Max-Planck-Institut für Mathematik Bonn

A note on Kuttler-Sigillito's inequalities

by

Asma Hassannezhad
Anna Siffert

Max-Planck-Institut für Mathematik
Preprint Series 2017 (59)

A note on Kuttler-Sigillito's inequalities

Asma Hassannezhad
Anna Siffert

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

University of Bristol
School of Mathematics
University Walk
Bristol BS8 1TW
UK

MPIM 17-59

A NOTE ON KUTTLER-SIGILLITO'S INEQUALITIES

ASMA HASSANNEZHAD AND ANNA SIFFERT

Abstract. We provide several inequalities between eigenvalues of some classical eigenvalue problems on domains with C^2 boundary in complete Riemannian manifolds. A key tool in the proof is the generalized Rellich identity on a Riemannian manifold. Our results in particular extend some inequalities due to Kutller and Sigillito from subsets of $\mathbb{R}^2$ to the manifold setting.

1. Introduction

The objective of this manuscript is to establish several inequalities between eigenvalues of the classical eigenvalue problems mentioned below. Let (M, g) be a complete Riemannian manifold of dimension $n \geq 2$ and Ω be a bounded domain in M with nonempty C^2 boundary $\partial\Omega$. The eigenvalue problems we consider include the Neumann and Dirichlet eigenvalue problems on Ω:

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{Dirichlet eigenvalue problem}, \tag{1.1}$$

$$\begin{cases} \Delta u + \mu u = 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{Neumann eigenvalue problem}, \tag{1.2}$$

where $\Delta = \text{div}\nabla$ is the Laplace–Beltrami operator, ν is the unit outward normal vector on $\partial\Omega$, and ∂_ν denotes the outward normal derivative. The Dirichlet eigenvalues describe the fundamental modes of vibration of an idealized drum, and the Neumann eigenvalues appear naturally in the study of the vibrations of a free membrane; see e.g. [2, 5].

We also consider the Steklov eigenvalue problem, which is an eigenvalue problem with the spectral parameter in the boundary conditions:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \partial_\nu u = \sigma u & \text{on } \partial\Omega, \end{cases} \qquad \text{Steklov eigenvalue problem}. \tag{1.3}$$

The Steklov eigenvalues encode the squares of the natural frequencies of vibration of a thin membrane with free frame, whose mass is uniformly distributed at the boundary; see the recent survey paper [8] and references therein.

The last set of eigenvalue problems we consider are the so-called Biharmonic Steklov problems:

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = \Delta u - \eta \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{Biharmonic Steklov problem I}; \tag{1.4}$$

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu \Delta u + \xi u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{Biharmonic Steklov problem II}. \tag{1.5}$$

The eigenvalues problems (1.4) and (1.5) play an important role in biharmonic analysis and elastic mechanics. We refer the reader to [7, 4, 14, 15] for some recent results on eigenvalue estimates of problem (1.4). Moreover, a physical interpretation of problem (1.4) can be found in [7, 14].

Date: September 29, 2017.
2010 *Mathematics Subject Classification.* 35P15,58C40,58J50.
Key words and phrases. Steklov eigenvalue problems, estimation of eigenvalues, Rellich identity.

Problem (1.5) was first studied in [10, 9] where the main focus was on the first nonzero eigenvalue, which appears as an optimal constant in a priori inequality; see [9] for more details.

It is well-known that the spectra of the eigenvalue problems (1.2)–(1.5) are discrete and nonnegative. We may thus arrange their eigenvalues in increasing order, where we repeat an eigenvalue as often as its multiplicity requires. The k-th eigenvalue of one of the above eigenvalue problems will be denoted by the corresponding letter for the eigenvalue with a subscript k, e.g. the k-th Neumann eigenvalue will be denoted by μ_k. Note that $\mu_1 = \sigma_1 = \xi_1 = 0$.

There is a variety of literature on the study of bounds on the eigenvalues of each problem mentioned above in terms of the geometry of the underlying space [12, 14, 19, 8]. However, instead of studying each eigenvalue problem individually, it is also interesting to explore relationships and inequalities between eigenvalues of different eigenvalue problems. Among this type of results, one can mention the relationships between the Laplace and Steklov eigenvalues studied in [21, 11, 18], and various inequalities between the first nonzero eigenvalue of problems (1.2)–(1.5) on bounded domains of $\mathbb{R}^2$ obtained by Kuttler and Sigilito in [10]; see Table 1 (Note that there was a misprint in Inequality VI in [10]. The correct version of the inequality is stated in Table 1.).

TABLE 1. Inequalities obtained by Kuttler and Sigillito in [10].

Inequalities	Conditions on $\Omega \subset \mathbb{R}^2$	Special case of
$\mu_2\sigma_2 \leq \xi_2$		Thm. 1.6
$\mu_2 h_{\min}/(1+\mu_2^{1/2} r_{\max}) \leq 2\sigma_2$	star-shaped with respect to a point	Thm. 1.7
$\eta_1 \leq \frac{1}{2}\lambda_1 h_{\max}$	star-shaped with respect to a point	Thm. 1.9 (i)
$\lambda_1^{1/2} \leq 2\eta_1 r_{\max}/h_{\min}$	star-shaped with respect to a point	Thm. 1.9 (i)
$\xi_2 \leq \mu_2^2 h_{\max}$	star-shaped with respect to its centroid	Thm. 1.9 (ii)

We extend Kuttler–Sigillito's results in two ways. Firstly, we consider domains Ω with C^2 boundary in a complete Riemannian manifold (M, g) of arbitrary dimension $n \geq 2$. Secondly, we also prove inequalities between higher-order eigenvalues.

Our first theorem provides lower bounds for ξ_k in terms of Neumann and Steklov eigenvalues.

Theorem 1.6. *For every $k \in \mathbb{N}$ we have (a) $\mu_k\sigma_2 \leq \xi_k$, and (b) $\mu_2\sigma_k \leq \xi_k$.*

Compared to inequality (b), inequality (a) gives a better lower bound for ξ_k for large k. For $k = 2$ and $\Omega \subset \mathbb{R}^2$, Theorem 1.6 was previously proved in [10]. Kuttler in [9] also obtained an inequality between some higher order eigenvalues ξ_k and μ_k for a rectangular domain in $\mathbb{R}^2$ using symmetries of the eigenfunctions.

In order to state our next results, we need to introduce some notation first. For any given $p \in M$, consider the distance function

$$d_p : \Omega \to [0, \infty), \quad d_p(x) := d(p, x),$$

and one half of the square of the distance function,

$$\rho_p(x) := \frac{1}{2} d_p(x)^2.$$

Furthermore, we set

$$r_{\max} := \max_{x\in\Omega} d_p(x) = \max_{x\in\partial\Omega} d_p(x),$$

$$h_{\max} := \max_{x\in\partial\Omega} \langle \nabla\rho_p, \nu\rangle, \quad \text{and} \quad h_{\min} := \min_{x\in\partial\Omega} \langle \nabla\rho_p, \nu\rangle,$$

where we borrowed the notation from [10].

We shall see that under the assumption of a lower Ricci curvature bound, there exists a lower bound on the first nonzero Steklov eigenvalue σ_2 in terms of μ_2 on star shaped domains.

Theorem 1.7. *Let the Ricci curvature* Ric_g *of the ambient space* M *be bounded from below*

$$\mathrm{Ric}_g \geq (n-1)\kappa,$$

and let $\Omega \subset M$ *be a bounded star shaped domain with respect to* $p \in \Omega$. *Then we have*

$$\sigma_2 \geq \frac{h_{\min}\mu_2}{2r_{\max}\mu_2^{1/2} + C_0}, \tag{1.8}$$

where $C_0 := C_0(n, \kappa, r_{\max})$ *is a positive constant depending only on* n, κ *and* $r_{\max}$.

When the ambient space M is Euclidean, inequality (1.8) was stated in [10] with $C_0 = 2$.

In the following theorem we provide several inequalities for eigenvalues of (1.2)–(1.5) on star shaped domains under the assumption of bounded sectional curvature. Here and hereafter, we make use of the notation

$$A \vee B := \max\{A, B\} \qquad \text{for all } A, B \in \mathbb{R},$$

and the convention $c/0 = +\infty$, $c \in \mathbb{R} \setminus \{0\}$.

