PSEUDO SYMPLECTIC GEOMETRY AS AN EXTENSION OF THE SYMPLECTIC GEOMETRY

Nik. Tyurin

Abstract. In [9] we've extended the Taubes result about nontriviality of the Seiberg — Witten invariants for symplectic manifolds to more general case, when a priori based manifold doesn't admit symplectic structure. In the present paper we discuss some properties which belong to pseudo symplectic 4- maniofids in the framework of the complex geometry.

§0. Introduction

In 1994 new invariants of smooth structures in 4- dimensional geometry were introduced by E.Witten in his seminal work [10]. These invariants are very closed to the Donaldson invariants (and a relationship was established in [5]). But so as it's much more easy to compute the Seiberg — Witten invariants a number of unsolved problems have been solved in the framework of the Seiberg — Witten theory. Two first results proved in [10] have been proved earlier in the framework of the Donaldson theory, namely

1. that for a connected sum $Y = X_1 \sharp X_2$ where $b_2^+(X_i) > 0$ the invariants vanish (as well as for the Donaldson invariants);
2. and that for a Kahler surface S the Seiberg — Witten invariant for the canonical class K_S equals to ± 1,

(see [10]).

Based on this facts Witten reproved that a priori Kahler surface S isn't diffeomorphic to its topomodel.

As in the Donaldson theory the invariants are defined if b_2^+ of based manifold bigger then 1 (for the Kahler case it just corresponds to $p_g(S) > 0$).

But really the Seiberg — Witten invariants use to be slightly more usefull then the Donaldson one. We have in mind the following important result due to Taubes

Theorem (Taubes, [6]). *Let X, ω be symplectic manifold with $b_2^+(X) > 1$. Then the Seiberg — Witten invariant of the canonical class $K_\omega \in H^2(X, \mathbb{Z})$, associated to given symplectic form, equals to ± 1.*

In other words, for symplectic manifold the canonical class is a basic class.

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But the inverse implication isn't true: in [3] the counterexample was found. The construction is as follows. One can start with a symplectic manifold X with $b_2^+(X) > 1$ and consider the connected sum of the following form

$$Y = X \sharp N,$$

where $b_1(N) = b_2^+(N) = 0$ but $\pi_1(N) \neq \{1\}$. Then Y doesn't admit symplectic structure but has nontrivial invartiant (see [3]).

The idea to find a criterion for nontrivialty of the invariants was quite reasonable, and the next step in this direction was done in [9]. In this paper the following generalization of the Taubes theorem above was established

Theorem ([9]). *Let X be quasi symplectic manifold with $b_2^+(X) > 1$, so there exists a hermitian triple (g, J, om) on X with nontrivial image in $H^2(X, \mathbb{R})$ by the canonical map τ. Then the invariant for $K_J \in H^2(X, \mathbb{Z})$ (which is, of course, the canonical class of J) equals to ± 1.*

Let us recall the definitions of pseudo symplectic manifolds and the canonical map τ. For a smooth 4 - manifold X one can consider the space of all hermitian triples

$$\mathcal{M}_X = \{(g, J, \omega)\},$$

where the first element is a riemannian metric on X, the second is an almost complex structure, compatible with the first element, and the third is the corresponding 2-form (which is called an almost Kahler form) such that

$$\omega(u, v) = g(u, Jv).$$

It's well known that this form ω is

1. self dual with respect to the conformal class $*_g$ and the orientation, choosen by J;
2. nondegenerated everywhere;
3. has the type (1,1) with respect to J.

We have to emphercise that this ω hasn't to be closed a priori. Because of this it is just *almost Kahler*.

The construction is pure local and it's well known from the linear algebra that in a hermitian triple every element can be reconstructed from two others.

Further, there exists the canonical map

$$\tau : \mathcal{M}_X \to K^+ \subset H^2(X, \mathbb{R}),$$

where K^+ is the inner part of isotropic cone (so consists of 2 - cohomology classes with positive squares), defined as follows. For a triple (g, J, ω) let us take the corresponding Hodge star operator

$$*_g : \Omega^i_X \to \Omega^{4-i}_X,$$

over X defined by the conformal class of our riemannian metric g and the orientation given by J. Then by the famous Hodge theorem ([2]) every form can be decomposed into three parts — harmonic part, exact and co- exact

$$\omega = \omega_h + d\rho_1 + d^*\rho_2, \tag{0.1}$$

where $\rho_1 \in \Omega^1_X, \rho_2 \in \Omega^3_X$.

Let us recall that d^*- operator on Ω^2 is equal to the following double combination

$$d^* : \Omega^2 \to \Omega^1 \quad | \quad d^* = *_g d *_g$$

of the ordinary d and two Hodge operators (in different dimensions).

But in our case ω is self dual, so one can apply $*_g$ to both sides of (??) and get

$$*_g \omega = \omega = *_g \omega_h + *_g d\rho_1 + d(*_g \rho_2).$$

From the uniqueness of the decomposition one imidiatelly gets that

$$\rho_1 = *_g \rho_2$$

and

$$*_g \omega_h = \omega_h,$$

hence one can rewrite the previous decomposition as follows

$$\omega = \omega_h + d^+ \rho_0, \quad \rho_0 \in \Omega^1_X \quad where \quad d^+ : \Omega^1_X \to \Omega^+_X, \tag{0.2}$$

and establish that the harmonic part ω_h is self dual too.

