WITTEN MULTIPLE ZETA VALUES ATTACHED TO $\mathfrak{sl}(4)$

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Abstract. In this paper we shall prove that every Witten multiple zeta value of weight $w > 3$ attached to $\mathfrak{sl}(4)$ at nonnegative integer arguments is a finite $\mathbb{Q}$-linear combination of MZVs of weight w and depth three or less, except for the nine irregular cases where the Riemann zeta value $\zeta(w-2)$ and the double zeta values of weight $w-1$ and depth < 3 are also needed.

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1. Introduction

It is well-known that suitably defined zeta and L-functions and their special values often play significant roles in many areas of mathematics. In [22] Witten studied one variable zeta functions attached to various Lie algebras and related their special values to the volumes of certain moduli spaces of vector bundles of curves. Zagier [23] (and independently Garoufalidis) gave direct proofs that such functions at positive even integers are rational multiples of powers of π. More recently Matsumoto and his collaborators [12, 13, 14, 15, 17, 21] defined multiple variable versions of these functions and began to investigate their analytical and arithmetical properties.

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any $d \in \mathbb{N}$ we let $[d] = (1, \dots, d)$ be a poset with the usual increasing order. By $\mathbf{i} = (i_1, \dots, i_\ell) \subseteq [d]$ we mean a nonempty subset of $[d]$ as a poset. We say the length of $\mathbf{i}$ is $\lg(\mathbf{i}) = \ell$ and the weight of $\mathbf{i}$ is $\mathrm{wt}(\mathbf{i}) = i_1 + \cdots + i_\ell$. We define the *generalized multiple zeta function* of depth d as

$$\zeta_d\big((s_{\mathbf{i}})_{\mathbf{i}\subseteq[d]}\big) := \sum_{m_1,\dots,m_d=1}^{\infty} \prod_{\mathbf{i}\subseteq[d]} \Big(\sum_{j=1}^{\lg(\mathbf{i})} m_{i_j}\Big)^{-s_{\mathbf{i}}}. \tag{1}$$

For example, the Euler-Zagier multiple zeta function [1, 23, 24]

$$\zeta(s_1, \dots, s_d) := \sum_{m_1 > \cdots > m_d \geq 1} m_1^{-s_1} m_2^{-s_2} \cdots m_d^{-s_d} \tag{2}$$

corresponds to the special case that $s_{\mathbf{i}}=0$ unless $\mathbf{i}=(1,2,\dots,\ell)$ for $\ell=1,\dots,d$. The Mordell-Tornheim multiple zeta function defined by (10) (see [18, 20]) is the case where $s_{\mathbf{i}}=0$ unless $\lg(\mathbf{i})=1$ or $\lg(\mathbf{i})=d$. If we set $s_{\mathbf{i}}=0$ unless $\mathbf{i}=(a,a+1,\dots,b)$ is a consecutive string of positive integers then we get exactly Witten multiple zeta function associated to the special linear Lie algebra $\zeta_{\mathfrak{sl}(d+1)}$ (see [17]):

$$(3)\quad \zeta_{\mathfrak{sl}(d+1)}\big((s_{i,j})_{1\le i\le j\le d}\big):=\sum_{m_1,\dots,m_d=1}^{\infty}\prod_{1\le i\le j\le d}(m_i+m_{i+1}+\cdots+m_j)^{-s_{i,j}}.$$

The generalized multiple zeta-functions defined by (1) are special cases of the functions studied by Essouabri [6], de Crisenoy [5], and Matsumoto [16]. In particular we know that $\zeta_d(\mathbf{s})$ has meromorphic continuation to the whole complex space $\mathbb{C}^{2^d-1}$. However, in the form (1) we may have better control of its arithmetical properties, namely, we may be able to compute explicitly their special values at nonnegative integers. As usual we say a value of the Euler-Zagier multiple zeta function at **positive** integers a *multiple zeta value* (MZV for short) if it is finite. Our major interest is to solve the following problem.

Main Problem. Suppose $\mathbf{s}\in\mathbb{N}^{2^d-1}$ (resp. $\mathbf{s}\in\mathbb{N}_0^{2^d-1}$) and $\zeta_d(\mathbf{s})$ converges. Is $\zeta_d(\mathbf{s})$ always a $\mathbb{Q}$-linear combination of MZVs of the same weight (resp. same or lower weights) and of depth d or less?

When $d=2$ the function in (1) becomes Mordell-Tornheim double zeta function. By the main result of [25] (see Prop. 2.4) we know the above Main Problem has an affirmative answer for all Mordell-Tornheim multiple zeta functions. In this paper we will consider $\zeta_{\mathfrak{sl}(4)}(\mathbf{s})$ which is essentially the case when $d=3$.

Throughout the rest of the paper whenever we say some special value with positive (resp. nonnegative) integer arguments is expressible by MZVs we mean that the value can be expressed as a finite $\mathbb{Q}$-linear combination of MZVs of the same (resp. same or lower) weights and the same or lower depths. Our main result is the following theorem which provides an affirmative answer to the above Main Problem for the case $d=3$.

Theorem 1.1. *Suppose $\mathbf{s}\in\mathbb{N}^7$. If $\zeta_3(\mathbf{s})$ converges then it is expressible by MZVs. Moreover, every Witten multiple zeta value of weight w attached to $\mathfrak{sl}(4)$ at nonnegative integers is a $\mathbb{Q}$-linear combination of MZVs of weight w and depth three or less, except for the nine irregular cases defined by* (13) *to* (17) *where the Riemann zeta value* $\zeta(w-2)$ *and the double zeta values of weight* $w-1$ *and depth* ≤ 2 *are also needed.*

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2. Some preliminary results

In this section we collect some useful facts which will be convenient for us to present our main result in later sections.

2.1. **Convergence domain.** We assume that all components of $\mathbf{s}$ are integers and derive the necessary and sufficient conditions for (1) to converge although the conditions are still sufficient if we allow complex variables and take the corresponding real parts of the variables in the conditions.

Recall that the MZV in (2) converges if and only if

$$\sum_{j=1}^{\ell} s_j > \ell \tag{4}$$

for all $\ell = 1, \dots, d$. It is straight-forward to see the same holds for the "star" version of the multiple zeta function:

$$\zeta^*(s_1, \dots, s_d) := \sum_{m_1 \ge \cdots \ge m_d \ge 1} m_1^{-s_1} m_2^{-s_2} \cdots m_d^{-s_d}. \tag{5}$$

Special values of (5) were studied in [9] and [19]. We can extend this convergence criterion easily to our generalized multiple zeta functions.