Theorem 1.9. *Let the sectional curvature* K_g *of the ambient space* M *satisfy* $\kappa_1 \leq K_g \leq \kappa_2$. *Moreover, let* $\Omega \subset M$ *be a star shaped domain with respect to* $p \in \Omega$ *which is contained in the complement of the cut locus of* p. *Then there exist constants* $C_i := C_i(n, \kappa_1, \kappa_2, r_{\max})$, *i=1,2, depending only on* n, κ_1, κ_2 *and* $r_{\max}$ *and* $C_3 = C_3(n, \kappa_1, r_{\max})$ *such that*

- *i)* $C_1\eta_m/h_{\max} \leq \lambda_k \leq \left(4r_{\max}^2\eta_k^2 - 2C_2h_{\min}\eta_k\right)/h_{\min}^2$, *where* m *is the multiplicity of* λ_k;
- *ii)* $\xi_{m+1} \leq h_{\max}\mu_k^2/\left((C_3 - n^{-1}\mathrm{vol}(\Omega)^{-1}\mu_k \int_\Omega d_p^2\, dv_g) \vee 0\right)$, *provided* $\kappa_2 \leq 0$.

Note that the constants C_i, $i = 1, 2, 3$ are not positive in general. However, there exists $r_0 := r_0(n, \kappa_1, \kappa_2) > 0$ such that for $r_{\max} \leq r_0$ these constants are positive; see Section 4 for details. In inequality *ii*), we have a non trivial upper bound only if

$$\mu_k < nC_3\mathrm{vol}(\Omega)\left(\int_\Omega d_p^2\, dv_g\right)^{-1}.$$

When Ω is a domain in $\mathbb{R}^n$, the quantity $\int_\Omega d_p^2 dv_g$ is called the second moment of inertia; see Example 4.10. The proof of Theorem 1.9 also leads to a non-sharp lower bound on η_1

$$\eta_1 \geq \frac{h_{\min}C_2}{r_{\max}^2}.$$

This in particular shows that the right-hand side of the inequality in part *i*) is always positive.

The proof of Theorem 1.6 is based on using the variational characterization of the eigenvalues and alternative formulations thereof. Apart from the Laplace and Hessian comparison theorems, and the variational characterization of the eigenvalues, the key tool in the proof of Theorems 1.7 and 1.9 is a generalization of the classical Rellich identity to the manifold setting. This is the content of the next theorem.

Theorem 1.10 (Generalized Rellich identity)**.** *Let $F : \Omega \to T\Omega$ be a Lipschitz vector field on Ω. Then for every $w \in C^2(\Omega)$ we have*

$$\begin{aligned}\int_\Omega (\Delta w + \lambda w)\langle F, \nabla w\rangle dv_g = &\int_{\partial\Omega} \partial_\nu w \langle F, \nabla w\rangle ds_g - \frac{1}{2}\int_{\partial\Omega} |\nabla w|^2 \langle F, \nu\rangle ds_g + \frac{\lambda}{2}\int_{\partial\Omega} w^2 \langle F, \nu\rangle ds_g \\ &+ \frac{1}{2}\int_\Omega \mathrm{div} F |\nabla w|^2 dv_g - \int_\Omega DF(\nabla w, \nabla w) dv_g - \frac{\lambda}{2}\int_\Omega w^2 \mathrm{div} F\, dv_g,\end{aligned}$$

where ν denotes the outward pointing normal and $\langle \cdot\,, \cdot \rangle = g(\cdot\,, \cdot)$.

The classical Rellich identity was first stated by Rellich in [20]. A special case of Theorem 3.1, called the generalized Pohozaev identity, was proved in [18, 22] in order to get some spectral inequalities between the Steklov and Laplace eigenvalues.

The paper is structured as follows. In Section 2, we recall tools needed in later sections, namely the Hessian and Laplace comparison theorems. Moreover, we give variational characterizations and alternative representations for the eigenvalues of problems (1.2)–(1.5). Section 3 contains the deduction of the Rellich identity on manifolds, as well as several applications thereof. Finally, we prove the main theorems in Section 4.

Acknowledgments. *The authors are grateful to Werner Ballmann and Henrik Matthiesen for valuable suggestions. Furthermore, the authors would like to thank the Max Planck Institute for Mathematics in Bonn (MPIM) for supporting a research visit of the first named author. This work was mainly completed when the first named author was an EPDI postdoctoral fellow at the Mittag-Leffler Institute. The first named author would like to thank the Mittag-Leffler Institute and the second named author would like to thank the MPIM for the support and for providing excellent working conditions.*

2. Preliminaries

In this section we provide the basic tools needed in later sections. Namely, we give the variational characterizations and alternative representations of the eigenvalues of problems (1.2)-(1.5) in the first subsection. In the second subsection, we recall the Hessian and Laplace comparison theorems.

2.1. Variational characterization and alternative representations.

Below, we list the variational characterization of eigenvalues of (1.2)–(1.5) and their alternative representations. For the special case $\Omega \subset \mathbb{R}^2$, the proofs are contained in [10]. The general proofs follow along the lines of these proofs and are therefore omitted.

Dirichlet eigenvalues:

$$\begin{aligned}\lambda_k &= \inf_{\substack{V\subset H_0^1(\Omega)\\ \dim V = k}} \sup_{0\neq u\in V} \frac{\int_\Omega |\nabla u|^2\, dv_g}{\int_\Omega u^2\, dv_g} \\ &= \inf_{\substack{V\subset H^2(\Omega)\cap H_0^1(\Omega)\\ \dim V = k}} \sup_{0\neq u\in V} \frac{\int_\Omega (\Delta u)^2\, dv_g}{\int_\Omega |\nabla u|^2\, dv_g}.\end{aligned} \tag{2.1}$$

Neumann eigenvalues:

$$\mu_k = \inf_{\substack{V\subset H^1(\Omega)\\ \dim V=k}} \sup_{0\neq u\in V} \frac{\int_\Omega |\nabla u|^2\, dv_g}{\int_\Omega u^2\, dv_g} = \inf_{\substack{V\subset H^2(\Omega)\\ \partial_\nu u=0 \text{ on } \partial\Omega\\ \dim V=k}} \sup_{0\neq u\in V} \frac{\int_\Omega (\Delta u)^2\, dv_g}{\int_\Omega |\nabla u|^2\, dv_g}. \tag{2.2}$$

Steklov eigenvalues:

$$\sigma_k = \inf_{\substack{V\subset H^1(\Omega)\\ \dim V=k}} \sup_{0\neq u\in V} \frac{\int_\Omega |\nabla u|^2\, dv_g}{\int_{\partial\Omega} u^2\, dv_g} = \inf_{\substack{V\subset \mathcal{H}(\Omega)\\ \dim V=k}} \sup_{0\neq u\in V} \frac{\int_{\partial\Omega} (\partial_\nu u)^2\, ds_g}{\int_\Omega |\nabla u|^2\, dv_g}, \tag{2.3}$$

where $\mathcal{H}(\Omega)$ is the space of harmonic functions on Ω.

Biharmonic Steklov I eigenvalues:

$$\eta_k = \inf_{\substack{V\subset H^2(\Omega)\cap H^1_0(\Omega)\\ \dim V=k}} \sup_{0\neq u\in V} \frac{\int_\Omega |\Delta u|^2\, dv_g}{\int_{\partial\Omega} (\partial_\nu u)^2\, ds_g}. \tag{2.4}$$

Biharmonic Steklov II eigenvalues:

$$\xi_k = \inf_{\substack{V\subset H^2_N(\Omega)\\ \dim V=k}} \sup_{0\neq u\in V} \frac{\int_\Omega |\Delta u|^2\, dv_g}{\int_{\partial\Omega} u^2\, ds_g}, \tag{2.5}$$

where $H^2_N(\Omega) := \{u \in H^2(\Omega) \,:\, \partial_\nu u = 0 \text{ on } \partial\Omega\}$.

2.2. **Hessian and Laplace comparison theorems.** The idea of comparison theorems is to compare a given geometric quantity on a Riemannian manifold with the corresponding quantity on a model space. Below we recall the Hessian and Laplace comparison theorems. For more details we refer the reader to [3, 6, 17] and [6, 17], respectively.

For any $\kappa \in \mathbb{R}$, denote by $H_\kappa : [0,\infty) \to \mathbb{R}$ the function satisfying the Riccati equation

$$H_\kappa' + H_\kappa^2 + \kappa = 0, \quad \text{with} \quad \lim_{r\to 0} \frac{rH_\kappa(r)}{n-1} = 1.$$

Clearly, we have

$$H_\kappa(r) = \begin{cases} (n-1)\sqrt{\kappa}\cot(\sqrt{\kappa}r) & \kappa > 0, \\ \frac{n-1}{r} & \kappa = 0, \\ (n-1)\sqrt{|\kappa|}\coth(\sqrt{|\kappa|}r) & \kappa < 0. \end{cases}$$

With this preparation at hand we can now state the Hessian comparison theorem.