Definition. *The image of the canonical map τ for given hermitian triple (g, J, ω) equals to the 2- cohomology class which is represented by the harmonic form ω_h defined in (0.1), (0.2)*

$$\tau(g, J, \omega) = [\omega_h] \in H^2(X, \mathbb{R}). \tag{0.3}$$

As we've seen above ω_h is self dual so it has positive square

$$\int_X \omega_h \wedge \omega_h = \int_X |\omega_h|^2_g dm_g > 0$$

unless the case when ω_h is vanishing everywhere. Hence the image $Im\tau$ lies in K^+.

Definition. *One can called based manifold X quasi simplectic if there exists a hermitian triple with nontrivial image in K^+.*

And it was proved that the canonical class of *quasi symplectic manifold* has the invariant equals to ± 1 as well as in the original symplectic case.

Example. First of all it's clear that every symplectic manifold is pseudo symplectic. For a given symplectic manifold X endowed with a symplectic form ω one gets that for an apropriate riemannian metric ω is harmonic itself so $\omega_h = \omega$ and hence $\tau(g, J, \omega) = [\omega] \neq 0$. But we'd like to recall an example of pseudo symplectic manifold which doesn't admit a symplectic structure. Namely, if one take the counterexample from [3] it isn't too hard to establish that this connected sum $Y = X \sharp N$ (see [3], [9]) admits a hermitian triple with nontrivial image in K^+. Such triple on whole Y can be constructed as an extension of the symplectic triple on the original symplectic manifold X (we'll discuss all details in the next section).

After all above is understood there is one absolutely reasonable quastion. It's well known that in our setup the Seiberg — Witten invariant depends only on the choice of $Spin^{\mathbb{C}}$ - structure so on the corresponding class $c \in H^2(X, \mathbb{Z})$ and doesn't depend on the choice of riemannian metric and almost complex structure. But from the first viewpoint two hermitian triples $(g_i, J_i, \omega_i), i = 1, 2$ with the same canonical class $K = K_{J_i}$ can have rather different images in K^+. The first can be nontrivial whether the second is trivial.

In this paper we proof the following

Main Theorem. *The nontriviality condition for image of τ is stable with respect to contineous deformations. It means that if $\tau(g_0, J_0, \omega_0) \neq [0] \in H^2(X, \mathbb{R})$ the same is true for every (g, J, ω) with $K_J = K_{J_0}$.*

On the other hand we can reformulate the statement with respect to the action of the group

$$\mathcal{G} = Aut(TX)$$

of all smooth fiberwise automorphisms of the tangent bundle over X. Namely there exists unique discrete invariant of triples — the corresponding canonical class $K_J \in H^2(X, \mathbb{Z})$. And it's more or less clear that if two triples $(g_i, J_i, \omega_i), i = 1, 2$, have the same canonical class then there exists an element $u \in \mathcal{G}$ which conjugates the first triple to the second. So as well one can say that the nontriviality of the image of τ *is stable with respect to the $\mathcal{G}$- action.*

§1. Pseudo symplectic manifolds in the framework of Seiberg — Witten theory

In this section we'll sketch the proof of the Taubes Theorem mentioned above and show in which point it uses symplectness of based manifolds.

First off all in absolutely general almost complex situation for a hermitian triple (g, J, ω) one has the following objects:

1. the decomposition on self dual and anti self dual forms represents as

$$\begin{aligned} \Lambda^+ =& (\Lambda^{2,0} \oplus \Lambda^{0,2})_{\mathbb{R}} \oplus \mathbb{R} < \omega >, \\ \Lambda^- =& (\mathbb{R} < \omega >)^{\perp} \subset \Lambda^{1,1}; \end{aligned} \tag{1.1}$$

2. spinor bundles correspond to $Spin^{\mathbb{C}}$ - structure $c_0 = -K_\omega \in H^2(X, \mathbb{Z})$ have the form

$$\begin{aligned} W^+ =& \Lambda^{0,0} \oplus \Lambda^{0,2} = I \oplus K^{-1}, \\ W^- =& \Lambda^{0,1} \end{aligned} \tag{1.2}$$

and our self dual form ω (as the section of $\Lambda^+ \cong adW^+$) acts on W^+ as diagonal operator with eigenvalues $2\imath$ and $-2\imath$ on the direct summands K^{-1} and I respectively;

3. the gauge group is

$$\mathcal{G} = Aut_h(detW^+) = Aut_h(K^{-1}); \tag{1.3}$$

4. the configuration space for the Seiberg — Witten system is represented by space of triples (a, α, β) where a is a hermitian connection on K^{-1}, α is a section of I so is a complex valued function and β is a section of K^{-1};

5. the Seiberg — Witten system in this setup reads as follows

$$\begin{aligned} &D_a(\alpha \oplus \beta) = \overline{\partial}_a \alpha + \overline{\partial}^*_a \beta = 0 \\ &\imath F^+_a = (|\beta|^2 - |\alpha|^2)\omega + \alpha\overline{\beta} + \overline{\alpha}\beta; \end{aligned} \tag{1.4}$$

and let us recall that in this case the moduli space of solutions is zero dimensional. (For all details see [4], [6], [9], [10].)

