ON $K_*(\mathbb{Z})$ AND CLASSICAL CONJECTURES IN THE ARITHMETIC OF CYCLOTOMIC FIELDS

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Abstract. This paper contains a short survey on the problem of computing K-groups of the integers and its relations to the classical conjectures in number theory, such as the Vandiver conjecture. The main point which we want to stress is that the topological methods of K-theory can be used in proving new theorems in the direction of the number theoretical conjectures.

0. Introduction

The algebraic K-groups of schemes introduced by Quillen are expected to carry deep arithmetic information, according to conjectures of Lichtenbaum, Beilinson, Bloch and Kato which relate K-theory to special values of L-functions. Despite the substantial progress which has been achieved, there are still many features of the relationship of the K-theory with number theory which remain to be understood.

One of the very fascinanting aspects of algebraic K-theory is the link between K-groups of $\mathbb{Z}$ and deep questions in number theory, such as the Vandiver conjecture. The conjecture asserts that for any prime number l, the class number h_l^+ of the maximal totally real subfield $\mathbb{Q}(\cos\frac{2\pi}{l})$ of the field of lth roots of unity, is prime to l. The statement of the conjecture is one of the most important questions in the theory of cyclotomic fields.

It comes as a surprise that the Vandiver conjecture - a classical problem in number theory, can be reformulated in terms of K-groups, whose definition and properties come from algebraic topology. Even more surprisingly, it turns out that using K-theory of $\mathbb{Z}$ one can prove new results in the direction of the conjecture. One of our goals in this paper is to present some of the recently proven results of this type, due to Banaszak, Kurihara, Soulé and the author (cf. Theorems: 4.4, 4.7, 4.8 and 5.5 below).

After recalling some basic facts on the K-theory of number fields, in Section 2 we review the current state of the knowledge on the K-groups of the integers,

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the conjecture of Kurihara on $K_*(\mathbb{Z})$ and relations to the conjectures of Quillen-Lichtenbaum and Bloch-Kato. In Section 3 we introduce the Beilinson element which is a nontorsion element of $K_{2n+1}(\mathbb{Z})$, where n is an even natural number. The Vandiver conjecture can be reformulated in terms of the Beilinson element. The conjecture is true if and only if for all n as above, the group $K_{2n+1}(\mathbb{Z})$ modulo 2-torsion, is generated by the Beilinson element. In Sections 4 and 5, in addition to the theorems mentioned above, we discuss other reformulations of the Vandiver conjecture and of the weaker form, due to Iwasawa.

In the moment one of the most important problems concerning K-groups of the integers is to construct nonzero elements (generators) in the groups $K_{2n}(\mathbb{Z})$ for n odd (cf. Remark 5.6). One can expect that such a construction can be provided by a topological method.

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1. Basic facts on K-groups of number fields

In the paper [Q1] Quillen introduced higher K-groups of a commutative ring R with unity as the homotopy groups:

$$K_m(R) = \pi_{m+1}(BQ\mathcal{P}(R))$$

for any $m \geq 0$ cf. [Q1, Def., p.103 and Example, p.104]. Here $\mathcal{P}(R)$ denotes the category of finitely generated, projective R-modules, $Q(-)$ stands for the categorical Q-construction and $B\mathcal{C}$ is the classifying space of a small category $\mathcal{C}$. Substituting m=0, 1, 2 into the definition we recover the groups: $K_0(R)$, $K_1(R)$ and $K_2(R)$ which were introduced earlier by Grothendieck, Bass and Milnor, respectively. Recall that

$$K_0(R) = F(R)/N(R),$$

where $F(R)$ is the free abelian group generated by the isomorphism classes of R-modules from $\mathcal{P}(R)$ and $N(R)$ denotes the subgroup generated by $[P \oplus Q] - [P] - [Q]$, for $P, Q \in \mathcal{P}(R)$. The group K_1 of Bass is by definition:

$$K_1(R) = GL(R)/E(R) = H_1(GL(R), \mathbb{Z})$$

the abelianization of the general linear group $GL(R) = \bigcup_{n \geq 0} GL_n(R)$, where we identify $Gl_n(R)$ with its image in $GL_{n+1}(R)$ by sending an invertible $n \times n$ matrix

A to $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. The abelianization of GL coincides with the first homology group, because for any discrete group G we have $\pi_1(BG){=}G$. The group $K_2(R)$ of Milnor (usually defined in terms of Steinberg symbols) is the second homology $H_2(E(R), \mathbb{Z})$ of the group of elementary matrices. Proofs of the following facts can be found in the textbooks on K-theory cf: [M] and [R].

Theorem 1.1.

(1) *If L is a field, then $K_0(L){=}\mathbb{Z}$ and $K_1(L){=}L^\times$ (the multiplicative group of L). Moreover we have (Matsumoto's theorem):*

$$K_2(L) = (L^\times \otimes_{\mathbb{Z}} L^\times)/N,$$

where

$$N = \langle x \otimes (1{-}x) : \quad x \in L, \quad x \neq 0, 1 \rangle$$

is the subgroup generated by the Steinberg relations.

(2) *For the ring of integers $\mathcal{O}_F$ of the number field F:*

$$K_0(\mathcal{O}_F){=}Cl(\mathcal{O}_F) \oplus \mathbb{Z} \qquad \textit{and} \qquad K_1(\mathcal{O}_F){=}\mathcal{O}_F^\times,$$

where $Cl(\mathcal{O}_F)$ is the ideal class group of the ring $\mathcal{O}_F$ and $\mathcal{O}_F^\times$ denotes its group of invertible elements (the group of global units of $\mathcal{O}_F$). The group $K_2(\mathcal{O}_F)$ is finite and can be explicitly computed in some special cases.

The class group and the group of units are two of the most important arithmetical invariants of the ring $\mathcal{O}_F$. The class group can be defined by the exact sequence:

$$(1.2) \qquad 1 \longrightarrow \mathcal{O}_F^\times \longrightarrow F^\times \xrightarrow{\partial} \bigoplus\nolimits_{\mathfrak{p}} \mathbb{Z} \longrightarrow Cl(\mathcal{O}_F) \longrightarrow 0$$

where the summation is taken over all prime ideals of $\mathcal{O}_F$. The map $\partial = \bigoplus \partial_{\mathfrak{p}}$ is defined by the valuations. For $c \in \mathcal{O}_F$ we define $\nu_{\mathfrak{p}}(c)$, to be the maximal integer such that the ideal $\mathfrak{p}^{\nu_{\mathfrak{p}}(c)}$ divides the principal ideal $c\mathcal{O}_F$. We put:

$$\partial_{\mathfrak{p}}(\frac{a}{b}) = \nu_{\mathfrak{p}}(a){-}\nu_{\mathfrak{p}}(b)$$

where a, $b \in \mathcal{O}_F$, and $b \neq 0$ (this clearly makes sense, because F is the field of fractions of the ring $\mathcal{O}_F$). Using Theorem 1.1 the exact sequence (1.2) can be identified with:

(1.3)

$$0 \longrightarrow K_1(\mathcal{O}_F) \longrightarrow K_1(F) \xrightarrow{\partial} \bigoplus\nolimits_{\mathfrak{p}} K_0(\kappa_{\mathfrak{p}}) \longrightarrow K_0(\mathcal{O}_F)/\mathbb{Z} \longrightarrow 0$$

where $\kappa_{\mathfrak{p}}{=}\mathcal{O}_F/\mathfrak{p}$ is the residue field at $\mathfrak{p}$. The residue field is finite. It follows by [Q1, Cor., p.113] that the sequence (1.3) extends to the exact *localization sequence*:

$$\longrightarrow K_m(\mathcal{O}_F) \longrightarrow K_m(F) \xrightarrow{\partial} \bigoplus\nolimits_{\mathfrak{p}} K_{m-1}(\kappa_{\mathfrak{p}}) \longrightarrow K_{m-1}(\mathcal{O}_F) \longrightarrow$$

for any $m \geq 1$. Using properties of the map ∂ and the vanishing of even K-groups of finite fields (proven by Quillen in [Q2]) Soulé showed that the localization sequence breaks down into short exact sequences.

