Modular Invariance of Manifolds with SU(n) Holonomy

Shi-shyr Roan

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

MPI/ 91-53

Introduction

Conformal field theories (CFT) in two dimensions have been studied extensively in the past few years, motivated mainly by their importance in constructing string vacua, and also because of their relation to the critical phenomena in two-dimensional statistical mechanical systems. This paper focuses on an important element in the Landau-Ginzburg model of $N = 2$ superconformal field theory: the modular invariance of manifolds with $SU(n)$ holonomy. When $n = 3$, the manifolds are the Calabi-Yau (CY) spaces, corresponding to $c = 9$ CFT, which are of considerable interest to string theorists. Here we propose a mathematical formulation for modular invariance of (2,2) CFT based upon the works of [4,6,7,10,15,23,25,26]. The Witten index and elliptic $\hat{A}$-genus of CY orbifolds in weighted projective 4-spaces will be our main concern. From the mathematical point of view, we shall study these topological invariances based upon the Hodge structure of some specific generators of $SU(n)$ cobordism class by the theory of Jacobi functions of modular group. The conclusions are derived from the representation theory of superconformal algebra. On the other hand, the mathematical results obtained here have justified much analysis which are recently expended by physicists on the study of Landau-Ginzburg models [4,10,23]. It seems that this is the proper context to understand the modular invariance of CY spaces from the geometry point of view. We now briefly describe the result of this paper.

For $K = (k_1, \ldots, k_N)$, k_j : positive integer, with

$$\frac{c(K)}{3}\left(\doteq \sum_{j=1}^{N} \frac{k_j}{k_j+2}\right) = N-2,$$

denote

$$f_K(Z_1, \ldots Z_N) = \sum_{j=1}^{N} Z_j^{k_j+2},$$

$$X_K = \left\{[Z_1, \ldots, Z_N] \in W\mathbb{P}^{N-1}_{(n_j)} | f_K(Z_1, \ldots, Z_N) = 0\right\},$$

here $W\mathbb{P}^{N-1}_{(n_j)}$ is the weighted projective space with

$$n_j = \frac{d}{k_j+2}, \qquad d \doteq \ell cm\,(k_j + 2 | 1 \leq j \leq N).$$

X_K is a V-manifold with (at most) cyclic quotient singularities, and has the trivial canonical sheaf. When $\dim_{\mathbb{C}} X_K (= N-2) \leq 3$, the "minimal" toroidal resolution $\hat{X}_K$ of X_K has the trivial canonical bundle. It is known that $\hat{X}_K$ is a $K3$ surface for $N = 4$, and CY space for $N = 5$ [9]. But the $c_1 = 0$ resolution of X_K is not known to exist for a general N. For the simplicity of the argument and also the application to $c = 9$ CFT, throughout this paper $\hat{X}_K$ will always be denoted by the manifold defined as follows:

$$\hat{X}_K = \begin{cases} \text{the minimal toroidal resolution of } X_K & \text{for } N \leq 5, \\ \text{the degree } N \text{ Fermat hypersurface } X_K \text{ in } \mathbb{P}^{N-1} \\ \qquad \text{(i.e. all } k_j = N-2) & \text{for } N > 5. \end{cases} \tag{1}$$

Consider the modular group Γ_θ defined by

$$\Gamma_\theta = \{M \in SL_2(\mathbb{Z}) | M \text{ congruent to } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ mod. } 2\}.$$

Note that Γ_θ is a conjugate of $\Gamma_1(2)$ in $SL_2(\mathbb{Z})$. The Jacobi group $\Gamma_\theta^J \left(\doteq \Gamma_\theta \rtimes \mathbb{Z}^2\right)$ acts on the space $\Gamma(\mathbb{C} \times \mathbb{H}, \mathcal{O})$ of holomorphic functions of $\mathbb{C} \times \mathbb{H}$ in a well-known manner, here $\mathbb{H}$ is the upper half plane [5]. An invariant function of Γ_θ^J is called a Jacobi function. We shall assign to each $\hat{X}_K$ a function $\mathcal{J}_K(z, \tau)$ which is a Jacobi function twisted by a certain character of Γ_θ. The method is through the representation theory. The values of $\mathcal{J}_K(z, \tau)$ at some special points z will give the topological invariances of $\hat{X}_K$, e.g. the Euler characteristic and the elliptic genus defined by Ochaine, Landweber and Stong [14]. In fact, the Euler number of $\hat{X}_K$ is expressed by

$$\chi\left(\hat{X}_K\right) = \mathcal{J}_K\left(\frac{\tau}{2} + \frac{1}{2}, \tau\right) \exp\left(\frac{c(K)}{3}\pi i\left(\frac{\tau}{4} + \frac{1}{2}\right)\right),$$

and $\mathcal{J}_K(0, \tau)$ determines a modular form of weight $\frac{c(K)}{3}$ for Γ_θ. This modular form equals to zero when $\dim_{\mathbb{C}} \hat{X}_K = \text{odd}$, and coincide with the elliptic genus of $K3$ surface when $\dim_{\mathbb{C}} \hat{X}_K = 2$. It is expected that the modular form obtained by $\mathcal{J}_K(0, \tau)$ is the elliptic genus of $\hat{X}_K$ for any K, and for $e^{2\pi i y}$ being roots of 1, $\mathcal{J}_K(y, \tau)$ should relate to the elliptic genera of higher level defined by Hirzebruch [11]. Work along this line is in progress. Before going any further, I shall explain first how the function $\mathcal{J}_K(y, \tau)$ comes from. For a given K, there associates a finite collection of highest weight representations of Neveu-Schwarz $N = 2$ superalgebra $\hat{A}$ with the central charge $c = c(K)$. It is obtained by the Gepner's construction [6], which is a specified procedure of selecting subrepresentations of a tensor product of unitary irreducible $c < 3$ highest weight modules (HWM) of the superalgebra $\hat{A}$. The selection is dictated by the modular invariance of characters of the involved HWM. These HWM form a finite dimensional $\mathbb{C}\left[\Gamma_\theta^J\right]$-module $\mathcal{M}(K)$, which can also be described purely from the abstract algebraic point of view. The algebraic construction of $\mathcal{M}(K)$ and its properties are given in section 2 and 3. In section 4, we discuss how the $\mathcal{M}(K)$ relates to HWM of $\hat{A}$, and through the characters of HWM, two Γ_θ^J-morphisms

$$NS,\ \tilde{R} : \mathcal{M}(K) \to \Gamma(\mathbb{C} \times \mathbb{H}, \mathcal{O})$$

are introduced for the discussion of modular invariance. There naturally associates a "Γ_θ^J-invariant" vector $w(K)$ in $\mathcal{M}(K)$ and the Jacobi function $\mathcal{J}_K(z, \tau)$ is then defined to be the image $NS_{w(K)}$ of $w(K)$ under the above morphism NS. The Witten index $Tr(-1)^F$ and elliptic $\hat{A}$ genus of the theory are given by

$$Tr(-1)^F = \tilde{R}_{w(K)}(0, \tau)\left(= \mathcal{J}_K\left(\frac{\tau}{2} + \frac{1}{2}, \tau\right) \exp\left(\frac{c(K)}{3}\pi i\left(\frac{\tau}{4} + \frac{1}{2}\right)\right)\right),$$

$$\text{elliptic } \hat{A} - \text{genus } = \mathcal{J}_K(0, \tau).$$

The general properties of Jacobi functions needed for our discussion are described in section 1. In section 5, the following equality is obtained:

$$\text{Euler number of } \hat{X}_K = \text{Witten index } Tr(-1)^F \text{of } \mathcal{M}(K).$$

We also give a rigorous mathematical justification of the Witten index formula obtained by Vafa in [23] based on physicist's reasoning. In this process, the cohomology group $H^*\left(\hat{X}_K, \mathbb{C}\right)$ of $\hat{X}_K$ can be identified with a certain space constructed from $\mathcal{M}(K)$, which correspond to the "massless" states of the CFT theory in physics literature. The dimension of a cohomology element is expressed by Witten index of its associated state. So the massless excitations of CFT are geometrically realized as the cohomology elements of the corresponding Calabi-Yau vacua. In section 6, we discuss the relation between $\mathcal{J}_K(0, \tau)$ and the topological elliptic genus of $\hat{X}_K$. The cases for $\dim_{\mathbb{C}} \hat{X}_K = 2$ or odd are treated and the equality of these two data are verified.

Although the modular invariance of superconformal algebras stems from theoretical physics, this paper is preoccupied primarily with its related problems in mathematics. Recent development on mirror CY spaces [28] [29] has further indicated that manifolds with $SU(n)$ holonomy are closely related to (2,2) superconformal theories. In this paper, we have put the analyses of modular invariance of Calabi-Yau vacua on a more firm mathematical footing. We have found that through the modular transform, the spectral flow is conjugate to the integral-charge operator. Using the former structure, C. Vafa derived the formula of Witten index of CFT in [23] by the physicist's argument. On the other hand, through the charge operator, the Euler number of a CY orbifold is also expressed by Vafa's formula in [21] from the topological method. In this paper, we also give a rigorous mathematical argument on Vafa's approach of the CY Fermat hypersurfaces in weighted projective spaces. This leads the speculation that the discussion in this paper should also apply to the more general hypersurfaces defined by superpotentials composed of A-D-E type singularities. Furthermore, we believe a deeper understand of the modular invariance will almost certainly clarify the meaning of certain invariances of manifolds with trivial canonical bundle, as indicated in the relationship between Witten index and Euler number of CY space. The study of mirror CY spaces is under consideration along these lines.

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§1. Jacobi function

Let Γ_θ to be the subgroup of $SL_2(\mathbb{Z})$ consisting of all the elements congruent to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (mod 2). The Jacobi group Γ_θ^J is the semi-direct product $\Gamma_\theta^J = \Gamma_\theta \ltimes \mathbb{Z}^2$ corresponding to

$$\left((\delta, \nu), \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \mapsto (\delta a + \nu c, \delta b + \nu d).$$

It is known that the substitutions in the variable $z \in \mathbb{C}$, $\tau \in \mathbb{H}$ (the upper half plane $Im\tau > 0$):

$$(z,\tau) \mapsto \left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right), \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta,$$

defines an action of Γ_θ on $\mathbb{C} \times \mathbb{H}$. Moreover, this action normalizes the lattice action on z, i.e. we have an action Γ_θ^J on $\mathbb{C} \times \mathbb{H}$, where $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\delta, \nu)\right)$ acts on $\mathbb{C} \times \mathbb{H}$ by

$$(z,\tau) \mapsto \left(\frac{z+\delta\tau+\nu}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right).$$

Let ρ be a character of Γ_θ^J to $\mathbb{C}^*$. The above action, together with ρ, induces a representation of Γ_θ^J on the vector space $\Gamma(\mathbb{C} \times \mathbb{H}, \mathcal{O})$ of holomorphic functions of $\mathbb{C} \times \mathbb{H}$, which is described by

$$\Gamma(\mathbb{C} \times \mathbb{H}, \mathcal{O}) \times \Gamma_\theta^J \to \Gamma(\mathbb{C} \times \mathbb{H}, \mathcal{O})$$

$$(\phi, *) \quad \mapsto \phi|*,$$

$$\begin{aligned} &(\phi|(\delta,\nu))(z,\tau) = \rho((\delta,\nu))e^{n\left[2\pi i\delta(z+\nu)+\pi i\delta^2\tau\right]}\phi(z+\delta\tau+\nu,\tau), \\ &\left(\phi|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(z,\tau) = \rho\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)e^{n\pi i\frac{-cz^2}{c\tau+d}}\phi\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right), \end{aligned} \tag{2}$$

here $(\delta,\nu) \in \mathbb{Z}^2$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta$. This representation is called the index $\frac{n}{2}$ representation of Γ_θ^J and its invariant function is the Jacobi function of index $\frac{n}{2}$ with character ρ. Let α, β, s, t be the elements of Γ_θ^J defined by

$$\alpha = (1,0),\ \beta = (0,1),\ s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ t = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

They satisfy the relations:

$$\begin{aligned} &\alpha\beta = \beta\alpha,\ s^4 = 1,\ s^2 t = t s^2, \\ &t^{-1}\alpha t = \alpha\beta^2,\ t^{-1}\beta t = \beta,\ s^{-1}\alpha s = \beta,\ s^{-1}\beta s = \alpha^{-1}. \end{aligned} \tag{3}$$

It is known that Γ_θ^J is characterized as the group generated by 4 elements α, β, s, t with the above relations. We define the following characters of Γ_θ^J which will appear later in this paper,

$$\rho_n, \mathcal{X}_n : \Gamma_\theta^J \to \mathbb{C}^* \qquad (n \in \mathbb{Z})$$

with

$$\begin{aligned} &\rho_n(\alpha) = \rho_n(\beta) = (-1)^n,\ \rho_n(s) = \rho_n(t) = (-1)^{\frac{-n}{2}}; \\ &\mathcal{X}_n(\alpha) = \mathcal{X}_n(\beta) = 1,\ \mathcal{X}_n(s) = \mathcal{X}_n(t) = (-1)^{\frac{n}{2}}. \end{aligned} \tag{4}$$

Consider the classical theta function

$$\vartheta(z,\tau) = \sum_{\delta\in\mathbb{Z}} e^{2\pi i\delta z+\pi i\delta^2\tau}, \quad (z,\tau) \in \mathbb{C} \times \mathbb{H}.$$

It is known [18] that $\vartheta(z,\tau)^2$ satisfies the relations:

$$\begin{aligned} &\vartheta(z+1,\tau)^2 = \vartheta(z,\tau)^2, \\ &e^{2(\pi i\tau + 2\pi i z)}\vartheta(z+\tau,\tau)^2 = \vartheta(z,\tau)^2, \\ &\varepsilon\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(c\tau+d)^{-1}e^{2\pi i\frac{-cz^2}{c\tau+d}}\vartheta\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)^2 = \vartheta(z,\tau)^2, \end{aligned} \tag{5}$$

here $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta$, ε : the character of Γ_θ with $\varepsilon(t) = 1$ and $\varepsilon(s) = i$.