The first step. In any case there exists unique up to gauge transformations hermitian connection $a_0 \in \mathcal{A}_h(K^{-1})$ such that the projection of the corresponding connection (on whole $W^+ = I \oplus K^{-1}$) to the first direct summand is equal to ordinary d

$$\nabla_{a_0}|_I = d \quad \text{on} \quad I. \tag{1.5}$$

So if one take spinor filed of the form $(s \oplus 0) \in I \oplus K^{-1}$ where s a constant function then

$$\nabla_{a_0}(s \oplus 0) = b \in \Gamma(K^{-1} \otimes T^*X) \tag{1.6}$$

and b essentually is *the torsion of our almost complex structure* J. So in the case of integrable complex structure one has $b = 0$ and hence $D_{a_0}(s \oplus 0)$ automatically equals to zero. But in the symplectice case Taubes observes that despite of nontriviality of $\nabla_{a_0}(s \oplus 0) = b$ the corresponding Dirac operator D_{a_0} vanishes on $s \oplus 0$ if and only if the form ω is closed (so is a symplectic form).

Taubes arguments on this step are extremly usefull and deep exercises in the theory. Namely he acts by the self dual form ω on the spinor field $s \oplus 0$

$$\omega(s \oplus 0) = -2\imath \cdot (s \oplus 0)$$

and then applies the corresponding coivariant deravitive to the both sides of the previous equality

$$\nabla_{a_0}(\omega(s \oplus 0)) = -2\imath b.$$

But on the right side we have by the Liebnitz rule

$$\nabla_{a_0}(\omega(s \oplus 0)) = C \cdot d^*\omega \otimes (s \oplus 0) + \omega(b) = C \cdot d^*\omega \otimes (s \oplus 0) + 2\imath b$$

(where C is an integer number) hence using Clifford multiplication one gets

$$D_{a_0}(s \oplus 0) = \frac{\imath C \cdot s}{4}(d^*\omega)^{0,1} \tag{1.7}$$

where the last term $(d^*\omega)^{0,1}$ can be equal to zero if and only if $d^*\omega$ is equal to zero so if and only if our form ω is closed and hence symplectic.

The second step. After it was established that the spinor field $s \oplus 0$ is harmonic with respect to the Dirac operator D_{a_0} Taubes considers some special perturbation of the original Seiberg — Witten system namely

$$\begin{aligned} &D_{a_0}(\alpha \oplus \beta) = 0 \\ &\imath F_a^+ = \imath F_{a_0}^+ + (|\beta|^2 - |\alpha|^2 + 1)\omega + \alpha\overline{\beta} + \overline{\alpha}\beta. \end{aligned} \tag{1.4'}$$

It's clear that the triple $(a = a_0, \alpha = 1, \beta = 0)$ is a solution of this perturbed system. But the point is that one can derive the Seiberg — Witten invariant from this perturbed system as well as from the original one. It's so since *there are no reducible solutions* for the perturbed system. Really for such solution (with trivial spinor part) one has

$$\imath F_a^+ = \imath F_{a_0}^+ + \omega$$

but these two curvature forms are cohomologically the same so

$$\omega = d^+\rho$$

where ρ is a real 1- form. But so as we are now in the symplectic cathegory it's impossibly hence there are no reducible solutions.

The third step. Then Taubes imposed an additional term to the right side of the second equation

$$\imath F_a^+ = \imath F_{a_0}^+ + (|\beta|^2 - |\alpha|^2 + 1)\omega + \alpha\overline{\beta} + \overline{\alpha}\beta + A(r, a_0, a, \alpha, \beta), A(r=0) = 0 \quad (1.8)$$

(for precise expression see [6], [4]) and established that for sufficienly large $r >> 0$ there exists unique up to gauge transformations solution for this perturbed system and hence the invariant equals to ± 1 (see [6], [4]).

We recalled more or less rigouresly just the first and the second steps because only on these steps Taubes uses *closedness* of the self dual 2- form ω.

So to generalize this imroptant result one has to avoid in these steps contradictions and to impose a condition on hermitian triple.

The first step. We've seen that in absolutely general situation (in the framework of the almost complex geometry)

$$D_{a_0}(1 \oplus 0) = \frac{\imath C}{4}(d^*\omega)^{0,1} \in \Omega_X^{0,1} = \Gamma(W^-)$$

supposing that $s = 1$. Then one can deform the connection a_0 such that the corresponding covariant derivative has the form

$$\nabla_{a_1} = \nabla_{a_0} + \begin{pmatrix} \frac{C}{4\imath} & 0 \\ 0 & \frac{C}{4\imath} \end{pmatrix} \otimes d^*\omega \in \mathcal{A}_h(W^+). \quad (1.9)$$

It corresponds to such hermitian connection $a_1 \in \mathcal{A}_h(detW^+)$ that

$$a_1 = a_0 + \frac{C}{4\imath} d^*\omega \quad (1.10)$$

and it's clear so as the last term is a pure imaginary 1- form that such deformation is well defined.

Then we can use this new connection a_1 instead of the original one in absolutely general case to establish that

$$D_{a_1}(1 \oplus 0) = 0$$

always (direct substitution of (1.9) to (1.7)).