Theorem 1.4. *Let $n > 0$.*

(1) *There exists a short exact sequence*

$$0 \longrightarrow K_{2n}(\mathcal{O}_F) \longrightarrow K_{2n}(F) \xrightarrow{\partial} \bigoplus_{\mathfrak{p}} K_{2n-1}(\kappa_{\mathfrak{p}}) \longrightarrow 0.$$

(2) *There is an isomorphism $K_{2n+1}(F) \cong K_{2n+1}(\mathcal{O}_F)$.*

Another important result of Quillen concerns the K-groups of algebraic integers.

Theorem 1.5. *[Q3]*
For any number field F the groups $K_m(\mathcal{O}_F)$ are finitely generated.

Borel computed ranks of these groups using cohomology of arithmetic groups.

Theorem 1.6. *[Bo]*

$$rank_{\mathbb{Z}}\, K_m(\mathcal{O}_F) = \begin{cases} 1 & m = 0 \\ r_1 + r_2 - 1 & m = 1 \\ 0 & m > 0,\ even \\ r_1 + r_2 & m > 1,\ and\ m \equiv 1 \ mod\ 4 \\ r_2 & m \equiv 3\ mod\ 4, \end{cases}$$

where r_1 ($2r_2$, resp.) denotes the number of real (complex, resp.) embeddings of the field F, hence $r_1+2r_2 = [F : \mathbb{Q}]$.

Note that the rank of the group $K_{2n+1}(\mathcal{O}_F)$ equals the order of vanishing of the Dedekind zeta function of F at $s = -n$. The celebrated Lichtenbaum conjecture [Li] postulates formulas for values of the Dedekind zeta at odd negative integers in terms of the K-groups of $\mathcal{O}_F$ for a totally real number field F (so then r_2=0). In the case of F=$\mathbb{Q}$ (which is of the main interest for us here) $\mathcal{O}_F$=$\mathbb{Z}$, r_1=1, r_2=0 and according to the Lichtenbaum conjecture we should have:

$$|\zeta(-n)| = 2\frac{\#K_{2n}(\mathbb{Z})}{\#K_{2n+1}(\mathbb{Z})}, \tag{1.7}$$

for any odd natural number n, where $\zeta(s)$ is the Riemann zeta function. This formula follows from the Quillen-Lichtenbaum conjecture (cf. section 3) as was shown by Kolster *et al.* in [KFN1] and [KFN2]. The Quillen-Lichtenbaum conjecture follows if the Bloch-Kato conjecture is true (cf. Remark 2.2 for the discussion).

Dealing with torsion in K-groups, it is very useful to have K-theory with finite coefficients introdued by Karoubi and Browder cf. [Br]. For natural numbers n and $m \geq 1$ we define

$$K_m(R,\, \mathbb{Z}/n) = \pi_{m+1}(BQ\mathcal{P}(R),\, \mathbb{Z}/n)$$

as the homotopy group with coefficients in $\mathbb{Z}/n$ cf. [N]. The homotopy group $\pi_m(X,\, \mathbb{Z}/n)$ of a topological space is the set of homotopy classes of maps from *the Moore space* $M^m(\mathbb{Z}/n)$ to X. The Moore space $M^m(\mathbb{Z}/n)$ is the mapping cone of the map $S^{m-1} \to S^{m-1}$ which induces mutliplication by n on π_m. We will use K-groups with coefficients in $\mathbb{Z}/l^k$, for an odd prime l and $k \geq 1$. The cofibration sequence in homotopy induces short exact sequences:

$$0 \longrightarrow K_m(R)/l^k \longrightarrow K_m(R, \mathbb{Z}/l^k) \xrightarrow{\mathcal{B}} K_{m-1}(R)[l^k] \longrightarrow 0$$

where we denote by $C[l^k]$ the subgroup of l^k-torsion elements of an abelian group C. The map $\mathcal{B}$ is called *the Bockstein map.* Clearly, the K-groups with $\mathbb{Z}/l^k$ are torsion groups and are anihilated by l^{2k}. Let R be a Dedekind domain, which contains a primitive root of unity ξ_{l^k} of order l^k, e.g., let $R = \mathbb{Z}[\xi_{l^k}]$. Consider the surjective Bockstein map:

$$\mathcal{B}\colon K_2(R, \mathbb{Z}/l^k) \to K_1(R)[l^k] \tag{1.8}$$

Observe that: $K_1(R)[l^k] = R^\times[l^k]$. Let μ_{l^k} denote the subgroup of l^kth roots of unity in $R^\times$. There exists *the Bott element* $\beta = \beta(\xi_{l^k}) \in K_2(R, \mathbb{Z}/l^k)$ (which depends only on ξ_{l^k}), such that $\mathcal{B}(\beta(\xi_{l^k})) = \xi_{l^k}$. By definition, $\beta(\xi_{l^k})$ is the image of ξ_{l^k} by the composition of maps:

$$\mu_{l^k} \xrightarrow{\cong} \pi_1(BGL_1(R))[l^k] \xrightarrow{\cong} \pi_2(BGL_1(R), \mathbb{Z}/l^k) \to K_2(R, \mathbb{Z}/l^k).$$

In K-theory and K-theory with finite coefficients there exist transfer maps similar to the transfer maps in group cohomology cf. [Q1, p. 111] and [So1]. In the case of number fields we have the localization sequence in K-theory with finite coefficients:

$$\begin{array}{ccccc} \longrightarrow & K_m(\mathcal{O}_F, \mathbb{Z}/l^k) & \longrightarrow & K_m(F, \mathbb{Z}/l^k) & \xrightarrow{\partial} \\ \longrightarrow & \bigoplus_{\mathfrak{p} \nmid l} K_{m-1}(\kappa_{\mathfrak{p}}, \mathbb{Z}/l^k) & \longrightarrow & K_{m-1}(\mathcal{O}_F, \mathbb{Z}/l^k) & \longrightarrow \end{array}$$

which doesn't split, in general. There are products maps:

$$\star : \; K_i(R) \otimes K_j(R) \to K_{i+j}(R)$$

which were constructed by Loday [Lo] (see also [We1]). When i=j=0, then the product is induced by the tensor product of R-modules. Product maps in K-theory are constructed using multiplicative properties of the space $BQ\mathcal{P}(R)$, which is the zero space of an E_∞-spectrum in the sense of algebraic topology cf. [Wal]. All expected properties of the products hold, which makes $\bigoplus_{i\geq 0} K_i(R)$ into a graded commutative ring. For K-groups with finite coefficients, the product maps:

$$\star : \; K_i(R, \mathbb{Z}/l^k) \otimes K_j(R, \mathbb{Z}/l^k) \to K_{i+j}(R, \mathbb{Z}/l^k),$$

(at least for l odd) were constructed by Browder in [Br]. Note that there are no well behaved product maps for $l = 2$ and $k \leq 3$ cf. *loc. cit.*

2. What we know about $K_*(\mathbb{Z})$ today

If $F = \mathbb{Q}$ and $m \geq 2$, then Borel's theorem implies:

$$\operatorname{rank} K_m(\mathbb{Z}) = \begin{cases} 1 & \text{if} \quad m \equiv 1 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

With exception for $K_0(\mathbb{Z}) = \mathbb{Z}$ and $K_1(\mathbb{Z}) = \mathbb{Z}/2$, the computation of the K-groups of the integers turned out to be quite a difficult problem. As for today the following K-groups of $\mathbb{Z}$ are known:

(a) $K_2(\mathbb{Z}) = \mathbb{Z}/2$ (see [M])

(b) $K_3(\mathbb{Z}) = \mathbb{Z}/48$ (Lee and Szczarba, [LS1])

(c) $K_4(\mathbb{Z}) = 0$ (Lee, Szczarba and Soulé considered the odd torsion cf. [LS2] and [So6]. Rognes showed that the group has no 2-torsion cf. [R])

(d) $K_5(\mathbb{Z}) = \mathbb{Z}$ (cf. [LS2] and [E-VGS])

(e) $K_m(\mathbb{Z}) \otimes \mathbb{Z}_2$, for $m \geq 6$, is known due to computations of Rognes and Weibel cf. [RW] and [We2], who used the Milnor conjecture proven by Voevodsky, [Vo1]. By $\mathbb{Z}_l$ we denote here the ring of l-adic integers, for a prime number l.

