(Y) = 1/3$ by Lemma 9.26. But the surface G is not a fiber of the smooth morphism
$$\omega_1 \circ \beta_2 \circ \tau \colon X \longrightarrow \mathbb{P}^1,$$
so we obtain a contradiction applying Lemma 2.26 to the morphism $\omega_1 \circ \beta_2 \circ \tau$. □

The proof of Lemma 10.8 implies the following.

Corollary 10.9. Suppose that $\daleth(X) = 5.2$. Then $\mathrm{lct}(X) = 1/3$.

Remark 10.10. Actually, the results of Lemma 10.8 and Corollary 10.9 are contained in Corollary 6.4, but we still prefer to give a detailed proof that may have further applications.

The following result is implied by Corollaries 9.19 and 10.9, Lemma 2.30 and Example 1.18.

Corollary 10.11. Suppose that $\rho \geqslant 5$. Then
$$\mathrm{lct}\big(X\big) = \begin{cases} 1/3 \text{ whenever } \daleth(X) \in \{5.1, 5.2\}, \\ 1/2 \text{ in the remaining cases.} \end{cases}$$

Lemma 10.12. Suppose that $\daleth(X) = 4.13$ and X is general. Then $\mathrm{lct}(X) = 1/2$.

Proof. Let $F_1 \cong F_2 \cong F_3 \cong \mathbb{P}^1 \times \mathbb{P}^1$ be fibers of three different projections
$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1,$$
respectively. There is a contraction $\alpha \colon X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of a surface $E \subset X$ to a curve
$$C \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$
such that $C \cdot F_1 = C \cdot F_2 = 1$ and $C \cdot F_3 = 3$. Then there is a smooth surface
$$\mathbb{P}^1 \times \mathbb{P}^1 \cong G \in \big|F_1 + F_2\big|$$
such that $C \subset G$. In particular, we see that
$$-K_X \sim 2\bar{G} + E + 2\bar{F}_3,$$
where $\bar{F}_3$ and $\bar{G}$ are proper transforms of F_3 and G, respectively. Hence $\mathrm{lct}(X) \leqslant 1/2$.

We suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$. Note that
$$\varnothing \ne \mathrm{LCS}\Big(X, \lambda D\Big) \subseteq E \cong \mathbb{F}_4,$$
because $\mathrm{lct}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = 1/2$ and $\alpha(D) \sim_{\mathbb{Q}} -K_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}$.

There are smooth surfaces $H_1 \in \big|3F_1 + F_3\big|$ and $H_2 \in \big|3F_2 + F_3\big|$ such that
$$C = G \cdot H_1 = G \cdot H_2,$$
and $H_1 \cong H_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $\bar{H}_i$ be a proper transform of H_i on the threefold X. Then
$$\bar{H}_1 \cap \bar{G} = \bar{H}_2 \cap \bar{G} = \varnothing.$$

There is a commutative diagram

$$\begin{tikzcd}
 & & X \arrow[dll, "\gamma_1"'] \arrow[drr, "\gamma_2"] \arrow[ddl, "\beta"'] \arrow[dd, "\alpha"] & & \\
U_1 \arrow[ddr, "\phi_1"'] & & & & U_2 \arrow[ddl, "\phi_2"] \\
 & V \arrow[dr, "\pi"] & \mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1 \arrow[dl, "\xi_1"'] \arrow[d, "\zeta"] \arrow[dr, "\xi_2"] & & \\
 & \mathbb{P}^1\times\mathbb{P}^1 & \mathbb{P}^1\times\mathbb{P}^1 & \mathbb{P}^1\times\mathbb{P}^1 &
\end{tikzcd}$$

such that β and γ_i are contractions of the surfaces $\bar{G}$ and $\bar{H}_i$ to a smooth curves, the morphisms π and ϕ_i are $\mathbb{P}^1$-bundles, the morphisms ζ and ξ_i are projections that are given by the linear systems $|F_1+F_2|$ and $|F_i+F_3|$, respectively.

It follows from $\bar{H}_1\cap\bar{G}=\varnothing$ that

- either the log pair $(V,\lambda\beta(D))$ is not log canonical,
- of the log pair $(U_1,\lambda\gamma_1(D))$ is not log canonical.

Applying Theorem 2.28 to $(V,\lambda\beta(D))$ or $(U_1,\lambda\gamma_1(D))$ (and the fibration π or ϕ_1) and using Theorem 2.7, we see that

$$\mathrm{LCS}\Big(X,\ \lambda D\Big)=\Gamma,$$

where Γ is a fiber of the natural projection $E\to C$.

We may assume that $\alpha(\Gamma)\in F_3$. Let $\bar{F}_3\subset X$ be the proper transform of the surface F_3. Put

$$D=\mu\bar{F}_3+\Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor on X such that $\bar{F}_3\not\subset\mathrm{Supp}(\Omega)$. Then

$$\mu F_3+\alpha\big(\Omega\big)\sim_{\mathbb{Q}}2\Big(F_1+F_2+F_3\Big),$$

which gives $\mu\leqslant 2$. The log pair $(\bar{F}_3,\lambda\Omega|_{\bar{F}_3})$ is not log canonical along $\Gamma\subset\bar{F}_3$ by Theorem 2.20. One has

$$\Omega\Big|_{\bar{F}_3}\sim_{\mathbb{Q}}-K_{\bar{F}_3},$$

and $\bar{F}_3$ is a del Pezzo surface such that $K_{\bar{F}_3}^2=5$. Note that $\bar{F}_3$ may be singular. Namely, we have

$$\mathrm{Sing}\big(\bar{F}_3\big)=\varnothing\iff\big|C\cap F_3\big|=F_3\cdot C=3,$$

and $\mathrm{Sing}(\bar{F}_3)\subset\Gamma$. The following cases are possible:

- the surface $\bar{F}_3$ is smooth and $|C\cap F_3|=3$;
- the surface $\bar{F}_3$ has one ordinary double point and $|C\cap F_3|=2$;
- the surface $\bar{F}_3$ has a singular point of type $\mathbb{A}_2$ and $|C\cap F_3|=1$.

We have $\mathrm{lct}(\bar{F}_3)\leqslant\lambda<1/2$. Thus, it follows from Examples 1.18 and 5.3 that $|C\cap F_3|=1$, which is impossible if the threefold X is sufficiently general. □

11. Upper bounds

We use the assumptions and notation introduced in section 1. The purpose of this section is to find upper bounds for the global log canonical thresholds of the varieties X with

$$\daleth(X)\in\Big\{1.1,1.2,\ldots,1.17,2.1,\ldots,2.36,3.1,\ldots,3.31,4.1,\ldots,4.13,5.1,\ldots,5.7,5.8\Big\}.$$

Lemma 11.1. Suppose that $\daleth(X)=1.8$. Then $\mathrm{lct}(X)\leqslant 6/7$.

Proof. The linear system $|-K_X|$ does not have base points and induces an embedding $X \subset \mathbb{P}^{10}$, and the threefold X contains a line $L \subset X$ (see [168], [178]).

It follows from [98, Theorem 4.3.3] (see [45], [178]) that there is a commutative diagram

$$\begin{tikzcd} U \arrow[r, dashed, "\rho"] \arrow[d, "\alpha"'] & W \arrow[d, "\beta"] \\ X \arrow[r, dashed, "\psi"'] & \mathbb{P}^3 \end{tikzcd}$$

where α is a blow up of the line L, the map ρ is a composition of flops, the morphism β is a blow up of a smooth curve of degree 7 and genus 3, and ψ is a double projection from L.