The second step. So on this level one can use a familiar perturbation of the original system which differs from the Taubes perturbation just in the second equation

$$\imath F_a^+ = \imath F_{a_1}^+ + (|\beta|^2 - |\alpha|^2 + 1)\omega + \alpha\overline{\beta} + \overline{\alpha}\beta. \quad (1.8')$$

Therefore again we have "primitive" solution which is now $(a = a_1, \alpha = 1, \beta = 0)$ and one has to establish non existance of reducible solutions for our perturbed system.

At this point our definition of pseudo symplectic triples has to be imposed. Namely let us suppose that a reducible solution exists for our choosen triple (g, J, ω). Then we get that

$$\imath F_a^+ - \imath F_{a_1}^+ = \omega$$

but again two curvature forms in the left side are cohomologically the same so in this case it should be a real 1- form ρ such that

$$\omega = d^+ \rho. \tag{1.9}$$

But such ρ can exist if and only if the corresponding harmonic part ω_h in the Hodge decomposition (0.1)

$$\omega = \omega_h + d\rho_1 + d^* \rho_2 = \omega_h + d^+ \rho_0$$

is trivial. So if our hermitian triple (g, J, ω) has nontrivial image $\tau(g, J, \omega) \neq [0]$ in $H^2(X, \mathbb{R})$ (see the Introduction) there are no reducible solutions for our perturbed system and one can exploit it to compute the invariants as well as in the original Taubes construction.

Let us continue with the example of pseudo symplectic but non symplectic manifolds mentioned in the Introduction. So X is a symplectic manifold with $b_2^+ > 1$ and symplectic form ω. Consider the following connected sum

$$Y = X \sharp N$$

where N is a manifold with $b_1(N) = b_2^+(N) = 0$. It was shown in [3] that such Y has the same invariants as the original X but doesn't admit symplectic structure if the fundamental group $\pi_1(N)$ admits nontrivial finite quotient. To establish that there are no symplectic structures on Y it's sufficiently to consider the universal covering of Y which is a smooth manifold of the following form

$$\widetilde{Y} = X \sharp X \sharp ... \sharp X \sharp N'$$

and here we have d copies of X where d is the corresponding order of the quotient. Since any symplectic structure on Y induces the corresponding symplectic structure on $\widetilde{Y}$ combinig this with the fact that every connected sum where summands have positive b_2^+ has trivial Seiberg — Witten invariants one gets that Y doesn't admit any symplectic structure.

But for our aims such examples are quite appropriate. Let us formulate the result as follows

Lemma. *Let X be symplectic 4- manifold endowed with symplectic form ω_0. Then for every 4- manifold N such that $b_1(N) = b_2^+(N) = 0$ the connected sum*

$$Y = X \sharp N$$

is a pseudo symplectic manifold.

To proof this Lemma one has to construct a pseudo symplectic structure on Y so to find a hermitian triple (g, J, ω) with nontrivial image in $H^2(Y, \mathbb{R})$.

Let us start with the given symplectic structure on X represented by a triple (g_0, J_0, ω_0) where the third element is our given symplectic form and we choose some compatible riemannian metric g_0 to form such triple. Then let us extend abritrary this original triple to the whole Y, getting some triple $(g_1, J_1, \omega_1) \in \mathcal{M}_Y$. First of all the corresponding canonical class

$$K_{J_1} \in H^2(Y, \mathbb{Z}) = H^2(X, \mathbb{Z}) \oplus H^2(N, \mathbb{Z})$$

has the following form

$$K_{J_1} = K_{J_0} + \sum_{1}^{b_2(N)} c_i \tag{1.10}$$

where c_i is the basis in $H^2(N, \mathbb{Z})$ in which the intersection form Q_N has standard form $(-1, ..., -1)$. Obviously we can extend all the elements if such choise is done. Really, let us realize our almost complex structure J_0 as a section of the corresponding spinor bundle $W^+_X \to X$ with the Chern classes $c_1(W^+_X) = K_{J_0}, c_2(W^+_X) = 0$. Then it's clear that over N the sum $\sum_1^{b_2(N)} c_i$ is a $Spin^{\mathbb{C}}$- structure (because it is a characteristic element of Q_N) and one can take the corresponding spinor bunle W^+ over N with the Chern classes $c_1(W^+_N) = \sum_1^{b_2(N)} c_i, c_2(W^+_N) = 1$ induced by some appropriate riemannian metric over N. Then over the whole Y we get the spinor bundle W^+_Y with the Chern classes $c_1(W^+_Y) = K_{J_1}, c_2(W^+_Y) = 0$ (it's easy to check it directly). So one can extend the choosen section of W^+_X corresponded to our given almost complex structure and then get a nonvanishing section of W^+_Y which would induce an almost complex structure on the whole Y.

So we claim now to deduce that the triple (g_1, J_1, ω_1) has nontrivial image $\tau(g_1, J_1, \omega_1)$ in $H^2(Y, \mathbb{R})$. For this let us take the harmonic with respect to $*_{g_1}$ 2- form ω_h such that

$$[\omega_h] = [\omega_0] \in H^2(Y, \mathbb{R}).$$

Then consider the following integral

$$\int_Y \omega_1 \wedge \omega_h. \tag{1.11}$$

Since both 2- forms are self dual with respect to $*_{g_1}$ this expression equals to *the inner product* so if it is nontrivial then the projection of ω_1 to the ray $\mathbb{R} < \omega_h >$ in the harmonic space is nontrivial and hence $\tau(g_1, J_1, \omega_1)$ has to be nontrivial too.