Proposition 1 Let n be an positive integer and $\phi(z,\tau)$ be a Jacobi function of index $\frac{n}{2}$ with character $\mathcal{X}_n$. Then

(i) When n is odd, $\phi(0,\tau) = 0$ for all $\tau \in \mathbb{H}$.

(ii) When n is even, the function of $\mathbb{H}$

$$\frac{\phi(0,\tau)}{\vartheta(0,\tau)^n}\eta(\tau)^{3n}$$

is a modular form of Γ_θ of weight n, here $\eta(\tau)$ is the Dedekind eta function defined by

$$\eta(\tau) = \exp\left(\frac{\pi i\tau}{12}\right)\prod_{\delta=1}^{n}(1-\exp(2\pi i\delta\tau)).$$

Proof: (i) By the equality

$$\mathcal{X}_n(s^2)(\phi|s^2) = \phi,$$

we have $(-1)^n\phi(-z,\tau) = \phi(z,\tau)$ for $(z,\tau) \in \mathbb{C}\times\mathbb{H}$. It follows that $\phi(z,\tau)$ is an odd function of the variable z for odd n, hence $\phi(0,\tau) \equiv 0$.

(ii) Assume n is an even positive integer. It is known [19] that $\eta(\tau)^{3n}$ is a modular form for Γ_θ of weight $\frac{3n}{2}$ with the character

$$\sigma : \Gamma_\theta \to \mathbb{C}^*,\ t \mapsto \exp\frac{n\pi i}{2},\ s \mapsto \exp\frac{-3n\pi i}{4},$$

i.e.

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)^{3n} = \sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(c\tau+d)^{\frac{3n}{2}}\eta(\tau)^{3n}.$$

By (2) and (5), the function

$$\psi(z,\tau) \doteq \frac{\phi(z,\tau)}{\vartheta(z,\tau)^n}\eta(\tau)^{3n}$$

satisfies the relation

$$\psi(z+\delta\tau+\nu,\tau) = \psi(z,\tau),$$

$$\psi\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^n \psi(z,\tau),$$

here $(z,\tau) \in \mathbb{C} \times \mathbb{H}$, $(\delta,\nu) \in \mathbb{Z}^2$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta$. As $\vartheta(0,\tau) \neq 0$ for $\tau \in \mathbb{H}$, $\psi(0,\tau)$ is a modular form for Γ_θ of weight n. q.e.d.

§2. Algebraic preliminary

For a positive integer k, define

$$L_k = \{(l,m) \in \mathbb{Z}^2 | 0 \leq l \leq k, \quad l-m \equiv 0 \pmod 2 \quad \},$$
$$r : L_k \to L_k,\ (l,m) \mapsto (k-l,\ m+k+2),$$
$$\mathfrak{B}_k = L_k / < r >,$$
$$\mathfrak{f} : \mathfrak{B}_k \to \mathfrak{B}_k,\ [l,m] \to [l, m-2],$$
$$\mathfrak{z} : \mathfrak{B}_k \to \mathbb{C}^*,\ [l,m] \mapsto \exp\left(\frac{2m\pi i}{k+2}\right),$$
$$\mathfrak{h} : \mathfrak{B}_k \to \mathbb{C}^*,\ [l,m] \mapsto \exp\left(\pi i\left(\frac{l^2+2l-m^2}{k+2} - \frac{k}{2k+4}\right)\right),$$

$$\mathfrak{s} : \mathfrak{B}_k \times \mathfrak{B}_k \to \mathbb{C},\ \left([l,m],[l',m']\right) \mapsto \frac{2}{k+2}\exp\left(\frac{-\pi i m m'}{k+2}\right) \sin\left(\frac{\pi(l+1)(l'+1)}{k+2}\right).$$

It is easy to see that the above maps $\mathfrak{f}, \mathfrak{z}, \mathfrak{h}, \mathfrak{s}$ are well defined.

Lemma 1. $\{(l,m) \in L_k | \, |m| \leq l\}$ is a set of complete representatives of $\mathfrak{B}_k$.

Proof:

Since $r^2(l,m) = (l, m+2k+4)$ for $(l,m) \in L_k$, every element of $L_k / < r^2 >$ can be uniquely represented by an element (l,m) in L_k with $-l \leq m < -l+2k+4$. For any two elements (l_i, m_i), $i = 1,2$, with $-l_i \leq m_i < -l_i + 2k + 4$,

$$r(l_1,m_1) = (l_2,m_2) \Leftrightarrow (k-l_1,\ m_1+k+2) = (l_2,m_2).$$

Then it is easy to see that every element of $\mathfrak{B}_k$ is uniquely represented by $(l,m) \in L_k$ with $|m| \leq l$. q.e.d.

Definition: An element (l,m) of L_k with $|m| \leq l$ is called the standard representative of the class $[l,m]$ in $\mathfrak{B}_k$.

Denote $V_k =$ the Hermitian vector space over $\mathbb{C}$ with $\mathfrak{B}_k$ as the orthonormal base. We shall identify an element λ of $\mathfrak{B}_k$ with the associated base element of V_k. We are interested in the following linear maps

$$\mathfrak{u}, \mathfrak{q}, H, S : V_k \to V_k$$

which acts on the vectors of V_k from the right, and their values for the base element $\lambda \in \mathfrak{B}_k$ are defined by:

$$\lambda|\mathfrak{u} = \mathfrak{f}(u),\ \lambda|\mathfrak{q} = \mathfrak{z}(\lambda)\lambda,\ \lambda|H = \mathfrak{h}(\lambda)\lambda,$$
$$\lambda|S = \sum_{\mu\in\mathfrak{B}_k} S^{\mu}_{\lambda}\mu \quad \text{with } S^{\mu}_{\lambda} = \mathfrak{s}(\lambda,\mu).$$

It is easy to see that $\mathfrak{u}, \mathfrak{q}, H$ are the unitary transformations of V_k.

Proposition 2. S is an order 4 symmetric, unitary transformation of V_k. S^2 is the linear map sending $[l,m] \in \mathfrak{B}_k$ $(|m| \le l)$ to $[l,-m] \in \mathfrak{B}_k$.

Proof:

The symmetric property follows from the definition of S. We need only to show the unitarity of S and the statement for S^2. The following identities are needed for the argument: For integers M, a, b with $M \ge 3$, $1 \le a,\ b \le M-1$,

$$\sum_{1\le j\le M-1} \sin\frac{ja\pi}{M}\sin\frac{jb\pi}{M} = \frac{M}{2}\delta_{a,b} \tag{6}$$

$$\sum_{1\le j\le M-1} (-1)^{j+1}\sin\frac{ja\pi}{M}\sin\frac{jb\pi}{M} = \frac{M}{2}\delta_{a+b,M}. \tag{7}$$

(A proof of the above equalities is given in the appendix.).

Let λ, μ be elements in $\mathfrak{B}_k$. The standard representatives of λ, μ are denoted by $(b,a), (\beta,\alpha)$ respectively in this proof.

For $0 \le \ell \le [\frac{k}{2}]$, we have

$$\sum_{\substack{(l,m)\in L_k \\ m\equiv 0(2k+4)}} \exp\left(\frac{-a+\alpha}{k+2}m\pi i\right) = \sum_{0\le j<k+2} \exp\left(\frac{-a+\alpha}{k+2}(l+2j)\pi i\right)$$
$$= \exp\left(\frac{-a+\alpha}{k+2}l\pi i\right)\sum_{0\le j<k+2}\exp\left(j\frac{-a+\alpha}{k+2}2\pi i\right)$$

$$= \begin{cases} k+2 & \text{if } a=\alpha \\ (-1)^l(k+2) & \text{if } a-\alpha = \pm(k+2) \\ 0 & \text{otherwise } (\because |-a+\alpha| \le 2k), \end{cases} \tag{8}$$

$$\sum_{\delta\in\mathfrak{B}_k} S_\lambda^\delta \bar{S}_\delta^\mu = \begin{cases} \left(\frac{2}{k+2}\right)^2 \sum\limits_{\substack{(l,m)\in L_k, 0\le l\le \frac{k-1}{2} \\ m\equiv 0(2k+4)}} \exp\left(\frac{-a+\alpha}{k+2} m\pi i\right) \sin\frac{\pi(b+1)(l+1)}{k+2} \sin\frac{\pi(\beta+1)(l+1)}{k+2} & (9) \\ \qquad\qquad \text{for } k = \text{odd}, \\ \left(\frac{2}{k+2}\right)^2 \sum\limits_{\substack{(l,m)\in L_k, 0\le l\le \frac{k}{2}-1 \\ m\equiv 0(2k+4)}} \exp\left(\frac{-a+\alpha}{k+2} m\pi i\right) \sin\frac{\pi(b+1)(l+1)}{k+2} \sin\frac{\pi(\beta+1)(l+1)}{k+2} \\ \quad + \left(\frac{2}{k+2}\right)^2 \cdot \frac{1}{2} \sum\limits_{\substack{(\frac{k}{2},m)\in L_k \\ m\equiv 0(2k+4)}} \exp\left(\frac{-a+\alpha}{k+2} m\pi i\right) \sin\frac{\pi(b+1)}{2} \sin\frac{\pi(\beta+1)}{2} \\ \qquad\qquad \text{for } k = \text{even}. & (10) \end{cases}$$

By (8), $\sum\limits_{\delta} S_\lambda^\delta \bar{S}_\delta^\mu = 0$ when $a-\alpha \neq 0$ or $\pm(k+2)$. Claim: When $a-\alpha = \pm(k+2)$, we have

$$b+\beta \neq k \quad \text{and} \quad \sum_\delta S_\lambda^\delta \bar{S}_\delta^\mu = 0.$$

In fact, if $b+\beta = k$ and $a-\alpha = k+2$,

$$|a| \le b \Rightarrow \alpha + k + 2 \le k - \beta \Rightarrow \alpha \le -\beta - 2.$$

This contradicts $|a| \le \beta$, so $b+\beta \neq k$. The same argument for the case $a-\alpha = -(k+2)$. By (8), (9), (10),

$$\sum_\delta S_\lambda^\delta \bar{S}_\delta^\mu = \begin{cases} \frac{4}{k+2} \sum\limits_{0\le l\le \frac{k-1}{2}} (-1)^l \sin\frac{\pi(b+1)(l+1)}{k+2} \sin\frac{\pi(\beta+1)(l+1)}{k+2} \\ \qquad\qquad \text{for } k = \text{odd}. \\ \frac{4}{k+2} \sum\limits_{0\le l\le \frac{k}{2}-1} (-1)^l \sin\frac{\pi(b+1)(l+1)}{k+2} \sin\frac{\pi(\beta+1)(l+1)}{k+2} + \\ \frac{2}{k+2}(-1)^{\frac{k}{2}} \sin\frac{\pi(b+1)(\frac{k}{2}+1)}{k+2} \sin\frac{\pi(\beta+1)(\frac{k}{2}+1)}{k+2} \\ \qquad\qquad \text{for } k = \text{even}. \end{cases}$$

Since $b \equiv a \equiv \alpha + k \equiv \beta + k \pmod 2$, $(-1)^k = (-1)^{b+\beta}$. Then it is easy to see that

$$\sum_\delta S_\lambda^\delta \bar{S}_\delta^\mu = \frac{2}{k+2} \sum_{1\le j\le k+1} (-1)^{j+1} \sin\frac{\pi(b+1)j}{k+2} \sin\frac{\pi(\beta+1)j}{k+2} = 0. \qquad \text{(By (7))}$$

Now we consider the case when $a = \alpha$. We have $b \equiv \beta \pmod 2$. By (8), (9), (10),

$$\sum_\delta S_\lambda^\delta \bar{S}_\delta^\mu = \frac{2}{k+2} \sum_{1\le j\le k+1} \sin\frac{\pi(b+1)j}{k+2} \sin\frac{\pi(\beta+1)j}{k+2} = \delta_{b,\beta}. \qquad \text{(By (6))}$$

Hence we obtain the unitarity of S. Replacing $\bar{S}_\delta^\mu$, α by S_δ^μ, $-\alpha$ respectively in the above argument. We have

$$\sum_\delta S_\lambda^\delta S_\delta^\mu = \begin{cases} 1 & \text{if } (b,a) = (\beta,-\alpha) \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof of this proposition. q.e.d.