There are precise conjectures concerning torsion of $K_m(\mathbb{Z})$ which were formulated by Kurihara cf. [K] and also, independently by Mitchell.

Kurihara Conjecture.
Up to 2-torsion, the following statements are true:

(1) $K_{2n}(\mathbb{Z}) = \mathbb{Z}/N_n$ *for* $n > 0$, *odd*

(2) $K_{2n+1}(\mathbb{Z}) = \mathbb{Z}/D_n$ *for* $n > 0$, *odd*

(3) $K_{2n}(\mathbb{Z}) = 0$ *for* $n > 0$, *even*

(4) $K_{2n+1}(\mathbb{Z}) = \mathbb{Z}$ *for* $n > 0$, *even.*

Here N_n and D_n are such relatively prime natural numbers that

$$|\zeta(-n)| = \frac{N_n}{D_n},$$

where $\zeta(s)$ is the Riemann zeta function. Euler showed that for positive, odd integers n

$$\zeta(-n) = -\frac{B_{n+1}}{n+1},$$

where B_k denote the Bernoulli numbers, which are defined by the generating series

$$\frac{t}{e^t-1} = \sum_{j\geq 0} \frac{B_j}{j!} t^j$$

and can be computed by the recursive formula:

$$\frac{B_m}{m!} = -\sum_{j=0}^{m-1} \frac{B_j}{j!} \frac{1}{(m-j+1)!}.$$

It is not difficult to check that $B_0{=}1$, $B_1{=}-\frac{1}{2}$ and $B_j{=}0$, for odd $j>1$. Moreover one computes (see the tables in [Wa]):

$$B_2=\tfrac{1}{6},\ B_4=-\tfrac{1}{30},\ B_6=\tfrac{1}{42},\ B_8=-\tfrac{1}{30},\ B_{10}=\tfrac{5}{66},\ B_{12}=-\tfrac{691}{2730}$$

$$B_{14}=\tfrac{7}{6},\ B_{16}=-\tfrac{3617}{510},\ B_{18}=\tfrac{43867}{798},\ B_{20}=-\tfrac{174611}{330},\ B_{22}=\tfrac{854513}{138}.$$

Thus the first numbers D_n are:

$$\begin{array}{l} D_1=12,\ D_3=120,\ D_5=252,\ D_7=240,\ D_9=132,\ D_{11}=32760,\\ D_{13}=12,\ D_{15}=8160,\ D_{17}=14364,\ D_{19}=6600,\ D_{21}=276. \end{array}$$

The denominators D_n can be computed completely due to the classical theorem of von Staudt [Wa, App.B, Th. B4], which says that a prime power p^a divides D_n, if and only if, $(p{-}1)p^{a-1}$ divides $n{+}1$. The numerators N_n are much more mysterious. For example, according to (1) of the Kurihara's conjecture $K_{38}(\mathbb{Z}){=}\mathbb{Z}/174611$ up to 2-torsion. Note that the statements (1) and (2) of the conjecture are consistent with the Lichtenbaum conjecture for $\mathbb{Q}$. According to (3) and (4) the groups:

$$\begin{array}{lll} K_8(\mathbb{Z}),\ K_{12}(\mathbb{Z}),\ K_{16}(\mathbb{Z}),\ K_{20}(\mathbb{Z}), & \dots & \text{should vanish,}\\ K_9(\mathbb{Z}),\ K_{13}(\mathbb{Z}),\ K_{17}(\mathbb{Z}),\ K_{21}(\mathbb{Z}), & \dots & \text{should be isomorphic to } \mathbb{Z}, \end{array}$$

if we ignore the 2-torsion.

K-groups of the integers map to Galois cohomology by Chern character maps.

Theorem 2.1. *[Soulé, Dwyer and Friedlander]*
For any odd prime l and any $n>0$, there exist epimorphisms of groups

(1) $K_{2n}(\mathbb{Z})\otimes\mathbb{Z}_l \longrightarrow H^2(G,\mathbb{Z}_l(n{+}1))$
(2) $K_{2n+1}(\mathbb{Z})\otimes\mathbb{Z}_l \longrightarrow H^1(G,\mathbb{Z}_l(n{+}1))$.

Here $G{=}Gal(\mathbb{Q}_S/\mathbb{Q})$ is the Galois group (with its natural profinite topology) of the maximal field extension $\mathbb{Q}_S/\mathbb{Q}$ contained in $\bar{\mathbb{Q}}$ and such that $\mathbb{Q}_S$ is unramified outside of the principal ideal (l). The group acts on all the l-primary roots of unity, since all the cyclotomic fields $\mathbb{Q}(\xi_{l^k})$ are contained in $\mathbb{Q}_S$. The coefficient module $\mathbb{Z}_l(n+1)$ is the $(n+1)$th tensor power of the G-module

$$\mathbb{Z}_l(1) = \lim_{\leftarrow} \mathbb{Z}/l^k(1),$$

which is the inverse limit of the G-modules of l^kth roots of unity. The cohomology is defined by continuous cochains using the topology of G and the compact G-module $\mathbb{Z}_l(n+1)$. The Dwyer-Friedlander maps were defined by methods of algebraic topology see [DF], Prop. 4.4. Surjectivity follows by [DF], Th. 8.7 and Rem. 8.8. Note that the maps constructed in [DF] take values in groups of continuous étale cohomology $H^i(\mathbb{Z}[\frac{1}{l}];\, \mathbb{Z}_l(n+1))$. We have replaced the étale cohomology by the isomorphic $H^i(G, \mathbb{Z}_l(n+1))$, (cf. [Mi, p. 209] or [Sch, p. 204]).

Quillen-Lichtenbaum Conjecture.
The maps from Theorem 2.1 are isomorphisms.

Remark 2.2. This is a special case of the conjecture which was formulated for the ring of integers of any number field. It was proven by Levine cf. [Le1] and [Le2] (see also [SV] and [GL]) that the Quillen-Lichtenbaum conjecture follows from the Bloch-Kato conjecture, which says that *the norm residue map*

$$K_m^M(L)/l \longrightarrow H^m(G_L,\, \mathbb{Z}/l(m))$$

from the Milnor K-theory to Galois cohomology is an isomorphism, for any field L of characteristic prime to l. The Milnor K-group $K_m^M(L)$ is defined as the quotient of the mth tensor power $(L^\times)^{\otimes m}$ by the submodule generated by Steinberg relations, similar to the subgroup N of Theorem 1.1 (1) for m=2. The norm residue map for m=1 is the isomorphism:

$$L^\times / L^{\times l} \xrightarrow{\cong} H^1(G_L,\, \mathbb{Z}/l(1)).$$

It is obtained when one takes cohomology of the exact sequence of G_L-modules:

$$0 \longrightarrow \mathbb{Z}/l(1) \longrightarrow \bar{L}^\times \xrightarrow{\times l} \bar{L}^\times \longrightarrow 1,$$

because $H^0(G_L,\, \bar{L}^\times) = L^\times$ and $H^1(G_L,\, \bar{L}^\times)$ vanishes by the Hilbert Theorem 90 cf. [La, Th. 10.1]. For $m>1$ the norm residue map is defined by sending an element $\{a_1, a_2, \ldots, a_m\}$ of $K_m^M(L)$ to the cup product $a_1 \cup a_2 \cup \cdots \cup a_m$ in Galois cohomology. There are strong reasons to believe that the proof of the Bloch-Kato conjecture for odd primes will be available soon. Then the Quillen-Lichtenbaum and the Lichtenbaum conjecture (at least for $\mathbb{Z}$) will be proven, too. The case of $l = 2$ of the Bloch-Kato conjecture, i.e., the Milnor conjecture, was proven by Voevodsky in [Vo1].