Proposition 2.1. *The generalized MZV*

$$\zeta_d(\mathbf{s}) := \sum_{m_1,\dots,m_d=1}^{\infty} \prod_{\mathbf{i}\subseteq[d]} \Big(\sum_{j=1}^{\lg(\mathbf{i})} m_{i_j}\Big)^{-s_{\mathbf{i}}} \tag{6}$$

converges if and only if for all $\ell = 1, \dots, d$ *and all* $\mathbf{i} = (i_1, \dots, i_\ell) \subseteq [d]$

$$\sum_{\mathbf{j} \text{ contains at least one of } i_1,\dots,i_\ell} s_{\mathbf{j}} > \ell. \tag{7}$$

Proof. The idea of the proof is similar to that of [25, Thm. 4]. First we observe that for any subset $\mathbf{i} = (i_1, \dots, i_\ell) \subseteq [d]$ we have

$$\max\{m_{i_j} : 1 \le j \le \ell\} \le \sum_{j=1}^{\ell} m_{i_j} \le \ell \max\{m_{i_j} : 1 \le j \le \ell\}.$$

Hence

(8)

$$\sum_{\pi\in\mathfrak{S}_d}\sum_{m_{\pi(1)}\ge\cdots\ge m_{\pi(d)}\ge 1}\prod_{\mathbf{i}\subseteq[d]}(m_{\pi(i_1)})^{-s_{\pi(\mathbf{i})}}\le\sum_{m_1,\dots,m_d=1}^{\infty}\prod_{\mathbf{i}\subseteq[d]}\Big(\sum_{j=1}^{\lg(\mathbf{i})}m_{i_j}\Big)^{-s_{\mathbf{i}}}\le d\cdot\text{LHS},$$

where LHS means the quantity at the extreme left of the above inequalities. Observe that for each fixed $\pi\in\mathfrak{S}_d$ the power of $m_{\pi(j)}$ in LHS is $-\sum_{\mathbf{i}}s_{\pi(\mathbf{i})}$ where $\mathbf{i}$ runs through all subsets of $[d]$ whose first component is j. Hence the criterion (4) implies that LHS and d(LHS) of (8) converges if and only if for each fixed $\pi\in\mathfrak{S}_d$

$$\sum_{i_1=1}^{\ell}\sum_{\mathbf{i}\text{: first component is } i_1}s_{\pi(\mathbf{i})}>\ell,$$

for all $\ell=1,\dots,d$. Let $i_j=\pi^{-1}(j)$ $(j=1,\dots,\ell)$. Then in the above sum $\mathbf{i}$ runs through all subset of $[d]$ containing at least some i_j $(j=1,\dots,\ell)$. This is exactly (7), as desired. □

Remark 2.2. It is easy to see that similar result holds if we replace the factors $\sum m_{i_j}$ in (6) by linear forms of $m_1,\dots,m_d$ with nonnegative integer coefficients.

2.2. **MZVs with arbitrary integer arguments.** In this paper we will mostly consider multiple zeta values with nonnegative integer arguments as long as they converge. However, we will prove a more general result as follows since the inductive proof forces us to do so.

Proposition 2.3. *Suppose $s_1,\dots,s_d\in\mathbb{Z}$. If $\zeta(s_1,\dots,s_d)$ converges then it can be expressed as a $\mathbb{Q}$-linear combination of MZVs (at* ***positive*** *integer arguments) of the same or lower weights and the same or lower depths.*

Proof. We prove the proposition by induction on the depth. When $d=1$ we have nothing to prove. Suppose the proposition holds for all MZVs of depth $d-1$. Suppose further

$$s_1+\cdots+s_j>j\quad\text{for all } j=1,\dots,d, \tag{9}$$

so that $\zeta(s_1,\dots,s_d)$ converges. Assume $-t=s_j\le 0$. Then by definition

$$\zeta(s_1,\dots,s_d)=\sum_{m_1>\cdots>m_{j-1}>m_{j+1}>\cdots>m_d}\frac{\Big(\sum_{m_j=1}^{m_{j-1}-1}-\sum_{m_j=1}^{m_{j+1}}\Big)m_j^t}{m_1^{s_1}\cdots m_{j-1}^{s_{j-1}}m_{j+1}^{s_{j+1}}\cdots m_d^{s_d}}.$$

Now by the well-known formula (see, for e.g., [11, p. 230, Thm. 1])

$$\sum_{m=1}^{n} m^t = \frac{1}{t+1}\Big(B_{t+1}(n+1) - B_{t+1}(0)\Big)$$

where $B_{t+1}(x)$ is the Bernoulli polynomial we immediately see that $\zeta(s_1,\dots,s_d)$ is a $\mathbb{Q}$-linear combination of MZVs of the forms $\zeta(s_1,\dots,s_{j-1}-s, s_{j+1},\dots,s_d)$ and $\zeta(s_1,\dots,s_{j-1}-s, s_{j+1},\dots,s_d)$ where $s=0,1,\dots,-s_j+1$. All of these MZVs are easily to be shown as convergent values by (9) and therefore the proposition follows from the induction assumption. □

2.3. **Mordell-Tornheim zeta functions.** They are defined by (see [18, 20])

$$\zeta_{\mathrm{MT}}(s_1,\dots,s_d;s) := \sum_{m_1,\dots,m_d=1}^{\infty} m_1^{-s_1} m_2^{-s_2}\cdots m_d^{-s_d}(m_1+\cdots+m_d)^{-s}. \tag{10}$$

The main result of [25] is the following

Proposition 2.4. ([25, Thm. 5]) *Let $s_1,\dots,s_d$ and s be nonnegative integers. If at most one of them is equal to 0 then the Mordell-Tornheim zeta value $\zeta_{\mathrm{MT}}(s_1,\dots,s_d;s)$ can be expressed as a $\mathbb{Q}$-linear combination of MZVs of the same weight and depth.*

In this paper we will only need this proposition when the depth is three.

2.4. **A combinatorial lemma.** The next lemma will be used heavily throughout the paper.

Lemma 2.5. ([25, Lemma 1]) *Let r and $n_1,\dots,n_r$ be positive integers, and let $x_1,\dots,x_r$ be non-zero real numbers such that $x_1+\cdots+x_r\neq 0$. Then*

$$\prod_{j=1}^{r}\frac{1}{x_j^{n_j}} = \sum_{j=1}^{r}\left(\prod_{\substack{k=1\\k\neq j}}^{r}\sum_{a_k=0}^{n_k-1}\right)\frac{M_j}{x^{n_j+A_j}}\prod_{\substack{k=1\\k\neq j}}^{r}\frac{1}{x_k^{n_k-a_k}},$$

where the multi-nomial coefficient

$$M_j = \frac{(n_j+A_j-1)!}{(n_j-1)!}\prod_{\substack{k=1\\k\neq j}}^{r}\frac{1}{a_k!} \qquad \textit{and} \qquad A_j=\sum_{\substack{k=1\\k\neq j}}^{r} a_k.$$

The notation $\displaystyle\prod_{\substack{k=1\\k\neq j}}^{r}\sum_{a_k=0}^{n_k-1}$ *means the multiple sum* $\displaystyle\sum_{a_1=0}^{n_1-1}\cdots\sum_{a_{j-1}=0}^{n_{j-1}-1}\sum_{a_{j+1}=0}^{n_{j+1}-1}\cdots\sum_{a_r=0}^{n_r-1}$.