Theorem 1. Let $V_k, \mathfrak{u}, \mathfrak{q}, H, S$ be the same as before. denote $c = \frac{3k}{k+2}$, $d = k+2$. Then $\mathfrak{u}, \mathfrak{q}, H, S$ are unitary transformations of V_k satisfying the following conditions:

$$\mathfrak{q}^d = \mathfrak{u}^d = H^{2d} = S^4 = id.$$

$$\mathfrak{u}\mathfrak{q} = \exp\Big(\frac{c}{3}2\pi i\Big)\mathfrak{q}\mathfrak{u},\ S^2H = HS^2,$$

$$S^{-1}\mathfrak{q}^{-1}S = \mathfrak{u},\ S^{-1}\mathfrak{u}S = \mathfrak{q},$$

$$H^{-1}\mathfrak{q}H = \mathfrak{q},\ H^{-1}\mathfrak{u}H = \exp\left(\frac{c}{3}2\pi i\right)\mathfrak{q}^2\mathfrak{u}.$$

Proof: It is easy to see that $\mathfrak{q}^d = \mathfrak{u}^d = H^{2d} = id.$, $H^{-1}\mathfrak{q}H = \mathfrak{q}$. By Proposition 2, $S^4 = id.$, $S^2H = HS^2$. For $\lambda = [l, m] \in \mathfrak{B}_k$,

$$(\lambda|\mathfrak{u})|\mathfrak{q} = [l, m-2]|\mathfrak{q} = \exp\Big(\frac{2(m-2)}{k+2}\pi i\Big)(\lambda|\mathfrak{u}) = \exp\Big(\frac{c}{3}2\pi i\Big)(\lambda|\mathfrak{q})|\mathfrak{u},$$

$$(\lambda|\mathfrak{u})|H = \exp\Big(\frac{\pi i}{k+2}\big(l^2 + 2l - (m-2)^2\big)\Big)(\lambda|\mathfrak{u}) = \exp\Big(\frac{c}{3}2\pi i\Big)(\lambda|H\mathfrak{q}^2\mathfrak{u}),$$

$$\lambda|\mathfrak{q}^{-1}S = \exp\Big(\frac{-2m\pi i}{k+2}\Big)\sum_\mu S^\mu_\lambda \cdot \mu = \sum_\mu S^{\mu|\mathfrak{u}^{-1}}_\lambda \mu = \lambda|S\mathfrak{u},$$

$$\lambda|\mathfrak{u}S = \sum_\mu S^\mu_{\lambda|\mathfrak{u}} \cdot \mu = \sum_\mu \mathfrak{z}(\mu) S^\mu_\lambda \cdot \mu = \lambda|S\mathfrak{q}.$$

q.e.d.

§3. Representation of Γ_θ

Let $k_1, \dots k_N$ be positive integers and $K = (k_1, \dots, k_N)$. Denote

$$c(K) = \sum_{1\le j\le N} \frac{3k_j}{k_j+2},$$

$$d(K) = lcm(k_1+2, \dots, k_N+2),$$

$$\mathfrak{S}(K) = \{\sigma : \text{permutation of } \{1, \dots, N\} \text{ with } k_{\sigma(j)} = k_j \quad \text{for } 1 \le j \le N\}.$$

$\mathcal{V}(K) = V_{k_1} \otimes \cdots \otimes V_{k_N}$ as the tensor product of Hermitian vector spaces.

Let $\mathfrak{u}_i, \mathfrak{q}_i, H_i, S_i$ be the unitary transformations of V_{k_i} as in the previous section. By tensor product, we have the unitary transformations $\mathfrak{u}, \mathfrak{q}, H, S$ of $\mathcal{V}(K)$ defined by $\mathfrak{u} = \mathfrak{u}_1 \otimes \cdots \otimes \mathfrak{u}_N$, $\mathfrak{q} = \mathfrak{q}_1 \otimes \cdots \otimes \mathfrak{q}_N$, $H = H_1 \otimes \cdots \otimes H_N$, $S = S_1 \otimes \cdots \otimes S_N$, which acts the vectors of $\mathcal{V}(K)$ from the right. On the other hand, $\mathfrak{S}(K)$ acts on $\mathcal{V}(K)$ from the left as unitary transformations by

$$(\sigma, v_1 \otimes \cdots \otimes v_N) \mapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(N)}.$$

It is easy to see that the action of $\mathfrak{S}(K)$ commutes with $\mathfrak{u}, \mathfrak{q}, H, S$, and we have the action:

$$\mathfrak{S}(K) \times \mathcal{V}(K) \times < \mathfrak{u}, \mathfrak{q}, H, S > \longrightarrow \mathcal{V}(K).$$

As before, Γ_θ^J is the Jacobi group for Γ_θ, which is generated by α, β, s, t with the relation (3). As a corollary of Theorem 1 we obtain the following result.

Proposition 3. When $\frac{c(K)}{3} \in \mathbb{Z}$, $\mathcal{V}(K)$ is a $\mathbb{C}[\Gamma_\theta^J]-$module under the following correspondence:

$$\alpha = (1,0) \mapsto \mathfrak{u}$$

$$\beta = (0,1) \mapsto \mathfrak{q}$$

$$t = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \mapsto H$$

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto S.$$

Remark: The above representation of Γ_θ^J can be factored through the representation of the finite group $\Gamma_{\theta,d}^J \doteq \Gamma_{\theta,2d} \ltimes (\mathbb{Z}/d\mathbb{Z})^2$, here $\Gamma_{\theta,2d} = \{M \in SL_2(\mathbb{Z}/2d\mathbb{Z}) | M$ congruent to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bmod 2\}$.

From now on, we shall always assume

$$\frac{c(K)}{3} = \text{integer}$$

for the rest of this paper, unless otherwise specified. In this case, $\mathfrak{q}$ and $\mathfrak{u}$ generate an abelian group isomorphic to $(\mathbb{Z}/_{d\mathbb{Z}})^2$ with $d = d(K)$. Identity $< \mathfrak{q}, \mathfrak{u} >$ with $(\mathbb{Z}/_{d\mathbb{Z}})^2$ via

$$< \mathfrak{q}, \mathfrak{u} >= \{(a_1, a_2) | a_i \in \mathbb{Z}/_{d\mathbb{Z}}\}, \ \mathfrak{q} \leftrightarrow (0,1) \quad \mathfrak{u} \leftrightarrow (1,0).$$

The group of characters of $< \mathfrak{q}, \mathfrak{u} >$ is

$$< \mathfrak{q}, \mathfrak{u} \overset{*}{>} = \left\{ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} | b_i \in \mathbb{Z}/_{d\mathbb{Z}} \right\}.$$

As $< \mathfrak{q}, \mathfrak{u} >$ is normalized by $< S, H >$, The eigenspaces of $\mathcal{V}(K)$ for $< \mathfrak{q}, \mathfrak{u} >$ are permuted by the action of $< S, H > (\simeq \Gamma_\theta)$. In fact, The eigenspace of $\mathcal{V}(K)$ with eigenvalue $\binom{b_1}{b_2}$ is mapped to the one with eigenvalue $M^{-1}\binom{b_1}{b_2}$(mod d) under the action of $M \in \Gamma_\theta$. The following $\Gamma_\theta-$ submodules of $\mathcal{V}(K)$ are needed for our discussion:

$\mathcal{M}(K) = \mathcal{V}(K)^{<\mathfrak{q},\mathfrak{u}>}, \mathcal{K}(K) = \mathcal{V}(K)^{<\mathfrak{S}(K),\mathfrak{q},\mathfrak{u}>}.$

It is clear that the above Γ_θ representation can be factored through $\Gamma_{\theta,2d.}$ We have the $\Gamma_\theta-$ morphism

$$\begin{aligned} \mathcal{M}(K) &\rightarrow \quad \mathcal{K}(K) \\ x &\mapsto \quad \frac{1}{|\mathfrak{S}(K)|} \sum_{\sigma \in \mathfrak{S}(K)} \sigma \cdot x, \end{aligned} \tag{11}$$

and we shall denote the image of $\lambda_1 \otimes \cdots \otimes \lambda_N$, $\lambda_j \in V_{k_j}$, by $\lambda_1 \cdots \lambda_N$. The following lemma is obvious.

Lemma 2. Let $\mathfrak{S}'$ be the $\mathfrak{S}(K)-$ isotropy subgroup of the element $\lambda_1 \otimes \cdots \otimes \lambda_N$ and θ be the $\mathfrak{S}(K)-$ orbit of $\lambda_1 \otimes \cdots \otimes \lambda_N$ in $\mathcal{V}(K)$. Then

$$|\theta| = \frac{|\mathfrak{S}(K)|}{|\mathfrak{S}'|},$$

$$\lambda_1 \cdots \lambda_N = \frac{1}{|\theta|} \sum_{x \in \theta} x,$$

$$||\lambda_1 \cdots \lambda_N||^2 = \frac{1}{|\theta|}.$$

Now we define an important notion, which corresponds to the chiral primary fields in CFT [15].

Definition. For $\lambda_i \in \mathfrak{B}_{k_i}$, $1 \leq i \leq N$, the element $\lambda = \lambda_1 \otimes \cdots \otimes \lambda_N$ in $\mathcal{V}(K)$ is called chiral (antichiral) if the standard representative of λ_i is the form $[l_i, l_i]$ ($[l_i - l_i]$ resp.) for each i.

It is obvious that the chiral and antichiral elements in $\mathcal{V}(K)$ are in one-one correspondence by $[l_i, l_i] \leftrightarrow [l_i, -l_i]$.

§4. Superconformal algebra and $\mathcal{J}_K(z, \tau)$

Having given the algebraic construction of the $\mathbb{C}[\Gamma_\theta]-$ module $\mathcal{M}(K)$, we are in the position to relate it to the representations of the Neveu-Schwarz $N = 2$ algebra. We here list some properties of the representations of $N = 2$ superconformal algebra needed for the discussion of this paper. Comprehensive descriptions can be found in [1], [2], [20] and references quoted there.

Definition. The Neveu-Schwarz $N=2$ subalgebra $\hat{A}$ is the complex Lie superalgebras generated by $\{L_m, J_n, G_p^{\pm} | m,n \in \mathbb{Z}, p \in \frac{1}{2}+\mathbb{Z}\}$ and a central element $\tilde{c}$ with the following super-Lie brackets:

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + \frac{\tilde{c}}{12}(m^3-m)\delta_{m+n,0} \\
&[L_m, J_n] = -nJ_{m+n} \\
&[J_m, J_n] = \frac{\tilde{c}}{3} m\delta_{m+n,0} \\
&[L_m, G_n^{\pm}] = \left(\frac{m}{2}-n\right)G_{m+n}^{\pm} \\
&[J_m, G_n^{\pm}] = \pm G_{m+n}^{\pm} \\
&\{G_n^+, G_m^-\} = 2L_{m+n} + (m-n)J_{m+n} + \frac{\tilde{c}}{3}\left(m^2-\frac{1}{4}\right)\delta_{m+n,0} \\
&\{G_m^+, G_n^+\} = \{G_m^-, G_n^-\} = 0.
\end{aligned}$$

Consider the standard decomposition of $\hat{A}$:

$$\hat{A} = N_+ \oplus H \oplus N_-$$

here

$$\begin{aligned}
H &=< \tilde{c}, L_0, J_0 >_{\mathbb{C}} \\
N_+ &=< L_m, J_n, G_p^{\pm} | m,n,p > 0 >_{\mathbb{C}} \\
N_- &=< L_m, J_n, G_p^{\pm} | m,n,p < 0 >_{\mathbb{C}} .
\end{aligned}$$

A highest weight module (HWM) over $\hat{A}$ is characterized by the highest weight $\lambda \in H^*$ and highest vector v_0 such that $Xv_0 = 0$ for $X \in N_+$ and $Xv_0 = \lambda(X)v_0$ for $X \in H$. Let $\lambda(\tilde{c}) = c$, $\lambda(L_0) = h$, $\lambda(J_0) = Q$. The largest HWM with weight λ is the Verma module $V^{c,h,Q}$. Denote by $L^{c,h,Q}$ the factor-module $V^{c,h,Q}/_{I^{c,h,Q}}$, where $I^{c,h,Q}$ is the maximal proper submodule of $V^{c,h,Q}$. Then every irreducible HWM over $\hat{A}$ is isomorphic to some $L^{c,h,Q}$. A HWM over $\hat{A}$ is called unitary if it satisfies

$$(L_m)^\dagger = L_{-m}, (J_n)^\dagger = J_{-n}, (G_p^{\pm})^\dagger = G_{-p}^{\mp}$$

For $(l,m) \in L_k$ (defined in §2), denote

$$h_{l,m} = \frac{1}{4(k+2)}(l^2+2l-m^2), \; Q_{l,m} = \frac{m}{k+2}.$$

It is known that for $0 < c < 3$, all the unitary irreducible HWM over $\hat{A}$ are labelled by

$$c = \frac{3k}{k+2} \quad (k = 1, 2, \ldots)$$

$$h = h_{l,m} \quad Q = Q_{l,m} \quad \text{for } (l,m) \in L_k \text{ and } |m| \leq l.$$

We are mainly concerned with the characters of HWM. For the latter discussion, we introduce the following notions.