Remark 2.3. The Galois cohomology groups which appear in Theorem 2.1 belong more to number theory than to topology. The current knowledge of these groups, especially of $H^2{:=}H^2(G,\mathbb{Z}_l(n{+}1))$, is limited.

- The group H^2 is finite. It is an immediate corollary of the finiteness of $K_{2n}(\mathbb{Z})$ and the surjectivity of the Dwyer-Friedlander map.
- If n is even and positive the group H^2 should vanish by (3) of the conjecture of Kurihara and Theorem 2.1. The vanishing is consistent with the Vandiver conjecture cf. section 4.
- If n is odd, it follows by the Main Conjecture in Iwasawa theory proven for $\mathbb{Q}$ by Mazur and Wiles in [MW], that the group H^2 has l^k elements, where l^k is the l-part of the number N_n. Theorem 2.1 and part (1) of the Kurihara conjecture imply that H^2 should be cyclic. The question about the cyclicity of H^2, for n odd, is related to another conjecture in number theory, which we will discuss in section 5.
- For odd n, the group $H^1{:=}H^1(G,\mathbb{Z}_l(n{+}1))$ has l^k elements and is cyclic, where l^k is the l-part of D_n. It is easy to prove this fact using standard Galois cohomology. In this case the Quillen-Lichtenbaum conjecture and the results on the image of J-homomorphism (see [Q4] and [Br, Th. 4.8]) give explicit generators of the group $K_{2n+1}(\mathbb{Z})$.

Remark 2.4. On the side of $K_*(\mathbb{Z})$, if the Dwyer-Friedlander maps are proven to be isomorphisms, then there is still plenty to do.

- As mentioned above, the part (2) of Kurihara's conjecture follow if the Quillen-Lichtenbaum conjecture is proven. To prove part (1) we will still need a generator in the group $K_{2n}(\mathbb{Z})$, for n odd.
- Not much is known in the moment about (3) and (4). Note that (3) and (4) are equivalent cf. Proposition 4.11, if the Quillen-Lichtenbaum conjecture holds true. Any results in the direction of statements: (1), (3) or (4) of the Kurihara conjecture will be of upmost interest for number theory.

Quillen in the classical paper [Q2] computed K-groups of finite fields using methods of algebraic topology.

Theorem 2.5.

$$K_m(\mathbb{F}_q) = \begin{cases} 0 & \textit{if} \quad m>0 \textit{ and } m \textit{ is even} \\ \mathbb{Z}/(q^{n+1}-1) & \textit{if} \quad m=2n{+}1, \end{cases}$$

where $\mathbb{F}_q$ denotes the field with q elements, and q is a power of a prime number.

3. Special elements in $K_{2n+1}(\mathbb{Z})$

The reader has seen already that it is quite difficult to compute K-groups of the integers. Using Quillen's calculation of K-groups of finite fields one can try to compare $K_*(\mathbb{Z})$ with the groups $K_*(\mathbb{F}_p)$. In this section we discuss such a comparision given by the map

$$K_{2n+1}(\mathbb{Z}) \longrightarrow K_{2n+1}(\mathbb{F}_p)$$

induced by reductions at prime numbers p, for n even, positive. In this case the conjecture of Kurihara predicts that $K_{2n+1}(\mathbb{Z})$ modulo 2-torsion is $\mathbb{Z}$. Sketching the proof of Theorem 3.2 below we introduce into discussion the Beilinson element which conjecturally generates the group $K_{2n+1}(\mathbb{Z})$. We also introduce certain cyclotomic numbers, which will be particularly useful in the next section when we talk about the Vandiver conjecture.

Definition 3.1.
Let $\mathcal{P}$ be a set of rational prime numbers. We say that $\mathcal{P}$ has natural density $d(\mathcal{P})$, if the limit

$$\lim_{x\to\infty} \frac{\#\{p \leq x: \quad p \in \mathcal{P}\}}{\#\{p \leq x\}}$$

exists and equals $d(\mathcal{P})$. It is clear from the definition that if $\mathcal{P}$ has density d, then $0 \leq d \leq 1$ and that every finite set of primes has density 0.

Theorem 3.2.
Let n be an even, positive integer. Assume that l is an odd prime which does not divide the number $(n+1)b$, where b is an integer (depending on n only) defined in the proof. Let k be a fixed positive integer. Denote by M_k the set of rational primes p which satisfy the following two conditions:

(1) *the l-torsion part of the group $K_{2n+1}(\mathbb{F}_p)$ is isomorphic to $\mathbb{Z}/l^k$*
(2) *the reduction map $K_{2n+1}(\mathbb{Z}) \longrightarrow K_{2n+1}(\mathbb{F}_p)_l$ is nontrivial.*

Then the set M_k has positive density, which equals $\frac{l^k-1}{l^{2k}}$.

Proof. **Step 1.** *Cyclotomic elements of Soulé and the element of Beilinson*
Recall that, since the K-groups of the integers are finitely generated, we have:

$$K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Z}_l \quad = \quad \lim_{\leftarrow} K_{2n+1}(\mathbb{Z}, \mathbb{Z}/l^k),$$

where the inverse limit on the right hand side is taken over the reduction of coefficients in K-groups with finite coefficients. In the beginning of the eigthies Soulé introduced the following construction cf. [So2]. Let ξ_{l^k} be a primitive root of unity of order l^k, e.g., $\xi_{l^k} = \exp \frac{2\pi i}{l^k}$. Consider the number $1-\xi_{l^k}$ which is a unit in the ring $R = \mathbb{Z}[\frac{1}{l}, \xi_{l^k}]$. There is the canonical Bott element $\beta = \beta(\xi_{l^k}) \in K_2(R, \mathbb{Z}/l^k)$ which depends only on the choice of ξ_{l^k}. Using products in the K-groups we can consider the nth power $\beta^n \in K_{2n}(R, \mathbb{Z}/l^k)$ of the Bott element and the product map

$$*: \; K_1(R) \otimes K_{2n}(R, \mathbb{Z}/l^k) \quad \longrightarrow \quad K_{2n+1}(R, \mathbb{Z}/l^k).$$

Since $K_1(R) = R^\times$ is the group of units, we can construct an element

$$(1 - \xi_{l^k}) * \beta^n \in K_{2n+1}(R, \mathbb{Z}/l^k).$$

Taking transfer map:

$$Tr:\, K_{2n+1}(R,\, \mathbb{Z}/l^k) \to K_{2n+1}(\mathbb{Z},\, \mathbb{Z}/l^k)$$

down to K-theory of the integers gives

$$c_{l^k} = Tr((1-\xi_{l^k}) * \beta^n) \in K_{2n+1}(\mathbb{Z}, \mathbb{Z}/l^k).$$

It can be checked using properties of the transfer that the elements c_{l^k} are compatible under the reductions of coefficients, hence they give a well defined element

$$c_l = \lim_{\leftarrow} c_{l^k} \in K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Z}_l \tag{3.3}$$

which is usually called *the Soulé cyclotomic element.* In this way we obtained one element c_l for any odd prime l. This simple construction turned out to be very powerful in applications. Note that to define c_l we have used products and transfers, which are encoded in the topology of the K-theory spectrum in the sense of algebraic topology. In the eighties there was yet another construction due to Beilinson which gave a nontrivial element in the rational K-group $K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Q}$. Recently it has been proven by Beilinson and Deligne [BD] (see also [HW] for a carefull check of details) that the element of Beilinson and the element of Soulé are compatible under the obvious map

$$K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Q} \quad \longrightarrow \quad K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Q}_l,$$

for any odd l. Since $K_{2n+1}(\mathbb{Z})$ is finitely generated, the compatibility implies that there exists a nontorsion element b_{2n+1} in $K_{2n+1}(\mathbb{Z})$, which for any odd l maps to c_l by the map induced by the imbedding of $\mathbb{Z}$ into $\mathbb{Z}_l$. Slightly abusing the terminology in what follows we will keep calling b_{2n+1} *the Beilinson element.* The number b from the assumption of the theorem is by definition the index of the subgroup of $K_{2n+1}(\mathbb{Z})$ generated by b_{2n+1}.