Let $S \subset X$ be the proper transform of the exceptional surface of β. Then

$$\mathrm{mult}_L\big(S\big) = 7$$

and $S \sim -3K_X$, which implies that $\mathrm{lct}(X) \leqslant 6/7$. □

Lemma 11.2. Suppose that $\beth(X) = 1.9$. Then $\mathrm{lct}(X) \leqslant 4/5$.

Proof. The linear system $|-K_X|$ does not have base points and induces an embedding $X \subset \mathbb{P}^{11}$, and the threefold X contains a line $L \subset X$ (see [168], [178]).

It follows from [98, Theorem 4.3.3] (see [45], [178]) that there is a commutative diagram

$$\begin{tikzcd} U \arrow[r, dashed, "\rho"] \arrow[d, "\alpha"'] & W \arrow[d, "\beta"] \\ X \arrow[r, dashed, "\psi"'] & Q \end{tikzcd}$$

where $Q \subset \mathbb{P}^4$ is a smooth quadric threefold, α is a blow up of the line L, the map ρ is a composition of flops, the morphism β is a blow up along a smooth curve of degree 7 and genus 2, and ψ is a double projection from the line L.

Let $S \subset X$ be the proper transform of the exceptional surface of β. Then

$$\mathrm{mult}_L\big(S\big) = 5$$

and $S \sim -2K_X$, which implies that $\mathrm{lct}(X) \leqslant 4/5$. □

Lemma 11.3. Suppose that $\beth(X) = 1.10$. Then $\mathrm{lct}(X) \leqslant 2/3$.

Proof. The linear system $|-K_X|$ does not have base points and induces an embedding $X \subset \mathbb{P}^{13}$, and the threefold X contains a line $L \subset X$ (see [168], [178]).

It follows from [98, Theorem 4.3.3] (see [45], [178]) that the diagram

$$\begin{tikzcd} U \arrow[r, dashed, "\rho"] \arrow[d, "\alpha"'] & W \arrow[d, "\beta"] \\ X \arrow[r, dashed, "\psi"'] & V_5 \end{tikzcd}$$

commutes, where V_5 is a smooth section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by a linear subspace of dimension 6, the morphism α is a blow up of the line L, the map ρ is a composition of flops, the morphism β is a blow up of a smooth rational curve of degree 5, and ψ is a double projection from L.

Let $S \subset X$ be the proper transform of the exceptional surface of β. Then

$$\mathrm{mult}_L\big(S\big) = 3$$

and $S \sim -K_X$, which implies that $\mathrm{lct}(X) \leqslant 2/3$. □

Lemma 11.4. Suppose that $\beth(X) = 2.2$. Then $\mathrm{lct}(X) \leqslant 13/14$.

Proof. There is a smooth divisor $B \subset \mathbb{P}^1 \times \mathbb{P}^2$ of bi-degree $(2,4)$ such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "\phi_1"'] \arrow[d, "\pi"] \arrow[dr, "\phi_2"] & \\ \mathbb{P}^1 & \mathbb{P}^1 \times \mathbb{P}^2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & \mathbb{P}^2 \end{tikzcd}$$

commutes, where π is a double cover branched along B, the morphisms π_1 and π_2 are natural projections, ϕ_1 is a fibration into del Pezzo surfaces of degree 2, and ϕ_2 is a conic bundle.

Let H_1 be a general fiber of ϕ_1. Put $\bar{H}_1 = \pi(H_1)$. Then the intersection

$$C = \bar{H}_1 \cap B \subset \bar{H}_1 \cong \mathbb{P}^2$$

is a smooth quartic curve.

There is a point $P \in C$ such that

$$\mathrm{mult}_P\Big(C \cdot L\Big) \geqslant 3,$$

where $L \subset \bar{H}_1 \cong \mathbb{P}^2$ is a line that is tangent to C at the point P.

The curve $\pi_2(L)$ is a line. Thus, there is a unique surface

$$H_2 \in \Big|\phi_2^*\Big(\mathcal{O}_{\mathbb{P}^2}(1)\Big)\Big|$$

such that $\phi_2(H_2) = \pi_2(L)$. Hence $-K_X \sim H_1 + H_2$.

Let us show that $\mathrm{lct}(X, H_1 + H_2) \leqslant 13/14$. Put $\bar{H}_2 = \pi(H_2)$. Then

$$\mathbb{LCS}\left(X,\ \frac{13}{14}\Big(H_1 + H_2\Big)\right) \neq \varnothing \iff \mathbb{LCS}\left(\mathbb{P}^1 \times \mathbb{P}^2,\ \frac{1}{2}B + \frac{13}{14}\Big(\bar{H}_1 + \bar{H}_2\Big)\right) \neq \varnothing$$

by [105, Proposition 3.16]. Let $\alpha\colon V \to \mathbb{P}^1 \times \mathbb{P}^2$ be a blow up of the curve C. Then

$$K_V + \frac{1}{2}\tilde{B} + \frac{13}{14}\Big(\tilde{H}_1 + \tilde{H}_2\Big) + \frac{3}{7}E \sim_{\mathbb{Q}} \alpha^*\left(K_{\mathbb{P}^1 \times \mathbb{P}^2} + \frac{1}{2}B + \frac{13}{14}\Big(\bar{H}_1 + \bar{H}_2\Big)\right),$$

where $\tilde{B}, \tilde{H}_1, \tilde{H}_2 \subset V$ are proper transforms of $B, \bar{H}_1, \bar{H}_2$, respectively. But the log pair

$$\left(V,\ \frac{13}{14}\tilde{H}_2 + \frac{3}{7}E\right)$$

is not log terminal along the fiber Γ of the morphism α such that $\alpha(\Gamma) = P$, because

$$\mathrm{mult}_\Gamma\Big(\tilde{H}_2 \cdot E\Big) = \mathrm{mult}_P\Big(C \cdot \bar{H}_2\Big) \geqslant \mathrm{mult}_P\Big(C \cdot L\Big) \geqslant 3$$

due to the choice of the fiber H_1. We see that

$$\Gamma \subseteq \mathrm{LCS}\left(V,\ \frac{13}{14}\tilde{H}_2 + \frac{3}{7}E\right) \subseteq \mathrm{LCS}\left(V,\ \frac{1}{2}\tilde{B} + \frac{13}{14}\Big(\tilde{H}_1 + \tilde{H}_2\Big) + \frac{3}{7}E\right),$$

which implies that $\mathrm{lct}(X, H_1 + H_2) \leqslant 13/14$. Hence the inequality $\mathrm{lct}(X) \leqslant 13/14$ holds. ☐

Remark 11.5. It follows from Lemmas 2.26 and 5.1 that $\mathrm{lct}(X) \geqslant 2/3$ if $\daleth(X) = 2.2$ and the threefold X satisfies the following generality condition: any fiber of ϕ_1 satisfies the hypotheses of Lemma 5.1.

Lemma 11.6. Suppose that $\daleth(X) = 2.7$. Then $\mathrm{lct}(X) \leqslant 2/3$.

Proof. There is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ Q \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^1 \end{tikzcd}$$

where $Q \subset \mathbb{P}^4$ is a smooth quadric threefold, α is a blow up of a smooth curve that is a complete intersection of two divisors

$$S_1, S_2 \in \Big|\mathcal{O}_{\mathbb{P}^4}(2)\Big|_Q\Big|,$$

the morphism β is a fibration into del Pezzo surfaces of degree 4, and ψ is a rational map that is induced by the pencil generated by the surfaces S_1 and S_2. Then $\mathrm{lct}(X) \leqslant 2/3$, because

$$-K_X \sim_{\mathbb{Q}} \frac{3}{2}\bar{S}_1 + \frac{1}{2}E,$$

where $\bar{S}_1 \subset X$ is a proper transform of the surface S_1, and E is the exceptional divisor of α. □

Lemma 11.7. Suppose that $\daleth(X) = 2.9$. Then $\mathrm{lct}(X) \leqslant 3/4$.