Definition. Denote $y = e^{2\pi i z}, q = e^{2\pi i \tau}$ for $z \in \mathbb{C}$, $\tau \in \mathbb{H}$.

(i) For $\lambda = [l,m] \in \mathfrak{B}_k$, k = positive integer,

$$NS_\lambda(z,\tau) := \varphi_A(z,\tau) q^{h_{l,m} - \frac{c}{24}} y^{Q_{l,m}} \gamma_{l,m}(z,\tau) \tag{12}$$

here

$$c = \frac{3k}{k+2},$$

$$\varphi_A(z,\tau) = \prod_{n=1}^{\infty} \frac{\left(1 + yq^{n-\frac{1}{2}}\right)\left(1 + y^{-1}q^{n-\frac{1}{2}}\right)}{(1-q^n)^2}$$

$$\gamma_{l,m}(z,\tau) = \prod_{n=1}^{\infty} \frac{\left(1 - q^{(k+2)(n-1)+l+1}\right)\left(1 - q^{(k+2)n-l-1}\right)\left(1 - q^{(k+2)n}\right)^2}{\left(1 + yq^{(k+2)n-j}\right)\left(1 + y^{-1}q^{(k+2)(n-1)+j}\right)\left(1 + y^{-1}q^{(k+2)n-i}\right)\left(1 + yq^{(k+2)(n-1)+i}\right)}$$

with

$$j = \frac{l+m+1}{2}, \; i = \frac{l-m+1}{2}.$$

(ii) For $\lambda = \lambda_1 \otimes \cdots \otimes \lambda_N, \lambda_j \in \mathfrak{B}_{k_j} (1 \leq j \leq N)$, $K = (k_1, \cdots k_N)$,

$$NS_\lambda(z,\tau) = \prod_{i=1}^{N} NS_{\lambda_i}(z,\tau)$$

$$\tilde{R}_\lambda(z,\tau) = NS_\lambda\left(z + \frac{\tau}{2} + \frac{1}{2}, \tau\right) \exp\left(\frac{c(K)}{3}\pi i \left(\frac{\tau}{4} + z + \frac{1}{2}\right)\right).$$

and the above definition is extended to the linear maps

$$\begin{aligned} NS &: \mathcal{V}(K) \to \Gamma(\mathbb{C} \times \mathbb{H}, \mathcal{O}), \\ \tilde{R} &: \mathcal{V}(K) \to \Gamma(\mathbb{C} \times \mathbb{H}, \mathcal{O}). \end{aligned} \tag{13}$$

Because of the following lemma, the above notions are well-defined.

Lemma 3. The $NS_\lambda(z,\tau)$ in the above definition (i) depends only on the class $\lambda = [l,m]$, i.e. independent of the choice of (l,m).

Proof: Let $(l,m), (l',m')$ be elements in L_k with $(k-l, m+k+2) = (l',m')$. We need only to show

$$q^{h_{l,m}} y^{Q_{l,m}} \gamma_{l,m}(z,\tau) = q^{h_{l',m'}} y^{Q_{l',m'}} \gamma_{l',m'}(z,\tau).$$

Let $(j,i) = \left(\frac{l+m+1}{2}, \frac{l-m+1}{2}\right)$, $(j',i') = \left(\frac{l'+m'+1}{2}, \frac{l'-m'+1}{2}\right)$. Then

$$(k+2-i, -j) = (j', i');$$

$(k+2)(n-1)+l+1=(k+2)n-l'-1$, $(k+2)n-l-1=(k+2)(n-1)+l'+1$;
$(k+2)(n-1)+i=(k+2)n-j'$, $(k+2)-i=(k+2)(n-1)+j'$,
$(k+2)n+j=(k+2)n-i'$, $(k+2)(n-1)-j=(k+2)(n-1)+i'$.

Hence

$$\gamma_{l',m'}(z,\tau)=\left(1+yq^{-j}\right)^{-1}\left(1+y^{-1}q^{j}\right)\gamma_{l,m}(z,\tau)=y^{-1}q^{j}\gamma_{l,m}(z,\tau),$$

and the result follows immediately. q.e.d.

For a positive integer k, we have the one-one correspondence between the following sets:

$$\left\{\text{irreducible unitary HWM of } \hat{A} \text{ with } c=\frac{3k}{k+2}\right\}\leftrightarrow \mathfrak{B}_k$$

$$L^{c,h_{l,m}Q_{l,m}}\longleftrightarrow \lambda,$$

here (l,m) is the standard representation of λ. It is known that $NS_\lambda(z,\tau)$ is equal to the character $Tr\left(q^{L_0-\frac{c}{24}}y^{J_0}\right)$ of the HWM $L^{c,h_{l,m}Q_{l,m}}$. Here several physicists have made contributions but the author is not familiar with the exact nature and extent of these. So we adher to three reference [2] [16] [20], which are most suitable for our purpose.

Lemma 4 [20] For a positive integer k, let $\mathfrak{z},\mathfrak{h},S_\lambda^\mu,\mathfrak{u}$ be the same as in §2, and $c=\frac{3k}{k+2}$. Then the following equalities hold:

$$\mathfrak{z}(\lambda)NS_\lambda(z,\tau)=NS_\lambda(z+1,\tau)$$
$$NS_{\mathfrak{u}(\lambda)}(z,\tau)=\exp\left(\pi i(\tau+2z)\frac{c}{3}\right)NS_\lambda(z+\tau,\tau)$$
$$\mathfrak{h}(\lambda)NS_\lambda(z,\tau)=NS_\lambda(z,\tau+2)$$
$$\sum_{\mu\in\mathfrak{B}_k}S_\lambda^\mu NS_\mu(z,\tau)=\exp\left(\frac{-c\pi i}{3}\frac{z^2}{\tau}\right)NS_\lambda\left(\frac{-z}{\tau},\frac{-1}{\tau}\right)$$
$$\mathfrak{z}(\lambda)\tilde{R}_\lambda(z,\tau)=\exp\left(\frac{-\pi ic}{3}\right)\tilde{R}_\lambda(z+1,\tau)$$
$$\tilde{R}_{\mathfrak{u}(\lambda)}(z,\tau)=\exp\left(\pi i(\tau+2z+1)\frac{c}{3}\right)\tilde{R}_\lambda(z,\tau)$$
$$\mathfrak{h}(\lambda)\tilde{R}_\lambda(z,\tau)=\exp\left(\frac{-c}{6}\pi i\right)\mathfrak{z}(\lambda)^{-1}\tilde{R}_\lambda(z,\tau+2)$$
$$\sum_{\mu\in\mathfrak{B}_k}S_\lambda^\mu\tilde{R}_\mu(z,\tau)=\mathfrak{z}(\lambda)^{-1}\exp\left(\frac{-c\pi i}{3}\left(\frac{z^2}{\tau}+\frac{1}{2}\right)\right)\tilde{R}_\lambda\left(\frac{-z}{\tau},\frac{-1}{\tau}\right).$$

Proof: The equalities for NS_λ follows from (2.4) of [4] and (26a), (26b), (28a) of [20]. Then the equalities for $\tilde{R}_\lambda$ follows from its definition. q.e.d.

Remark: For $\lambda\in\mathfrak{B}_k$, and $[l,m]$ =standard representative of λ, from (2.3) of [4], $\tilde{R}_\lambda(z,\tau)$ is equal to the quantity $ch_{l,m}^{(k+2,\frac{1}{2})}(\tau,z+\frac{1}{2})$of [20].

Definition. Let $K=(k_1,\cdots,k_N)$ and $\lambda=\lambda_1\otimes\cdots\otimes\lambda_N\in\mathcal{V}(K)$ with $\lambda_j\in\mathfrak{B}_{k_j}$. Let (l_j,m_j) be the standard representative of λ_j. The charge of λ is defined by

$$Q_\lambda=\sum_{j=1}^{N}\frac{m_j}{k_j+2}.$$

Proposition 4 Let NS, $\tilde{R}$ be the linear map from $\mathcal{V}(K)$ to $\Gamma(\mathbb{C}\times\mathbb{H},\mathcal{O})$ in (13), and $\lambda=\lambda_1\otimes\cdots\otimes\lambda_N\in\mathcal{V}(K)$ for $\lambda_j\in\mathfrak{B}_{k_j}$. Then

(i)

$$\tilde{R}_\lambda(0,\tau)\equiv\begin{cases}(-1)^{Q_\lambda-\frac{c(K)}{6}} & \text{if } \lambda=\text{a chiral element},\\ 0 & \text{otherwise.}\end{cases}$$

(ii)

$$\lim_{Im\tau\to\infty} NS_\lambda(0,\tau)\exp\left(\frac{c(K)}{24}2\pi i\tau\right)=\begin{cases}1 & \text{if}\,\lambda_j=[0,0] \quad \text{for all } j\\ 0 & \text{otherwise.}\end{cases}$$

Proof:

(i) It suffices to show that for a positive integer k, and $\lambda\in\mathfrak{B}_k$,

$$\tilde{R}_\lambda(0,\tau)\equiv\begin{cases}(-1)^{Q_\lambda-\frac{k}{2(k+2)}} & \text{if } \lambda=\text{chiral}\\ 0 & \text{otherwise.}\end{cases}$$

Let (l,m) be the standard representative of λ, and $j'=\frac{l+m}{2}+1$, $i'=\frac{l-m}{2}$. By the remark of Lemma 4, and (4b), (5b), (7), (10) of [20], we have

$$\tilde{R}_\lambda(z,\tau)=\varphi_P\left(z+\frac{1}{2},\tau\right)q^{\frac{j'i'}{k+2}}(-y)^{\frac{j'-i'}{k+2}}[*],$$

here $\varphi_P(z,\tau)=\left(y^{1/2}+y^{-1/2}\right)\prod_{n=1}^{\infty}\frac{(1+yq^n)(1+y^{-1}q^n)}{(1-q^n)^2}$,

$$[*]=\prod_{n=1}^{\infty}\frac{\left(1-q^{(k+2)(n-1)+j+i}\right)\left(1-q^{(k+2)n-j-i}\right)\left(1-q^{(k+2)n}\right)^2}{\left(1-yq^{(k+2)n-j'}\right)\left(1-y^{-1}q^{(k+2)(n-1)+j'}\right)\left(1-y^{-1}q^{(k+2)n-i'}\right)\left(1-yq^{(k+2)(n-1)+i'}\right)}.$$

By $0\le j'-1,\ i',\ j'+i'\le k+1$,

$$\tilde{R}_\lambda(z,\tau)=\left((-y)^{1/2}+(-y)^{-1/2}\right)\left(1-yq^{i}\right)^{-1}q^{\frac{j'i'}{k+2}}(-y)^{\frac{j'-i'}{k+2}}q\{\text{some series in } \mathbb{C}[[y,y^{-1},q]]\}$$

Hence $\tilde{R}_\lambda(0,\tau)=0$ if $i'\ne 0$, i.e. $l\ne m$. When $i'=0$,

$$\tilde{R}_\lambda(0,\tau)=(-1)^{-\frac{1}{2}}(-1)^{\frac{j'}{k+2}}=(-1)^{Q_\lambda-\frac{k}{2(k+2)}}.$$

(ii) It suffices to show that for $\lambda\in\mathfrak{B}_k$, $k=$ positive integer,

$$\lim_{Im\tau\to\infty}NS_\lambda(0,\tau)q^{\frac{k}{8(k+2)}}=\begin{cases}1 & \text{if}\,\lambda=[0,0],\\ 0 & \text{otherwise.}\end{cases}$$

Let (l,m) be the standard representative λ, $\varphi_A(z,\tau)$ and $\gamma_{l,m}(z,\tau)$ the functions in (12). By

$$NS_\lambda(0,\tau)q^{\frac{k}{8(k+2)}}=\varphi_A(0,\tau)q^{h_{l,m}}\gamma_{l,m}(0,\tau),$$

the conclusion follows from

$$\lim_{Im\tau\to\infty}\varphi_A(0,\tau)=\lim_{Im\tau\to\infty}\gamma_{l,m}(0,\tau)=1,$$

$$\lim_{Im\tau\to\infty} q^{h_{l,m}}=\begin{cases}1 & \text{if } (l,m)=(0,0)\\ 0 & \text{otherwise.}\end{cases}$$

q.e.d.

<u>**Theorem 2**</u> For $K=(k_1,\cdots,k_N)$, let

$$n=\text{the integer}\ \frac{c(K)}{3}.$$

$\mathcal{V}(K),\ \mathcal{M}(K)=$ the $\mathbb{C}[\Gamma_\theta^J]-$module in §3,
$\Gamma(\mathbb{C}\times\mathbb{H},\mathcal{O})_{\frac{n}{2},\rho}=\Gamma(\mathbb{C}\times\mathbb{H},\mathcal{O})$ with the index $\frac{n}{2}$ representation of Γ_J^θ for a character ρ.
$\rho_j:\Gamma_\theta^J\to\mathbb{C}^*$, the character in (4) for $j\in\mathbb{Z}$.
Then
(i) The linear maps of (13) define the Γ_θ^J-morphism:

$$NS:\mathcal{V}(K)\to\Gamma(\mathbb{C}\times\mathbb{H},\mathcal{O})_{\frac{n}{2},\rho_0},$$
$$\tilde{R}:\mathcal{V}(K)\to\Gamma(\mathbb{C}\times\mathbb{H},\mathcal{O})_{\frac{n}{2},\rho_n}.$$

(ii) For $v\in\mathcal{M}(K)$, let

$$\phi(z,\tau)=NS_v\ \text{or}\ \tilde{R}_v.$$

Then for fixed $\tau\in\mathbb{H}$, the function $z\mapsto\phi(z,\tau)$, if not identically zero, has exactly $\frac{c}{3}$ zeros (counting multiplicity) in any fundamental domain for the action of the lattice $\mathbb{Z}\tau+\mathbb{Z}$ on $\mathbb{C}$.