3. Proof of Theorem 1.1

By definition (1)

(11)
$$\zeta_3(s_1,\dots,s_7) := \sum_{m_1,\dots,m_3=1}^{\infty} \frac{(m_1+m_2+m_3)^{-s_7}}{m_1^{s_1}m_2^{s_2}m_3^{s_3}(m_1+m_2)^{s_4}(m_2+m_3)^{s_5}(m_1+m_3)^{s_6}}.$$

To guarantee convergence by (7) we need to assume:

(12)
$$\begin{cases} \mathbf{i}=(1): & s_1+s_4+s_6+s_7>1,\\ \mathbf{i}=(1): & s_2+s_4+s_5+s_7>1,\\ \mathbf{i}=(1): & s_3+s_5+s_6+s_7>1,\\ \mathbf{i}=(1,2): & s_1+s_2+s_4+s_5+s_6+s_7>2,\\ \mathbf{i}=(1,3): & s_1+s_3+s_4+s_5+s_6+s_7>2,\\ \mathbf{i}=(2,3): & s_2+s_3+s_4+s_5+s_6+s_7>2,\\ \mathbf{i}=(1,2,3): & s_1+s_2+s_3+s_4+s_5+s_6+s_7>3. \end{cases}$$

We now use a series of reductions to prove the theorem. All of the steps will be explicitly given so that one may carry out the computation of (11) by following them.

Step (i). Since $2(m_1+m_2+m_3)=(m_1+m_2)+(m_2+m_3)+(m_1+m_3)$ we see clearly that we can assume either $s_4=0$ or $s_5=0$ or $s_6=0$. In fact in Lemma 2.5 taking $r=3$, $x_1=m_1+m_2$, $x_2=m_2+m_3$, $x_3=m_1+m_3$, $n_1=s_4$, $n_2=s_5$ and $n_3=s_6$ we get (with $\mathbf{s}=(s_1,s_2,s_3)$ fixed):

$$\begin{aligned} &\zeta_3(\mathbf{s},s_4,s_5,s_6,s_7)\\ =&\sum_{a_4=0}^{s_4-1}\sum_{a_5=0}^{s_5-1}\frac{(s_6+a_4+a_5-1)!\zeta_3(\mathbf{s},s_4-a_4,s_5-a_5,0,s_7+s_6+a_4+a_5)}{2^{s_7+a_4+a_5}(s_6-1)!a_4!a_5!}\\ +&\sum_{a_4=0}^{s_4-1}\sum_{a_6=0}^{s_6-1}\frac{(s_5+a_4+a_6-1)!\zeta_3(\mathbf{s},s_4-a_4,0,s_6-a_6,s_7+s_5+a_4+a_6)}{2^{s_7+a_4+a_6}(s_5-1)!a_4!a_6!}\\ +&\sum_{a_5=0}^{s_5-1}\sum_{a_6=0}^{s_6-1}\frac{(s_4+a_5+a_6-1)!\zeta_3(\mathbf{s},0,s_5-a_5,s_6-a_6,s_7+s_4+a_5+a_6)}{2^{s_7+a_5+a_6}(s_4-1)!a_5!a_6!}. \end{aligned}$$

We recommend the interested reader to check the convergence of the above values by (12). The rule of thumb is that if we apply Lemma 2.5 with each x_j a positive combination of indices then the convergence is automatically guaranteed. In each of the following steps we often omit this convergence checking since it is straight-forward in most cases. The only exception is (26) which in fact poses the most difficulty.

By symmetry we only need to show that $\zeta_3(s_1,\dots,s_5,0,s_6)$ is expressible by MZVs. This is nothing but the Matsumoto's version of Witten multiple zeta function of depth 3 associated to the special linear Lie algebra $\mathfrak{sl}(4)$ (see (3)):

$$\zeta_{\mathfrak{sl}(4)}(s_1,\dots,s_6) := \sum_{m_1,\dots,m_3=1}^{\infty} \frac{(m_1+m_2+m_3)^{-s_6}}{m_1^{s_1}m_2^{s_2}m_3^{s_3}(m_1+m_2)^{s_4}(m_2+m_3)^{s_5}}.$$

Before going on we need to define the so called *regular* and *irregular* special values of $\zeta_{\mathfrak{sl}(4)}(s_1,\dots,s_6)$. Clearly the following special values are expressible by MZVs of mixed weights: ($\{0\}_k$ means to repeat 0 k times)

(13)

$$\begin{aligned}&\zeta_{\mathfrak{sl}(4)}(\{0\}_3,b,0,a)=\zeta_{\mathfrak{sl}(4)}(\{0\}_4,b,a)\\ =&\zeta(a,b,0)=\begin{cases}\zeta(a,b-1)-\zeta(a,b), & a>1,b>1,\\ \zeta(a-1)-\zeta(a)-\zeta(a,1), & a>1,b=1,\end{cases}\end{aligned}$$

(14)

$$\begin{aligned}&\zeta_{\mathfrak{sl}(4)}(b,\{0\}_4,a)=\zeta_{\mathfrak{sl}(4)}(0,b,\{0\}_3,a)=\zeta_{\mathfrak{sl}(4)}(0,0,b,0,0,a)\\ =&\zeta(a,0,b)=\begin{cases}\zeta(a-1,b)-\zeta(a,b-1)-\zeta(a,b), & a>2,b>1,\\ \zeta(a-1,1)-\zeta(a-1)+\zeta(a)-\zeta(a,1), & a>2,b=1,\end{cases}\end{aligned}$$

$$\zeta_{\mathfrak{sl}(4)}(\{0\}_5,a)=\zeta(a,0,0)=\frac{1}{2}\zeta(a-2)-\frac{3}{2}\zeta(a-1)+\zeta(a), \quad a>3. \tag{15}$$

By Step (ii) we will see that special values

$$\zeta_{\mathfrak{sl}(4)}(0,0,s_3,s_4,0,s_6) \quad \text{and} \quad \zeta_{\mathfrak{sl}(4)}(s_1,0,0,0,s_5,s_6) \tag{16}$$

are also expressible by MZVs of mixed weights. Further, if $a,b\ge 2$ then we have

(17)

$$\begin{aligned}\zeta_{\mathfrak{sl}(4)}(0,0,0,a,b,0)=&\sum_{m_1,m_2,m_3=1}^{\infty}\frac{1}{(m_1+m_2)^a(m_2+m_3)^b}\\ =&\sum_{m_2=1}^{\infty}\left(\sum_{m_1<m_3}+\sum_{m_1=m_3}+\sum_{m_1>m_3}\right)\frac{1}{(m_1+m_2)^a(m_2+m_3)^b}\\ =&\zeta(a,b,0)+\zeta(b,a,0)+\zeta(a+b,0)\\ =&\zeta(b,a-1)+\zeta(a,b-1)+\zeta(a+b-1)-\zeta(a)\zeta(b).\end{aligned}$$

We call the values appearing in the nine cases from (13) to (17) *irregular* values. Otherwise $\zeta_{\mathfrak{sl}(4)}(s_1,\dots,s_6)$ is called a *regular* value.