Proof. There is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathbb{P}^3 \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^2 \end{tikzcd}$$

where α is a blow up of a smooth curve $C \subset \mathbb{P}^3$ of degree 7 and genus 5 that is a scheme-theoretic intersection of cubic surfaces in $\mathbb{P}^3$, the morphism β is a conic bundle, and ψ is a rational map that is given by the linear system of cubic surfaces that contain C. One has

$$-K_X \sim_{\mathbb{Q}} \frac{4}{3}S + \frac{1}{3}E,$$

where $S \in |\beta^*(\mathcal{O}_{\mathbb{P}^2}(1))|$, and E is the exceptional divisor of α. We see that $\mathrm{lct}(X) \leqslant 3/4$. □

Lemma 11.8. Suppose that $\daleth(X) = 2.12$. Then $\mathrm{lct}(X) \leqslant 3/4$.

Proof. There is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathbb{P}^3 \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^3 \end{tikzcd}$$

where α and β are blow ups of smooth curves $C \subset \mathbb{P}^3$ and $Z \subset \mathbb{P}^3$ of degree 6 and genus 3 that are scheme-theoretic intersections of cubic surfaces in $\mathbb{P}^3$, and ψ is a birational map that is given by the linear system of cubic surfaces that contain C. Then

$$-K_X \sim_{\mathbb{Q}} \frac{4}{3}S + \frac{1}{3}E,$$

where $S \in |\beta^*(\mathcal{O}_{\mathbb{P}^3}(1))|$, and E is the exceptional divisor of α. We see that $\mathrm{lct}(X) \leqslant 3/4$. □

Lemma 11.9. Suppose that $\daleth(X) = 2.13$. Then $\mathrm{lct}(X) \leqslant 2/3$.

Proof. There is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ Q \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^2 \end{tikzcd}$$

where $Q \subset \mathbb{P}^4$ is a smooth quadric threefold, α is a blow up of a smooth curve $C \subset Q$ of degree 6 and genus 2, the morphism β is a conic bundle, and ψ is a rational map that is given by the linear system of surfaces in $|\mathcal{O}_{\mathbb{P}^4}(2)|_Q|$ that contain the curve C. One has

$$-K_X \sim_{\mathbb{Q}} \frac{3}{2}S + \frac{1}{2}E,$$

where $S \in |\beta^*(\mathcal{O}_{\mathbb{P}^2}(1))|$, and E is the exceptional divisor of α. We see that $\mathrm{lct}(X) \leqslant 2/3$. □

Lemma 11.10. Suppose that $\daleth(X) = 2.16$. Then $\mathrm{lct}(X) \leqslant 1/2$.

Proof. There is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ V_4 \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^2 \end{tikzcd}$$

where $V_4 \subset \mathbb{P}^5$ is a smooth complete intersection of two quadric hypersurfaces, α is a blow up of an irreducible conic $C \subset V_4$, the morphism β is a conic bundle, and ψ is a rational map that is given by the linear system of surfaces in $|\mathcal{O}_{\mathbb{P}^5}(1)|_{V_4}|$ that contain C. One has

$$-K_X \sim 2S + E,$$

where $S \in |\beta^*(\mathcal{O}_{\mathbb{P}^2}(1))|$, and E is the exceptional divisor of α. We see that $\mathrm{lct}(X) \leqslant 1/2$. $\square$

Lemma 11.11. Suppose that $\daleth(X) = 2.17$. Then $\mathrm{lct}(X) \leqslant 2/3$.

Proof. There is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ Q \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^3 \end{tikzcd}$$

where $Q \subset \mathbb{P}^4$ is a smooth quadric threefold, the morphisms α and β are blow ups of smooth elliptic curves $C \subset Q$ and $Z \subset \mathbb{P}^3$ of degree 5, respectively, and the map ψ is given by the linear system of surfaces in $|\mathcal{O}_{\mathbb{P}^4}(2)|_Q|$ that contain the curve C. One has

$$-K_X \sim_{\mathbb{Q}} \frac{3}{2}S + \frac{1}{2}E,$$

where $S \in |\beta^*(\mathcal{O}_{\mathbb{P}^3}(1))|$, and E is the exceptional divisor of α. We see that $\mathrm{lct}(X) \leqslant 2/3$. $\square$

Lemma 11.12. Suppose that $\daleth(X) = 2.20$. Then $\mathrm{lct}(X) \leqslant 1/2$.

Proof. There is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ V_5 \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^2 \end{tikzcd}$$

where $V_5 \subset \mathbb{P}^6$ is a smooth intersection of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ with a linear subspace of dimension 6, the morphism α is a blow up of a cubic curve $\mathbb{P}^1 \cong C \subset V_5$, the morphism β is a conic bundle, and ψ is given by the linear system of surfaces in $|\mathcal{O}_{\mathbb{P}^6}(1)|_{V_5}|$ that contain the curve C. One has

$$-K_X \sim 2S + E,$$

where $S \in |\beta^*(\mathcal{O}_{\mathbb{P}^2}(1))|$, and E is the exceptional divisor of α. We see that $\mathrm{lct}(X) \leqslant 1/2$. $\square$

Lemma 11.13. Suppose that $\daleth(X) = 2.21$. Then $\mathrm{lct}(X) \leqslant 2/3$.

Proof. There is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ Q \arrow[rr, dashed, "\psi"'] & & Q \end{tikzcd}$$

where $Q \subset \mathbb{P}^4$ is a smooth quadric threefold, the morphisms α and β are blow ups of smooth rational curves $C \subset Q$ and $Z \subset Q$ of degree 4, and ψ is a birational map that is given by the linear system of surfaces in $|\mathcal{O}_{\mathbb{P}^4}(2)|_Q|$ that contain the curve C. One has

$$-K_X \sim_{\mathbb{Q}} \frac{3}{2}S + \frac{1}{2}E,$$

where $S \in |\beta^*(\mathcal{O}_{\mathbb{P}^4}(1))|_Q|$, and E is the exceptional divisor of α. We see that $\mathrm{lct}(X) \leqslant 2/3$. $\square$

Lemma 11.14. Suppose that $\gimel(X) = 2.22$. Then $\mathrm{lct}(X) \leqslant 1/2$.

Proof. There is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ V_5 \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^3 \end{tikzcd}$$

where $V_5 \subset \mathbb{P}^6$ is a smooth intersection of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ with a linear subspace of dimension 6, the morphisms α and β are blow ups of a conic $C \subset V_5$ and a rational (not linearly normal) quartic $Z \subset \mathbb{P}^3$, respectively, and ψ is given by the linear system of surfaces in $|\mathcal{O}_{\mathbb{P}^6}(1)|_{V_5}|$ that contain the curve C. One has

$$-K_X \sim 2S + E,$$

where $S \in |\beta^*(\mathcal{O}_{\mathbb{P}^3}(1))|$, and E is the exceptional divisor of α. We see that $\mathrm{lct}(X) \leqslant 1/2$. □

Lemma 11.15. Suppose that $\gimel(X) = 3.13$. Then $\mathrm{lct}(X) \leqslant 1/2$.