As usual for connected sums let's divide the integral (1.11) into three parts

$$\int_Y \omega_1 \wedge \omega_h = \int_{X \backslash B_1} \omega_1 \wedge \omega_h + \int_{\text{Neck}} \omega_1 \wedge \omega_h + \int_{N \backslash B_2} \omega_1 \wedge \omega_h$$

where B_i are small balls using for the glueing procedure.

Then since over punctured $X \backslash B_1$ these two 2- forms are very closed to the original symplectic form we get that

$$\int_{X \backslash B_1} = 2 Vol X + o(r),$$

where r is the radius of the balls. So as the Neck is conformal flat one gets

$$\int_{\text{Neck}} \omega_1 \wedge \omega_h = o(r),$$

and since there are no self dual harmonic form over N the same expression can be obtained for the third summand

$$\int N\backslash B_2\omega_1 \wedge \omega_h = o(r).$$

Taking all together we see that

$$\int_Y \omega_1 \wedge \omega_h = 2VolX + o(r) \tag{1.12}$$

so shrinking the Neck (standard trick, see f.e. [1]) we get that this integral is nontrivial and positive.

The Lemma proved above gives us a number of examples of pseudo symplectic but non symplectic 4- manifolds.

This story represents the motivation to introduce the definition of pseudo symplectic manifolds. Now we'd like to repeat the natural conjecture which was stated above: we know that the invariants doesn't depend of the choice of riemannian metric and depends only on the corresponding class in $H^2(X,\mathbb{Z})$. Is the same true for nontriviality of the image of our canonical map τ?

The answer (included in our Main Theorem) is positive and in the following sections we prove this statement.

§2. The structure of the space $\mathcal{M}_X$ for a given X

In this section we'll describe the structure of the hermitian triple space for a given smooth 4 - manifold X.

First of all let's consider the discrete invariants, which divides all the $\mathcal{M}_X$ into discrete set of subspaces.

The most generic division of $\mathcal{M}_X$ corresponds to the orientation. For an orientable X there are two choices of orientation. And if one fix an almost complex structure then the corresponding orientation is fixed. So one can divide $\mathcal{M}_X$ into two pieces

$$\mathcal{M}_X = \mathcal{M}_X^+ \cup \mathcal{M}_X^-, \tag{2.1}$$

where triples from the first subspace are compatible with one orientation and triples from the second one are compatible with the other. There are not any preferences in the choice of the orientations but let us assign $+$ and $-$ arbitrary to distinguish the two orientations.

The next (and the last) discrete data are the labelling of $\mathcal{M}_X$ by the corresponding canonical classes. Namely let's consider the following obvious map

$$Can : \mathcal{M}_X \to K_X \subset H^2(X,\mathbb{Z}) \tag{2.2}$$

defining by

$$Can(g,J,\omega) = K_J \in H^2(X,\mathbb{Z})$$

where K_J is the canonical class of J. There are two aproaches to define this canonical class. First of all, the tangent bundle TX together with our choosen almost complex structure J can be regarded as rank two complex bundle $T^{1,0}_{\mathbb{C}}X$, so one can take the first Chern class of this bundle and

$$K_J = -c_1(T^{1,0}_{\mathbb{C}}X).$$

The second approach is to consider the direct sum

$$\Lambda^2(T^*X) = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^{\pm}$ are real rank- 3 bundles on self dual (respectively anti self dual) 2- forms with respect to the conformal class $*_g$ and the orientation defined by J. It's well known that there is trivial real subbundle $\mathbb{R} < \omega > \subset \Lambda^+$ and in the presistance of the riemannina metric g one gets the ortogonal to $\mathbb{R} < \omega >$ subbundle $(\mathbb{R} < \omega >)^{\perp} \subset \Lambda^+$. Moreover, the vector multiplication by our form ω defines a complex structure on $(\mathbb{R} < \omega >)^{\perp}$ and one can take the first Chern class of this line complex bundle and it is our canonical class again.

It's easy to decribe the image of Can in $H^2(X, \mathbb{Z})$. Let us fix an orientation on X, then for any $K_0 \in ImCan \subset H^2(X, \mathbb{Z})$ one has

$$K_0 = w_2(X)(mod 2), \quad K_0^2 = 2\chi + 3\sigma, \tag{2.3}$$

where $w_2(X)$ is the second Stiefel - Whitney class, χ is the euler characteristic (and these elements are independent of the choice of orientation) and σ is the signature, corresponding to the choosen orientation.

Further, for any $K_0 \in ImCan$ one has that $-K_0 \in ImCanK$. Namely for any triple (g, J, ω) with K_0 we have $(g, -J, -\omega)$ with $-K_0$. So the space $\mathcal{M}_X$ has *standard real structure* Θ_X without real points. For any point $(g, J, \omega) \in \mathcal{M}_X$ we have the conjugated point

$$\Theta_X(g, J, \omega) = (g, -J, -\omega) \in \mathcal{M}_X \tag{2.4}$$

and obviuosly our real structure Θ_X preserves both $\mathcal{M}^{\pm}_X$ and induces just the multiplication by -1 on $K_X \subset H^2(X, \mathbb{Z})$.