(iii) Let $\mathcal{X}$ be a character of Γ_θ^J with $\mathcal{X}|_{<\mathfrak{u},\mathfrak{g}>}=$ trivial. If v is an eigenvector in $\mathcal{M}(K)$ for Γ_θ with eigenvalue $\mathcal{X}$, then NS_v is a Jacobi function of index $\frac{n}{2}$ with character $\mathcal{X}^{-1}$.

<u>**Proof:**</u> (i) follows from Lemma 4 and the structure of Γ_θ^J-module $\mathcal{V}(K)$. For $v\in\mathcal{M}(K)$, the $<\mathfrak{u},\mathfrak{q}>-$invariant property of v implies the function $\phi(z,\tau)$ in (ii) is a theta function for $z\in\mathbb{C}$ with lattice $\mathbb{Z}\tau+\mathbb{Z}$. Then the result follows from the standard argument of contour integral over its fundamental domain. (iii) is obvious. q.e.d.

For each K there is an element $w(K)$ in $\mathcal{M}(K)$ with the property (ii) in the above theorem. The construction is as follows. Consider the subset of the base of $\mathcal{V}(K)$, $\{\lambda=\lambda_1\otimes\cdots\otimes\lambda_N|\lambda_j\in\mathfrak{B}_{k_j},\mathfrak{q}(\lambda)=\lambda\}$. It is stable under $<\mathfrak{u}>-$action, and let

$$\{\lambda=\lambda_1\otimes\cdots\otimes\lambda_N|\lambda_j\in\mathfrak{B}_{k_j},\mathfrak{q}(\lambda)=\lambda\}=\coprod_{1\le i\le\delta}\mathfrak{s}_i \tag{14}$$

be the $<\mathfrak{u}>-$orbit decomposition. Define

$$v_i=\sum_{\lambda\in\mathfrak{s}_i}\lambda,\ v^i=\frac{d(K)}{|\mathfrak{s}_i|}v_i\quad\text{for } 1\le i\le\delta.$$

Then $\{v_i | 1 \leq i \leq \delta\}$ is an orthogonal base of $\mathcal{M}(K)$ with $||v_i||^2 = |\mathfrak{s}_i|$, and

$$< v_i, v^i >= d(K)\delta_i^j \quad \text{for } 1 \leq i,j \leq \delta.$$

By Proposition 4, $\tilde{R}_{v_i}(0,\tau) = (-1)^{\frac{c(K)}{6}} \sum_{\substack{\lambda \in \mathfrak{s}_i \\ \lambda : \text{chiral}}} (-1)^{Q_\lambda}.$

Definition.

(i) $w(K) = \sum_i \tilde{R}_{v_i}(0,\tau)v^i \in \mathcal{M}(K)$.

(ii) $\mathcal{J}_K(z,\tau) = NS_{w(K)}(z,\tau) \in \Gamma(\mathbb{C} \times \mathbb{H}, \mathcal{O})$.

(iii) Witten index of $\mathcal{M}(K), Tr(-1)^F = \tilde{R}_{w(K)}(0,\tau)\Big(= \mathcal{J}_K(\frac{\tau}{2}+\frac{1}{2},\tau)\exp\Big(\frac{c(K)}{3}\pi i(\frac{\tau}{4}+\frac{1}{2})\Big)\Big)$.

(iv) Elliptic $\widehat{A}-$ genus of $\mathcal{M}(K) = \mathcal{J}_K(0,\tau)$.

Theorem 3. Let $\mathcal{X}_*$ be the character of Γ_θ in (4) for $* \in \mathbb{Z}$.

(i) $Tr(-1)^F$ of $\mathcal{M}(K)$ is equal to

$$\sum_i \frac{d(K)}{||v_i||^2}\tilde{R}_{v_i}(0,\tau)^2 = (-1)^{\frac{c(K)}{3}} \sum_{(\lambda,\lambda')} (-1)^{Q_\lambda + Q_{\lambda'}} \frac{d(K)}{|< u > -\text{orbit of } \lambda|},$$

here the λ, λ' in the above index run over all the values of chiral elements of $\mathcal{V}(K)$ with the same $< u > -$ orbit.

(ii) The element $w(K)$ of $\mathcal{M}(K)$ is a $\Gamma_\theta-$ eigenvector with eigenvalue $\mathcal{X}_{-c(K)/3}$.

(iii) $\mathcal{J}_K(z,\tau)$ is a Jacobi function of index $\frac{c(K)}{6}$ with character $\mathcal{X}_{c(K)/3}$, and the elliptic $\hat{A}-$ genus of $\mathcal{M}(K)$ is a $\Gamma_\theta-$ eigenfunction with eigenvalue $\mathcal{X}_{c(K)/3}$.

Proof: Let $\coprod_{1\leq i\leq \delta} \mathfrak{s}_i$ be the same in (14). We have

$$\begin{aligned} Tr(-1)^F &= \sum_i \tilde{R}_{v_i}(0,\tau)\tilde{R}_{v^i}(0,\tau) \\ &= \sum_i \frac{d(K)}{||v_i||^2}\tilde{R}_{v_i}(0,\tau)^2 \\ &= \sum_i \frac{d(K)}{||v_i||^2} \sum_{(\lambda,\lambda')\in \mathfrak{s}_i \times \mathfrak{s}_i} \tilde{R}_\lambda(0,\tau)\tilde{R}_{\lambda'}(0,\tau), \end{aligned}$$

and (i) follows from Proposition 4.

Let a_i^j, b_i, $1 \leq i \leq \delta$, be the entries of $\delta \times \delta$ matrices defined by

$$\begin{pmatrix} v_1|S \\ \vdots \\ v_\delta|S \end{pmatrix} = \left(a_i^j\right)\begin{pmatrix} v_1 \\ \vdots \\ v_\delta \end{pmatrix},$$

$$\begin{pmatrix} v_1|H \\ \vdots \\ v_\delta|H \end{pmatrix} = \left(b_i\delta_i^j\right)\begin{pmatrix} v_1 \\ \vdots \\ v_\delta \end{pmatrix}.$$

Then

$$\left(v^1|S^\dagger,\cdots,v^\delta|S^\dagger\right) = \left(v^1,\cdots v^\delta\right)\left(\bar{a}_i^j\right),$$
$$\left(v^1|H^\dagger,\cdots,v^\delta|H^\dagger\right) = \left(v^1,\cdots,v^\delta\right)\left(\bar{b}_i\delta_i^j\right).$$

By Theorem 2 (i) and Proposition 4, we have

$$\exp\left(\frac{c(K)}{6}\pi i\right)\begin{pmatrix}\tilde{R}_{v_1}(0,\tau)\\ \vdots \\ \tilde{R}_{v_\delta}(0,\tau)\end{pmatrix} = \left(\bar{a}_i^j\right)\begin{pmatrix}\tilde{R}_{v_1}(0,\tau)\\ \vdots \\ \tilde{R}_{v_\delta}(0,\tau)\end{pmatrix},$$
$$\exp\left(\frac{c(K)}{6}\pi i\right)\begin{pmatrix}\tilde{R}_{v_1}(0,\tau)\\ \vdots \\ \tilde{R}_{v_\delta}(0,\tau)\end{pmatrix} = \begin{pmatrix}\bar{b}_1\tilde{R}_{v_1}(0,\tau)\\ \vdots \\ \bar{b}_\delta\tilde{R}_{v_\delta}(0,\tau)\end{pmatrix}$$

By $S^\dagger = S^{-1}$, $H^\dagger = H^{-1}$, we obtain

$$w(K)|S^{-1} = \exp\left(\frac{c(K)}{6}\pi i\right)w(K),\ w(K)|H^{-1} = \exp\left(\frac{c(K)}{6}\pi i\right)w(K),$$

which implies (ii). (iii) follows from Theorem 2 (iii). q.e.d.

The following lemma is convenient for the computation of $Tr(-1)^F$ and elliptic $\hat{A}-$ genus of $\mathcal{M}(K)$.

Lemma 5. [4] Let $\{\alpha_1,\ldots,\alpha_m\}$ be the images of $\{v_1,\ldots,v_\delta\}$ under the projection $\mathcal{M}(K) \to \mathcal{K}(K)$ of (11). Denote

$$D_i = \frac{d(K)}{||\alpha_i||^2||v_{j(i)}||^2}, \quad 1 \le i \le m,$$

here $v_{j(i)}$ is an element whose image under (11) equals to α_i. Then

$$w(K) = \sum_{1\le i\le m} D_i\tilde{R}_{\alpha_i}(0,\tau)\alpha_i.$$

Proof: By Lemma 2, for each i, $|\{v_j|v_j \mapsto \alpha_i\}| = ||\alpha_i||^{-2}$. If v_j and $v_{j'}$ have the same image α_i, $||v_j||^2 = ||v_{j'}||^2$ and

$$\tilde{R}_{v_{j'}}(0,\tau) = \tilde{R}_{v_j}(0,\tau) = \tilde{R}_{\alpha_i}(0,\tau).$$

Then the result follows easily from the definition of D_i and $||\alpha_i||^2$. q.e.d.

We now give some examples for the expression of $w(K)$.

Example (i). $c(K) = 3$.

By Theorem 2 (ii), for $v \in \mathcal{M}(K)$, $NS_v(z,\tau) = \vartheta(z,\tau)\cdot$ (some function of τ), and $\tilde{R}_v(0,\tau) = \vartheta(\frac{\tau}{2}+\frac{1}{2},\tau)$(some function of τ) $= 0$. Therefore, $w(K) = 0$, which implies $Tr(-1)^F$ = elliptic $\hat{A} - genus = 0$.

Example (ii). $K = (2,2,2,2)$ (by S-K Yang).

We have $c(K) = 6$, $d(K) = 4$, and the chiral elements of $\mathfrak{B}_2$ are

$$a = [0,0],\ b = [2,2],\ e = [1,1].$$

For $x_i \in \mathfrak{B}_2$, $1 \le i \le 4$, with $\mathfrak{q}(x_1 \cdot x_2 \cdot x_3 \cdot x_4) = x_1 \cdot x_2 \cdot x_3 \cdot x_4$, denote

$$[x_1 \cdot x_2 \cdot x_3 \cdot x_4] = \sum (< \mathrm{u} > -\text{orbit of } x_1 \cdot x_2 \cdot x_3 \cdot x_4) \in \mathcal{K}(K).$$

By Lemma 5 and Proposition 4, we have

$$w(K) = -2[a \cdot a \cdot a \cdot a] + 2[e \cdot e \cdot e \cdot e] + 12[a \cdot b \cdot e \cdot e] + 6[a \cdot a \cdot b \cdot b],$$

hence

$$Tr(-1)^F = 2 \cdot 2 + 2 + 12 + 6 = 24.$$

Example (iii). $K = (2, 2, 2, 6, 6)$. we have $c(K) = 9$ and $d(K) = 8$. Let a, b, e be the elements of $\mathfrak{B}_2$ as in Example (ii). The chiral elements of $\mathfrak{B}_6$ are:

$$A = [0,0],\ B = [1,1],\ C = [2,2],\ D = [3,3],\ E = [4,4],\ F = [5,5],\ G = [6,6].$$

For $x_1, x_2, x_3 \in \mathfrak{B}_2$, $y_4, y_5 \in \mathfrak{B}_6$ with $\mathfrak{q}(x_1 \cdot x_2 \cdot x_3 \cdot y_4 \cdot y_5) = x_1 \cdot x_2 \cdot x_3 \cdot y_4 \cdot y_5$, denote

$$[x_1 \cdot x_2 \cdot x_3 \cdot y_4 \cdot y_5] = \sum (< \mathrm{u} > -\text{orbit of } x_1 \cdot x_2 \cdot x_3 \cdot y_4 \cdot y_5) \in \mathcal{K}(K).$$