Proof. There is a commutative diagram

$$\begin{tikzcd} & \mathbb{P}^2 & \\ W_2 \arrow[ur, "\alpha_2"] \arrow[dd, "\beta_2"'] & & W_3 \arrow[ul, "\beta_3"'] \arrow[dd, "\alpha_3"] \\ & X \arrow[uu, "\phi_1"'] \arrow[ul, "\pi_2"] \arrow[ur, "\pi_3"'] \arrow[dl, "\phi_3"'] \arrow[dr, "\phi_2"] \arrow[dd, "\pi_1"] & \\ \mathbb{P}^2 & & \mathbb{P}^2 \\ & W_1 \arrow[ul, "\alpha_1"] \arrow[ur, "\beta_1"'] & \end{tikzcd}$$

such that $W_i \subset \mathbb{P}^2 \times \mathbb{P}^2$ is a divisor of bi-degree $(1,1)$, the morphisms α_i and β_i are $\mathbb{P}^1$-bundles, π_i is a blow up of a smooth curve $C_i \subset W_i$ of bi-degree $(2,2)$ such that

$$\alpha_i\big(C_i\big) \subset \mathbb{P}^2 \supset \beta_i\big(C_i\big)$$

are irreducible conics, and ϕ_i is a conic bundle. Let E_i be the exceptional divisor of π_i. Then

$$-K_X \sim 2H_1 + E_1 \sim 2H_2 + E_2 \sim 2H_3 + E_3 \sim E_1 + E_2 + E_3,$$

where $H_i \in |\phi_i^*(\mathcal{O}_{\mathbb{P}^2}(1))|$. We see that $\mathrm{lct}(X) \leqslant 1/2$. □

Remark 11.16. Let us use the notation of the proof of Lemma 11.15 and assume that $\mathrm{lct}(X) < 1/2$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$. Since $\mathrm{lct}(W_i) = 1/2$ by Theorem 7.1, one has

$$\varnothing \neq \mathrm{LCS}(X, \lambda D) \subset E_1 \cap E_2 \cap E_3.$$

In particular, by Theorem 2.7 the locus $\mathrm{LCS}(X, \lambda D)$ consists of a single point P; note that P is an intersection $P = F_1 \cap F_2 \cap F_3$ of three curves F_i such that $F_2 \cup F_3$ (resp., $F_1 \cup F_3$, $F_1 \cup F_2$) is a reducible fiber of the conic bundle ϕ_1 (resp., ϕ_2, ϕ_3).

Appendix A. By Jean-Pierre Demailly. On Tian's invariant and log canonical thresholds

The goal of this appendix is to relate log canonical thresholds with the α-invariant introduced by G. Tian [179] for the study of the existence of Kähler–Einstein metrics. The approximation technique of closed positive $(1,1)$-currents introduced in [48] is used to show that the α-invariant actually coincides with the log canonical threshold.

Algebraic geometers have been aware of this fact after [49] appeared, and several papers have used it consistently in the latter years (see e.g. [81], [12]). However, it turns out that the required result is stated only in a local analytic form in [49], in a language which may not be easily recognizable by algebraically minded people. Therefore, we will repair here the lack of a proper reference by stating and proving the statements required for the applications to projective varieties, e.g. existence of Kähler–Einstein metrics on Fano varieties and Fano orbifolds.

Usually, in these applications, only the case of the anticanonical line bundle $L = -K_X$ is considered. Here we will consider more generally the case of an arbitrary line bundle L (or $\mathbb{Q}$-line bundle L) on a complex manifold X, with some additional restrictions which will be stated later.

Assume that L is equipped with a *singular hermitian metric* h (see e.g. [47]). Locally, L admits trivializations $\theta : L_{|U} \simeq U \times \mathbb{C}$, and on U the metric h is given by a weight function φ such that

$$\|\xi\|_h^2 = |\xi|^2 e^{-2\varphi(z)} \text{ for all } z \in U, \xi \in L_z,$$

when $\xi \in L_z$ is identified with a complex number. We are interested in the case where φ is (at the very least) a locally integrable function for the Lebesgue measure, since it is then possible to compute the curvature form

$$\Theta_{L,h} = \frac{i}{\pi}\partial\overline{\partial}\varphi$$

in the sense of distributions. We have $\Theta_{L,h} \geqslant 0$ as a $(1,1)$-current if and only if the weights φ are plurisubharmonic functions. In the sequel we will be interested only in that case.

Let us first introduce the concept of complex singularity exponent for singular hermitian metrics, following e.g. [184], [185], [4] and [49].

Definition A.1. If K is a compact subset of X, we define the complex singularity exponent $c_K(h)$ of the metric h, written locally as $h = e^{-2\varphi}$, to be the supremum of all positive numbers c such that $h^c = e^{-2c\varphi}$ is integrable in a neighborhood of every point $z_0 \in K$, with respect to the Lebesgue measure in holomorphic coordinates centered at z_0.

Now, we introduce a generalized version of Tian's invariant α, as defined in [179] (see also [173]).

Definition A.2. Assume that X is a compact manifold and that L is a pseudo-effective line bundle, i. e. L admits a singular hermitian metric h_0 with $\Theta_{L,h_0} \geqslant 0$. If K is a compact subset of X, we put

$$\alpha_K(L) = \inf_{\{h,\,\Theta_{L,h}\geqslant 0\}} c_K(h)$$

where h runs over all singular hermitian metrics on L such that $\Theta_{L,h} \geqslant 0$.

In algebraic geometry, it is more usual to look instead at linear systems defined by a family of linearly independent sections $\sigma_0, \sigma_1, \ldots, \sigma_N \in H^0(X, L^{\otimes m})$. We denote by Σ the vector subspace generated by these sections and by

$$|\Sigma| := P(\Sigma) \subset |mL| := P(H^0(X, L^{\otimes m}))$$

the corresponding linear system. Such an $(N+1)$-tuple of sections $\sigma = (\sigma_j)_{0\leqslant j\leqslant N}$ defines a singular hermitian metric h on L by putting in any trivialization

$$\|\xi\|_h^2 = \frac{|\xi|^2}{\left(\sum_j |\sigma_j(z)|^2\right)^{1/m}} = \frac{|\xi|^2}{|\sigma(z)|^{2/m}} \text{ for } \xi \in L_z,$$

hence $h(z) = |\sigma(z)|^{-2/m}$ with

$$\varphi(z) = \frac{1}{m}\log|\sigma(z)| = \frac{1}{2m}\log\sum_j |\sigma_j(z)|^2$$

as the associated weight function. Therefore, we are interested in the number $c_K(|\sigma|^{-2/m})$. In the case of a single section σ_0 (corresponding to a linear system containing a single divisor), this is the same as the log canonical threshold $\mathrm{lct}_K(X, \frac{1}{m}D)$ of the where D is a divisor corresponding to σ_0. We will also use the formal notation $\mathrm{lct}_K(X, \frac{1}{m}|\Sigma|)$ in the case of a higher dimensional linear system $|\Sigma| \subset |mL|$.