Hence our space $\mathcal{M}_X$ is labelled by the discrete image of the map Can and we have

$$\mathcal{M}_X/\Theta_X = \bigcup_i \mathcal{M}^{K_i}_X,$$

where $\mathcal{M}^{K_i}_X = Can^{-1}(K_i)$.

To describe the structure of the space $\mathcal{M}_X$ we consider a triple (g, J, ω) with fixed canonical class as the corresponding pair g, J. Then there are two natural projections: to the first and the second components

$$\pi_1(g, J) = g, \quad \pi_2(g, J) = J. \tag{2.5}$$

The images lie in the homogeneous space of all riemannian metrics $\mathcal{H}$, compatible with the given smooth structure, and in the space of all almost complex structures with the fixed canonical class. Now we want to describe the fibers of these two projections. But for this at a moment we'll use not the space $\mathcal{M}^{K_i}$ but the space of pairs $(*_g, J)$ where $*_g$ is the conformal class of a riemannian metric. Let us denote this space as $\widetilde{\mathcal{M}^{K_i}}$, so we have the following natural fibration

$$con : \mathcal{M}_X^{K_i} \to \widetilde{\mathcal{M}_X^{K_i}}$$

which is a principal $e^{\mathbb{R}}$- bundle.

To describe the projections above it's very usefull to translate all constructions to the language of projectivization. In this setup the conformal class $*_g$ of a metric g corresponds to the following object. Over each point $p \in X$ one has in

$$\mathbb{CP}^3 = \mathbb{P}(T_p^{\mathbb{C}} X)$$

non- degenerated quadric $Q_p \subset \mathbb{CP}^3$ together with standard real stucture

$$\Theta_p : \mathbb{CP}^3 \to \mathbb{CP}^3, \quad \Theta_p^2 = id$$

such that Q_p is real but without real points. Then any almost complex structure J compatible with this conformal class corresponds to a pair of projective lines l_p^1, l_p^2 on the quadric. It's well known that there are two families of projective lines on a nondegenerated quadric. And our two lines lie in the same family because

$$\Theta_p(l_p^1) = l_p^2$$

and if l_p^1, l_p^2 will be in two different families then the intersection point would be real. Our nondegenerated quadric Q_p is the direct product of two projective lines

$$Q_p = \mathbb{P}^+ \times \mathbb{P}^-$$

(since the orientation is choosen by the fixing of K_i one can distinguish these two projective lines by the signs). So by the definition l_p^i is represented by points in $\mathbb{P}^+$ (for details see [10]).

In this language the description of π_i is quite obviuos.

The first projection. Now we're ready to describe a fiber of

$$\pi_1 : \mathcal{M}^{K_i} \to \mathcal{H},$$

where $\mathcal{H}$ is the homogeneous space of all riemannian metrics compatible with the given smooth structure.

Let us fix a riemannian metric g and a canonical class K_i. Then one gets the following lifting of the corresponding projective bundle $\mathbb{P}^+$. Since our X is orientable (and the corresponding orientation is fixed by our canonical class) the lifting is defined by the choice of the first Chern class (which is called the choice of $Spin^{\mathbb{C}}$ - structure). So if we choose K_i as the first Chern class we immediately get the corresponding vector bundle W^+ which is called *spinor bundle.* The topological type of the bundle is defined by our choosen K_i

$$c_1(W^+) = K_i, \quad c_2(W^+) = 0.$$

Then one can consider the space of smooth everywhere nonvanishing sections:

$$\Gamma_*(W^+) = \{\phi \in \Gamma(W^+) \quad | \quad (\phi_0)|_0 = \emptyset\}. \tag{2.6}$$

Then any section $\phi \in \Gamma_*(W+)$ defines an almost complex structure as follows. There is natural pairing $\phi \to (\phi \otimes \overline{\phi})_0$ and in the composition with well- known isomorphism

$$adW^+ = \Lambda^+$$

one gets the corresponding self dual 2- form ω_ϕ which is nondegenerated everywhere. So from g and ω_ϕ one can reconstruct the second element in hermitian triple and it is our almost complex structure (see, f.e., [8]).

But it's clear that two nonvanishing sections define the same complex structure if and only if they are *gauge equivalent* with respect to the gauge group

$$\mathcal{G}_0 = Aut_h(detW^+)$$

of fiberwise transformations of the determinant line bundle $detW^+$. So a fiber of π_1 above is naturally isomorphic to

$$\pi_1^{-1} = \Gamma_*(W^+)/\mathcal{G}_0. \tag{2.7}$$

Equivalently one can describe the first projection in terms of the space of nonvanishing self dual 2- forms. Again, since we have choosen orientation and riemannian metric g, we can decompose the space Ω^2_X of all smooth 2- forms into self dual and anti self dual parts

$$\Omega^2_X = \Omega^+ \oplus \Omega^-.$$

Then let us derive the subset $\Omega^+_* \subset \Omega^+$ of nonvanishing everywhere self dual forms. Let us recall that for self dual 2- forms there are just two possibilities: to have rank 0 or rank 4 (full rank). So the subset Ω^+_* is defined by just one condition: to be nonvanishing. Then the subset Ω^+_* consists of a number of disconnected components, labelled as above by the corresponding canonical classes. In this sense the present picture is more universal then the picture above (with the spinor bundle) because now all the components of $\pi_1^{-1}(g)$ are defined on the whole space $\mathcal{M}_X$. So the corresponding connected component of Ω^+_* is the fiber of $\pi^{-1} \in \mathcal{M}^{K_i}_X$.