Step (ii). Taking $x_1 = m_1$ and $x_2 = m_2 + m_3$ in Lemma 2.5 we get

(18)
$$\zeta_{\mathfrak{sl}(4)}(s_1, \dots, s_6) = \sum_{a_5=0}^{s_5-1} \binom{s_1 + a_5 - 1}{a_5} \zeta_{\mathfrak{sl}(4)}(0, s_2, s_3, s_4, s_5 - a_5, s_6 + s_1 + a_5)$$
$$+ \sum_{a_1=0}^{s_1-1} \binom{s_5 + a_1 - 1}{a_1} \zeta_{\mathfrak{sl}(4)}(s_1 - a_1, s_2, s_3, s_4, 0, s_6 + s_5 + a_1).$$

We see clearly that we can assume either (ii.1): $s_5 = 0$ or (ii.2): $s_1 = 0$ and $s_5 \geq 1$. Moreover, by taking $s_2 = s_3 = s_4 = 0$ in (18) we see that $\zeta_{\mathfrak{sl}(4)}(s_1, 0, 0, 0, s_5, s_6)$ in (16) can be expressed by $\mathbb{Q}$-linear combinations of MZVs appeared in the (13) and (14). The argument is similar for $\zeta_{\mathfrak{sl}(4)}(0, 0, s_3, s_4, 0, s_6)$ in (16). Therefore in what follows we assume that $\zeta_{\mathfrak{sl}(4)}(s_1, \dots, s_6)$ are always regular and show that they can be expressed by $\mathbb{Q}$-linear combinations of MZVs of the same weight and depth three or less.

Step (ii.1). Let $s_5 = 0$. Then we must have either $s_1 \geq 1$ or $s_2 \geq 1$ since we assume

$$\zeta_{\mathfrak{sl}(4)}(s_1, \dots, s_4, 0, s_6) = \sum_{m_1, \dots, m_3=1}^{\infty} \frac{1}{m_1^{s_1} m_2^{s_2} m_3^{s_3} (m_1 + m_2)^{s_4} (m_1 + m_2 + m_3)^{s_6}}$$

is regular. Then we may use $x_1 = m_1$ and $x_2 = m_2$ in Lemma 2.5 to get

$$\begin{aligned}
&\zeta_{\mathfrak{sl}(4)}(s_1, \dots, s_4, 0, s_6) \\
=& \sum_{a_1=0}^{s_1-1} \binom{s_2 + a_1 - 1}{a_1} \zeta_{\mathfrak{sl}(4)}(s_1 - a_1, 0, s_3, s_2 + s_4 + a_1, 0, s_6) \\
+& \sum_{a_2=0}^{s_2-1} \binom{s_1 + a_2 - 1}{a_2} \zeta_{\mathfrak{sl}(4)}(0, s_2 - a_2, s_3, s_1 + s_4 + a_2, 0, s_6).
\end{aligned}$$

By symmetry we only need to consider

(19)
$$\zeta_{\mathfrak{sl}(4)}(0, s_2, s_3, s_4, 0, s_6) = \sum_{m_1, \dots, m_3=1}^{\infty} \frac{(m_1 + m_2 + m_3)^{-s_6}}{m_2^{s_2} m_3^{s_3} (m_1 + m_2)^{s_4}}$$

where $s_2 \geq 1$. But now we may take $x_1 = m_1 + m_2$ and $x_2 = m_3$ in Lemma 2.5 to reduce it to either $s_3 = 0, s_4 \geq 1$ or $s_4 = 0, s_3 \geq 1$. Then we get the following two kind of values:

$$\zeta_{\mathfrak{sl}(4)}(0, s_2, 0, s_4, 0, s_6) = \zeta(s_6, s_4, s_2), \ \zeta_{\mathfrak{sl}(4)}(0, s_2, s_3, 0, 0, s_6) = \zeta_{\mathrm{MT}}(s_2, s_4, 0; s_6),$$

where $s_2, s_3, s_4, s_6 \geq 1$. But $\zeta_{\mathrm{MT}}(s_2, s_4, 0; s_6)$ is expressible by MZVs of the same weight and of depth three by Prop. 2.4. Case (ii.1) is proved.

Step (ii.2). Let $s_1 = 0$ and $s_5 \geq 1$. Consider

$$\zeta_{\mathfrak{sl}(4)}(0, s_2, \dots, s_6) = \sum_{m_1,\dots,m_3=1}^{\infty} \frac{(m_1+m_2+m_3)^{-s_6}}{m_2^{s_2} m_3^{s_3} (m_1+m_2)^{s_4} (m_2+m_3)^{s_5}}.$$

To guarantee convergence we must have $s_4 + s_6 > 1$, $s_3 + s_5 + s_6 > 1$, $s_2 + s_4 + s_5 + s_6 > 2$, $s_3 + s_4 + s_5 + s_6 > 2$, and $s_2 + s_3 + s_4 + s_5 + s_6 > 3$. Taking $x_1 = m_1 + m_2$ and $x_2 = m_3$ in Lemma 2.5 we get

$$\zeta_{\mathfrak{sl}(4)}(0, s_2, \dots, s_6)$$

$$= \sum_{a_3=0}^{s_3-1} \binom{s_4 + a_3 - 1}{a_3} \zeta_{\mathfrak{sl}(4)}(0, s_2, s_3 - a_3, 0, s_5, s_4 + s_6 + a_3) \tag{20}$$

$$+ \sum_{a_4=0}^{s_4-1} \binom{s_3 + a_4 - 1}{a_4} \zeta_{\mathfrak{sl}(4)}(0, s_2, 0, s_4 - a_4, s_5, s_3 + s_6 + a_4). \tag{21}$$

So we may assume that either (ii.2.1): $s_1 = s_4 = 0, s_5, s_6 \geq 1$ or (ii.2.2): $s_1 = s_3 = 0$ and $s_2, s_4, s_5 \geq 1$, or (ii.2.3): $s_1 = s_2 = s_3 = 0$ and $s_4, s_5 \geq 1$. Here (ii.2.1) comes from (20) while (ii.2.2) and (ii.2.3) come from (21).

Step (ii.2.1). Let $s_1 = s_4 = 0, s_5, s_6 \geq 1$. Then we must have either $s_2 \geq 1$ or $s_3 \geq 1$ in

$$\zeta_{\mathfrak{sl}(4)}(0, s_2, s_3, 0, s_5, s_6) = \sum_{m_1,\dots,m_3=1}^{\infty} \frac{1}{m_2^{s_2} m_3^{s_3} (m_2+m_3)^{s_5} (m_1+m_2+m_3)^{s_6}}.$$

since we consider only regular values only. Thus we may put $x_1 = m_2$ and $x_2 = m_3$ in Lemma 2.5 to get

$$\begin{aligned}
&\zeta_{\mathfrak{sl}(4)}(0, s_2, s_3, 0, s_5, s_6)\\
&= \sum_{a_2=0}^{s_2-1} \binom{s_3 + a_2 - 1}{a_2} \zeta_{\mathfrak{sl}(4)}(0, s_2 - a_2, 0, 0, s_3 + s_5 + a_2, s_6)\\
&+ \sum_{a_3=0}^{s_3-1} \binom{s_2 + a_3 - 1}{a_3} \zeta_{\mathfrak{sl}(4)}(0, 0, s_3 - a_3, 0, s_2 + s_5 + a_3, s_6)\\
&= \sum_{a_2=0}^{s_2-1} \binom{s_3 + a_2 - 1}{a_2} \zeta(s_6, s_3 + s_5 + a_2, s_2 - a_2)\\
&+ \sum_{a_3=0}^{s_3-1} \binom{s_2 + a_3 - 1}{a_3} \zeta(s_6, s_2 + s_5 + a_3, s_3 - a_3).
\end{aligned}$$

It is easy to see that all the triple zeta values above have the same weight. We remind the reader that to determine the weight of a MZV it is not enough just to add up all the components. One also needs to check that every component is positive.