Now, recall that the line bundle L is said to be *big* if the Kodaira–Iitaka dimension $\varkappa(L)$ equals $n = \dim_{\mathbb{C}}(X)$. The main result of this appendix is

Theorem A.3. Let L be a big line bundle on a compact complex manifold X. Then for every compact set K in X we have

$$\alpha_K(L) = \inf_{\{h,\,\Theta_{L,h}\geqslant 0\}} c_K(h) = \inf_{m\in\mathbb{Z}_{>0}} \inf_{D\in|mL|} \mathrm{lct}_K\Big(X, \frac{1}{m}D\Big).$$

Observe that the inequality

$$\inf_{m\in\mathbb{Z}_{>0}} \inf_{D\in|mL|} \mathrm{lct}_K\Big(X, \frac{1}{m}D\Big) \geq \inf_{\{h,\,\Theta_{L,h}\geqslant 0\}} c_K(h)$$

is trivial, since any divisor $D \in |mL|$ gives rise to a singular hermitian metric h. The converse inequality will follow from the approximation technique of [48] and some elementary analysis. The proof is parallel to the proof of [49, Theorem 4.2], although the language used there was somewhat different. In any case, we use again the crucial concept of multiplier ideal sheaves: if h is a singular hermitian metric with local plurisubharmonic weights φ, the multiplier ideal sheaf $\mathcal{I}(h) \subset \mathcal{O}_X$ (also denoted by $\mathcal{I}(\varphi)$) is the ideal sheaf defined by

$$\mathcal{I}(h)_z = \Big\{ f \in \mathcal{O}_{X,z} \mid \text{there exists a neighborhood } V \ni z, \text{ such that } \int_V |f(x)|^2 e^{-2\varphi(x)} d\lambda(x) < +\infty \Big\},$$

where λ is the Lebesgue measure. By Nadel (see [132]), this is a coherent analytic sheaf on X. Theorem A.3 has a more precise version which can be stated as follows.

Theorem A.4. Let L be a line bundle on a compact complex manifold X possessing a singular hermitian metric h with $\Theta_{L,h} \geqslant \varepsilon\omega$ for some $\varepsilon > 0$ and some smooth positive definite hermitian $(1,1)$-form ω on X. For every real number $m > 0$, consider the space $\mathcal{H}_m = H^0(X, L^{\otimes m} \otimes \mathcal{I}(h^m))$ of holomorphic sections σ of $L^{\otimes m}$ on X such that

$$\int_X |\sigma|^2_{h^m} dV_\omega = \int_X |\sigma|^2 e^{-2m\varphi} dV_\omega < +\infty,$$

where $dV_\omega = \frac{1}{m!}\omega^m$ is the hermitian volume form. Then for $m \gg 1$, $\mathcal{H}_m$ is a non zero finite dimensional Hilbert space and we consider the closed positive $(1,1)$-current

$$T_m = \frac{i}{2\pi}\partial\overline{\partial}\Big(\frac{1}{2m}\log\sum_k |g_{m,k}|^2\Big) = \frac{i}{2\pi}\partial\overline{\partial}\Big(\frac{1}{2m}\log\sum_k |g_{m,k}|^2_h\Big) + \Theta_{L,h}$$

where $(g_{m,k})_{1\leqslant k\leqslant N(m)}$ is an orthonormal basis of $\mathcal{H}_m$. The following statements hold.

(i) For every trivialization $L_{|U} \simeq U \times \mathbb{C}$ on a cordinate open set U of X and every compact set $K \subset U$, there are constants $C_1, C_2 > 0$ independent of m and φ such that

$$\varphi(z) - \frac{C_1}{m} \leqslant \psi_m(z) := \frac{1}{2m}\log\sum_k |g_{m,k}(z)|^2 \leqslant \sup_{|x-z|<r} \varphi(x) + \frac{1}{m}\log\frac{C_2}{r^n}$$

for every $z \in K$ and $r \leqslant \frac{1}{2}d(K, \partial U)$. In particular, ψ_m converges to φ pointwise and in L^1_{loc} topology on Ω when $m \to +\infty$, hence T_m converges weakly to $T = \Theta_{L,h}$.

(ii) The Lelong numbers $\nu(T, z) = \nu(\varphi, z)$ and $\nu(T_m, z) = \nu(\psi_m, z)$ are related by

$$\nu(T, z) - \frac{n}{m} \leqslant \nu(T_m, z) \leqslant \nu(T, z) \qquad \text{for every } z \in X.$$

(iii) For every compact set $K \subset X$, the complex singularity exponents of the metrics given locally by $h = e^{-2\varphi}$ and $h_m = e^{-2\psi_m}$ satisfy

$$c_K(h)^{-1} - \frac{1}{m} \leqslant c_K(h_m)^{-1} \leqslant c_K(h)^{-1}.$$

Proof. The major part of the proof is a small variation of the arguments already explained in [48] (see also [49, Theorem 4.2]). We give them here in some detail for the convenience of the reader.

(i) We note that $\sum |g_{m,k}(z)|^2$ is the square of the norm of the evaluation linear form $\sigma \mapsto \sigma(z)$ on $\mathcal{H}_m$, hence

$$\psi_m(z) = \sup_{\sigma \in B(1)} \frac{1}{m} \log |\sigma(z)|$$

where $B(1)$ is the unit ball of $\mathcal{H}_m$. For $r \leqslant \frac{1}{2} d(K, \partial\Omega)$, the mean value inequality applied to the plurisubharmonic function $|\sigma|^2$ implies

$$|\sigma(z)|^2 \leqslant \frac{1}{\pi^n r^{2n}/n!} \int_{|x-z|<r} |\sigma(x)|^2 d\lambda(x) \leqslant$$

$$\leqslant \frac{1}{\pi^n r^{2n}/n!} \exp\Big(2m \sup_{|x-z|<r} \varphi(x)\Big) \int_\Omega |\sigma|^2 e^{-2m\varphi} d\lambda.$$

If we take the supremum over all $\sigma \in B(1)$ we get

$$\psi_m(z) \leqslant \sup_{|x-z|<r} \varphi(x) + \frac{1}{2m} \log \frac{1}{\pi^n r^{2n}/n!}$$

and the right hand inequality in (i) is proved. Conversely, the Ohsawa–Takegoshi extension theorem [134], [135] applied to the 0-dimensional subvariety $\{z\} \subset U$ shows that for any $a \in \mathbb{C}$ there is a holomorphic function f on U such that $f(z) = a$ and

$$\int_U |f|^2 e^{-2m\varphi} d\lambda \leqslant C_3 |a|^2 e^{-2m\varphi(z)},$$

where C_3 only depends on n and $\mathrm{diam}(U)$. Now, provided a remains in a compact set $K \subset U$, we can use a cut-off function θ with support in U and equal to 1 in a neighborhood of a, and solve the $\overline{\partial}$-equation $\overline{\partial} g = \overline{\partial}(\theta f)$ in the L^2 space associated with the weight $2m\varphi + 2(n+1)\log|z-a|$, that is, the singular hermitian metric $h(z)^m |z-a|^{-2(n+1)}$ on $L^{\otimes m}$. For this, we apply the standard Andreotti–Vesentini–Hörmander L^2 estimates (see e.g. [46] for the required version). This is possible for $m \geqslant m_0$ thanks to the hypothesis that $\Theta_{L,h} \geqslant \varepsilon\omega > 0$, even if X is non Kähler (X is in any event a Moishezon variety from our assumptions). The bound m_0 depends only on ε and the geometry of a finite covering of X by compact sets $K_j \subset U_j$, where U_j are coordinate balls (say); it is independent of the point a and even of the metric h. It follows that $g(a) = 0$ and therefore $\sigma = \theta f - g$ is a holomorphic section of $L^{\otimes m}$ such that

$$\int_X |\sigma|^2_{h^m} dV_\omega = \int_X |\sigma|^2 e^{-2m\varphi} dV_\omega \leqslant C_4 \int_U |f|^2 e^{-2m\varphi} dV_\omega \leqslant C_5 |a|^2 e^{-2m\varphi(z)},$$

in particular, $\sigma \in \mathcal{H}_m = H^0(X, L^{\otimes m} \otimes \mathcal{I}(h^m))$. We fix a such that the right hand side of the latter inequality is 1. This gives the inequality

$$\psi_m(z) \geqslant \frac{1}{m} \log |a| = \varphi(z) - \frac{\log C_5}{2m}$$

which is the left hand part of statement (i).