Theorem 2.6 (Hessian comparison theorem)**.** *Let $\gamma : [0, L] \to M$ be a minimizing geodesic starting from $p \in M$, such that its image is disjoint to the cut locus of p. Assume furthermore that*

$$\kappa_1 \le K_g(X, \dot\gamma(t)) \le \kappa_2$$

for all $t \in [0, L]$ and $X \in T_{\gamma(t)}M$ perpendicular to $\dot\gamma(t)$. Then *(a) d_p satisfies the inequalities*

$$\nabla^2 d_p(X,X) \le \frac{H_{\kappa_1}(t)}{n-1} g(X,X), \qquad \forall t \in [0,L], \quad X \in \langle \dot\gamma(t)\rangle^\perp \subset T_{\gamma(t)}M,$$
$$\nabla^2 d_p(X,X) \ge \frac{H_{\kappa_2}(t)}{n-1} g(X,X), \qquad \forall t \in [0, L \wedge \frac{\pi}{2\sqrt{\kappa_2 \vee 0}}], \quad X \in \langle \dot\gamma(t)\rangle^\perp \subset T_{\gamma(t)}M.$$

Furthermore, we have

$$\nabla^2 d_p(\dot\gamma(t), \dot\gamma(t)) = 0, \qquad \forall t \in [0,L].$$

Here $A \wedge B := \min\{A,B\}$ and $A \vee B := \max\{A,B\}$ for $A, B \in \mathbb{R}$.

(b) ρ_p satisfies the inequalities

$$\nabla^2 \rho_p(X,X) \le \frac{tH_{\kappa_1}(t)}{n-1} g(X,X), \qquad \forall t \in [0,L], \quad X \in \langle \dot\gamma(t)\rangle^\perp \subset T_{\gamma(t)}M,$$
$$\nabla^2 \rho_p(X,X) \ge \frac{tH_{\kappa_2}(t)}{n-1} g(X,X), \qquad \forall t \in [0, L \wedge \frac{\pi}{2\sqrt{\kappa_2 \vee 0}}], \quad X \in \langle \dot\gamma(t)\rangle^\perp \subset T_{\gamma(t)}M,$$

and

$$\nabla^2 \rho_p(\dot\gamma(t), \dot\gamma(t)) = 1, \qquad \forall t \in [0,L].$$

Next, we state the Laplace comparison theorem.

Theorem 2.7 (Laplace comparison theorem)**.** *The distance function d_p and the squared distance function satisfy the following.*

(a) Let $\mathrm{Ric}_g \ge (n-1)\kappa$, $\kappa \in \mathbb{R}$. Then for every $p \in M$ the inequalities

$$\Delta d_p(x) \le H_\kappa(d_p(x)), \quad \text{and} \quad \Delta \rho_p(x) \le 1 + d_p(x) H_\kappa(d_p(x))$$

hold at smooth points of d_p. Moreover the above inequalities hold on the whole manifold in the sense of distribution.

(b) Under the same assumption and notations of Theorem 2.6, the following inequalities hold.

(i) For every $t \in [0,L]$

$$\Delta d_p(\gamma(t)) \le H_{\kappa_1}(t), \quad \text{and} \quad \Delta \rho_p(\gamma(t)) \le 1 + tH_{\kappa_1}(t)\,;$$

(ii) For every $t \in [0, L \wedge \frac{\pi}{2\sqrt{\kappa_2 \vee 0}}]$

$$\Delta d_p(\gamma(t)) \ge H_{\kappa_2}(t), \quad \text{and} \quad \Delta \rho_p(\gamma(t)) \ge 1 + tH_{\kappa_2}(t).$$

Notice that part (b) in the above theorems is an immediate consequence of part (a), since the distance function d_p and one half of the square of the distance function ρ_p satisfy

$$\nabla^2 \rho_p = d_p \nabla^2 d_p + \nabla d_p \otimes \nabla d_p, \qquad \Delta \rho_p = |\nabla d_p|^2 + d_p \Delta d_p.$$

3. Generalized Rellich identity

An important identity which is used in the study of eigenvalue problems is the Rellich identity. To our knowledge it was first stated and used by Rellich [20] in the study of the eigenvalue problem. Some versions of the Rellich identity are also referred to as Pohozaev identity; see [18, 22]. In this section, we provide the generalized Rellich identity on Riemannian manifolds, i.e. Theorem 1.10, and its higher order version. Applications of this result can be found in the last subsection and in Section 4.

3.1. **Rellich identity on manifolds.** The next theorem states the Rellich identity on Riemannian manifolds.

Theorem 3.1 (Generalized Rellich identity for manifolds)**.** *Let (Ω, g) be a Riemannian manifold with piecewise smooth boundary. Let $F : \Omega \to T\Omega$ be a Lipschitz vector field on Ω. Then for every $w \in C^2(\Omega)$ we have*

$$\begin{aligned}\int_\Omega (\Delta w + \lambda w)\langle F, \nabla w\rangle dv_g = &\int_{\partial\Omega} \partial_\nu w \langle F, \nabla w\rangle ds_g - \frac{1}{2}\int_{\partial\Omega} |\nabla w|^2 \langle F, \nu\rangle ds_g + \frac{\lambda}{2}\int_{\partial\Omega} w^2 \langle F, \nu\rangle ds_g \\ &+ \frac{1}{2}\int_\Omega \mathrm{div} F |\nabla w|^2 dv_g - \int_\Omega DF(\nabla w, \nabla w) dv_g - \frac{\lambda}{2}\int_\Omega w^2 \mathrm{div} F\, dv_g,\end{aligned}$$

where ν denotes the outward pointing normal and $\langle\,\cdot\,,\,\cdot\,\rangle = g(\,\cdot\,,\,\cdot\,)$.

In [18, 22], the authors proved the above identity when w is harmonic and $\lambda = 0$. The proof of the general version follows the same line of argument. For the sake of completeness we give the whole argument.

Proof of Theorem 3.1. We calculate $\int_\Omega \Delta w \langle F, \nabla w\rangle dv_g$ and $\int_\Omega \lambda w \langle F, \nabla w\rangle dv_g$ separately. In order to calculate the latter, we apply the divergence theorem to obtain

$$\int_{\partial\Omega} w^2 \langle F, \nu\rangle\, ds_g = \int_\Omega \mathrm{div}(w^2 F) dv_g = \int_\Omega \big(2w\langle F, \nabla w\rangle + w^2 \mathrm{div} F\big)\, dv_g.$$

Thus, we get

$$\int_\Omega \lambda w \langle F, \nabla w\rangle dv_g = \frac{\lambda}{2}\left(\int_{\partial\Omega} w^2 \langle F, \nu\rangle ds_g - \int_\Omega w^2 \mathrm{div} F\, dv_g\right).$$

For the other term, using integration by parts, we obtain

$$\begin{aligned}\text{(3.2)}\quad \int_\Omega \Delta w \langle F, \nabla w\rangle dv_g &= \int_{\partial\Omega} \langle F, \nabla w\rangle \partial_\nu w ds_g - \int_\Omega \langle \nabla\langle F, \nabla w\rangle, \nabla w\rangle dv_g \\ &= \int_{\partial\Omega} \langle F, \nabla w\rangle \partial_\nu w ds_g - \int_\Omega \langle \nabla_{\nabla w} F, \nabla w\rangle dv_g - \int_\Omega \langle \nabla_{\nabla w} \nabla w, F\rangle dv_g \\ &= \int_{\partial\Omega} \langle F, \nabla w\rangle \partial_\nu w ds_g - \int_\Omega DF(\nabla w, \nabla w) dv_g - \int_\Omega \nabla^2 w(\nabla w, F) dv_g.\end{aligned}$$

For further simplification, we observe that

$$\begin{aligned}2\int_\Omega \nabla^2 w(\nabla w, F) dv_g &= \int_\Omega \mathrm{div}(F|\nabla w|^2) dv_g - \int_\Omega \mathrm{div} F |\nabla w|^2 dv_g \\ &= \int_{\partial\Omega} |\nabla w|^2 F ds_g - \int_\Omega \mathrm{div} F |\nabla w|^2 dv_g.\end{aligned}$$

Plugging this identity into (3.2) we get

$$\begin{aligned}\int_\Omega \Delta w \langle F, \nabla w\rangle dv_g = &\int_{\partial\Omega} \partial_\nu w \langle F, \nabla w\rangle ds_g - \frac{1}{2}\int_{\partial\Omega} |\nabla w|^2 \langle F, \nu\rangle ds_g \\ &+ \frac{1}{2}\int_\Omega \mathrm{div} F |\nabla w|^2 dv_g - \int_\Omega DF(\nabla w, \nabla w) dv_g.\end{aligned}$$

This completes the proof. ∎

3.2. **Higher order Rellich identities.** In this section we provide a higher order Rellich identity.