Step (ii.2.2). Let $s_1 = s_3 = 0$ and $s_2, s_4, s_5 \geq 1$. By (12), to guarantee convergence of

$$\zeta_{\mathfrak{sl}(4)}(0, s_2, 0, s_4, s_5, s_6) = \sum_{m_1,\dots,m_3=1}^{\infty} \frac{(-1)^{s_2}(m_1+m_2+m_3)^{-s_6}}{(-m_2)^{s_2}(m_1+m_2)^{s_4}(m_2+m_3)^{s_5}}$$

we need to assume

$$s_4+s_6>1,\ \ s_5+s_6>1,\ \ s_4+s_5+s_6>2,\ \ s_2+s_4+s_5+s_6>3. \tag{22}$$

Moreover, since $\zeta_{\mathfrak{sl}(4)}(0, s_2, 0, s_4, s_5, s_6)$ is regular we must have $s_2 + s_6 \geq 1$. Putting $x_1 = -m_2$, $x_2 = m_1 + m_2$ and $x_3 = m_2 + m_3$ in Lemma 2.5 we get

(23)

$$\begin{aligned}
&\zeta_{\mathfrak{sl}(4)}(0, s_2, 0, s_4, s_5, s_6)\\
=&\sum_{a_4=0}^{s_4-1}\sum_{a_5=0}^{s_5-1} \frac{(s_2+a_4+a_5-1)!}{(-1)^{s_2}(s_2-1)!a_4!a_5!}\zeta_{\mathfrak{sl}(4)}(0,0,0,s_4-a_4,s_5-a_5,s_6+s_2+a_4+a_5)\\
+&\sum_{a_2=0}^{s_2-1}\sum_{a_5=0}^{s_5-1} \frac{(s_4+a_2+a_5-1)!}{(-1)^{a_2}(s_4-1)!a_2!a_5!}\zeta_{\mathfrak{sl}(4)}(0,s_2-a_2,0,0,s_5-a_5,s_6+s_4+a_2+a_5)\\
+&\sum_{a_2=0}^{s_2-1}\sum_{a_4=0}^{s_4-1} \frac{(s_5+a_2+a_4-1)!}{(-1)^{a_2}(s_5-1)!a_2!a_4!}\zeta_{\mathfrak{sl}(4)}(0,s_2-a_2,0,s_4-a_4,0,s_6+s_5+a_2+a_4),
\end{aligned}$$

We point out that since we have used $x_1 = -m_2$ we need to check the convergence of all the above three kinds of values under the assumption (22) even though the checking itself is trivial.

Returning to the reduction of (23) we see that the last two sums are expressible by MZVs so we only need to consider those values appearing in the first sum, namely, those of the form in the next case.

Step (ii.2.3). Let $s_1 = s_2 = s_3 = 0$ and $s_4, s_5 \geq 1$. Then we must have $s_6 \geq 1$ since we only consider regular values. To guarantee convergence of

$$\zeta_{\mathfrak{sl}(4)}(0, 0, 0, s_4, s_5, s_6) = \sum_{m_1,\dots,m_3=1}^{\infty} \frac{(m_1+m_2+m_3)^{-s_6}}{(m_1+m_2)^{s_4}(m_2+m_3)^{s_5}} \tag{24}$$

we need to assume $s_4 + s_6 > 1$, $s_5 + s_6 > 1$, and $s_4 + s_5 + s_6 > 3$. If either $s_4 > 1$ or $s_5 > 1$ then we may further assume that $s_4 > 1$ by change of index $m_1 \leftrightarrow m_3$ in (24). We may set $x_1 = -(m_2 + m_3)$ and $x_2 = m_1 + m_2 + m_3$

in Lemma 2.5 to get

$$\begin{aligned}
&\zeta_{\mathfrak{sl}(4)}(0,0,0,s_4,s_5,s_6)\\
&= \sum_{a_5=0}^{s_5-2} \binom{s_6+a_5-1}{a_5} (-1)^{a_5} \zeta_{\mathfrak{sl}(4)}(s_6+a_5,0,0,s_4,s_5-a_5,0) \qquad (25)\\
&+ \sum_{a_6=0}^{s_6-2} \binom{s_5+a_6-1}{a_6} (-1)^{s_5} \zeta_{\mathfrak{sl}(4)}(s_5+a_6,0,0,s_4,0,s_6-a_6)\\
&+ \binom{s_6+s_5-2}{s_5-1} (-1)^{s_5} \lim_{N\to\infty} \Big(\zeta^{(N)}_{\mathfrak{sl}(4)}(s_6+s_5-1,0,0,s_4,0,1) \qquad (26)\\
&\qquad - \zeta^{(N)}_{\mathfrak{sl}(4)}(s_6+s_5-1,0,0,s_4,1,0) \Big),
\end{aligned}$$

where $\zeta^{(N)}$ is the partial sum of (1) when each index m_{i_j} goes from 1 to N. Observe that all the values in the first two lines above are convergent and the second line is already expressible by MZVs of the same weight. So we are reduced to consider the following two kinds of values:

(A). $\zeta_{\mathfrak{sl}(4)}(s_1,0,0,s_4,t,0)$ (with $s_1, s_4 \ge 1$, $t > 1$) from first line above and
(B). The limit (26) (with $s := s_5 + s_6 - 1 \ge 1$).

We now need to divide into two subcases to compute (25) (leading to (A)) and (26) (leading to (B)): (ii.2.3.1) $s_1 = s_2 = s_3 = 0$, either $s_4 > 1$ or $s_5 > 1$, and (ii.2.3.2) $s_1 = s_2 = s_3 = 0$ and $s_4 = s_5 = 1$. Before treating the two cases separately we first deform the expression inside the limit (26) as follows:

$$\begin{aligned}
&\Big(\zeta^{(N)}_{\mathfrak{sl}(4)}(s,0,0,s_4,0,1) - \zeta^{(N)}_{\mathfrak{sl}(4)}(s,0,0,s_4,1,0) \Big)\\
&= \sum_{m_1,m_2=1}^{N} \left(\sum_{n=1}^{m_1+m_2+N} - \sum_{n=1}^{m_1+m_2} \right) \frac{1}{m_1^s (m_1+m_2)^{s_4} n}\\
&\quad - \sum_{m_1,m_2=1}^{N} \left(\sum_{n=1}^{m_2+N} - \sum_{n=1}^{m_2} \right) \frac{1}{m_1^s (m_1+m_2)^{s_4} n}\\
&= - \sum_{m_1,m_2=1}^{N} \sum_{n=1+m_2}^{m_1+m_2} \frac{1}{m_1^s (m_1+m_2)^{s_4} n} \qquad (27)\\
&\quad + \sum_{m_1,m_2=1}^{N} \sum_{n=1+m_2+N}^{m_1+m_2+N} \frac{1}{m_1^s (m_1+m_2)^{s_4} n}.
\end{aligned}$$

Step (ii.2.3.1). Let $s_1 = s_2 = s_3 = 0$, $s_6 \ge 1$, either $s_4 > 1$ or $s_5 > 1$. Without loss of generality assume $s_4 > 1$ as we have remarked after (24).