(ii) The first inequality in (i) implies $\nu(\psi_m, z) \leqslant \nu(\varphi, z)$. In the opposite direction, we find

$$\sup_{|x-z|<r} \psi_m(x) \leqslant \sup_{|x-z|<2r} \varphi(x) + \frac{1}{m} \log \frac{C_2}{r^n}.$$

Divide by $\log r < 0$ and take the limit as r tends to 0. The quotient by $\log r$ of the supremum of a psh function over $B(x, r)$ tends to the Lelong number at x. Thus we obtain

$$\nu(\psi_m, x) \geqslant \nu(\varphi, x) - \frac{n}{m}.$$

(iii) Again, the first inequality in (i) immediately yields $h_m \leqslant C_6 h$, hence $c_K(h_m) \geqslant c_K(h)$. For the converse inequality, since we have $c_{\cup K_j}(h) = \min_j c_{K_j}(h)$, we can assume without loss of generality that K is contained in a trivializing open patch U of L. Let us take $c < c_K(\psi_m)$. Then, by definition, if $V \subset X$ is a sufficiently small open neighborhood of K, the Hölder inequality for the conjugate exponents $p = 1 + mc^{-1}$ and $q = 1 + m^{-1}c$ implies, thanks to equality $\frac{1}{p} = \frac{c}{mq}$,

$$\int_V e^{-2(m/p)\varphi} dV_\omega = \int_V \Big(\sum_{1 \leqslant k \leqslant N(m)} |g_{m,k}|^2 e^{-2m\varphi}\Big)^{1/p} \Big(\sum_{1 \leqslant k \leqslant N(m)} |g_{m,k}|^2\Big)^{-c/mq} dV_\omega \leqslant$$

$$\leqslant \left(\int_X \sum_{1 \leqslant k \leqslant N(m)} |g_{m,k}|^2 e^{-2m\varphi} dV_\omega\right)^{1/p} \left(\int_V \Big(\sum_{1 \leqslant k \leqslant N(m)} |g_{m,k}|^2\Big)^{-c/m} dV_\omega\right)^{1/q} =$$

$$= N(m)^{1/p} \left(\int_V \Big(\sum_{1 \leqslant k \leqslant N(m)} |g_{m,k}|^2\Big)^{-c/m} dV_\omega\right)^{1/q} < +\infty.$$

From this we infer $c_K(h) \geqslant m/p$, i.e., $c_K(h)^{-1} \leqslant p/m = 1/m + c^{-1}$. As $c < c_K(\psi_m)$ was arbitrary, we get $c_K(h)^{-1} \leqslant 1/m + c_K(h_m)^{-1}$ and the inequalities of (iii) are proved. □

Proof of Theorem A.3. Given a big line bundle L on X, there exists a modification $\mu : \tilde{X} \to X$ of X such that $\tilde{X}$ is projective and $\mu^* L = \mathcal{O}(A + E)$ where A is an ample divisor and E an effective divisor with rational coefficients. By pushing forward by μ a smooth metric h_A with positive curvature on A, we get a singular hermitian metric h_1 on L such that

$$\Theta_{L,h_1} \geqslant \mu_* \Theta_{A,h_A} \geqslant \varepsilon\omega$$

on X. Then for any $\delta > 0$ and any singular hermitian metric h on L with $\Theta_{L,h} \geqslant 0$, the interpolated metric $h_\delta = h_1^\delta h^{1-\delta}$ satisfies $\Theta_{L,h_\delta} \geqslant \delta\varepsilon\omega$. Since h_1 is bounded away from 0, it follows that $c_K(h) \geqslant (1-\delta) c_K(h_\delta)$ by monotonicity. By Theorem A.4 (iii) applied to h_δ, we infer

$$c_K(h_\delta) = \lim_{m \to +\infty} c_K(h_{\delta,m}),$$

and we also have

$$c_K(h_{\delta,m}) \geqslant \mathrm{lct}_K\Big(\frac{1}{m} D_{\delta,m}\Big)$$

for any divisor $D_{\delta,m}$ associated with a section $\sigma \in H^0(X, L^{\otimes m} \otimes \mathcal{I}(h_\delta^m))$, since the metric $h_{\delta,m}$ is given by $h_{\delta,m} = (\sum_k |g_{m,k}|^2)^{-1/m}$ for an orthornormal basis of such sections. This clearly implies

$$c_K(h) \geqslant \liminf_{\delta \to 0} \liminf_{m \to +\infty} \mathrm{lct}_K\Big(\frac{1}{m} D_{\delta,m}\Big) \geqslant \inf_{m \in \mathbb{Z}_{>0}} \inf_{D \in |mL|} \mathrm{lct}_K\Big(\frac{1}{m} D\Big). \qquad \square$$

In the applications, it is frequent to have a finite or compact group G of automorphisms of X and to look at G-invariant objects, namely G-equivariant metrics on G-equivariant line bundles L; in the case of a reductive algebraic group G we simply consider a compact real form $G^{\mathbb{R}}$ instead of G itself.

One then gets an α invariant $\alpha_{G,K}(L)$ by looking only at G-equivariant metrics in Definition A.2. All contructions made are then G-equivariant, especially $\mathcal{H}_m \subset |mL|$ is a G-invariant linear system. For every G-invariant compact set K in X, we thus infer

$$\text{(A.5)} \qquad \alpha_{G,K}(L) = \inf_{\{h \text{ is } G-\text{equivariant}, \Theta_{L,h} \geqslant 0\}} c_K(h) = \inf_{m \in \mathbb{Z}_{>0}} \inf_{|\Sigma| \subset |mL|,\ \Sigma^G = \Sigma} \mathrm{lct}_K\Big(\frac{1}{m}|\Sigma|\Big).$$

When G is a finite group, one can pick for m large enough a G-invariant divisor $D_{\delta,m}$ associated with a G-invariant section σ, possibly after multiplying m by the order of G. One then gets the slightly simpler equality

$$\text{(A.6)} \qquad \alpha_{G,K}(L) = \inf_{m \in \mathbb{Z}_{>0}} \inf_{D \in |mL|^G} \mathrm{lct}_K\Big(\frac{1}{m} D\Big).$$

In a similar manner, one can work on an orbifold X rather than on a non singular variety. The L^2 techniques work in this setting with almost no change (L^2 estimates are essentially insensitive to singularities, since one can just use an orbifold metric on the open set of regular points). □

Appendix B. The Big Table

This appendix contains the list of nonsingular Fano threefolds. We follow the notation and the numeration of these in [98], [126], [127]. We also assume the following conventions

- the symbol V_i denotes a smooth Fano threefold such that $-K_X \sim 2H$ and
$$\mathrm{Pic}\big(V_i\big) = \mathbb{Z}\big[H\big],$$
where H is a Cartier divisor on V_i, and $H^3 = 8i \in \{8, 16, \ldots, 40\}$,
- the symbol W denotes a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,1)$ (or, that is the same, the variety $\mathbb{P}(T_{\mathbb{P}^2})$),
- the symbol V_7 denotes a blow up of $\mathbb{P}^3$ at a point (or, that is the same, the variety $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$),
- the symbol Q denotes a smooth quadric hypersurface in $\mathbb{P}^4$,
- the symbol S_i denotes a smooth del Pezzo surface such that
$$K_{S_i}^2 = i \in \big\{1, \ldots, 8\big\},$$
where $S_8 \not\cong \mathbb{P}^1 \times \mathbb{P}^1$.