The following preparatory lemma is a simple consequence from Theorem 3.1. For the special case $M=\mathbb{R}^n$, the identity stated in the lemma was first proven by Mitidieri in [16].

Lemma 3.3. *For $u,v\in C^2(\Omega)$ we have*

$$\int_\Omega \Delta w\langle F,\nabla v\rangle+\Delta v\langle F,\nabla w\rangle dv_g=\int_{\partial\Omega}\{\partial_\nu w\langle F,\nabla v\rangle+\partial_\nu v\langle F,\nabla w\rangle\}ds_g-\int_{\partial\Omega}\langle\nabla w,\nabla v\rangle\langle F,\nu\rangle ds_g$$
$$+\int_\Omega \mathrm{div}F\langle\nabla w,\nabla v\rangle dv_g-2\int_\Omega DF(\nabla w,\nabla v)dv_g.$$

Proof. Replacing w by $w+v$ in Theorem 3.1 and set $\lambda=0$ we get the identity. □

The following theorem states the higher order Rellich identity.

Theorem 3.4. *For $w\in C^4(\Omega)$ we have*

$$\begin{aligned}\int_\Omega(\Delta^2w+\lambda\Delta w)\langle F,\nabla w\rangle dv_g&=\frac12\int_\Omega \mathrm{div}F(\Delta w)^2dv_g-\frac12\int_{\partial\Omega}(\Delta w)^2\langle F,\nu\rangle dv_g\\&+\int_{\partial\Omega}\{\partial_\nu w\langle F,\nabla\Delta w\rangle+\partial_\nu\Delta w\langle F,\nabla w\rangle\}ds_g-\int_{\partial\Omega}\langle\nabla w,\nabla\Delta w\rangle\langle F,\nu\rangle ds_g\\&+\int_\Omega \mathrm{div}F\langle\nabla w,\nabla\Delta w\rangle dv_g-2\int_\Omega DF(\nabla w,\nabla\Delta w)dv_g+\lambda\int_{\partial\Omega}\partial_\nu w\langle F,\nabla w\rangle ds_g\\&-\frac{\lambda}{2}\int_{\partial\Omega}|\nabla w|^2\langle F,\nu\rangle ds_g+\frac{\lambda}{2}\int_\Omega \mathrm{div}F|\nabla w|^2dv_g-\lambda\int_\Omega DF(\nabla w,\nabla w)dv_g.\end{aligned}$$

Proof. If we choose $v=\Delta w$ in Lemma 3.3, we obtain

$$\begin{aligned}\int_\Omega\Delta^2w\langle F,\nabla w\rangle dv_g&=-\int_\Omega\Delta w\langle F,\nabla\Delta w\rangle dv_g\\&\quad+\int_{\partial\Omega}\{\partial_\nu w\langle F,\nabla\Delta w\rangle+\partial_\nu\Delta w\langle F,\nabla w\rangle\}ds_g-\int_{\partial\Omega}\langle\nabla w,\nabla\Delta w\rangle\langle F,\nu\rangle ds_g\\&\quad+\int_\Omega \mathrm{div}F\langle\nabla w,\nabla\Delta w\rangle dv_g-2\int_\Omega DF(\nabla w,\nabla\Delta w)dv_g.\end{aligned}$$

By the divergence theorem we have

$$\begin{aligned}\int_\Omega\Delta w\langle F,\nabla\Delta w\rangle dv_g&=\frac12\int_\Omega\langle F,\nabla(\Delta w)^2\rangle dv_g\\&=-\frac12\int_\Omega \mathrm{div}F(\Delta w)^2dv_g+\frac12\int_{\partial\Omega}(\Delta w)^2\langle F,\nu\rangle dv_g,\end{aligned}$$

which together with Theorem 3.1 establishes the claim. □

For the special case $M=\mathbb{R}^n$ and $\lambda=0$, the statement of Theorem 3.4 is contained in [16].

3.3. **Applications of the Rellich identities.** In 1940 Rellich [20] dealt with the Dirichlet eigenvalue problem on sets $\Omega\subset\mathbb{R}^n$. For this special case he used the identity derived in Theorem 3.1 to express the Dirichlet eigenvalues in terms of an integral over the boundary. One decade ago, Liu [13] extended Rellich's result to the Neumann eigenvalue problem, the clamped plate eigenvalue problem and the buckling eigenvalue problem, each on sets $\Omega\subset\mathbb{R}^n$. In the latter two cases Liu (implicitly) applied the higher order Rellich identity.

Recall that for any bounded domain $\Omega \subset M$ with C^2 boundary $\partial\Omega$ the clamped plate eigenvalue problem and the buckling eigenvalue problem are given by

$$(3.5)\qquad \begin{cases} \Delta^2 u + \Lambda \Delta u = 0 & \text{in } \Omega, \\ u = \partial_\nu u = 0 & \text{on } \partial\Omega; \end{cases} \qquad \text{Buckling problem},$$

$$(3.6)\qquad \begin{cases} \Delta^2 u - \Gamma^2 u = 0 & \text{in } \Omega, \\ u = \partial_\nu u = 0 & \text{on } \partial\Omega; \end{cases} \qquad \text{Clamped plate},$$

respectively.

Below we reprove the result of Liu for the case of the buckling eigenvalue problem. Note there is no new idea for the proof, however, our proof is shorter and clearer since we do not carry out the calculations in coordinates. One can proceed similarly for the clamped plate eigenvalue problem.

Lemma 3.7 ([13]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary.*

(i) Let w be an eigenfunction corresponding to the eigenvalue Λ of the buckling eigenvalue problem. Then we have

$$\Lambda = \frac{\int_{\partial\Omega} (\partial^2_{\nu\nu} w)^2 \partial_\nu (r^2) ds_g}{4 \int_\Omega |\nabla w|^2 dv_g},$$

where $r^2 = x_1^2 + \cdots + x_n^2$ and x_i are Euclidean coordinates.

(ii) Let w be an eigenfunction corresponding to the eigenvalue Γ of the clamped plate eigenvalue problem. Then we have

$$\Gamma = \frac{\int_{\partial\Omega} (\partial^2_{\nu\nu} w)^2 \partial_\nu (r^2) ds_g}{8 \int_\Omega w^2 dv_g}.$$

Proof. In order to prove (i) we apply Theorem 3.4 for the special case $\Omega \subset \mathbb{R}^n$ and where F is given by the gradient of the distance function. In this case we have $DF(\,\cdot\,,\,\cdot\,) = g(\,\cdot\,,\,\cdot\,)$ and $\mathrm{div} F = n$. Note furthermore that $w_{|\partial\Omega} = 0$ implies $\nabla w = \partial_\nu w \nu$ on $\partial\Omega$. Since we have $\partial_\nu w_{|\partial\Omega} = 0$ by assumption, ∇w vanishes along the boundary of Ω.

Plugging the above information into the higher order Rellich identity we get

$$0 = \int_\Omega (\Delta^2 w + \lambda \Delta w) \langle F, \nabla w \rangle dv_g = \frac{n}{2} \int_\Omega (\Delta w)^2 dv_g - \frac{1}{2} \int_{\partial\Omega} (\Delta w)^2 \langle F, \nu \rangle dv_g$$
$$+ (n-2) \int_\Omega \langle \nabla w, \nabla \Delta w \rangle dv_g + \Lambda (\frac{n}{2} - 1) \int_\Omega |\nabla w|^2 dv_g.$$

Applying the divergence theorem once more, we thus obtain

$$\Lambda (\frac{n}{2} - 1) \int_\Omega |\nabla w|^2 dv_g = \frac{1}{2} \int_{\partial\Omega} (\Delta w)^2 \langle F, \nu \rangle ds_g - (2 - \frac{n}{2}) \int_\Omega (\Delta w)^2 dv_g.$$

The variational characterization of Λ asserts that for an eigenfunction w corresponding to Λ we have

$$(3.8)\qquad \int_\Omega (\Delta w)^2 dv_g - \Lambda \int_\Omega |\nabla w|^2 dv_g = 0.$$

Furthermore, the identities

$$\langle F, \nu \rangle = \sum_{i=1}^n x_i \partial_\nu x_i = \frac{1}{2} \partial_\nu (r^2)$$

and $\Delta w = \partial^2_{\nu\nu} w$ hold on the boundary of Ω. Thus the claim is established.