(A). Since $s_1 \geq 1, s_4, t > 1$ we have

$$\begin{aligned}
&\zeta_{\mathfrak{sl}(4)}(s_1, 0, 0, s_4, t, 0)\\
=&\sum_{m_1,m_2,m_3=1}^{\infty} \frac{1}{m_1^{s_1}(m_1+m_2)^{s_4}(m_2+m_3)^t}\\
=&\sum_{m_1,m_2=1}^{\infty} \sum_{n>m_2} \frac{1}{m_1^{s_1}(m_1+m_2)^{s_4} n^t}\\
=&\zeta(t)\zeta(s_4,s_1) - \zeta_{\mathrm{MT}}(s_1,t;s_4) - \sum_{m_1,m_2=1}^{\infty} \sum_{n<m_2} \frac{1}{m_1^{s_1}(m_1+m_2)^{s_4} n^t}\\
=&\zeta(t)\zeta(s_4,s_1) - \zeta_{\mathrm{MT}}(s_1,t;s_4) - \sum_{m_1,n,m_3=1}^{\infty} \frac{1}{m_1^{s_1}(m_1+n+m_3)^{s_4} n^t}
\end{aligned}$$

by setting $m_2 = n + m_3$. But the last sum is just $\zeta_{\mathrm{MT}}(s_1, t, 0; s_4)$ which is expressible by MZVs of the same weight and of depth three by Prop. 2.4.

(B). The limit (26). This has been reduced to (27) with $s \geq 1$ and $s_4 \geq 2$. Then the second sum in (27) is bounded by

$$\frac{1}{1+N} \sum_{m_1,m_2=1}^{N} \frac{1}{m_1 m_2} = O(\log^2 N/N) \to 0, \quad \text{as } N \to \infty.$$

By setting $m_2 = m_3 + n$ in the first sum in (27) we get

$$\begin{aligned}
&\lim_{N\to\infty} \Big(\zeta^{(N)}_{\mathfrak{sl}(4)}(s,0,0,s_4,0,1) - \zeta^{(N)}_{\mathfrak{sl}(4)}(s,0,0,s_4,1,0) \Big)\\
=&-\sum_{m_1}^{\infty}\sum_{n=1}^{\infty} \left(\sum_{n=1}^{m_1+m_2} - \sum_{n=1}^{m_2} \right) \frac{1}{m_1^s (m_1+m_2)^{s_4} n}\\
=&-\sum_{m_1,m_2=1}^{\infty} \left(\sum_{n=1}^{m_1-1} + \sum_{m_1\leq n\leq m_1+m_2} - \sum_{n=1}^{m_2} \right) \frac{1}{m_1^s (m_1+m_2)^{s_4} n} \qquad (28)\\
=&\zeta_{\mathrm{MT}}(s,1;s_4) - \big(\zeta(s_4,s,1) + \zeta(s_4,s+1) + \zeta(s_4,1,s) + \zeta(s_4+1,s)\big)\\
&+\sum_{m_1,m_2=1}^{\infty} \sum_{n=1}^{m_2-1} \frac{m_1^{-s} n^{-1}}{(m_1+m_2)^{s_4}}.
\end{aligned}$$

In the last sum setting $m_2 = n + m_3$ then we get

$$\sum_{m_1,m_2=1}^{\infty} \sum_{n=1}^{m_2-1} \frac{(m_1+m_2)^{-s_4}}{m_1^s n} = \sum_{n,m_1,m_3=1}^{\infty} \frac{(m_1+m_3+n)^{-s_4}}{m_1^s n} = \zeta_{\mathrm{MT}}(s,1,0;s_4)$$

which is expressible by MZVs of the same weight and of depth three by Prop. 2.4.

Step (ii.2.3.2). Let $s_1 = s_2 = s_3 = 0$ and $s_4 = s_5 = 1$. Then we must have $s_6 \geq 2$ in order to have convergent values. Moreover, (25) is vacuous so we don't need to consider (A).

(B). Observe that the second term in (27) is bounded by

$$\frac{1}{1+N} \sum_{m_1,m_2=1}^{N} \frac{1}{m_1^{s-1} m_2} = O(\log^2(N)/N) \to 0, \quad \text{as } N \to \infty$$

since $s \geq 2$. So

$$\begin{aligned} &\lim_{N\to\infty} \Big(\zeta^{(N)}_{\mathfrak{sl}(4)}(s,0,0,1,0,1) - \zeta^{(N)}_{\mathfrak{sl}(4)}(s,0,0,1,1,0) \Big) \\ &= - \lim_{N\to\infty} \sum_{m_1,m_2=1}^{N} \sum_{n=1+m_2}^{m_1+m_2} \frac{1}{m_1^s (m_1+m_2) n} \end{aligned}$$

is expressible by MVZs by Lemma 4.2 (with $t = 1$) in §4.

This finishes the proof of the theorem. □

4. Regularized MZVs and a technical lemma

We need to derive several formulas concerning MVZs. They are best understood as consequences of the stuffle (also called quasi-shuffle or harmonic product) relations for the MZVs including the divergent ones suitably regularized.

By [10] one may define a regularized value $\bar{\zeta}(\mathbf{s})$ (denoted by $Z^*_{\mathbf{s}}(T)$ in [10]) if $\mathbf{s} = (s_1, \dots, s_d)$ with $s_1 = 1$ such that the following stuffle relation holds:

$$\bar{\zeta}(\mathbf{s}_1)\bar{\zeta}(\mathbf{s}_2) = \bar{\zeta}(\mathbf{s}_1 * \mathbf{s}_2) \tag{29}$$

where the the product $*$ is the stuffle product (see [9] or [10, §1]).

Lemma 4.1. *Let s and t be two positive integers greater than 1. Then*
(30)

$$\zeta(s)\zeta(t) = \sum_{a=0}^{s} \binom{t-1+a}{a} \zeta(t+a, s-a) + \sum_{b=0}^{t} \binom{s-1+b}{b} \zeta(s+b, t-b).$$

Further

$$\zeta(s+1) = \sum_{a=1}^{s-1} \zeta(1+a, s-a). \tag{31}$$

Proof. Equation (30) is the famous Euler's decomposition formula [7] (or see [3]). To derive (31) we use the regularized MZVs as follows. For any positive integer N we have by Lemma 2.5

$$\sum_{m=1}^{N} \sum_{n=1}^{N-m} \frac{1}{mn^s} = \sum_{m=1}^{N} \sum_{n=1}^{N-m} \left(\frac{1}{m(m+n)^s} + \sum_{a=0}^{s-1} \frac{1}{n^{s-a}(m+n)^{a+1}} \right).$$

Taking the regularization we get

$$\bar{\zeta}(1)\zeta(s) = \bar{\zeta}(1,s) + \zeta(s,1) + \sum_{a=1}^{s-1} \zeta(1+a, s-a).$$

On the other hand by stuffle (29)

$$\bar{\zeta}(1)\zeta(s) = \bar{\zeta}(1,s) + \zeta(s,1) + \zeta(s+1).$$

Equation (31) follows immediately. $\square$

Lemma 4.2. *For all positive integers s and t such that $s+t \geq 3$ the value*

$$\sum_{m_1,m_2=1}^{\infty} \sum_{n=1+m_2}^{m_1+m_2} \frac{1}{m_1^s(m_1+m_2)n^t} \tag{32}$$

is expressible by MVZs of weight $s+t+1$ and depth three or less.