The fourth column of Table 1 contains the values of global log canonical thresholds of the corresponding Fano varieties. The symbol $\star$ near a number means that $\mathrm{lct}(X)$ is calculated for a general X with a given deformation type. If we know only the upper bound $\mathrm{lct}(X) \leqslant \alpha$, we write $\leqslant \alpha$ instead of the exact value of $\mathrm{lct}(X)$, and the symbol ? means that we don't know any reasonable upper bound (apart from a trivial $\mathrm{lct}(X) \leqslant 1$).

Table 1: Smooth Fano threefolds

$\daleth(X)$	$-K_X^3$	Brief description	$\mathrm{lct}(X)$
1.1	2	a hypersurface in $\mathbb{P}(1,1,1,1,3)$ of degree 6	$1\star$
1.2	4	a hypersurface in $\mathbb{P}^4$ of degree 4 or a double cover of smooth quadric in $\mathbb{P}^4$ branched over a surface of degree 8	?
1.3	6	a complete intersection of a quadric and a cubic in $\mathbb{P}^5$	?
1.4	8	a complete intersection of three quadrics $\mathbb{P}^6$	?
1.5	10	a section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by quadric and linear subspace of dimension 7	?
1.6	12	a section of the Hermitian symmetric space $M = G/P \subset \mathbb{P}^{15}$ of type DIII by a linear subspace of dimension 8	?
1.7	14	a section of $\mathrm{Gr}(2,6) \subset \mathbb{P}^{14}$ by a linear subspace of codimension 5	?
1.8	16	a section of the Hermitian symmetric space $M = G/P \subset \mathbb{P}^{19}$ of type CI by a linear subspace of dimension 10	$\leqslant 6/7$
1.9	18	a section of the 5-dimensional rational homogeneous contact manifold $G_2/P \subset \mathbb{P}^{13}$ by a linear subspace of dimension 11	$\leqslant 4/5$
1.10	22	a zero locus of three sections of the rank 3 vector bundle $\bigwedge^2 \mathcal{Q}$, where $\mathcal{Q}$ is the universal quotient bundle on $\mathrm{Gr}(7,3)$	$\leqslant 2/3$
1.11	8	V_1 that is a hypersurface in $\mathbb{P}(1,1,1,2,3)$ of degree 6	1/2
1.12	16	V_2 that is a hypersurface in $\mathbb{P}(1,1,1,1,2)$ of degree 4	1/2
1.13	24	V_3 that is a hypersurface in $\mathbb{P}^4$ of degree 3	1/2
1.14	32	V_4 that is a complete intersection of two quadrics in $\mathbb{P}^5$	1/2
1.15	40	V_5 that is a section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by linear subspace of codimension 3	1/2
1.16	54	Q that is a hypersurface in $\mathbb{P}^4$ of degree 2	1/3

1.17	64	$\mathbb{P}^3$	$1/4$
2.1	4	a blow up of the Fano threefold V_1 along an elliptic curve that is an intersection of two divisors from $\lvert -\frac{1}{2}K_{V_1}\rvert$	$1/2$
2.2	6	a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ whose branch locus is a divisor of bidegree $(2,4)$	$\leqslant 13/14$
2.3	8	the blow up of the Fano threefold V_2 along an elliptic curve that is an intersection of two divisors from $\lvert -\frac{1}{2}K_{V_2}\rvert$	$1/2$
2.4	10	the blow up of $\mathbb{P}^3$ along an intersection of two cubics	$3/4\star$
2.5	12	the blow up of the threefold $V_3 \subset \mathbb{P}^4$ along a plane cubic	$1/2\star$
2.6	12	a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(2,2)$ or a double cover of W whose branch locus is a surface in $\lvert -K_W\rvert$	?
2.7	14	the blow up of Q along the intersection of two divisors from $\lvert\mathcal{O}_Q(2)\rvert$	$\leqslant 2/3$
2.8	14	a double cover of V_7 whose branch locus is a surface in $\lvert -K_{V_7}\rvert$	$1/2\star$
2.9	16	the blow up of $\mathbb{P}^3$ along a curve of degree 7 and genus 5 which is an intersection of cubics	$\leqslant 3/4$
2.10	16	the blow up of $V_4 \subset \mathbb{P}^5$ along an elliptic curve which is an intersection of two hyperplane sections	$1/2\star$
2.11	18	the blow up of V_3 along a line	$1/2\star$
2.12	20	the blow up of $\mathbb{P}^3$ along a curve of degree 6 and genus 3 which is an intersection of cubics	$\leqslant 3/4$
2.13	20	the blow up of $Q \subset \mathbb{P}^4$ along a curve of degree 6 and genus 2	$\leqslant 2/3$
2.14	20	the blow up of $V_5 \subset \mathbb{P}^6$ along an elliptic curve which is an intersection of two hyperplane sections	$1/2\star$
2.15	22	the blow up of $\mathbb{P}^3$ along the intersection of a quadric and a cubic surfaces	$1/2\star$
2.16	22	the blow up of $V_4 \subset \mathbb{P}^5$ along a conic	$\leqslant 1/2$
2.17	24	the blow up of $Q \subset \mathbb{P}^4$ along an elliptic curve of degree 5	$\leqslant 2/3$
2.18	24	a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ whose branch locus is a divisor of bidegree $(2,2)$	$1/2$
2.19	26	the blow up of $V_4 \subset \mathbb{P}^5$ along a line	$1/2\star$
2.20	26	the blow up of $V_5 \subset \mathbb{P}^6$ along a twisted cubic	$\leqslant 1/2$
2.21	28	the blow up of $Q \subset \mathbb{P}^4$ along a twisted quartic	$\leqslant 2/3$
2.22	30	the blow up of $V_5 \subset \mathbb{P}^6$ along a conic	$\leqslant 1/2$
2.23	30	the blow up of $Q \subset \mathbb{P}^4$ along a curve of degree 4 that is an intersection of a surface in $\lvert\mathcal{O}_{\mathbb{P}^4}(1)\vert_Q\rvert$ and a surface in $\lvert\mathcal{O}_{\mathbb{P}^4}(2)\vert_Q\rvert$	$1/3\star$
2.24	30	a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,2)$	$1/2\star$
2.25	32	the blow up of $\mathbb{P}^3$ along an elliptic curve which is an intersection of two quadrics	$1/2$
2.26	34	the blow up of the threefold $V_5 \subset \mathbb{P}^6$ along a line	$1/2\star$
2.27	38	the blow up of $\mathbb{P}^3$ along a twisted cubic	$1/2$
2.28	40	the blow up of $\mathbb{P}^3$ along a plane cubic	$1/4$
2.29	40	the blow up of $Q \subset \mathbb{P}^4$ along a conic	$1/3$
2.30	46	the blow up of $\mathbb{P}^3$ along a conic	$1/4$