On the other hand we can realize the situation globally using the following global construction. For our fixed riemannian metric g we consider pointwise all self dual 2- forms which correspond to almost Kahler forms. Namely over each point p we have a point ω_p in the fiber Λ^+_p. But one has to unify all these forms so that each ω_p has the norm equals to $\sqrt{2}$. Hence we have 2- sphere $S^3_p \subset \Lambda^+_p$ consists of locally compatible forms. After globalization one gets the following fibration

$$s : \mathcal{S} \to X \tag{2.8}$$

with a fiber $s^{-1}(p) = S^2$. It's clear that

a) $\mathcal{S}$ is smooth compact orientable 6- dimensional manifold;

b) the topological type of $\mathcal{S}$ is fixed by the topological type of X;

c) every global everywhere nonvanishing self dual 2- form ω defines smooth inclusion

$$i_\omega X \hookrightarrow \mathcal{S}$$

such that the image $i_\omega(X)$ intersects all the fibers transversally;

d) the 4- homology classes of the images $[i_{\omega_i}] \in H_4(\mathcal{S}, \mathbb{Z})$ are in one - to - one correspondence with the canonical classes defined by ω_i.

Moreover since every 2- sphere fiber is endowed with the natural riemannian metric one gets the corresponding riemannian metric G_g on whole $\mathcal{S}$. So:

e) the restriction $G|_{i_\omega(X)}$ is isomorphic to g on X.

And we can define an almost complex structure $\mathcal{J}_g$ on $\mathcal{S}$ as follows. Over each point $t \in \mathcal{S}$ there is ortogonal decomposition of the tangent space induced by our metric G_g

$$T\mathcal{S}_t = TS^2 \oplus TX$$

and we take the standard complex structure induced by the vector multiplication by ω on the first summand and the tautological almost complex structure defined by g and ω on the second summand.

Really the space $\mathcal{S}$ is well known in differential geometry as twistor space but originally it was defined for conformal classes instead of riemannian metrics themselfs. But the correspondence is absolutely clear and will be used further in this paper.

The second projection. Now let's consider the second projection

$$\pi_2 : \mathcal{M}_X^{K_i} \to \mathcal{K}_{K_i}$$

to the space of almost complex structure with the same canonical classes.

For this we'd like to present the procedure to reconstruct riemannian metrics compatible with a given almost complex structure from some additional data.

First of all let us recall that our standard real structure Θ_p on $\mathbb{CP}^3_p$ induces quaternionic real structures on our projective lines $\mathbb{P}^\pm_p$ which parametrize the two families on the quadric Q_p.

Suppose that we already fixed an almost complex structure J and got over each point $p \in X$ two projective lines l^i_p. The same picture exists over each point and one can see that J defines an inclusion

$$\rho_J : \mathbb{P}^- \hookrightarrow \mathbb{CP}^3$$

of projective bundle (global!) $\mathbb{P}^- \to X$ to our projective bundle $\mathbb{CP}^3 \to X$ so the image is a projective subbundle of $\mathbb{CP}^3$. Over each point

$$\rho_J(\mathbb{P}^-)|_p = l^1_p. \tag{2.9}$$

Then we can reconstruct a quadric Q_p from the corresponding quaternionic real structure θ_p on $\mathbb{P}^-_p$ by the following procedure. Together with the identification (2.9) above one gets the corresponding quaternionic real structure

$$\theta'_p = (\rho_J)_*(\theta_p)$$

on l_p^1. We define for each point $s^1 \in l_p^1$ a point $s^2 \in l_p^2$ as

$$\Theta_p(\theta'_p(s^1)) = s^2.$$

Then if we take all projective lines of the form

$$< s^1, s^2 >$$

we'll get a nondegenerated quadric which is of course our Q_p. An extra datum for the reconstruction procedure is the following: over each point we have a global pintwise quaternionic structure on $\mathbb{P}^-$. So the space of all conformal classes (we have to emphercise thet we get just a conformal class) compatible with a given almost complex structure J can be regarded as the space of all global sections of the corresponding principal $PGL(2, \mathbb{C})$- bundle. After local consideration we again insure that $Aut(TX)$ acts transitively on $\mathcal{M}_X^{K_i}$. But it's clear that for every global θ there is nontrivial stibilizer in $PGL(2, \mathbb{C})$.

Let us note that really the picture of the reconstruction is more *universal* in the following sense. For a fixed quaternionic real structure on the projective bundle $\mathbb{P}^-$ one can get different metrics using different almost complex structures with different (ad hoc) canonical classes. So the unique fixed additional datum corresponds to the subset in the whole $\mathcal{M}_X$.