Proof. Let's consider the partial sum

$$\sum_{1\leq m_1<m_3\leq N} \sum_{n=1+m_3-m_1}^{m_3} \frac{1}{m_1^s m_3 n^t} = \sum_{1\leq m_1<m_3\leq N} \left(\sum_{n=1}^{m_3} - \sum_{n=1}^{m_3-m_1} \right) \frac{1}{m_1^s m_3^t n}$$

whose limit when $N \to \infty$ is clearly (32). Using regularization and playing similar trick of breaking the sum as we did in (28) we find that

$$\begin{aligned}
(32) = \mathrm{Reg}&\left\{ \sum_{1\leq m_1<m_3\leq N} \left(\sum_{n=1}^{m_3} - \sum_{n=1}^{m_3-m_1} \right) \frac{1}{m_1^s m_3^t n} \right\},\\
=&\bar{\zeta}(1,t,s) + \bar{\zeta}(1,s+t) + \bar{\zeta}(1,s,t) + \zeta(t+1,s)\\
&- \mathrm{Reg}\left\{ \sum_{1\leq m_1<m_3\leq N} \sum_{n=1}^{m_3-m_1} \frac{1}{m_1^s m_3^t n} \right\}.
\end{aligned} \tag{33}$$

Now the untreated sum can be deformed as follows: taking $m_2 = m_3 - m_1$ we get

(34)

$$\begin{aligned}
\sum_{1\leq m_1<m_1+m_2\leq N} \sum_{n=1}^{m_2-1} \frac{1}{m_1^s(m_1+m_2)n^t} + \sum_{1\leq m_1<m_1+m_2\leq N} \frac{1}{m_1^s m_2^t (m_1+m_2)}\\
= \sum_{m_1=1}^{N} \sum_{n=1}^{N-m_1-1} \sum_{n_3=1}^{N-m_1-n} \frac{1}{m_1^s n^t (m_1+n_3+n)} + \zeta_{\mathrm{MT}}^{(N)}(s,t;1)
\end{aligned}$$

by setting $n_3 = m_2 - n$. Taking $x_1 = m_1$ and $x_2 = n$ in Lemma 2.5 we see that (34) becomes

$$\sum_{m_1=1}^{N} \sum_{n=1}^{N-m_1-1} \sum_{n_3=1}^{N-m_1-n} \left\{ \sum_{a=0}^{s-1} \binom{t+a-1}{a} \frac{1}{m_1^{s-a}(m_1+n)^{t+a}(m_1+n_3+n)} + \sum_{b=0}^{t-1} \binom{s+b-1}{b} \frac{1}{n^{t-b}(m_1+n)^{s+b}(m_1+n_3+n)} \right\} + \zeta_{\mathrm{MT}}^{(N)}(s,t;1).$$

Taking regularization and combining with (33) yields

$$\begin{aligned}
(32) &= \bar\zeta(1,t,s) + \bar\zeta(1,s+t) + \bar\zeta(1,s,t) + \zeta(t+1,s) - \zeta_{\mathrm{MT}}(s,t;1) \\
&\quad - \left\{ \sum_{a=0}^{s-1} \binom{t+a-1}{a} \bar\zeta(1,t+a,s-a) + \sum_{b=0}^{t-1} \binom{s+b-1}{b} \bar\zeta(1,s+b,t-b) \right\} \\
&= \bar\zeta(1,s+t) + \zeta(t+1,s) - \zeta_{\mathrm{MT}}(s,t;1) \\
&\quad - \left\{ \sum_{a=1}^{s-1} \binom{t+a-1}{a} \bar\zeta(1,t+a,s-a) + \sum_{b=1}^{t-1} \binom{s+b-1}{b} \bar\zeta(1,s+b,t-b) \right\}.
\end{aligned}$$

Applying the stuffle relations (29) we get

$$\begin{aligned}
(32) =& \bar\zeta(1)\zeta(s+t) - \zeta(s+t+1) - \zeta(s+t,1) + \zeta(t+1,s) - \zeta_{\mathrm{MT}}(s,t;1) \\
& - \bar\zeta(1) \left\{ \sum_{a=1}^{s-1} \binom{t+a-1}{a} \bar\zeta(t+a,s-a) + \sum_{b=1}^{t-1} \binom{s+b-1}{b} \bar\zeta(s+b,t-b) \right\} \\
& + \sum_{a=1}^{s-1} \binom{t+a-1}{a} \Big\{ \zeta(t+a+1,s-a) + \zeta(t+a,1,s-a) \\
& \qquad + \zeta(t+a,s-a+1) + \zeta(t+a,s-a,1) \Big\} \\
& + \sum_{b=1}^{t-1} \binom{s+b-1}{b} \Big\{ \zeta(s+b+1,t-b) + \zeta(s+b,1,t-b) \\
& \qquad + \zeta(s+b,t-b+1) + \zeta(s+b,t-b,1) \Big\}.
\end{aligned}$$

We can cancel all the terms involving $\bar\zeta(1)$ in the two cases $t > 1$ and $t = 1$ by applying (30) and (31) of Lemma 4.1, respectively. This finishes the proof of the lemma. $\square$

5. Some examples

Using Maple we have verified all the formulas on [17, p. 1502] by our general approach. We can compute any special values of $\zeta_3(s_1,\dots,s_7)$ and in particular $\zeta_{\mathfrak{sl}(4)}(s_1,\dots,s_6)$ at nonnegative integers whenever they satisfy the convergence conditions given by (12). For example, by Maple computation there are 32 weight four convergent $\zeta_{\mathfrak{sl}(4)}(s_1,\dots,s_6)$ values at nonnegative integers, with 15 distinct ones as follows:

$$\begin{aligned}
\zeta_{\mathfrak{sl}(4)}(0,0,0,0,0,4) &= \tfrac{1}{2}\zeta(2)-\tfrac{3}{2}\zeta(3)+\tfrac{2}{5}\zeta(2)^2,\\
\zeta_{\mathfrak{sl}(4)}(0,0,0,2,2,0) &= 3\zeta(3)-\zeta(2)^2,\\
\zeta_{\mathfrak{sl}(4)}(0,0,0,0,1,3) &= \zeta_{\mathfrak{sl}(4)}(0,0,0,1,0,3)=\zeta(2)-\zeta(3)-\tfrac{1}{10}\zeta(2)^2,\\
\zeta_{\mathfrak{sl}(4)}(0,0,0,0,2,2) &= \zeta_{\mathfrak{sl}(4)}(0,0,0,2,0,2)=\zeta(3)-\tfrac{3}{10}\zeta(2)^2,\\
\zeta_{\mathfrak{sl}(4)}(1,0,0,0,1,2) &= \zeta_{\mathfrak{sl}(4)}(0,0,1,1,0,2)=\zeta(3)-\tfrac{1}{5}\zeta(2)^2,\\
\zeta_{\mathfrak{sl}(4)}(1,0,0,0,2,1) &= \zeta_{\mathfrak{sl}(4)}(0,0,1,2,0,1)=2\zeta(3)-\tfrac{1}{2}\zeta(2)^2,\\
\zeta_{\mathfrak{sl}(4)}(0,0,1,0,0,3) &= \zeta_{\mathfrak{sl}(4)}(0,1,0,0,0,3)\\
&= \zeta_{\mathfrak{sl}(4)}(1,0,0,0,0,3)=2\zeta(3)-\zeta(2)-\tfrac{1}{10}\zeta(2)^2,\\
\zeta_{\mathfrak{sl}(4)}(0,1,0,1,1,1) &= \tfrac{7}{10}\zeta(2)^2, \qquad \zeta_{\mathfrak{sl}(4)}(1,1,1,0,0,1)=\tfrac{12}{5}\zeta(2)^2,\\
\zeta_{\mathfrak{sl}(4)}(0,0,0,1,1,2) &= \tfrac{1}{10}\zeta(2)^2, \qquad \zeta_{\mathfrak{sl}(4)}(1,0,1,1,1,0)=\tfrac{17}{10}\zeta(2)^2,\\
\zeta_{\mathfrak{sl}(4)}(1,0,1,0,0,2) &= \zeta_{\mathfrak{sl}(4)}(0,1,1,0,0,2)=\zeta_{\mathfrak{sl}(4)}(1,1,0,0,0,2)=\tfrac{4}{5}\zeta(2)^2,\\
\zeta_{\mathfrak{sl}(4)}(1,0,0,1,1,1) &= \zeta_{\mathfrak{sl}(4)}(1,0,0,1,2,0)=\zeta_{\mathfrak{sl}(4)}(0,0,1,2,1,0)\\
&= \zeta_{\mathfrak{sl}(4)}(0,0,1,1,1,1)=\tfrac{1}{2}\zeta(2)^2,\\
\zeta_{\mathfrak{sl}(4)}(0,1,0,1,0,2) &= \zeta_{\mathfrak{sl}(4)}(1,0,0,1,0,2)=\zeta_{\mathfrak{sl}(4)}(0,1,0,0,1,2)\\
&= \zeta_{\mathfrak{sl}(4)}(0,0,1,0,1,2)=\tfrac{2}{5}\zeta(2)^2,\\
\zeta_{\mathfrak{sl}(4)}(1,1,0,0,1,1) &= \zeta_{\mathfrak{sl}(4)}(1,0,1,1,0,1)=\zeta_{\mathfrak{sl}(4)}(1,0,1,0,1,1)\\
&= \zeta_{\mathfrak{sl}(4)}(0,1,1,1,0,1)=\tfrac{6}{5}\zeta(2)^2.
\end{aligned}$$

Notice that in the first seven values the weights are mixed which correspond to the nine cases of irregular values (13) to (17). In higher weight cases if we only consider regular (pure weight) values then we have the following examples.

$$\begin{aligned}
\zeta_{\mathfrak{sl}(4)}(1,1,0,1,1,1) &= \zeta_{\mathfrak{sl}(4)}(0,1,1,1,1,1)=\tfrac{5}{2}\zeta(5)-\zeta(2)\zeta(3),\\
\zeta_{\mathfrak{sl}(4)}(1,0,1,1,1,1) &= -\tfrac{3}{2}\zeta(5)+\zeta(2)\zeta(3),\\
\zeta_{\mathfrak{sl}(4)}(1,1,1,1,1,1) &= -\tfrac{62}{105}\zeta(2)^3+2\zeta(3)^2,\\
\zeta_3(1,1,1,1,1,1,1) &= \tfrac{3}{2}\zeta_{\mathfrak{sl}(4)}(1,1,1,1,1,2)=\tfrac{21}{8}\zeta(7)-\tfrac{3}{2}\zeta(2)\zeta(5),\\
\zeta_{\mathfrak{sl}(4)}(2,2,2,2,2,2) &= \tfrac{368}{875875}\zeta(2)^6.
\end{aligned}$$

The last value of the above agrees with [12, (4.29)] and the first value in [8, Table 4] by the relation

$$\zeta_{\mathfrak{sl}(4)}(2m,2m,2m,2m,2m,2m)=12^{-2m}\zeta_{\mathfrak{sl}(4)}(2m),$$

where $\zeta_{\mathfrak{sl}(4)}$ on the LHS is the multiple variable function being studied in the current paper and $\zeta_{\mathfrak{sl}(4)}$ on the RHS is the original single variable. Witten zeta function (see [17, Prop. 2.1]). As two intriguing computation we have

$$\begin{aligned}\zeta_{\mathfrak{sl}(4)}(1,2,3,3,2,1) = {}& \zeta(2)\zeta(8,2) + \tfrac{811324}{238875}\zeta(2)^6 - \tfrac{5}{2}\zeta(2)\zeta(5)^2 - \tfrac{37}{2}\zeta(3)\zeta(9) \\ & -35\zeta(5)\zeta(7) - 2\zeta(7)\zeta(2)\zeta(3) + \tfrac{37}{4}\zeta(10,2) = .0129650292\ldots, \\ \zeta_{\mathfrak{sl}(4)}(3,2,1,1,2,3) = {}& 10\zeta(2)\zeta(8,2) - \tfrac{120112}{53625}\zeta(2)^6 - 6\zeta(2)\zeta(5)^2 + 44\zeta(3)\zeta(9) \\ & +40\zeta(5)\zeta(7) - 20\zeta(7)\zeta(2)\zeta(3) - 22\zeta(10,2) = .0056078053\ldots.\end{aligned}$$

We have also verified all the above computation numerically with Maple and EZface [4].

We conclude our paper with a conjecture. Let $\mathcal{MZV}(w, \le l)$ be the $\mathbb{Q}$-vector space generated by MZVs of weight w and depth $\le l$. Then by Theorem 1.1 the space generated by special values of $\zeta_{\mathfrak{sl}(4)}$ of weight $w > 3$ over $\mathbb{Q}$ is included in the space $\mathcal{MZV}(w, \le 3) + \mathcal{MZV}(w-1, \le 2) + \mathcal{MZV}(w-2, 1)$.

Conjecture 5.1. *The space generated by special values of $\zeta_{\mathfrak{sl}(4)}$ of weight $w > 3$ at nonnegative integers over $\mathbb{Q}$ is*

$$\mathcal{MZV}(w, \le 3) \oplus \mathcal{MZV}(w-1, \le 2) \oplus \mathcal{MZV}(w-2, 1).$$

Using the MZV table [2] and assuming the sum is direct we have verified the conjecture for all weights up to 12.

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