Step 2. *A map from K-theory down to cyclotomic units*

We will denote by E the cyclotomic field $\mathbb{Q}(\xi_{l^k})$ for $k > 0$. Note that E is the field of fractions of the ring $\mathcal{O}_E$. For a finite set S of prime ideals of $\mathcal{O}_E$ by $\mathcal{O}_{E,S}$ we denote the ring of S-integers in E. By definition, the ring $\mathcal{O}_{E,S}$ consists of fractions from E with denominators divisible by prime ideals from S. The symbol $\mathcal{O}_{E,S}^{\times}$ is the notation for the group of units of the ring. The map

$$\alpha\colon K_{2n+1}(\mathbb{Z}) \quad \longrightarrow \quad \mathcal{O}_{E,S_k}^{\times}/\mathcal{O}_{E,S_k}^{\times l^k} \otimes \mathbb{Z}/l^k(n) \tag{3.4}$$

which we are about to define takes values in twisted units of the cyclotomic field E. We use étale cohomology for conventional reasons but the reader should keep in mind that in what follows it can be replaced by the continuous cochains cohomology of Galois groups similar to the group G from Theorem 2.1. The map α is the composition of the following maps:

$$\begin{array}{c}
K_{2n+1}(\mathbb{Z}) \\
\alpha_0 \big\downarrow \\
(K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Z}_l)/l^k \\
\alpha_1 \big\downarrow \\
H^1(\mathbb{Z}[\frac{1}{l}]; \mathbb{Z}_l(n+1))/l^k \\
\alpha_2 \big\downarrow \\
H^1(\mathcal{O}_{E,S_k}; \mathbb{Z}_l(n+1))/l^k \\
\alpha_3 \big\downarrow \\
H^1(\mathcal{O}_{E,S_k}; \mathbb{Z}/l^k(n+1)) \\
\alpha_4 \big\downarrow \cong \\
\mathcal{O}_{E,S_k}^{\times}/\mathcal{O}_{E,S_k}^{\times l^k} \otimes \mathbb{Z}/l^k(n)
\end{array} \tag{3.5}$$

The map labeled α_0 is induced by the obvious

$$K_{2n+1}(\mathbb{Z}) \to K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Z}_l$$

which sends x to $x \otimes 1$. The map α_1 is induced by the Dwyer-Friedlander map from K-theory to étale cohomology. Note that α_1 is an isomorphism for prime numbers l such that $K_{2n+1}(\mathbb{Z}) = 0$ has no l-torsion by our assumption on l. To define α_2 we choose S_k to be any finite set of primes of E such that it contains primes over l and that the class group of the ring $\mathcal{O}_{E,S_k}$ vanishes. This can be done because the class group of every field is zero and $Cl(E) = \lim_{\rightarrow} Cl(\mathcal{O}_{E,S})$, where the direct system is over all finite sets S of prime ideals of $\mathcal{O}_E$. The arrow α_2 is the injection which comes from the Hochschild-Serre spectral sequence of the monomorphism $\mathbb{Z}[\frac{1}{l}] \to \mathcal{O}_{E,S_k}$. The map α_3 is an imbedding which comes from the exact sequence in cohomology induced by the exact sequence

$$0 \longrightarrow \mathbb{Z}_l(n+1) \xrightarrow{\times l^k} \mathbb{Z}_l(n+1) \longrightarrow \mathbb{Z}/l^k(n+1) \longrightarrow 0.$$

Finally, the map α_4 is the inverse of the isomorphism provided by tensoring *the Kummer exact sequence*

$$0 \longrightarrow \mathcal{O}_{E,S_k}^{\times}/\mathcal{O}_{E,S_k}^{\times l^k} \longrightarrow H^1(\mathcal{O}_{E,S_k}; \mathbb{Z}/l^k(1)) \longrightarrow Cl(\mathcal{O}_{E,S_k}) \longrightarrow 0,$$

by $\mathbb{Z}/l^k(n)$, since $Cl(\mathcal{O}_{E,S_k}) = 0$.

Lemma 3.6. *The map*

$$\alpha\colon K_{2n+1}(\mathbb{Z}) \quad\longrightarrow\quad \mathcal{O}^{\times}_{E,S_k}/\mathcal{O}^{\times l^k}_{E,S_k}\otimes \mathbb{Z}/l^k(n)$$

sends the Beilinson element b_{2n+1} *onto* $u_k^{(n)}\otimes\xi_{l^k}^{\otimes n}$, *where* $u_k^{(n)}$ *is the number*

$$u_k^{(n)} = \prod_{1\leq a<l^k,\,(a,l)=1} (1-\xi^a_{l^k})^{a^n}.$$

Proof of Lemma 3.6 is a routine exercise. The key point is the well-known property of the transfer map: Tr when composed with the map α_2 gives the product over the elements of the Galois group $Gal(E/\mathbb{Q})$. The exponents a in the formula for $u_k^{(n)}$ come from the Galois action on the units in $\mathcal{O}_{E,S_k}$, while the a^n's indicate the Galois action on the module $\mathbb{Z}/l^k(n)=\mathbb{Z}/l^k(1)^{\otimes n}$. The numbers $u_k^{(n)}$ provide the link between $K_*(\mathbb{Z})$ and the conjectures in number theory which we are going to discuss in the next section.

Step 3. *Application of the theorem of Chebotarev*
We use the Chebotarev density theorem which enables us to count the density of the set M_k. We prepare the setup accordingly. The map α which we have defined by the diagram (3.5) appears as the left vertical arrow in the following commutative diagram.

$$\begin{tikzcd}
K_{2n+1}(\mathbb{Z}) \arrow[r, "\phi_p"] \arrow[d] & K_{2n+1}(\mathbb{F}_p)_l=\mathbb{Z}/l^k \arrow[d, "="] \\
(K_{2n+1}(\mathbb{Z})\otimes\mathbb{Z}_l)/l^k \arrow[r] \arrow[d] & (K_{2n+1}(\mathbb{F}_p)\otimes\mathbb{Z}_l)/l^k \arrow[d] \\
\mathcal{O}^{\times}_{E,S_k}/\mathcal{O}^{\times l^k}_{E,S_k}\otimes\mathbb{Z}/l^k(n) \arrow[r] & \kappa_w^{\times}/\kappa_w^{\times l^k}\otimes\mathbb{Z}/l^k(n)
\end{tikzcd} \tag{3.7}$$

The horizontal arrows in the diagram are induced by the reduction at p. Here $w\notin S_k$ is a prime of $\mathcal{O}_E$ which divides the principal ideal (p) and $\kappa_w=\mathcal{O}_{E,S_k}/w$ denotes its residue field. It is a finite field containing $\mathbb{F}_p$. The lower horizontal arrow is the natural map induced by the projection $\mathcal{O}_{E,S_k}\longrightarrow\kappa_w$. We will need a bit more arithmetic of the numbers $u_k^{(n)}$. Let $L=E(\sqrt[l^k]{u_k})$ be the field extension obtained by adding to E the l^kth root of $u_k^{(n)}$. The field is well defined because the l^kth roots of unity are in E. It is a cyclic extension of E with the Galois group $G(L/E)=\mathbb{Z}/l^k$. We choose a prime ideal $\widetilde{w}$ in $\mathcal{O}_L$ dividing w. The situation is shown at the picture below, where at the left hand side stands the tower of the field extensions. At the right we have indicated the chosen prime ideals such that $\widetilde{w}|w$ and $w|(p)$.

$$\begin{array}{ccc} L = E(\sqrt[l^k]{u_k}) & \qquad & \widetilde{w} \\ | & & | \\ E = \mathbb{Q}(\xi_{l^k}) & & w \\ | & & | \\ \mathbb{Q} & & (p) \end{array}$$

Inside the Galois group $G(L/E)$ there exists the canonical automorphism Fr_w, called the Frobenius automorphism, which generates the cyclic group $Gal(\kappa_{\tilde{w}}/\kappa_w) \subset G(L/E)$. We have the following result.