2.31	46	the blow up of $Q \subset \mathbb{P}^4$ along a line	1/3
2.32	48	W that is a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,1)$	1/2
2.33	54	the blow up of $\mathbb{P}^3$ along a line	1/4
2.34	54	$\mathbb{P}^1 \times \mathbb{P}^2$	1/3
2.35	56	$V_7 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$	1/4
2.36	62	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$	1/5
3.1	12	a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched in a divisor of tridegree $(2,2,2)$	$3/4\star$
3.2	14	a divisor on a $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1))$ such that $X \in \lvert L^{\otimes 2} \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,3)\rvert$, where L is the tautological line bundle	$1/2\star$
3.3	18	a divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of tridegree $(1,1,2)$	$2/3\star$
3.4	18	the blow up of the Fano threefold Y with $\daleth(Y) = 2.18$ along a smooth fiber of the composition $Y \to \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ of the double cover with the projection	1/2
3.5	20	the blow up of $\mathbb{P}^1 \times \mathbb{P}^2$ along a curve C of bidegree $(5,2)$ such that the composition $C \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ is an embedding	$1/2\star$
3.6	22	the blow up of $\mathbb{P}^3$ along a disjoint union of a line and an elliptic curve of degree 4	$1/2\star$
3.7	24	the blow up of the threefold W along an elliptic curve that is an intersection of two divisors from $\lvert -\frac{1}{2}K_W\rvert$	$1/2\star$
3.8	24	a divisor in $\lvert(\alpha \circ \pi_1)^*(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}^2}(2))\rvert$, where $\pi_1\colon \mathbb{F}_1 \times \mathbb{P}^2 \to \mathbb{F}_1$ and $\pi_2\colon \mathbb{F}_1 \times \mathbb{P}^2 \to \mathbb{P}^2$ are projections, and $\alpha\colon \mathbb{F}_1 \to \mathbb{P}^2$ is a blow up of a point	$1/2\star$
3.9	26	the blow up of a cone $W_4 \subset \mathbb{P}^6$ over the Veronese surface $R_4 \subset \mathbb{P}^5$ with center in a disjoint union of the vertex and a quartic on $R_4 \cong \mathbb{P}^2$	1/3
3.10	26	the blow up of $Q \subset \mathbb{P}^4$ along a disjoint union of two conics	1/2
3.11	28	the blow up of the threefold V_7 along an elliptic curve that is an intersection of two divisors from $\lvert -\frac{1}{2}K_{V_7}\rvert$	1/2
3.12	28	the blow up of $\mathbb{P}^3$ along a disjoint union of a line and a twisted cubic	1/2
3.13	30	the blow up of $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ along a curve C of bidegree $(2,2)$ such that $\pi_1(C) \subset \mathbb{P}^2$ and $\pi_2(C) \subset \mathbb{P}^2$ are irreducible conics, where $\pi_1\colon W \to \mathbb{P}^2$ and $\pi_2\colon W \to \mathbb{P}^2$ are natural projections	$\leqslant 1/2$
3.14	32	the blow up of $\mathbb{P}^3$ along a disjoint union of a plane cubic curve that is contained in a plane $\Pi \subset \mathbb{P}^3$ and a point that is not contained in Π	1/2
3.15	32	the blow up of $Q \subset \mathbb{P}^4$ along a disjoint union of a line and a conic	1/2
3.16	34	the blow up of V_7 along a proper transform via the blow up $\alpha\colon V_7 \to \mathbb{P}^3$ of a twisted cubic passing through the center of the blow up α	1/2
3.17	36	a divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of tridegree $(1,1,1)$	1/2
3.18	36	the blow up of $\mathbb{P}^3$ along a disjoint union of a line and a conic	1/3
3.19	38	the blow up of $Q \subset \mathbb{P}^4$ at two non-collinear points	1/3
3.20	38	the blow up of $Q \subset \mathbb{P}^4$ along a disjoint union of two lines	1/3
3.21	38	the blow up of $\mathbb{P}^1 \times \mathbb{P}^2$ along a curve of bidegree $(2,1)$	1/3
3.22	40	the blow up of $\mathbb{P}^1 \times \mathbb{P}^2$ along a conic in a fiber of the projection $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$	1/3

3.23	42	the blow up of V_7 along a proper transform via the blow up $\alpha\colon V_7 \to \mathbb{P}^3$ of an irreducible conic passing through the center of the blow up α	1/4
3.24	42	$W \times_{\mathbb{P}^2} \mathbb{F}_1$, where $W \to \mathbb{P}^2$ is a $\mathbb{P}^1$-bundle and $\mathbb{F}_1 \to \mathbb{P}^2$ is the blow up	1/3
3.25	44	the blow up of $\mathbb{P}^3$ along a disjoint union of two lines	1/3
3.26	46	the blow up of $\mathbb{P}^3$ with center in a disjoint union of a point and a line	1/4
3.27	48	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	1/2
3.28	48	$\mathbb{P}^1 \times \mathbb{F}_1$	1/3
3.29	50	the blow up of the Fano threefold V_7 along a line in $E \cong \mathbb{P}^2$, where E is the exceptional divisor of the blow up $V_7 \to \mathbb{P}^3$	1/5
3.30	50	the blow up of V_7 along a proper transform via the blow up $\alpha\colon V_7 \to \mathbb{P}^3$ of a line that passes through the center of the blow up α	1/4
3.31	52	the blow up of a cone over a smooth quadric in $\mathbb{P}^3$ at the vertex	1/3
4.1	24	divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of multidegree $(1,1,1,1)$	1/2
4.2	28	the blow up of the cone over a smooth quadric $S \subset \mathbb{P}^3$ along a disjoint union of the vertex and an elliptic curve on S	1/2
4.3	30	the blow up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a curve of tridegree $(1,1,2)$	1/2
4.4	32	the blow up of the smooth Fano threefold Y with $\daleth(Y) = 3.19$ along the proper transform of a conic on the quadric $Q \subset \mathbb{P}^4$ that passes through the both centers of the blow up $Y \to Q$	1/3
4.5	32	the blow up of $\mathbb{P}^1 \times \mathbb{P}^2$ along a disjoint union of two irreducible curves of bidegree $(2,1)$ and $(1,0)$	3/7
4.6	34	the blow up of $\mathbb{P}^3$ along a disjoint union of three lines	1/2
4.7	36	the blow up of $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ along a disjoint union of two curves of bidegree $(0,1)$ and $(1,0)$	1/2
4.8	38	the blow up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a curve of tridegree $(0,1,1)$	1/3
4.9	40	the blow up of the smooth Fano threefold Y with $\daleth(Y) = 3.25$ along a curve that is contracted by the blow up $Y \to \mathbb{P}^3$	1/3
4.10	42	$\mathbb{P}^1 \times S_7$	1/3
4.11	44	the blow up of $\mathbb{P}^1 \times \mathbb{F}_1$ along a curve $C \cong \mathbb{P}^1$ such that C is contained in a fiber $F \cong \mathbb{F}_1$ of the projection $\mathbb{P}^1 \times \mathbb{F}_1 \to \mathbb{P}^1$ and $C \cdot C = -1$ on F	1/3
4.12	46	the blow up of the smooth Fano threefold Y with $\daleth(Y) = 2.33$ along two curves that are contracted by the blow up $Y \to \mathbb{P}^3$	1/4
4.13	26	the blow up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a curve of tridegree $(1,1,3)$	1/2$\star$
5.1	28	the blow up of the smooth Fano threefold Y with $\daleth(Y) = 2.29$ along three curves that are contracted by the blow up $Y \to Q$	1/3
5.2	36	the blow up of the smooth Fano threefold Y with $\daleth(Y) = 3.25$ along two curves $C_1 \neq C_2$ that are contracted by the blow up $\phi\colon Y \to \mathbb{P}^3$ and that are contained in the same exceptional divisor of the blow up ϕ	1/3
5.3	36	$\mathbb{P}^1 \times S_6$	1/2
5.4	30	$\mathbb{P}^1 \times S_5$	1/2
5.5	24	$\mathbb{P}^1 \times S_4$	1/2

5.6	18	$\mathbb{P}^1 \times S_3$	1/2
5.7	12	$\mathbb{P}^1 \times S_2$	1/2
5.8	6	$\mathbb{P}^1 \times S_1$	1/2

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