The proof of (ii) is omitted since it is similar to the one of (i). □

Remark 3.9. In Lemma 3.7 (i), when normalizing the eigenfunction w such that $\int_\Omega |\nabla w|^2 dv_g = 1$, we obtain

$$\Lambda = \frac{1}{4}\int_{\partial\Omega} (\partial^2_{\nu\nu} w)^2 \partial_\nu (r^2) ds_g;$$

i.e. Λ is expressed in terms of an integral over the boundary. A similar remark holds for Lemma 3.7 (ii).

Finally we use the Rellich identities to get some estimates on eigenvalues. Note that from now on we do not assume anymore that Ω is a subset of the Euclidean space. However, we assume that Ω is a manifold with smooth boundary and that there exists a vector field F on Ω satisfying the following properties:

A) $0 < c_1 \leq \mathrm{div} F \leq c_2$, for some positive constants $c_1, c_2 \in \mathbb{R}_+$,
B) $DF(X, X) \geq \alpha g(X, X)$ for some positive constant $\alpha \in \mathbb{R}_+$,
C) $\langle F, \nu \rangle \geq 0$ on $\partial\Omega$.

Remark 3.10. Domains in Hadamard manifolds, and free boundary minimal hypersufaces in the unit ball in $\mathbb{R}^{n+1}$ provide examples for which conditions A-C for the gradient of the distance function on Ω are satisfied. For the latter see Example 4.11 in which condition A with $c_1 = c_2$ holds.

The following lemma is an easy consequence of Theorem 3.1 and Theorem 3.4, respectively. It establishes upper estimates for eigenvalues in terms of integrals over the boundary $\partial\Omega$ and α.

Lemma 3.11. *Assume that there exists a vector field F on $\Omega \subset M^n$ satisfying properties A-C above. Then*

(i) the eigenvalue λ corresponding to eigenfunction w of the Dirichlet eigenvalue problem satisfies

$$\lambda \leq \frac{\int_{\partial\Omega} (\partial_\nu w)^2 \langle F, \nu \rangle ds_g}{(2\alpha + c_1 - c_2) \int_{\Omega w^2 dv_g}};$$

(ii) the eigenvalue Λ corresponding to eigenfunction w of the buckling eigenvalue problem satisfies

$$\frac{\int_{\partial\Omega} (\Delta w)^2 \langle F, \nu \rangle dv_g}{2\alpha \int_\Omega |\nabla w|^2 dv_g} \leq \Lambda$$

provided $c_1 = c_2 =: c$ in property A.

Proof. We start by proving (i). Theorem 3.1 implies

$$0 = \int_\Omega (\Delta w + \lambda w)\langle F, \nabla w\rangle dv_g \leq \int_{\partial\Omega} \partial_\nu w \langle F, \nabla w \rangle ds_g - \frac{1}{2}\int_{\partial\Omega} |\nabla w|^2 \langle F, \nu\rangle ds_g$$
$$+ \frac{c_2}{2}\int_\Omega |\nabla w|^2 dv_g - \int_\Omega DF(\nabla w, \nabla w) dv_g - \frac{\lambda c_1}{2}\int_\Omega w^2\, dv_g.$$

Since $w \equiv 0$ on $\partial\Omega$ we have $\nabla w = \partial_\nu w \nu$ on $\partial\Omega$. Thus, we obtain

$$\frac{\lambda c_1}{2}\int_\Omega w^2 dv_g \leq \frac{1}{2}\int_{\partial\Omega} (\partial_\nu w)^2 \langle F, \nu\rangle ds_g + (\frac{\lambda c_2}{2} - \alpha\lambda)\int_\Omega w^2\, dv_g.$$

The latter inequality implies the claim.

Below we prove (ii). Theorem 3.4 implies

$$\begin{aligned}0 &= \frac{c}{2}\int_\Omega (\Delta w)^2 dv_g - \frac{1}{2}\int_{\partial\Omega}(\Delta w)^2\langle F,\nu\rangle dv_g + c\int_\Omega \langle \nabla w, \nabla\Delta w\rangle dv_g \\ &\quad - 2\int_\Omega DF(\nabla w, \nabla \Delta w) dv_g + \frac{c\Lambda}{2}\int_\Omega |\nabla w|^2 dv_g - \Lambda\int_\Omega DF(\nabla w,\nabla w) dv_g \\ &\leq (2\alpha - \frac{c}{2})\int_\Omega (\Delta w)^2 dv_g - \frac{1}{2}\int_{\partial\Omega}(\Delta w)^2\langle F,\nu\rangle dv_g + (\frac{c\Lambda}{2} - \Lambda\alpha)\int_\Omega |\nabla w|^2 dv_g,\end{aligned}$$

where we made use of

$$\int_\Omega \langle \nabla w, \nabla \Delta w\rangle dv_g = -\int_\Omega (\Delta w)^2 dv_g,$$

what is a consequence of the divergence theorem. Applying (3.8) yields

$$0 \leq -\frac{1}{2}\int_{\partial\Omega}(\Delta w)^2 \langle F,\nu\rangle dv_g + \Lambda\alpha\int_\Omega |\nabla w|^2 dv_g,$$

and thus the claim is established. □

4. Proof of the Main Theorems

Proof of Theorem 1.6. Inequalities (a) and (b) are an immediate consequence of the variational characterizations of μ_k, σ_k and ξ_k given in (2.2), (2.3) and (2.5). Indeed, let V be the space generated by eigenfunctions associated with $\xi_2,\dots,\xi_k$ with $\int_{\partial\Omega} u = 0$, for every $u\in V$. Then by the variational characterization (2.2) we get

$$\begin{aligned}\mu_k &\leq \sup_{0\neq u\in V}\frac{\int_\Omega (\Delta u)^2\, dv_g}{\int_\Omega |\nabla u|^2\, dv_g} \leq \xi_k \sup_{0\neq u\in V}\frac{\int_{\partial\Omega} u^2\, dv_g}{\int_\Omega |\nabla u|^2\, dv_g} \\ &= \xi_k\left(\inf_{0\neq u\in V}\frac{\int_\Omega |\nabla u|^2\, dv_g}{\int_{\partial\Omega} u^2\, dv_g}\right)^{-1} \leq \frac{\xi_k}{\sigma_2}.\end{aligned}$$

The proof of part (b) is similar and we leave it to the reader. □

Proof of Theorem 1.7. We use the following identity

$$\frac{1}{2}\int_{\partial\Omega} w^2\langle \nu, \nabla\rho_p\rangle ds_g = \int_\Omega w\langle \nabla w, \nabla \rho_p\rangle dv_g + \frac{1}{2}\int_\Omega w^2 \Delta\rho_p dv_g$$

which follows easily from integration by parts. Using the Laplace comparison theorem, we thus get

$$\frac{1}{2}\int_{\partial\Omega} w^2\langle \nu,\nabla\rho_p\rangle ds_g \leq \int_\Omega w\langle\nabla w,\nabla\rho_p\rangle dv_g + \frac{1}{2}\max_{x\in\Omega}(1 + d_p(x)H_{\kappa_1}(d_p(x)))\int_\Omega w^2 dv_g. \tag{4.1}$$

The Cauchy Schwarz inequality yields

$$\Big(\int_\Omega w\langle \nabla w, \nabla\rho_p\rangle dv_g\Big)^2 \leq r_{\max}^2 \int_\Omega w^2 dv_g \int_\Omega |\nabla w|^2 dv_g.$$

Assuming $\int_\Omega w dv_g = 0$ and using the variational characterisation of μ_2 we get

$$\int_\Omega w\langle\nabla w,\nabla\rho_p\rangle dv_g \leq r_{\max}\mu_2^{-1/2}\int_\Omega |\nabla w|^2 dv_g.$$

Thus, from inequality (4.1), we get

$$\frac{1}{2}\int_{\partial\Omega} w^2\langle\nu,\nabla\rho_p\rangle ds_g \leq (r_{\max}\mu_2^{-1/2} + \frac{1}{2}\max_{x\in\Omega}(1 + d_p(x)H_\kappa(d_p(x)))\mu_2^{-1})\int_\Omega |\nabla w|^2 dv_g. \tag{4.2}$$

Let u be an eigenfunction associated to the eigenvalue σ_2 and choose w to be

$$w := u - \mathrm{vol}(\Omega)^{-1}\int_\Omega u dv_g.$$

Then we have

$$\int_\Omega |\nabla w|^2 dv_g = \int_\Omega |\nabla u|^2 dv_g = \sigma_2 \int_{\partial\Omega} u^2 ds_g \leq \sigma_2 \int_{\partial\Omega} w^2 ds_g.$$