Lemma 3.8. *In this setting, the number of elements of the image of the reduction map ϕ_p equals the order of the Frobenius element Fr_w.*

Theorem of Chebotarev. *Let L/K be a finite extension of number fields with Galois group G. Assume that the element $\sigma \in G$ has c conjugates in G. Let P_σ denote the set of prime ideals of $\mathcal{O}_K$ which have a prime divisor in $\mathcal{O}_L$ whose Frobenius automorphism is σ. Then the set P_σ has the density which equals $\frac{c}{\#G}$.*

To finish the proof of Theorem 3.2 one uses the Chebotarev theorem for the extension $L/\mathbb{Q}$. The Galois group $G(L/\mathbb{Q})$ is the semidirect product of two cyclic groups: $Gal(E/\mathbb{Q}) = (\mathbb{Z}/l^k)^\times$ and $G(L/E) = \mathbb{Z}/l^k$. It is possible to compute the number of conjugates in $G(L/\mathbb{Q})$ of the elements of a given order. Together with the Chebotarev theorem and Lemma 3.8 this implies the claim on the density of M_k. For more details we refer the interested reader to [G]. Proofs of the theorem of Chebotarev which can be found in textbooks on number theory, e.g., [Ne] usually rely on analytic methods. We recommend the survey paper [SL] which contains a sketch of the very first proof (purely algebraic) together with some interesting historical remarks about Chebotarev and his theorem.

4. Vandiver conjecture and K-theory

Let A denote the l-Sylow subgroup of the class group of the ring $\mathbb{Z}[\xi_l]$. It has an action of the Galois group $\Delta := Gal(\mathbb{Q}(\xi_l)/\mathbb{Q}) \cong (\mathbb{Z}/l)^\times$. We define the Teichmüller character

$$\omega : \Delta \longrightarrow (\mathbb{Z}/l)^\times$$

by $\sigma(\xi_l) = \xi_l^{\omega(\sigma)}$, for $\sigma \in \Delta$. Note that ω generates the character group

$$\mathrm{Hom}(\Delta, \mathbb{C}^\times) \cong \mathbb{Z}/(l-1)$$

if we identify $(\mathbb{Z}/l)^\times$ with the group of $(l-1)$th roots of unity $\mu_{l-1} \subset \mathbb{C}^\times$. Since Δ has order prime to l, we have the canonical decomposition cf. [Wa, section 6.3]

$$A = \bigoplus_{0\le i\le l-2} A^{[i]}, \tag{4.1}$$

where

$$A^{[i]} = \{a \in A : \quad \sigma(a) = \omega^i(\sigma)a \quad \text{for every } \sigma \in \Delta\}$$

is the ith eigensubmodule of A. Note that $A^{[i]} = e_i A$, where

$$e_i = \frac{1}{l-1} \sum_{\sigma\in\Delta} \omega^{-i}(\sigma)\sigma$$

is the orthogonal idempotent of the ring $\mathbb{Z}_l[\Delta]$. It is well known that $A^{[0]} = A^{[1]} = 0$ cf. [W, Prop. 6.16].

Vandiver Conjecture.
$A^{[i]} = 0$ for every even integer i such that $2 \le i \le l-3$.

This statement was formulated by Vandiver about 70 years ago. Vandiver writes in [Va] that the conjecture had appeared to him much earlier, around 1912. It is also being attributed to Kummer, who considered the same statement on class numbers of cyclotomic fields in a letter to Kronecker dated on the 28th of December 1848, cf. [Ku, p. 84]. The Vandiver conjecture has been checked numerically for many primes. The most recent published account [BCEMS] shows that there is no counterexample for the conjecture for primes smaller than $12,000,000$. Note that there exists an heuristic argument against the conjecture based on certain assumption on the divisibility properties of the numbers N_n by prime numbers [Wa, p. 158].

The numbers $u_k^{(n)}$ which have been introduced in the previous section are of interest for number theorists because of their direct relation to the Vandiver conjecture.

Proposition 4.2. *Let l be an odd prime and let n be an even interger such that $2 \le n \le l-3$. Let k be a positive integer. The following statements are equivalent.*

(1) *The number*

$$u_k^{(n)} = \prod_{(a,l)=1;\, 1\le a<l^k} (1-\xi_{l^k}^a)^{a^n}$$

can not be written as $u_k^{(n)} = \eta^l$, for any $\eta \in \mathbb{Z}[\xi_{l^k}]$.

(2) *The eigensubmodule $A^{[l-1-n]}$ is trivial.*

Proof. We show how to reduce the proof to the similar statement for $u_1^{(n)}$ which was obtained earlier by Thaine [Th]. To simplify notation a little bit, let $u_k{=}u_k^{(n)}$. For every $k{>}1$ the following three statements are equivalent

(1) u_1 is an lth power in the group $\mathbb{Q}(\xi_l)^\times$
(2) u_1 is an lth power in the group $\mathbb{Q}(\xi_{l^k})^\times$
(3) u_k is an lth power in the group $\mathbb{Q}(\xi_{l^k})^\times$.

In order to see this, observe that for every n and k we have the following equalities in the group $\mathbb{Q}(\xi_{l^k})^\times/\mathbb{Q}(\xi_{l^k})^{\times l}$.

$$\begin{aligned} u_k &\equiv \prod_{1\le b<l} [\prod_{a\equiv b \mod l;\, 1\le a<l^k} (1-\xi_{l^k}^a)]^{b^n} \\ &\equiv \prod_{1\le b<l} [\prod_{1\le j\le l^{k-1}} (1-\xi_{l^k}^{b+lj})]^{b^n} \\ &\equiv \prod_{1\le b<l} [\prod_{\lambda^{l^{k-1}}=1} (1-\lambda\xi_{l^k}^b)]^{b^n} \qquad (4.3) \\ &\equiv \prod_{1\le b<l} [(1-(\xi_{l^k}^b)^{l^{k-1}})]^{b^n} \\ &\equiv \prod_{1\le b<l} (1-\xi_l^b)^{b^n} = u_1 \end{aligned}$$

This shows that (2) and (3) are equivalent. Also, (1) clearly implies (2). To check that (2) implies (1), let us assume that $u_1 \notin \mathbb{Q}(\xi_l)^{\times l}$. Consider the Kummer extension $L=\mathbb{Q}(\xi_l, \sqrt[l]{u_1})$. By assumption it has degree l over $\mathbb{Q}(\xi_l)$. It is not difficult to check that the Galois group $G(L/\mathbb{Q})$ is not abelian. Since $G(\mathbb{Q}(\xi_{l^k})/\mathbb{Q})$ is obviously abelian, we see that $L \not\subset \mathbb{Q}(\xi_{l^k})$, hence $\sqrt[l]{u_1} \notin \mathbb{Q}(\xi_{l^k})$. $\square$

Using properties of the map α from (3.5) one can show the following result.

Theorem 4.4. *[G, Th. 2.14]*
Assume that the l-torsion part of the group $K_{2n+1}(\mathbb{Z})$ is trivial. Let l^{k_0} be the largest power of l which divides the index of the subgroup of $K_{2n+1}(\mathbb{Z})$ generated by the Beilinson element. Let k be a natural number which is not smaller than k_0. If $u_k^{(n)} = \eta^{l^{k_1}}$, for some $\eta \in \mathbb{Z}[\xi_{l^k}]$, then $k_1 \le k_0$.

Observe that the statement of Theorem 4.4 relies on topology, because in order to define the Soulé elements one needs products and transfers in K-groups, which come from the multiplicative structure of the K-theory ring spectrum. To the best of the author's knowledge no bouds on the divisibility of the numbers $u_k^{(n)}$ has been obtained by other methods.

Corollary 4.5.
Fix a positive, even integer n. If the Quillen-Lichtenbaum conjecture is true, then the following two statements are equivalent.