Then

$$\begin{aligned}
&(-1)^{\frac{-3}{2}} w(K) = \\
&2(-[a \cdot a \cdot a \cdot C \cdot G] - [a \cdot a \cdot a \cdot D \cdot F] - [e \cdot e \cdot e \cdot A \cdot C] + [e \cdot e \cdot e \cdot E \cdot G] \\
&+ [b \cdot b \cdot b \cdot A \cdot E] + [b \cdot b \cdot b \cdot B \cdot D]) + 3(-[a \cdot a \cdot b \cdot C \cdot C] - [a \cdot e \cdot e \cdot C \cdot C] \\
&- [a \cdot b \cdot b \cdot A \cdot A] - [e \cdot e \cdot b \cdot A \cdot A] + [a \cdot a \cdot b \cdot G \cdot G] + [a \cdot e \cdot e \cdot G \cdot G] \\
&+ [a \cdot b \cdot b \cdot E \cdot E] + [e \cdot e \cdot b \cdot E \cdot E]) + 6(-[a \cdot a \cdot e \cdot A \cdot G] - [a \cdot a \cdot e \cdot B \cdot F] \\
&- [a \cdot a \cdot e \cdot C \cdot E] - [a \cdot a \cdot e \cdot D \cdot D] - [a \cdot a \cdot b \cdot A \cdot E] - [a \cdot a \cdot b \cdot B \cdot D] \\
&- [a \cdot e \cdot e \cdot A \cdot E] - [a \cdot e \cdot e \cdot B \cdot D] - [a \cdot e \cdot b \cdot B \cdot B] + [a \cdot e \cdot b \cdot F \cdot F] \\
&+ [a \cdot b \cdot b \cdot C \cdot G] + [a \cdot b \cdot b \cdot D \cdot F] + [e \cdot e \cdot b \cdot C \cdot G] + [e \cdot e \cdot b \cdot D \cdot F] + \\
&[e \cdot b \cdot b \cdot A \cdot G] + [e \cdot b \cdot b \cdot B \cdot F] + [e \cdot b \cdot b \cdot C \cdot E] + [e \cdot b \cdot b \cdot D \cdot D]) \\
&+ 12([a \cdot e \cdot b \cdot A \cdot C] + [a \cdot e \cdot b \cdot E \cdot G]),
\end{aligned}$$

$$Tr(-1)^F = -12 - 24 - 108 - 24 = -168.$$

The Witten index of the above examples have the following topological interpretation:

$Tr(-1)^F$ of (ii) = Euler number of $K3$ surface;

$Tr(-1)^F$ of (iii) = Euler number of the Calabi-Yau resolution of the hypersurface

$Z_1^4 + Z_2^4 + Z_3^4 + Z_4^8 + Z_5^8 = 0$ in $W\mathbb{P}^4_{(2,2,2,1,1)}$ (Example (I) in [21]).

The above relations illustrate the general property of the equality of Witten index and the Euler number of Calabi-Yau orbifolds, which are discussed in the next section.

§5. Witten index of manifolds with $c_1 = 0$

As before, $K = (k_1, \cdots, k_N)$, k_j = positive integer. For the rest of this paper, we shall always assume

$$\frac{c(K)}{3} = N - 2,$$

which is equivalent to

$$\sum_{j=1}^{N} \frac{1}{k_j + 2} = 1.$$

Let $\hat{X}_K$ be the manifold defined in (1). In this section, we shall show that the Euler number of $\hat{X}_K$ is equal to the Witten index of $\mathcal{M}(K)$.

Definition Let λ be an element in $\{\lambda_1 \otimes \cdots \otimes \lambda_N | \lambda_j \in \mathfrak{B}_{k_j}\}$, and $\mathfrak{u}, \mathfrak{q} : \mathcal{V}(K) \to \mathcal{V}(K)$ the same as before.

(i)

$$P(K) = \{(f, \lambda) | \lambda : \text{chiral with } \mathfrak{q}(\lambda) = \lambda,\ f \in <\mathfrak{u}>\},$$

$$\mathcal{P}(K) = \text{The Hermitian vector space with } P(K) \text{ as an othonormal basis.}$$

(ii)

$$CP(K) = \{(f, \lambda) \in P(K) |\ \lambda \text{ and } \lambda | f \text{ are chiral in } \mathcal{V}(K)\}.$$

$$C\mathcal{P}(K) = \text{The subspace of } \mathcal{P}(K) \text{ generated by } CP(K).$$

(iii) $(-1)^F_p = \tilde{R}_{\lambda|f}(0, \tau)\tilde{R}_\lambda(0, \tau)$ for $p = (f, \lambda) \in P(K)$.

Remark: The elements of $CP(K)$ are corresponding to the chiral primary fields of the (2,2) CFT in [23].

By Proposition 4 (i), we have

$$(-1)^F_p = \begin{cases} 0 & \text{if } p \notin CP(K), \\ (-1)^{\frac{c(K)}{3}}(-1)^{Q_{\lambda|f}+Q_\lambda} & \text{if } p=(f,\lambda) \in CP(K). \end{cases}$$

Lemma 6 $Tr(-1)^F$ of $\mathcal{M}(K) = \sum_{p \in CP(K)} (-1)^F_p$.

Proof: For a chiral element λ in $\mathcal{V}(K)$, we have

$$(-1)^{\frac{c(K)}{3}} \sum_{\substack{\lambda' \in <\mathfrak{u}> \lambda \\ \lambda' : \text{chiral}}} \frac{d(K)}{|<\mathfrak{u}>\lambda|}(-1)^{Q_{\lambda'}+Q_\lambda} = \sum_{\substack{p \in CP(K) \\ p = (f, \lambda)}} (-1)^F_p.$$

Then the result follows from Theorem 3 (i). q.e.d.

Lemma 7. Let W be a quasi-smooth hypersurface in $W\mathbb{P}^{N-1}_{(m_i)}$ defined by a quasi-homogenous polynomial $g(z_1, \cdots, z_N) = 0$ of degree d. Assume $qcd(m_i|i \neq j) = 1$ for each j, and $d = \sum_{i=1}^{N} m_i$.

(i) For an element y of W with the coordinate $y = [y_1, \cdots, y_N]$,

$$y \in \text{Sing}\ (W) \Leftrightarrow qcd(m_i|1 \leq i \leq N,\ y_i \neq 0) > 1$$

(ii) When

$$g(z_1, \cdots, z_N) = Z_1^{d_1} + \cdots + Z_N^{d_N},$$

the following equality holds

$$(-1)^{N-2} h^{N-2}(w)_0 = \frac{1}{d} \sum_{r=o}^{d-1} \prod_{rq_i \in \mathbb{Z}} \left(1 - \frac{1}{q_i}\right),$$

here $q_i = \frac{1}{d_j}$, $\prod_{rq_i \in \mathbb{Z}} \left(1 - \frac{1}{q_i}\right) \doteq 1$ if no q_i with $rq_i \in \mathbb{Z}$,

$h^{N-2}(W)_0 = \dim_{\mathbb{C}} \left(\text{primitive part of } H^{N-2}(W, \mathbb{C})\right)$.

Proof:

(i) We may assume the $N-th$ homogeneous coordinate of y equal to 1 and denote

$$h(z) = h(z_1, \cdots, z_{N-1}) \doteq g(z_1, \cdots, z_{N-1}, 1),$$

$$G \doteq \{\lambda \in \mathbb{C}^* | \lambda^{m_i} y_i = y_i \text{ for all } i\}.$$

Then the order $|G|$ of G divides d and equals to gcd qed $(m_i|1 \leq i \leq N,\ y_i \neq o)$. Consider the linear action of G on $\mathbb{C}^{N-1}$,

$$(\lambda, z) \mapsto \lambda \bullet z \doteq (\lambda^{m_i} z_1, \cdots, \lambda^{m_{N-1}} z_{N-1}), \quad (\lambda, z) \in G \times \mathbb{C}^{N-1}.$$

By $d = \sum_{i=1}^{N} m_i$, G is a subgroup of $SU(N-1)$ and $h(z)$ is a $G-$invariant function. $\{h(z) = 0\}$ is a non-singular hypersurface passing through the point $\bar{y} := (y_1, \ldots y_{N-1})$. Then the following spaces are isomorphic as germs of analytic spaces:

$$\begin{aligned}(W, y) &\simeq \left({}^{\{h(z)=0\}}/_{G}, \bar{y}\right) \\ &\simeq \left({}^{\mathbb{C}^{N-2}}/_{\mu}, 0\right),\end{aligned}$$

here μ is a small cyclic subgroup of $SU(N-2)$ with order$= |G|$. Hence y is singular if $\mu \neq id$, and the conclusion follows immediately.

(ii) The following relation holds between Euler numbers of W and $W'\left(\doteq W\mathbb{P}^{N-1} - W\right)$:

$$\chi(W) + \chi(W') = \chi\left(W\mathbb{P}^{N-1}\right) = N.$$

By [3], $\chi(W) = (-1)^N h^{N-2}(W)_0 + N - 1$, hence $(-1)^N h^{N-2}(W)_0 = 1 - \chi(W')$. It is easy to see that

$$W' = {}^F/_{<\sigma>}$$

here

$$F = \left\{(Z_1, \dots, Z_N) \in \mathbb{C}^N | g(Z) = 1\right\},$$

$$\sigma : F \to F,\ (Z_1, \dots, Z_N) \mapsto \left(w^{m_1} Z_1, \dots, w^{m_N} Z_N\right), \text{ with } w = \exp\left(\frac{2\pi i}{d}\right).$$

Then the conclusion follows from the following formula in [17]:

$$1 - \chi(W') = \frac{1}{d}\sum_{r=0}^{d-1} \prod_{rq_i \in \mathbb{Z}} \left(1 - \frac{1}{q_i}\right)$$

q.e.d.

For the rest of this section, we are going to prove the following result.

Theorem 4 Let $\hat{X}_K$ be a projective manifold defined in (1). Then there is a $\mathbb{C}-$ isomorphism between $\mathcal{CP}(K)$ and the cohomology space $H^*\left(\hat{X}_K, \mathbb{C}\right)\left(= \bigoplus_r H^r\left(\hat{X}_K, \mathbb{C}\right)\right)$,

$$\varphi : \mathcal{CP}(K) \to H^*\left(\hat{X}_K, \mathbb{C}\right)$$

such that for $p \in CP(K)$, $\varphi(p)$ is an element in $H^{r(p)}\left(\hat{X}_K, \mathbb{C}\right)$ for some $r(p)$ with the property

$$(-1)^F_p = (-1)^{r(p)}. \tag{15}$$

As a consequence, Witten index of $\mathcal{M}(K)$ equals to the Euler number of $\hat{X}_K$,

$$Tr(-1)^F \text{of } \mathcal{M}(K) = \chi\left(\hat{X}_K\right).$$

Proof: The last statement follows from the rest by Lemma 6. We are going to define the map φ. Denote

$n = N - 2$.

$q_j = \frac{1}{k_j+2}$, $\mathrm{u}_j =$ the linear automorphism u of V_{k_j} in section 1 for $1 \le j \le N$.

$J_K = \{m = (m_1, \dots m_N) | m_j \in \mathbb{Z}, 0 \le m_j \le k_j\}$.

$Q(m) = \sum_{i=1}^{N} m_i q_i$, $Z^m = Z_1^{m_1} \dots Z_N^{m_N}$ for $m = (m_1, \dots, m_N) \in J_K$.

$I_K = \{m \in J_K | Q(m) \in \mathbb{Z}\}$.

Then $\{Z^m | m \in J_K\}$ forms a base of the Jacobian ring $\mathbb{C}[Z]/ < \partial f_K(Z) >$ of the polynomial $f_K(Z)$. By [22], the subspace of ${}^{\mathbb{C}[Z]}/_{(\partial f_K)}$ generated by $\{Z^m | m \in I_K\}$ is isomophic to the primitive $n-th$ cohomology group $H^n(X_K, \mathbb{C})_0$ of X_K. We shall identify these spaces and the cohomology element in $H^n(X_K, \mathbb{C})_0$ corresponding to Z^m will be denoted by $[Z^m]$ for $m \in I_K$.

For a positive integer k, the chiral elements in $\mathfrak{B}_k$ are given by $[l, l]$, $1 \leq l \leq k$. We have the following one-one correspondence:

$$J_K \leftrightarrow \{\text{chiral elements in } \mathcal{V}(K)\}$$

$$m = (m_1, \ldots m_N) \leftrightarrow \lambda_m \doteq [m_1, m_1] \otimes \ldots \otimes [m_N, m_N],$$

and $Q(m)$ is the charge Q_{λ_m} of λ_m. Hence under the above correspondence,

$$I_K \leftrightarrow \{\lambda : \text{chiral element of } \mathcal{V}(K),\ \mathfrak{q}(\lambda) = \lambda\}.$$

Now the set $P(K)$ can be identified with $< \mathrm{u} > \times I_K$. Then

$$CP(K) = \coprod_{\beta \in < \mathrm{u} >} T_\beta$$

here

$$T_\beta = \{(\beta, m) | m \in I_K,\ (\lambda_m | \beta) = \text{chiral}\} \text{ for } \beta \in < \mathrm{u} > .$$

We are going to define the map φ on each T_β. For $\beta =$ the identity element 1,

$$\varphi : T_1 \to H^n(X_K, \mathbb{C})_0 \subseteq H^*\left(\hat{X}_K, \mathbb{C}\right)$$

is defined by $\varphi(1, m) = [Z^m]$. By

$$(-1)^F_{(1,\lambda_m)} = (-1)^{\frac{c(K)}{3}} = (-1)^n \quad \text{for } m \in I_K,$$

T_1 is bijective to a basis of $H^n(X_K, \mathbb{C})_0$ via φ which satisfies (15). Now we consider the case for $\beta \neq 1$.