Combining this inequality with (4.2), we finally get

$$\begin{aligned}\frac{1}{2}h_{\min}\int_{\partial\Omega} w^2 ds_g &\leq \frac{1}{2}\int_{\partial\Omega} w^2\langle\nu,\nabla\rho_p\rangle ds_g\\ &\leq (r_{\max}\mu_2^{-1/2} + \frac{1}{2}\max_{x\in\Omega}(1+d_p(x)H_\kappa(d_p(x)))\mu_2^{-1})\int_\Omega |\nabla w|^2 dv_g\\ &\leq (r_{\max}\mu_2^{-1/2} + \frac{1}{2}\max_{x\in\Omega}(1+d_p(x)H_\kappa(d_p(x)))\mu_2^{-1})\sigma_2\int_{\partial\Omega} w^2 ds_g.\end{aligned}$$

Thus the claim is established. □

Proof of Theorem 1.9. Throughout the proof we repeatedly use the Hessian and Laplace comparison theorems as well as the generalized Rellich identity, i.e. Theorem 3.1.

i) We start by proving the first inequality in *i)*, namely $C_1\eta_m/h_{\max} \leq \lambda_k$. Let E_k be the eigenspace associated with λ_k and let $u_1, \cdots, u_m$ be an orthonormal basis for E_k.
The functions $\partial_\nu u_1, \cdots, \partial_\nu u_m$ are linearly independent on $\partial\Omega$. Indeed, if there exist $u \in \mathrm{Span}(\partial_\nu u_1, \cdots, \partial_\nu u_m)$ such that $\partial_\nu u = 0$, then we define

$$v(x) = \begin{cases} u(x) & \text{if } x \in \Omega,\\ 0 & \text{if } x \in M\setminus\Omega.\end{cases}$$

Clearly, we have $v \in H^1(M)$. Furthermore, v satisfies the identity $\Delta v = \lambda_k v$. Since $v \equiv 0$ on $M\setminus\Omega$ we get $v \equiv 0$ on M by the unique continuation theorem. Thus, we can consider E_k as a test functional space in (2.4).
Let $h_{\max} = \sup_{x\in\partial\Omega}\langle\nabla\rho_p,\nu\rangle$. Since $0 < \frac{1}{h_{\max}}\langle\nabla\rho_p,\nu\rangle \leq 1$, we get

$$\eta_m \leq \sup_{u\in E_k}\frac{\int_\Omega |\Delta u|^2\, dv_g}{\int_{\partial\Omega}(\partial_\nu u)^2\, ds_g} \leq h_{\max}\lambda_k^2 \sup_{u\in E_k}\frac{\int_\Omega u^2\, dv_g}{\int_{\partial\Omega}\langle\nabla\rho_p,\nu\rangle(\partial_\nu u)^2\, ds_g}.$$

Next we bound the denominator from below. Applying Theorem 3.1 with $\lambda = 0$ and $F = \nabla\rho_p$ yields

$$\int_{\partial\Omega}\langle\nabla\rho_p,\nu\rangle(\partial_\nu u)^2\, ds_g = 2\int_\Omega \Delta u\langle\nabla\rho_p,\nabla u\rangle dv_g - \int_\Omega \Delta\rho_p|\nabla u|^2 dv_g + 2\int_\Omega \nabla^2\rho_p(\nabla u,\nabla u)dv_g.$$

Using $u \in E_k$ and integration by parts we get

$$2\int_\Omega \Delta u\langle\nabla\rho_p,\nabla u\rangle dv_g = -\lambda_k\int_\Omega\langle\nabla\rho_p,\nabla u^2\rangle dv_g = \lambda_k\int_\Omega u^2\Delta\rho_p dv_g.$$

Consequently, we have

$$\begin{aligned}
\int_{\partial\Omega} \langle \nabla\rho_p, \nu\rangle (\partial_\nu u)^2\, ds_g &= \lambda_k \int_\Omega u^2 \Delta\rho_p dv_g - \int_\Omega \Delta\rho_p |\nabla u|^2 dv_g + 2\int_\Omega \nabla^2\rho_p(\nabla u, \nabla u) dv_g \\
&\geq \lambda_k \left(1 + \min_{x\in\Omega} d_p(x) H_{\kappa_2}(d_p(x))\right) \int_\Omega u^2 dv_g \\
&\quad - \left(1 + \max_{x\in\Omega} d_p(x) H_{\kappa_1}(d_p(x))\right) \int_\Omega |\nabla u|^2 dv_g \\
&\quad + 2 \min_{x\in\Omega} \frac{d_p(x) H_{\kappa_2}(d_p(x))}{n-1} \int_\Omega |\nabla u|^2 dv_g \\
&= \lambda_k C_1 \int_\Omega u^2 dv_g.
\end{aligned}$$

In the second line we used the Hessian and Laplace comparison theorems; see Section 2. Here C_1 is

$$C_1 := \left(1 + \frac{2}{n-1}\right) \min_{r\in[0,r_{\max})} r H_{\kappa_2}(r) - \max_{r\in[0,r_{\max})} r H_{\kappa_1}(r). \tag{4.3}$$

Therefore, we get

$$C_1 \eta_m \leq h_{\max} \lambda_k.$$

We conclude the proof of the first inequality with a remark on the sign of C_1. The function $rH_\kappa(r)$ is constant if $\kappa = 0$, increasing on $[0,\infty)$ if $\kappa < 0$, and decreasing on $[0,\infty)$ if $\kappa > 0$. Thus we calculate C_1 considering the following different cases:

(a) If $\kappa_1 = \kappa_2 = 0$, then $C_1 = 2$.
(b) If $\kappa_1 \leq \kappa_2 \leq 0$, then $C_1 = n + 1 - r_{\max} H_{\kappa_1}(r_{\max})$.
(c) If $0 \leq \kappa_1 \leq \kappa_2$, then $C_1 = \left(1 + \frac{2}{n-1}\right) r_{\max} H_{\kappa_2}(r_{\max}) - (n-1)$.
(d) If $\kappa_1 \leq 0 \leq \kappa_2$, then $C_1 = \left(1 + \frac{2}{n-1}\right) r_{\max} H_{\kappa_2}(r_{\max}) - r_{\max} H_{\kappa_1}(r_{\max})$.

Of course when $C_1 \leq 0$, we only get a trivial bound. However, depending on κ_1 and κ_2, in all cases, there exists $r_0 \in (0,\infty]$ such that for $r_{\max} < r_0$, C_1 is positive.

We proceed with the proof of the second inequality of part i). Let $u_1, \cdots, u_k \in H^2(\Omega)$ be a family of eigenfunctions associated to $\eta_1, \cdots, \eta_k$. We can choose $u_1, \cdots, u_k$ such that $\partial_\nu u_1, \cdots, \partial_\nu u_k$ are orthonormal in $L^2(\partial\Omega)$. Then, due to (2.1) and (2.4), we have

$$\lambda_k \leq \eta_k \sup_{u\in E_k} \frac{\int_{\partial\Omega} (\partial_\nu u)^2\, ds_g}{\int_\Omega |\nabla u|^2\, dv_g}, \tag{4.4}$$

where $E_k := \mathrm{Span}(u_1, \cdots, u_k)$. Applying Theorem 3.1 with $\lambda = 0$ and $F = \nabla\rho_p$ we get

$$\begin{aligned}
\int_{\partial\Omega} \langle \nabla\rho_p, \nu\rangle (\partial_\nu u)^2\, ds_g &= 2\int_\Omega \Delta u \langle \nabla\rho_p, \nabla u\rangle dv_g - \int_\Omega \Delta\rho_p |\nabla u|^2 dv_g + 2\int_\Omega \nabla^2\rho_p(\nabla u, \nabla u) dv_g \\
&\leq 2 \max_{x\in\Omega} |\nabla\rho_p| \left(\int_\Omega (\Delta u)^2 dv_g \int_\Omega |\nabla u|^2 dv_g\right)^{1/2} \\
&\quad + \left(-1 - \min_{x\in\Omega} d_p(x) H_{\kappa_2}(d_p(x)) + 2 \max_{x\in\Omega} \frac{d_p(x) H_{\kappa_1}(d_p(x))}{n-1}\right) \int_\Omega |\nabla u|^2 dv_g \\
&\leq 2 r_{\max} \eta_k^{\frac{1}{2}} \left(\int_{\partial\Omega} (\partial_\nu u)^2\, ds_g \int_\Omega |\nabla u|^2 dv_g\right)^{1/2} - C_2 \int_\Omega |\nabla u|^2 dv_g,
\end{aligned}$$

where

$$C_2 := 1 + \min_{x\in\Omega} d_p(x) H_{\kappa_2}(d_p(x)) - 2 \max_{x\in\Omega} \frac{d_p(x) H_{\kappa_1}(d_p(x))}{n-1}. \tag{4.5}$$