(1) *The group $A^{[l-1-n]}=0$, for every odd prime l.*

(2) *The Beilinson element* b_{2n+1} *generates the group* $K_{2n+1}(\mathbb{Z})$ *modulo 2-torsion.*

The class groups of the cyclotomic field are also connected with the even dimensional K-groups, because of Theorem 2.1 and the following observation due to Kurihara, [K, Cor.1.5].

Proposition 4.6.
Let l *be an odd prime. If* $n > 0$ *is even, then* $A^{[l-1-n]}=0$ *iff* $H^2(G, \mathbb{Z}_l(n+1))=0$.

Note that the Galois twisting of the G-module $\mathbb{Z}_l(n+1)$ forced the unusual indexing of the eigensubmodules of A. Section 6 contains the proof of Proposition 4.6. Since $K_4(\mathbb{Z})=0$, and the group surjects onto $H^2(G, \mathbb{Z}_l(3))$ by Theorem 2.1, we have

Corollary 4.7. *[K, Cor.3.8]*
For any odd l *the eigensubmodule* $A^{[l-3]}$ *is zero.*

This has not been proven by another method. A couple of years ago Soulé was able to extend the result of Kurihara cf. [So4].

Theorem 4.8.
Assume that $m > 1$ *is odd.*[1] *If* $\log l > m^{224m^4}$, *then* $A^{[l-m]} = 0$.

The starting point of Soulé's proof of Theorem 4.8 is as in [K]. By Theorem 2.1 we know that to control $A^{[l-m]}$ it is enough to bound the torsion of the group $K_{2m-2}(\mathbb{Z})$. Then Soulé uses the Hurewicz map and the stabilization theorem for the homology of the general linear group to get the map

$$K_{2m-2}(\mathbb{Z}) \quad \longrightarrow \quad H_{2m-2}(SL_N(\mathbb{Z})\,,\,\mathbb{Z})$$

with N sufficiently large. It follows by a result of Arlettaz [A] that the kernel of the latter map has the exponent divisible only by primes smaller than m. Thus it is enough to bound the torsion of the homology of the $SL_N(\mathbb{Z})$. The classical Voronoi "reduction theory" gives an explicit cell decomposition of the compactification of the locally symmetric space attached to the group $SL_N(\mathbb{Z})$. With this in hand, Soulé implements the following brilliant observation of Gabber: there is an upper bound on the torsion in the homology of a finite CW-complex. The bound depends only on the data associated to the chain complex $C_*(X)$ of X, such as the number of cells of a fixed dimension and the number of faces of each cell. This yields the explicit bound on the torsion in $K_{2m-2}(\mathbb{Z})$ and consequently the double exponential bound in Theorem 4.8. In the paper [So5, 5.5] Soulé sketches a way to prove that $A^{[l-5]}=0$, by bounding from below the value of the Borel regulator on $K_9(\mathbb{Z})$.

As far as we are aware Theorem 4.4, Corollary 4.7 and Theorem 4.8 are the first results on the Vandiver conjecture of the general nature.

We have two additional reformulations of the Vandiver conjecture in terms of K-theory.

[1] Put $m = n+1$ to return to our previous indexing.

Theorem 4.9. *[BG1, section 5.2]*
The following statements are equivalent for a fixed odd prime l

1. *The Vandiver conjecture holds true for l.*
2. *The groups of infinitely l-divisible elements $D_{n+1}(\mathbb{Q})_l := \bigcap_{k\geq 1} l^k K_{2n}(\mathbb{Q})$ of the K-groups of $\mathbb{Q}$ vanish for all n, even and positive.*
3. *The étale K-theory group $K_4^{et}(R)$ is trivial, where $R=\mathbb{Z}[\frac{1}{l}, \xi_l+\xi_l^{-1}]$ is the ring obtained from $\mathbb{Z}$ by inverting l and adding the number $\xi_l+\xi_l^{-1}=2\cos\frac{2\pi}{l}$.*

If the Quillen-Lichtenbaum holds true, then $D_{n+1}(\mathbb{Q})_l=K_{2n}(\mathbb{Z})_l$ in (2) and one can skip the superscript ét in the statement of (3). The l-torsion part of an abelian group C is denoted here by C_l.

Remark 4.10. The subgroup $D_{n+1}(F)_l = \bigcap_{k\geq 1} l^k K_{2n}(F)$ of infinitely l-divisible elements of K-groups of a number field F, for any $n > 0$ was introduced and investigated by G.Banaszak in [Ba1] and [Ba2]. Theorem 1.4 (1) implies that the group is contained in $K_{2n}(\mathcal{O}_F)$, hence it is finite. For k large enough the group is isomorphic to the cokernel of the boundary map ∂ in the localization sequence for the K-theory with finite coefficients:

$$K_{2n+1}(F,\, \mathbb{Z}/l^k) \xrightarrow{\ \partial\ } \bigoplus_{\mathfrak{p}\not{|}l} K_{2n}(\kappa_{\mathfrak{p}},\, \mathbb{Z}/l^k) \longrightarrow D_{n+1}(F) \longrightarrow 0$$

(compare this sequence with the sequence (1.2) defining $Cl(\mathcal{O}_F)$). Because of the properties the group $D_{n+1}(F)$ can be considered as an anologue in higher K-theory of $\mathcal{O}_F$ of the class group. For more on the group of infinitely divisible elements and its rich arithmetic the reader is refered to the papers: [Ba1], [Ba2], [BG1], [BG2] and [BGKZ]. Note that in the paper [BGKZ] the Quillen-Lichtenbaum conjecture was reformulated in terms of divisibility properties of elementary matrices.

If the Quillen-Lichtenbaum conjecture is proven, then the groups $K_{2n+1}(\mathbb{Z})$ and $K_{2n}(\mathbb{Z})$, with $n>0$ even, will be related like units and the class group of the cyclotomic field in a classical formula of Kummer. The formula of Kummer relates the index of the cyclotomic units in the group of units to the class number of the cyclotomic field $\mathbb{Q}(\xi_l)$ cf. [Wa, p. 145].

Proposition 4.11.
Assume that the Quillen-Lichtenbaum conjecture holds true. Let $n > 0$ be an even integer and denote by $\langle b_{2n+1}\rangle$ the subgroup in $K_{2n+1}(\mathbb{Z})$ generated by the Beilinson element b_{2n+1}. Then up to a power of 2 the following index equality

$$[K_{2n+1}(\mathbb{Z}) : \langle b_{2n+1}\rangle] \quad = \quad \#K_{2n}(\mathbb{Z})$$

holds true.

Proposition 4.11 is a consequence of the Main Conjecture cf. [BK, (6.6), p.385]. Observe that it shows that two approaches toward the Vandiver conjecture which were discussed in this section are in principle equivalent, if the Quillen-Lichtenbaum conjecture is true.

5. Iwasawa cyclicity conjecture

Theorem 5.1. *[Herbrand-Ribet]*
Fix an odd prime number l. Let i be an odd integer such that $1 \le i \le l-2$. Then the following two statements are equivalent.

1. *The group $A^{[i]}$ is nontrivial.*
2. *The prime l divides the number N_{l-i-1}.*

Recall that in our notation N_n is the numerator of the divided Bernoulli number $\frac{B_{n+1}}{n+1}$. The implication (1) $\Rightarrow$ (2) was known classically, [Wa, Thm. 6.17]. The converse and its proof due to Ribet is the cornerstone of the arithmetic of cyclotomic fields [Ri]. Nowadays this can be proven using the important techniques of Euler systems due to Kolyvagin and Rubin. The original proof of Ribet and its reinterpretation by Wiles [Wi] was a significant clue in the work of Mazur and Wiles on the Main Conjecture in [MW]. This in turn has direct consequences for the eigen-submodules $A^{[i]}$. For n as above we have by [Wa, Cor.5.15] $\frac{B_{l-i}}{l-i} \equiv B_{1,\omega^{-i}} \bmod l$, where

$$B_{1,\omega^{-i}} = \frac{1}{l}\sum_{a=1}^{l-1} a\omega^{-i}(\sigma_a)$$

is a certain l-adic integer (*the generalized Bernoulli number* attached to the character ω^{-i}) and $\sigma_a \in \Delta$ is the automorphism maped by ω onto $a \in \mathbb{Z}/(l-1)$. The following result was proven by Wiles in [Wi].