For a chiral element $[l, l]$, $0 \leq l \leq k_j$, of $\mathfrak{B}_{k_j}$,

$$| < \mathrm{u}_j > -\text{orbit of } [l, l]| = \begin{cases} \frac{k_j+2}{2} & \text{if } k_j = \text{even and } l = \frac{k_j}{2}, \\ k_j + 2 & \text{otherwise,} \end{cases}$$

and

$$[l, l] | \mathrm{u}_j^r = \text{chiral} \Leftrightarrow r = 0,\ l + 1.$$

For an element $\beta \in < \mathrm{u} >$, we denote $r(\beta)$ the element in $\mathbb{Z}^N$ whose $i-th$ coordinate $r(\beta)_i$ satisfies the equation:

$$\begin{cases} l \equiv r(\beta)_i \pmod{k_i + 2},\ 0 \leq r(\beta)_i < k_i + 2, \\ \beta = \mathrm{u}^l. \end{cases}$$

Define $F(\beta) = \{i|1 \le i \le N,\ r(\beta)_i = 0\}$. We now process the proof of this theorem in the following steps.

Step (I). Claim: For $\beta \neq 1$, we have the following description of the elements (β, m) of T_β for $m = (m_1, \ldots m_N) \in I_K$.

(i) When $F(\beta) = \phi$ and $T_\beta \neq \phi$, T_β consists of only one element (β, m) with $m_j = r(\beta)_j - 1$ for all j. Conversely, if $\beta \in < \mathrm{u} >$ and $m \in I_K$ satisfy the relation $m_j = r(\beta)_j - 1$ for all j, then $F(\beta) = \phi$ and $T_\beta = \{(\beta, m)\}$.

(ii) When $F(\beta) \neq \phi$, we have

$$(\beta, m) \in T_\beta \Leftrightarrow \begin{cases} m_j = r(\beta)_j - 1 & \text{for } j \notin F(\beta) \\ \sum_{i \in F(\beta)} (m_i + 1) q_i \in \mathbb{Z}. \end{cases}$$

Furthermore,

$$T_\beta \neq \phi \Longleftrightarrow |F(\beta)| \ge 2,$$

in which situation, $X_K \cap (Z_j = 0 | j \notin F(\beta))$ is a non-empty subset contained in $\mathrm{Sing}(X_K)$. For $(\beta, m) \in T_\beta$ and $m' = (m'_1, \ldots, m'_N) \in I_K$ with $\lambda_m | \beta = \lambda_{m'}$, we have

$$m'_j = m_j \quad \text{or } k_j - m_j,$$

and

$$m'_j = k_j - m_j \Leftrightarrow r(\beta)_j - 1 = m_j.$$

For $j \notin F(\beta)$, if $m'_j = m_j$, then $\left[m'_j, m'_j\right] = [m_j, m_j] | \mathrm{u}_j^{r(\beta)_j} = [m_j, m_j]$, which implies $r(\beta)_j - 1 = m_j = m'_j = \frac{k_j}{2}$. Therefore we obtain

$$m_j = r(\beta)_j - 1 \text{ for } j \notin F(\beta). \tag{16}$$

When $F(\beta) = \phi$, we have $m'_j = k_j - m_j$ for all j. Then (i) is obvious. Write $\beta = \mathrm{u}^l$ for some $0 < l < d$. When $F(\beta) \neq \phi$, l is divided by $k_i + 2$ for all $i \in F(\beta)$. Hence

$$lcm(k_i + 2 | i \in F(\beta)) < d,$$

which is equivalent to

$$qcd(n_i | i \in F(\beta)) > 1.$$

For $(\beta, m) \in T_\beta$, by $\sum_{j=1}^{N} q_i = 1$, we have

$$\begin{aligned} Q(m) &\equiv \sum_{i \in F(\beta)} (m_i + 1) q_i + \sum_{j \notin F(\beta)} r(\beta)_j q_j \\ &\equiv \sum_{i \in F(\beta)} (m_i + 1) q_i \qquad \pmod{\mathbb{Z}}. \end{aligned} \qquad \text{(by (16))}$$

Therefore $\sum_{i \in F(\beta)} (m_i + 1) q_i \in \mathbb{Z}$, and $|F(\beta)| \ge 2$. By Lemma 7, $X_K \cap (Z_j = 0 | j \notin F(\beta)) \subseteq \mathrm{Sing}(X_K)$. By reversing the above argument, we obtain the conclusion of (ii).

Step (II). Claim: Denote

$$B_l = \frac{1}{d} \sum_{0 \le r \le d-1} \prod_{lq_i, rq_i \in \mathbb{Z}} \left(1 - \frac{1}{q_i}\right) \quad \text{for } 0 \le l \le d-1.$$

Then (i) $\sum_{p \in T_{u^l}} (-1)^F_p = B_l$ for all l.

(ii) Witten index $Tr(-1)^F$ of $\mathcal{M}(K) = \sum_{l=0}^{d-1} B_l$

When $F(u^l) = \phi$, T_{u^l} consists of only one element (β, m) with $\lambda_m|\beta = \lambda_{K-m}$ by Step (I) (i), then the conclusion follows immediately. We now consider the case when $F(\beta) \neq \phi$. For a subset $I \subseteq \{1, \ldots, N\}$, define

$$\begin{aligned} X_K(I) &= X_K \cap (Z_j = 0 | j \in I) \\ e(I) &= gcd(n_j | j \notin I). \end{aligned} \tag{17}$$

By Step (I) (ii), there is an one-one correspondence between the following sets:

$$\iota : T_\beta \leftrightarrow \left\{ \prod_{i \in F(\beta)} Z_i^{\alpha_i} | \sum_{i \in F(\beta)} (\alpha_i + 1) q_i \in \mathbb{Z} \right\} \tag{18}$$

$$(\beta, m) \mapsto \prod_{i \in F(\beta)} Z_i^{m_i}.$$

By [22], the monomials in the right hand side form a basis of

$$H^{\dim X_K(I)}(X_K(I), \mathbb{C})_0 \quad \text{with } I = \{1, \ldots, N\} - F(\beta).$$

For $(\beta, m) \in T_\beta$, we have

$$(-1)^F_{(\beta,m)} = (-1)^N (-1)^{N - |F(\beta)| + 2 \sum_{i \in F(\beta)} (m_i + 1) q_i} = (-1)^{\dim X_K(I)}. \tag{19}$$

So (i) follows from Lemma 7 (ii). By Lemma 6,

$$Tr(-1)^F = \sum_{p \in CP(K)} (-1)^F_p = \sum_{0 \le l \le d-1} \sum_{p \in T_{u^l}} (-1)^F_p = \sum_{l=0}^{d-1} B_l.$$

hence we obtain (ii).

Step (III). We have the equality

$$Tr(-1)^F \text{of } \mathcal{M}(K) = \text{Euler number } \chi\left(\hat{X}_K\right) \text{ of } \hat{X}_K.$$

When $n = 3$, by Theorem 1 of [21], we have

$$\chi\left(\hat{X}_K\right) = \sum_{l=0}^{d-1} B_l.$$

With the same proof given there , the above equality holds also for $\dim \hat{X}_K = 2$ or $\hat{X}_K = X_K$. Hence the conclusion follows from Step (II) (ii).

Step (IV). We are going to define a $\mathbb{C}-$isomorphism

$$\varphi : \mathcal{CP}(K) \to H^*\left(\hat{X}_K, \mathbb{C}\right).$$

We shall assign a cohomology class in $H^{r(p)}\left(\hat{X}_K, \mathbb{C}\right)$ with the property (15) for an element p of T_β, $\beta \neq 1$. Denote

$$\wedge^j = \begin{cases} \text{The image of the standard generator of } H^{2j}\left(W\mathbb{P}^{N-1}_{(n_j)}, \mathbb{C}\right) \text{in } H^{2j}(X_K, \mathbb{C}) \text{for } 2j \leq n, \\ \text{The Poincare dual of } \wedge^{n-j} \text{ in } H^{2j}(X_K, \mathbb{C}) \quad \text{for } 2j > n. \end{cases}$$

By [3], $H^{2j}(X_K, \mathbb{C})$ is a 1–dimensional space with base $\wedge^j$, and $H^{2j+1}(X_K, \mathbb{C}) = 0$ for $2j + 1 \neq n$. For the convenience, we shall divide the $\hat{X}_K$ in the following cases.

Case (i). $X_K =$ the degree N Fermat hypersurface in $\mathbb{P}^{N-1}$,

$$X_K : Z_1^N + \ldots, Z_N^N = 0.$$

For $0 \leq l \leq N-2$, we have $F(\mathrm{u}^{l+1}) = \phi$. By Step (I) (i), $T_{\mathrm{u}^{l+1}} = \left\{\left(\mathrm{u}^{l+1}, [l,l] \otimes \cdots \otimes [l,l]\right)\right\}$. Define

$$\varphi\left(\mathrm{u}^{l+1}, [l,l] \otimes \cdots \otimes [l,l]\right) = \wedge^l \quad \text{for } 0 \leq l \leq N-2,$$

Since the value $(-1)^F$ of $\left(\mathrm{u}^{l+1}, [l,l] \otimes \cdots \otimes [l,l]\right) = 1$, the induced $\mathbb{C}-$linear map

$$\varphi : \mathcal{CP}(K) \to H^*(X_K, \mathbb{C})$$

is an isomorphism with the property (15).

Case (ii). The case for $n = 1$.

$\hat{X}_K = X_K$ is a non-singluar elliptic curve. I_K consists of only two elements: $0 = (0, \ldots, 0)$, $K = (k_1, \ldots, k_N)$. Hence $CP(K) - T_1 = T_{\mathrm{u}} \coprod T_{\mathrm{u}^{-1}}$. Then

$$T_{\mathrm{u}} = \{(\mathrm{u}, 0)\}, \; T_{\mathrm{u}^{-1}} = \left\{\left(\mathrm{u}^{-1}, K\right)\right\},$$

and $(-1)^F_{(\mathrm{u},0)} = (-1)^F_{(\mathrm{u}^{-1},K)} = 1$. Then the correspondence

$$\begin{array}{l} (\mathrm{u}, 0) \mapsto \wedge^\circ \\ \left(\mathrm{u}^{-1}, K\right) \mapsto \wedge^1 \end{array}$$

defines the isomormorphism φ.

Case (iii). The cases for $N = 4, 5$ (hence $\hat{X}_K$ is a $K3$ or CY space respectively).

$H^1\left(\hat{X}_K, \mathbb{C}\right) = H^{2n-1}\left(\hat{X}_K, \mathbb{C}\right) = 0$. In the following, I always denote a subset of $\{1, \ldots, N\}$ with $|I| \leq N-2$ and $X_K(I)$, $e_K(I)$ the same as in (17).

Denote $\mathcal{S} = \begin{cases} \{I \mid |I|=2,\, e(I)>1\} & \text{for } n=2 \\ \{I \mid |I|=2,3,\, e(I)>1\} & \text{for } n=3. \end{cases}$

Then $Sing(X_K) = \cup\{X_K(I) | \in \mathcal{S}\}$. Denote the birational morphism from $\hat{X}_K$ to X_K by

$$\sigma : \hat{X}_K \to X_K.$$

The exceptional divisors in $\hat{X}_K$ over $X_K(I)$, $I \in \mathcal{S}$, can be described as follows [9]:
For $n = 2$, $* \in X_K(I)$,

$$\sigma^{-1}(*) = \text{a union of } e(I)' \text{exceptional } \mathbb{P}^1 - \text{curves with } e(I)' \doteq e(I) - 1.$$

For $n = 3$,

$$\sigma^{-1}(\gamma) = \text{a union of } e(I)' \text{ ruled surfaces over an irreducible component } \gamma \text{ of } X_K(I) \text{ for } |I| = 2 \text{ with } e(I)' \doteq e(I) - 1,$$

$$\sigma^{-1}(*) = \text{a union of } e(J)' \text{rational surfaces over an element } * \in X_K(J) \text{ for } |J| = 3 \text{ with}$$
$$2e(J)' \doteq e(J) - 1 - \sum \{e(I) - 1 | I \in \mathcal{S},\ I \subseteq J, I \neq J\}.$$

We have the following natural isomorphisms:

For $n = 2$

$$H^2\left(\hat{X}_K, \mathbb{C}\right) \simeq H^2(X_K, \mathbb{C}) \bigoplus \bigoplus_{I \in \mathcal{S}} H^0(X_K(I), \mathbb{C})^{\oplus e(I)'}$$

For $n = 3$ [21],

$$H^2\left(\hat{X}_K, \mathbb{C}\right) \simeq H^2(X_K, \mathbb{C}) \oplus \left(\bigoplus_{\substack{|I| = 2 \\ I \in \mathcal{S}}} H^0(X_K(I), \mathbb{C})^{\oplus e(I)'} \oplus \bigoplus_{\substack{|J| = 3 \\ J \in \mathcal{S}}} H^0(X_K(J), \mathbb{C})^{\oplus e(J)'} \right),$$

$$H^3\left(\hat{X}_K, \mathbb{C}\right) \simeq H^3(X_K, \mathbb{C}) \oplus \left(\bigoplus_{\substack{|I| = 2 \\ I \in \mathcal{S}}} H^1(X_K(I), \mathbb{C})^{\oplus e(I)'} \right).$$

For the simplicity of notations, we shall make the above identifications in what follow.