Let $A^2 := \frac{\int_{\partial\Omega} (\partial_\nu u)^2\, ds_g}{\int_\Omega |\nabla u|^2 dv_g}$. From the above inequality, A satisfies

$$h_{\min} A^2 \leq 2 r_{\max} \eta_k^{\frac{1}{2}} A - C_2.$$

This implies

$$r_{\max}^2 \eta_k - h_{\min} C_2 \geq 0,$$

Remark that since this is true for every k, we get in particular

$$\eta_1 \geq \frac{h_{\min} C_2}{r_{\max}^2}. \tag{4.6}$$

We now obtain the following upper bound on A^2

$$A^2 \leq \frac{\left(r_{\max}\eta_k^{\frac{1}{2}} + \sqrt{r_{\max}^2 \eta_k - C_2 h_{\min}}\right)^2}{h_{\min}^2} \leq \frac{4 r_{\max}^2 \eta_k - 2 C_2 h_{\min}}{h_{\min}^2}.$$

Replacing in (4.4) we conclude

$$\lambda_k \leq \frac{4 r_{\max}^2 \eta_k^2 - 2 C_2 h_{\min} \eta_k}{h_{\min}^2}.$$

Remark 4.7. The function $rH_\kappa(r)$ is constant if $\kappa = 0$, increasing on $[0,\infty)$ if $\kappa < 0$, and decreasing on $[0,\infty)$ if $\kappa > 0$. We calculate C_2 considering different cases:
(a) If $\kappa_1 = \kappa_2 = 0$, then $C_2 = n-2$.
(b) If $\kappa_1 \leq \kappa_2 \leq 0$, then $C_2 = n - 2\frac{r_{\max} H_{\kappa_1}(r_{\max})}{n-1}$.
(c) $0 \leq \kappa_1 \leq \kappa_2$. Then $C_2 = r_{\max} H_{\kappa_2}(r_{\max}) - 1$.
(d) $\kappa_1 \leq 0 \leq \kappa_2$. Then $C_2 = 1 + r_{\max} H_{\kappa_2}(r_{\max}) - 2\frac{r_{\max} H_{\kappa_1}(r_{\max})}{n-1}$.
Depending on κ_1 and κ_2, in all cases , there exists $r_0 \in (0,\infty]$ so that when $r_{\max} < r_0$, then C_2 is positive.

ii) Let $\phi > 0$ be a continuous function on $\partial\Omega$. For every $l \in \mathbb{N}$ set

$$\xi_{l+1}(\phi) \ := \inf_{\substack{V \subset \tilde{H}^2_{N,\phi}(\Omega) \\ \dim V = l}} \sup_{u\in V} \frac{\int_\Omega |\Delta u|^2\, dv_g}{\int_{\partial\Omega} u^2 \phi\, ds_g}, \qquad \xi_1(\phi) = 0,$$

where $\tilde{H}^2_{N,\phi}(\Omega) := \{u \in H^2(\Omega) \,:\, \partial_\nu u = 0 \text{ on } \partial\Omega \text{ and } \int_{\partial\Omega} \phi u ds_g = 0\}$. The following relation between ξ_l and $\xi_l(\phi)$ holds:

$$\xi_l \leq \|\phi\|_\infty \xi_l(\phi). \tag{4.8}$$

Indeed, let $V = \mathrm{Span}(v_1, \cdots, v_l)$ be a subspace of $\tilde{H}^2_{N,\phi}(\Omega)$ of dimension l. The functional space $W = \mathrm{Span}(w_1, \cdots, w_l)$, where $w_j = v_j - \frac{1}{\mathrm{vol}(\partial\Omega)} \int v_j\, ds_g$, is an l-dimensional subspace of $\tilde{H}^2_N(\Omega) := \{u \in H^2(\Omega) \,:\, \partial_\nu u = 0 \text{ on } \partial\Omega \text{ and } \int_{\partial\Omega} u ds_g = 0\}$ since $1 \notin V$. It is easy to check that for every $v \in \tilde{H}^2_{N,\phi}(\Omega)$ and $w = v - \frac{1}{\mathrm{vol}(\partial\Omega)} \int v\, ds_g$ we have

$$\frac{\int_\Omega |\Delta w|^2\, dv_g}{\|\phi\|_\infty \int_{\partial\Omega} w^2\, ds_g} \leq \frac{\int_\Omega |\Delta v|^2\, dv_g}{\int_{\partial\Omega} v^2 \phi\, ds_g},$$

and inequality (4.8) follows. Later on we take $\phi := \langle \nabla\rho_p, \nu\rangle$. Thus, it is enough to show that

$$\xi_{m+1}(\phi) \leq \frac{\mu_k^2}{(C_3 - n^{-1}\mu_k r_{\rm in}^2)\vee 0},$$

for some constants C_3. Let E_k be the eigenspace associated with μ_k, $k\geq 2$, and $u_1,\cdots,u_m$ be an orthonormal basis for E_k. Let F be a vector field on Ω satisfying properties A–C on page 10. Consider

$$v_j := u_j - \frac{1}{\int_\Omega {\rm div}F\,dv_g}\int_{\partial\Omega} u_j\langle F,\nu\rangle ds_g, \qquad j=1,\cdots,m.$$

The functional space $V = {\rm Span}(v_1,\ldots,v_m)$ forms an m-dimensional subspace of $\tilde H^2_{N,\phi}(\Omega)$, where $\phi := \langle F,\nu\rangle$.

$$\xi_{m+1}(\phi) \leq \sup_{v\in V}\frac{\int_\Omega|\Delta v|^2\,dv_g}{\int_{\partial\Omega} v^2\langle F,\nu\rangle\,ds_g} = \sup_{u\in E_k}\frac{\mu_k^2\int_\Omega u^2\,dv_g}{\int_{\partial\Omega}u^2\langle F,\nu\rangle\,ds_g - (\int_\Omega {\rm div}F\,dv_g)^{-1}\left(\int_{\partial\Omega}u\langle F,\nu\rangle ds_g\right)^2}.$$

By the Green formula and Theorem 3.1, we get

$$\begin{aligned}
\int_{\partial\Omega} u^2\langle F,\nu\rangle\,ds_g &= 2\int_\Omega u\langle\nabla u,F\rangle dv_g + \int_\Omega u^2{\rm div}F dv_g\\
&= 2\mu_k^{-1}\int_\Omega \Delta u\langle\nabla u,F\rangle dv_g + \int_\Omega u^2{\rm div}F dv_g\\
&= \mu_k^{-1}\left(\int_{\partial\Omega}|\nabla u|^2\langle F,\nu\rangle ds_g - \int_\Omega {\rm div}F|\nabla u|^2 dv_g + 2\int_\Omega DF(\nabla u,\nabla u)dv_g\right)\\
&\quad + \int_\Omega u^2{\rm div}F dv_g\\
&\geq \mu_k^{-1}\int_{\partial\Omega}|\nabla u|^2\langle F,\nu\rangle ds_g + (c_1 - c_2 + 2\alpha)\int_\Omega u^2\,dv_g\\
&\geq (c_1 - c_2 + 2\alpha)\int_\Omega u^2\,dv_g.
\end{aligned}$$

We also have

$$\begin{aligned}
\left(\int_{\partial\Omega} u\langle F,\nu\rangle ds_g\right)^2 &= \left(\int_\Omega\langle F,\nabla u\rangle\,dv_g\right)^2 \leq \int_\Omega |F|^2\,dv_g\int_\Omega|\nabla u|^2\,dv_g\\
&= \mu_k\int_\Omega|F|^2\,dv_g\int_\Omega u^2\,dv_g.
\end{aligned}$$

Therefore,

$$\xi_{m+1}(\phi) \leq \frac{\mu_k^2}{((c_1 - c_2 + 2\alpha) - c_1^{-1}{\rm vol}(\Omega)^{-1}\mu_k\int_\Omega|F|^2\,dv_g)\vee 0}.$$

Thanks to the Laplace and Hessian comparison theorem, the vector field $F = \nabla\rho_p$ satisfies properties $A - C$ (see page 10) on Ω with $\alpha = 1$, and

$$c_1 = n, \qquad c_2 = 1 + \max_{r\in[0,r_{\max})} rH_\kappa(r) = 1 + r_{\max}H_\kappa(r_{\max}).$$

Taking

$$C_3 := n + 1 - r_{\max}H_\kappa(r_{\max}), \tag{4.9}$$

we get

$$\xi_{m+1}(\phi) \le \frac{\mu_k^2}{(C_3 - n^{-1}\mathrm{vol}(\Omega)^{-1}\mu_k \int_\Omega d_p^2 \, dv_g) \vee 0}$$

which completes the proof.