Theorem 5.2. *[Wiles]*
Let i be an odd integer such that $1 \le i \le l-2$. Then $A^{[i]}$ has exactly l^{k_i} elements, where l^{k_i} is the maximal power of l dividing $B_{1,\omega^{-i}}$.

Iwasawa Cyclicity Conjecture.
For every odd integer i such that $1 \le i \le l-2$ the group $A^{[i]} \cong \mathbb{Z}/l^{k_i}$.

This conjecture appeared as an assumption of a theorem of Iwasawa cf. [I1, p.78]. The claim of the theorem concerned Galois module properties of class groups of towers of cyclotomic fields and was reformulated by Iwasawa in [I2] as a conjecture, later named the Main Conjecture in Iwasawa theory. The Main Conjecture for $\mathbb{Q}$ was proven by Mazur and Wiles in [MW] by considering reductions of modular curves. There exists more elementary proof of the Main Conjecture based on the method of Euler systems which is due to Kolyvagin and Rubin. We refer the interested reader to [C] and [Gr] for the introduction to the Iwasawa theory and to [Wa, Chapter 15] for the discussion of the Rubin's proof of the Main Conjecture. It is well-known that the Vandiver conjecture implies the Iwasawa cyclicity conjecture [Wa, Cor. 10.15]. Contrary to the case of the Vandiver conjecture, there is no doubt among the experts that the cyclicity conjecture should hold true. When proven, it will simplify the arithmetic of cyclotomic fields considerably.

As far as the connection with Galois cohomology and K-theory is concerned we have the observation of Kurihara (for the proof see Section 6).

Proposition 5.3. *[K, Cor. 1.5]*
Let l be an odd prime. If $n > 0$ is odd, then $A^{[l-1-n]}$ is a cyclic group iff $H^2(G, \mathbb{Z}_l(n+1))$ is a cyclic group.

Remark 5.4. In [BG1] new elements in K-groups of $\mathbb{Q}$ were constructed using certain Gauss sums introduced by Coates [C]. The elements have the Euler system property in the sense of Kolyvagin, which made them useful for computing the order of the subgroup $D_{n+1}(\mathbb{Q}) \subset K_{2n}(\mathbb{Z})_l$, for n odd cf. [BG1, Th C]. In [BG2, Th. 2.4] the same elements were used to construct nonzero cohomology classes in the groups $H^2(G, \mathbb{Z}_l(n+1))$, when n is odd. Assuming the Quillen-Lichtenbaum conjecture we obtain the following statement about the group $K_{2n}(\mathbb{Z})$.

Theorem 5.5.
Assume that l and n are as in Proposition 5.3 and that the Quillen-Lichtenbaum conjecture is true. Let k be so large that $K_{2n}(\mathbb{Z})_l = K_{2n}(\mathbb{Z})[l^k]$. Then every nonzero element $x \in K_{2n}(\mathbb{Z})_l$ can be written in the form (note the resemblance to the Soulé's element c_l):

$$x \quad = \quad \mathcal{B}[Tr(\lambda_x * \beta^n)],$$

where $\beta = \beta(\xi_{l^k}) \in K_2(\mathbb{Z}[\xi_{l^k}], \mathbb{Z}/l^k)$ is the Bott element, Tr denotes the transfer map in K-theory and

$$\mathcal{B}\colon K_{2n+1}(\mathbb{Z}, \mathbb{Z}/l^k) \longrightarrow K_{2n}(\mathbb{Z})[l^k]$$

is the Bockstein map in K-theory with finite coefficients. Here λ_x is an element of $K_1(\mathbb{Q}(\xi_{l^k}))$, which depends on x. It was defined using Gauss sums cf. [BG2, p.11] .

Remark 5.6. Let n and l be odd. In the moment one of the most important problems concerning $K_*(\mathbb{Z})$ is to construct new nonzero elements - possibly generators, in the groups $K_{2n}(\mathbb{Z})_l$. In this respect Theorem 5.5 is not satisfactory enough because the Gauss sum λ_x and its arithmetical properties depend strongly on x. In the case of $K_{2n+1}(\mathbb{Z})_l$, (n and l odd) the nonzero elements came from algebraic topology, or more precisely from the image of the J-homomorphism cf. Remark 2.3. It is possible that in the case of $K_{2n}(\mathbb{Z})_l$, the expected nonzero elements will be constructed by a topological method. With the development of motivic cohomology and related theories (for example the algebraic cobordism theory of Voevodsky cf. [Vo2] and [LM]), such a construction may be accomplished in the future.

6. Proof of (4.6) and (5.3)

Proposition 6.1. *Let l be an odd prime.*

(1) *If $n>0$ is even, then $A^{[l-1-n]}=0$ iff $H^2(G,\mathbb{Z}_l(n+1))=0$.*
(2) *If $n>0$ is odd, then $A^{[l-1-n]}$ is a cyclic group iff $H^2(G,\mathbb{Z}_l(n+1))$ is a cyclic group.*

Proof. Since the groups in question are finite l-groups, it is enough to show that for any l and n

$$A^{[l-1-n]}/l \quad \cong \quad H^2(G,\mathbb{Z}_l(n+1))/l. \tag{6.2}$$

Taking the long exact sequence in Galois cohomology associated to the sequence of Galois modules

$$0 \longrightarrow \mathbb{Z}_l(n+1) \xrightarrow{\times l} \mathbb{Z}_l(n+1) \longrightarrow \mathbb{Z}/l(n+1) \longrightarrow 0$$

we obtain that $H^2(G,\mathbb{Z}_l(n+1))/l = H^2(G,\mathbb{Z}/l(n+1))$, becuase $H^3(G,\mathbb{Z}_l(n+1))=0$, by the cohomological dimension. In the remaining part of the proof it is more convenient to replace the Galois cohomology by the isomorphic etale cohomology groups. In particular, we are going to prove that

$$A^{[l-1-n]}/l \quad \cong \quad H^2(\mathbb{Z}[\frac{1}{l}];\, \mathbb{Z}/l(n+1)). \tag{6.3}$$

Let R be the ring $\mathbb{Z}[\frac{1}{l},\xi_l]$ and let F denote the cyclotomic field $\mathbb{Q}(\xi_l)$. Consider the exact sequence in etale cohomology

$$0 \longrightarrow H^1(R,\mathbb{G}_m)/l \longrightarrow H^2(R,\mu_l) \longrightarrow H^2(R,\mathbb{G}_m)[l] \longrightarrow 0$$

which comes from the exact sequence of group schemes over $Spec\, R$

$$1 \longrightarrow \mu_l \longrightarrow \mathbb{G}_m \xrightarrow{\times l} \mathbb{G}_m \longrightarrow 1.$$

Note that $\mu_l=\mathbb{Z}/l(1)$. By the basics of etale cohomology cf. [Mi] we know that $H^1(R,\mathbb{G}_m)=Pic(R)$ is the class group of the ring R and $H^2(R,\mathbb{G}_m)=Br(R)$ is its Brauer group. It follows by the class field theory that $Br(R)[l]=0$, the reason being that there is a single prime ideal in R above l (recall that in R we have the equality of ideals $(l)=(1-\xi_l)^{l-1}$). Hence we get

$$H^2(R,\mu_l) = H^1(R,\mathbb{G}_m)/l = A/l = Pic(R)/l.$$

Tensoring with $\mathbb{Z}/l(n)$ and taking invariants under the action of Δ gives

$$(A\otimes \mathbb{Z}/l(n))^{\Delta} \cong H^2(R,\mathbb{Z}/l(n+1))^{\Delta}.$$

Since Δ has order prime to l, it follows by the Hochschild-Serre spectral sequence that the group on the right hand side is isomorphic to $H^2(\mathbb{Z}[\frac{1}{l}];\,\mathbb{Z}/l(n+1))$. The group on the left hand side equals $A^{[l-1-n]}/l$, what can be checked by a simple computation. □

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