For $\beta \neq 1$ with $F(\beta) \neq \phi$, the image of an element (β, m) in T_β under the map (18) determines a cohomology class in $H^{\dim X_K(I_\beta)}(X_K(I_\beta), \mathbb{C})_0$ with $I_\beta \doteq \{1, \cdots, N\} - F(\beta)$, hence an element of $H^*(X_K, \mathbb{C})$ through the above identification of cohomology spaces. Through this procedure, φ is defined on the T_β for $\beta \neq 1$ and $F(\beta) \neq \phi$:

For $n = 2$.

$$\varphi : T_\beta \to H^0(X_K(I_\beta), \mathbb{C})_0,$$

hence

$$\varphi : \coprod \{T_\beta | \beta \neq 1, F(\beta) \neq \phi\} \hookrightarrow \bigoplus_{I \in \mathcal{S}} H^0(X_K(I), \mathbb{C})^{\oplus e(I)'} \hookrightarrow H^2\left(\hat{X}_K, \mathbb{C}\right);$$

For $n = 3$,

$$\varphi : T_\beta \to H^1\left(X_K\left(I_\beta\right), \mathbb{C}\right) \text{ for } |F(\beta)| = 3,$$

$\varphi : T_\beta \to H^0\left(X_K\left(I_\beta\right), \mathbb{C}\right)_0$ for $\beta = \mathrm{u}^l$, $2l < d$, $|F(\beta)| = 2$,

hence

$$\varphi : \coprod \{T_\beta | \beta \neq 1, |F(\beta)| = 3\} \hookrightarrow \bigoplus_{\substack{|I| = 2 \\ I \in S}} H^1(X_K(I), \mathbb{C})^{\oplus e(I)'} \hookrightarrow H^3\left(\hat{X}_K, \mathbb{C}\right),$$

$$\varphi : \coprod \left\{T_\beta | \begin{matrix} \beta = \mathrm{u}^l & \text{with} \\ |F(\beta)| = 2, & l < d - l \end{matrix}\right\} \hookrightarrow \bigoplus_{\substack{|J| = 3 \\ J \in S}} H^0(X_K(J), \mathbb{C})^{\oplus e(J)'} \hookrightarrow H^2\left(\hat{X}_K, \mathbb{C}\right)$$

and we define

$$\varphi : \coprod \left\{T_{\beta^{-1}} | \beta = \mathrm{u}^l, |F(\beta)| = 2, l < d - l\right\} \to H^4\left(\hat{X}_K, \mathbb{C}\right)$$

by requiring that $\varphi\left(T_\beta\right)$ and $\varphi\left(T_{\beta^{-1}}\right)$ are the Poincaré dual in $H^*\left(\hat{X}_K, \mathbb{C}\right)$ corresponding to the pairing

$T_\beta \in (\beta, m) \leftrightarrow \left(\beta^{-1}, m|\beta\right) \in T_{\beta^{-1}}$.

By (19), φ satisfies the property (15).

We now define φ on T_β with $F(\beta) = \phi$. By Step (I) (i), T_β consists of only one element whenever it is non-empty. Define φ on the following T_β's :

$$T_{\mathrm{u}} = \{(\mathrm{u}, 0)\}, \quad \varphi(\mathrm{u}, 0) \doteq \wedge^0 \in H^0(X_K, \mathbb{C});$$

$$T_{\mathrm{u}^{-1}} = \left\{\left(\mathrm{u}^{-1}, K\right)\right\}, \quad \varphi\left(\mathrm{u}^{-1}, K\right) \doteq \wedge^n \in H^{2n}(X_K, \mathbb{C});$$

$$T_{\mathrm{u}^2} = \left\{\left(\mathrm{u}^2, \mathbb{1}\right)\right\}, \ \varphi\left(\mathrm{u}^2, \mathbb{1}\right) \doteq \wedge^1 \in H^2(X_K, \mathbb{C}), \quad \text{here } \mathbb{1} \doteq (1, 1, \cdots, 1).$$

For $I \subseteq \{1, \cdots, N\}$ with $|I| \leq N - 2$, $e(I)' > 0$, and $1 \leq j \leq e(I)'$, we denote

$$d(I) = d/_{e(I)},$$

$$\beta(I, j) = \mathrm{u}^{jd(I)+1}.$$

Then $T_{\beta(I,j)} = \{(\beta(I, j), m(jd(I)))\}$, here $m(jd(I)) =$ the element in I_K with th $i-$ the coordinate m_i defined by the equation

$$\begin{cases} jd(I) \equiv m_i & (\text{mod. } k_i + 2) \\ 0 \leq m_i < k_i + 2 & \end{cases}$$

(which implies $0 \leq m_i \leq k_i$). We define

$\varphi(\beta(I,j)),\ m(jd(I))$ = the base element of the complement of $H^0(X_K(I),\mathbb{C})_0$ in $H^0(X_K(I),\mathbb{C})$ which is identified with the $I-$th factor of $H^0(X_K(I),\mathbb{C})^{\oplus e(I)'} \hookrightarrow H^2\left(\hat{X}_K,\mathbb{C}\right)$.

For $n = 3$, we need to consider the following $T'_\beta s$. Note that $T_{\beta(I,j)^{-1}} = \left\{\left(\beta(I,i)^{-1}, K - m(jd(I))\right)\right\}$. By the relation

$$Q(m(jd(I))) + Q(K - m(jd(I))) = 3$$

and

$$Q(m(jd(I))),\ Q(K - m(jd(I))) \in \mathbb{Z}_{>0},$$

$\beta(I,j)^{-1}$ is not any one of the elements we have considered before. We define

$\varphi\left(\beta(I,j)^{-1},\ K - m(jd(I))\right)$ = the Poincare dual of $\varphi(\beta(I,j),\ m(jd(I)))$ in $H^4\left(\hat{X}_K,\mathbb{C}\right)$,

$\varphi(\mathrm{u}^{-2},\ K - \mathbb{1}) = \wedge^2 \in H^4(X_K,\mathbb{C}),\ \ (T_{\mathrm{u}^{-2}} = \{(\mathrm{u}^{-2},\ K - \mathbb{1})\})$.

It can be verified that the defined values of φ satisfy the property (15).

Denote

$$R \doteq \begin{cases} \{\beta \in <\mathrm{u}> | F(\beta) = \phi\} - \{\mathrm{u}^{\pm 1}, \mathrm{u}^2, \beta(I,j) \text{ for } e(I) > 1, 1 \leq j < e(I)'\} & \text{for } n = 2 \\ \{\beta \in <\mathrm{u}> | F(\beta) = \phi\} - \left\{\mathrm{u}^{\pm 1}, \mathrm{u}^{\pm 2}, \beta(I,j)^{\pm 1} \text{for } e(I) > 1, 1 \leq j \leq e(I)'\right\} & \text{for } n = 3 \end{cases}$$

By the above construction, we have defined a $\mathbb{C}-$isomorphism

$$\varphi : \oplus\left\{\mathbb{C}p | p \in \mathrm{CP}(K) - \bigcup_{\beta \in R} T_\beta\right\} \xrightarrow{\sim} H^*\left(\hat{X}_K,\mathbb{C}\right)$$

satisfying the property (15).

hence $\chi\left(\hat{X}_K\right) = \sum\left\{(-1)^F_p | p \in \mathrm{CP}(K) - \bigcup_{\beta \in R} T_\beta\right\}$. By Step (III),

$$\chi\left(\hat{X}_K\right) = \sum_{p \in \mathrm{CP}(K)} (-1)^F_p,$$

which implies

$$0 = \sum\left\{(-1)^F_p | p \in \bigcup_{\beta \in R} T_\beta\right\} = \sum_{\beta \in R} |T_\beta|$$

by Step (I) (i). Therefore $T_\beta = \phi$ for $\beta \in R$, and the map φ is the isomorphism from $\mathcal{CP}(K)$ to $H^*\left(\hat{X}_K,\mathbb{C}\right)$ with the property (15). q.e.d.

Remark: The Step (II) in the above proof corresponds to the physicist's argument employed by Vafa in [23]. The conclusion in Step (III) is the mathematical argument for the equality of the Witten index of CFT and Euler number of CY orbifold $\hat{X}_K$. The map φ we have constructed here illustrates the explicit correspondence between twisted sectors and blowing-up modes for the Calabi-Yau orbifolds in the physics literature.

§6. Elliptic genus of manifolds with $c_1 = 0$

By Theorem 3, $\mathcal{J}_K(z,\tau)$ is a Jacobi function of index $\frac{c(K)}{6}$ with character $\mathcal{X}_{c(K)/3}$. By Proposition 1, the elliptic $\hat{A}-$ genus of $\mathcal{M}(K)$, $\mathcal{J}_K(0,\tau)$, is zero for $\frac{c(K)}{3}$ = odd, which corresponds to the vanishing (topological) elliptic genus (of level 2) of $\hat{X}_K$ when $\dim \hat{X}_K$ = odd. We now consider the case when $\dim \hat{X}_K = 2$, and we shall describe the relation between the $\hat{A}-$ genus $\mathcal{J}_K(0,\tau)$ and elliptic genus of the $K3$ surface $\hat{X}_K$. By Proposition 1 (ii),

$$\mathcal{E}_K(\tau) \doteq -\frac{\mathcal{J}_K(0,\tau)}{\vartheta(0,\tau)^2}\eta(\tau)^6$$

is a modular form of Γ_θ of weight 2. By the definition,

$$\begin{aligned}\mathcal{J}_K(0,\tau) &= NS_{w(K)}(0,\tau)\\ &= \sum_i \tilde{R}_{v_i}(0,\tau) NS_{v_i}(0,\tau)\frac{d(K)}{||v_i||^2}.\end{aligned}$$

We may assume $v_1 =< u > -$ orbit of $[0,0]\otimes\cdots\otimes[0,0]$. As $\lim_{Im\tau\to\infty}\vartheta(0,\tau) = 1$, by Proposition 4,

$$\lim_{Im\tau\to\infty}\mathcal{E}_K(\tau) = -\lim_{Im\tau\to\infty}\tilde{R}_{v_1}(0,\tau) = 2,$$

which equals to the $\hat{A}-$ genus of the $K3$ surface $\hat{X}_K$. Since the dimension of space of modular forms for Γ_θ of weight 2 is equal to 1, $\mathcal{E}_K(\tau)$ is the elliptic genus of $\hat{X}_K$. Therefore we have shown the following result.

Theorem 5 The elliptic $\hat{A}-$ genus of $\mathcal{M}(K)$ is corresponding to the topological elliptic genus (of level 2) of $\hat{X}_K$ when $\dim \hat{X}_K = 2$ or odd.

Appendix We are going to prove the following equalities:

For positive integers M, a, b with $M \geq 3$, $1 \leq a,\ b \leq M-1$,

$$\sum_{1\leq j\leq M-1} \sin\frac{ja\pi}{M}\sin\frac{jb\pi}{M} = \frac{M}{2}\delta_{ab},$$

$$\sum_{1\leq j\leq M-1} (-1)^{j+1}\sin\frac{ja\pi}{M}\sin\frac{jb\pi}{M} = \frac{M}{2}\delta_{a+b,M}.$$

Proof: For an integer d, we have

$$\begin{aligned}\sum_{j=1}^{M-1}\cos\frac{jd\pi}{M} &= \frac{1}{2}\sum_{j=1}^{M-1}\left(\exp\frac{jd\pi i}{M} + \exp\frac{-jd\pi i}{M}\right)\\ &= \frac{1}{2}\left\{-1-(-1)^d + \sum_{j=0}^{2M-1}\exp\frac{jd\pi i}{M}\right\}\\ &= \begin{cases} M-1 & \text{if } d\equiv 0 \pmod{2M}\\ \frac{-1}{2}\left(1+(-1)^d\right) & \text{if } d\not\equiv 0 \pmod{2M}.\end{cases}\end{aligned}$$

$$\sum_{j=1}^{M-1} \sin\frac{ja\pi}{M}\sin\frac{jb\pi}{M} = \frac{-1}{2}\sum_{j=1}^{M-1}\left[\cos\frac{j(a+b)\pi i}{M} - \cos\frac{j(a-b)\pi i}{M}\right]$$

$$= \frac{1}{4}\left(1+(-1)^{a+b}\right) + \frac{1}{2}\sum_{j=1}^{M-1}\cos\frac{j(a-b)\pi i}{M} \qquad (\because 2 \le a+b \le 2M-2)$$

$$= \begin{cases} \frac{1}{4}\left(1+(-1)^{a+b}\right) + \frac{1}{2}\left(\frac{-1}{2}\right)\left(1+(-1)^{a-b}\right) = 0 & \text{if } a \neq b \\ \frac{1}{4}\left(1+(-1)^{a+b}\right) + \frac{1}{2}(M-1) = \frac{M}{2} & \text{if } a = b \end{cases}$$

$$(\because a \neq b \quad a-b \not\equiv 0 \pmod{2M}).$$

Hence we obtain the first equality. the second equality follows by substituting a by $M-a$ in the first one. q.e.d.

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