□

Finally, we provide examples for Theorem 1.9 (*ii*) in which vector fields satisfying conditions A-C arise naturally. The first example is just a special case of Theorem 1.9 (*ii*).

Example 4.10. Let Ω be a star-shaped domain Ω in $\mathbb{R}^n$ with respect to the origin. Thus $F(x) = x$ satisfies properties $A - C$ above on Ω for $\alpha = 1$ and $c_1 = c_2 = n$. Then by Theorem 1.9 part *ii* we have

$$\xi_{m+1} \le \frac{\max_{x\in\partial\Omega}\langle x, \nu\rangle \mu_k^2}{(2 - n^{-1}\mathrm{vol}(\Omega)^{-1}\mu_k I_2(\Omega)) \vee 0},$$

where m is the multiplicity of μ_k and $I_2(\Omega) = \int_\Omega |x|^2 \, dv_g$ is the second moment of inertia. If in addition the origin is also the centroid of Ω, i.e. $\int_\Omega x dv_g = 0$, then we have

$$\xi_{m_0+1} \le \max_{x\in\partial\Omega}\langle x, \nu\rangle \mu_2^2,$$

where m_0 denotes the multiplicity of μ_2. Combining this inequality with Theorem 1.6 (*b*) we get

$$\sigma_{m_0+1} \le \max_{x\in\partial\Omega}\langle x, \nu\rangle \mu_2.$$

These two last inequalities has been previously obtained in [10] for the special case $n = 2$.

Example 4.11. Let $\mathbf{B^{n+1}}$ be the unit ball in $\mathbb{R}^{n+1}$ centered at the origin, and Ω be a free boundary minimal hypersurface in $\mathbf{B^{n+1}}$. Consider $F(x) = x$, or equivalently $\rho_0(x) = \rho(x) = \frac{|x|^2}{2}$. It is well-known that the coordinate functions of $\mathbb{R}^{n+1}$ are harmonic on Ω. Hence

$$\mathrm{div}F = \Delta\rho = n.$$

Also, by the definition of a free boundary minimal hypersurface, we have $\langle \nabla\rho, \nu\rangle = 1$ on $\partial\Omega$. Thus, conditions A and C on page 10 are satisfied. To verify condition B, one can show that the eigenvalues of $\nabla^2\rho$ at point $x \in \Omega$ are given by $1 - \kappa_i\langle x, N(x)\rangle$, $i = 1, \cdots, n$, where $N(x)$ is the unit normal to the Ω such that $N|_{\partial\Omega} = \nu$, and κ_i are principal curvatures. Indeed, let $X, Y \in T_x\Omega$. Then we have

$$\begin{aligned}
\nabla^2\rho(x)(X,Y) &= X \cdot (Y \cdot \rho(x)) - \nabla_X Y \cdot \rho(x) \\
&= X\langle x, Y\rangle - \langle x, \nabla_X Y\rangle \\
&= \langle X, Y\rangle + \langle x, D_X Y\rangle - \langle x, \nabla_X Y\rangle \\
&= \langle X, Y\rangle - \langle x, \langle S(X), Y\rangle N(x)\rangle \\
&= \langle X - S(X), Y\rangle\langle x, N(x)\rangle,
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the Euclidian inner product, ∇ is the induced connection on Ω, D is the Euclidean connection (or simply the differentiation) on $\mathbb{R}^{n+1}$, and $S(x)$ is the shape operator

$$S : T_x\Omega \to T_x\Omega, \quad X \mapsto \nabla_X N.$$

Then the eigenvalues of $\nabla^2\rho(x)$ are of the form $1 - \kappa_i(x)\langle x, N(x)\rangle$, $i = 1, \ldots, n$. Define

$$\alpha := \min_{\substack{i=1,\ldots,n \\ x\in\Omega}} (1 - \kappa_i\langle x, N(x)\rangle).$$

When $\alpha > 0$, then Ω with vector field F as above satisfies all three conditions A-C on page 10. In particular, following the proof of Theorem 1.9 *ii*, we get

$$\xi_{m+1} \leq \frac{\mu_k^2}{(2\alpha - n^{-1}\mathrm{vol}(\Omega)^{-1}\mu_k \int_\Omega |x|^2\, dv_g) \vee 0}.$$

In dimension two, $\alpha > 0$ is equivalent to $|\kappa_i|\langle x, N(x)\rangle < 1$. By results in [1], if $|\kappa_i|\langle x, N(x)\rangle < 1$ then $\langle x, N(x)\rangle \equiv 0$ on Ω, and Ω is the equilateral disk. Hence, there is no nontrivial 2-dimensional minimal surface satisfying condition A-C. It is an intriguing question whether there are non-trivial minimal hypersurfaces with $\alpha > 0$ in higher dimensions.

References

[1] L. Ambrozio and I. Nunes. A gap theorem for free boundary minimal surfaces in the three-ball. arXiv:1608.05689.

[2] P. H. Bérard. Spectral geometry: direct and inverse problems, volume 41 of Monografías de Matemática [Mathematical Monographs]. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 1986. With appendices by Gérard Besson, Bérard and Marcel Berger.

[3] G. P. Bessa and J. F Montenegro. Eigenvalue estimates for submanifolds with locally bounded mean curvature. Ann. Global Anal. Geom., 24(3):279–290, 2003.

[4] D. Bucur, A. Ferrero, and F. Gazzola. On the first eigenvalue of a fourth order Steklov problem. Calc. Var. Partial Differential Equations, 35(1):103–131, 2009.

[5] I. Chavel. Eigenvalues in Riemannian geometry, volume 115 of Pure and Applied Mathematics. Academic Press, Inc., Orlando, FL, 1984. Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk.

[6] B. Chow, P. Lu, and L. Ni. Hamilton's Ricci flow, volume 77 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI; Science Press Beijing, New York, 2006.

[7] A. Ferrero, F. Gazzola, and T. Weth. On a fourth order Steklov eigenvalue problem. Analysis (Munich), 25(4):315–332, 2005.

[8] A. Girouard and I. Polterovich. Spectral geometry of the Steklov problem (survey article). J. Spectr. Theory, 7(2):321–359, 2017.

[9] J. R. Kuttler. Bounds for Stekloff eigenvalues. SIAM J. Numer. Anal., 19(1):121–125, 1982.

[10] J. R. Kuttler and V. G. Sigillito. Inequalities for membrane and Stekloff eigenvalues. J. Math. Anal. Appl., 23:148–160, 1968.

[11] P. D. Lamberti and L. Provenzano. Viewing the Steklov eigenvalues of the Laplace operator as critical Neumann eigenvalues. In Current trends in analysis and its applications, Trends Math., pages 171–178. Birkhäuser/Springer, Cham, 2015.

[12] P. Li and S. T. Yau. Estimates of eigenvalues of a compact Riemannian manifold. In Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, pages 205–239. Amer. Math. Soc., Providence, R.I., 1980.

[13] G. Liu. Rellich type identities for eigenvalue problems and application to the Pompeiu problem. J. Math. Anal. Appl., 330(2):963–975, 2007.

[14] G. Liu. The Weyl-type asymptotic formula for biharmonic Steklov eigenvalues on Riemannian manifolds. Adv. Math., 228(4):2162–2217, 2011.

[15] G. Liu. On asymptotic properties of biharmonic Steklov eigenvalues. J. Differential Equations, 261(9):4729–4757, 2016.

[16] E. Mitidieri. A Rellich type identity and applications. Comm. Partial Differential Equations, 18(1-2):125–151, 1993.

[17] P. Petersen. Riemannian geometry, volume 171 of Graduate Texts in Mathematics. Springer, New York, second edition, 2006.

[18] L. Provenzano and J. Stubbe. Weyl–type bounds for Steklov eigenvalues. arXiv:1611.00929.

[19] S. Raulot and A. Savo. Sharp bounds for the first eigenvalue of a fourth-order Steklov problem. J. Geom. Anal., 25(3):1602–1619, 2015.

[20] F. Rellich. Darstellung der Eigenwerte von $\delta u + \lambda u = 0$ durch ein Randintegral. Math. Z., 46:635–637, 1940.

[21] Q. Wang and C. Xia. Sharp bounds for the first non-zero Stekloff eigenvalues. J. Funct. Anal., 257(8):2635–2644, 2009.

[22] C. Xiong. Comparison of Steklov eigenvalues on a domain and Laplacian eigenvalues on its boundary in Riemannian manifolds. arXiv:1704.02073.

Asma Hassannezhad: University of Bristol, School of Mathematics, University Walk, Bristol BS8 1TW, UK
E-mail address: asma.hassannezhad@bristol.ac.uk

Anna Siffert: Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: siffert@mpim-bonn.mpg.de