The existance of the stabilizers gives us the following conclusion. The space $\mathcal{M}_X$ has no structure of a principal bundle but is a homogeneous space. This is easy to see from the consideration of any of our two projections. So let's now consider the fibration discribed above

$$Can : \mathcal{M}_X \to K_X.$$

Then we have the action of $Aut(TX)$ on each fiber. But for every point $(g, J, \omega) \in \mathcal{M}_X$ there is nontrivial stabilizer which is the space of sections of a principal $U(2)$-bundle. We'd like to postpone more precise description to a future since our goal now is just to study the pseudo symplecticity condition.

§3. Inclusions into twistor space

We can discribe a fiber of the first projection π_1 as it was outlined above using smooth inclusions and turning to an analogy with complete linear systems in the algebraic geometry.

Let us fix a conformal class $*_g$ and consider the corresponding fiber of π_1. Then over each point we have $\mathbb{P}_p^+$ projective lines so that our quadric Q_p is the direct product of two projective lines parametrizing two families of projective lines on Q_p. Let us globalize this picture over whole X to get a projective bundle $\mathbb{P}^+ \to X$. The total space of this bundle is *twistor space* of the conformal class which we'll denote as Y. The topological type of this Y is fixed by the topological type of X and moreover it has a smooth structure and an almost complex structure $\mathcal{J}_g$ both defined by the smooth structure on X and our conformal class. We have

$$\pi : Y \to X \tag{3.1}$$

such that $\pi^{-1}(p) = \mathbb{CP}^1$ which is our projective line above. So for any compatible almost complex structure J on X one has an inclusion

$$i_J : X \to Y \tag{3.2}$$

such that

1. i_J is smooth inclusion;
2. $\mathcal{J}_g|_{i_J(X)} = J$ hence $i_J(X)$ is *pseudo holomorphic* with respect to $\mathcal{J}_g$;
3. the corresponding 4- homological class $[i_J(X)] \in H^4(Y, \mathbb{Z})$ is Poincare dual to $\pi^*(K_J) \in H^2(Y, Z) = H^2(X, \mathbb{Z}) \oplus \mathbb{Z}[H]$ where H is tautological line bundle. One can compare this picture with the construction of $\mathcal{S}$ above. Of course, the ambient spaces are the same, and one can reduce the problem from conformal classes to metrics by fixing some appropriate nondegenerated positive (with respect to the orientation) 4- form and then for every conformal class there is unique riemannian metric from this class which has this fixed 4- form as volume form. So one can represent a fiber of the projection π_1 as follows. Let one has the twistor space $Y, \mathcal{J}_g$ with almost complex structure defined by our conformal class. Then

$$\pi_1^{-1}(*_g) = \{X' \subset Y | X' \quad satisfies :\}$$

1. X' is a smooth representation of the class $(\pi^*(K_i))^* \in H^4(Y, \mathbb{Z})$
2. X' is pseudoholomorphic
3. X' intersects all the fibers transversally.

Due to this twistor picture one can prove the following

Proposition 1. *Let* $(g_0, J, \omega) \in \mathcal{M}_X^{K_i}$ *has nontirivial image* $\tau(g_0, J, \omega) \in K^+$ *so*

$$\tau(g_0, J, \omega) \neq [0].$$

Then the same is true for any (g_0, J_1, ω_1).

To prove this Proposition one can consider for a fixed metric g_0 the corresponding twistor space

$$\pi : Y \to X$$

with the almost complex structure $\mathcal{J}_{g_0}$ defined by the conformal class of g_0. But moreover using the standard metric of $\mathbb{CP}^1 = \pi^{-1}(p)$, given metric g_0 and a connection on the $PU(2)$- bundle corresponding to the Levi - Chivitae connection of the metric g_0 one gets as well *a riemannian metric* G_{g_0} on the whole Y (it is much more clear in the setup of the construction of $\mathcal{S}$ (2.8) above — all the necessary definitions were given there). So our riemannian metric g_0 defines *a hermitian triple* $(G_{g_0}, \mathcal{J}_{g_0}, \Omega_{g_0})$ on Y and we have the following inclusion

$$h : \mathcal{H}_X \to \mathcal{M}_Y$$

and it's easy to see that the image lies in the following component

$$Imh \subset \mathcal{M}_Y^{K_{g_0}} \subset \mathcal{M}_Y$$

and

$$K_{g_0} = -2H$$

where $H \in H^2(Y, \mathbb{Z})$ is the tautological line bundle.

Furhter, let us consider the second element of the original triple J. It corresponds to an inclusion of X to Y as described above.

It's obvious that

$$\begin{aligned} \mathcal{J}_{g_0}|_{i_J(X)} &= J \\ G_{g_0}|_{i_J(X)} &= g_0 \\ \Omega_{g_0}|_{i_J(X)} &= \omega, \end{aligned} \tag{3.3}$$

and the triviality or nontrivialty of $\tau(g_0, J, \omega)$ depends on the triviality or nontriviality of the corresponding class

$$\widetilde{\tau}(G_{g_0}, \mathcal{J}_{g_0}, \Omega_{g_0}) \in H^2(Y, \mathbb{R}), \tag{3.4}$$

restricted on the submanifold $i_J(X) \subset Y$. Here the map

$$\widetilde{\tau} : \mathcal{M}_Y \to H^2(Y, \mathbb{R})$$

has the same definition as the map τ above (but really there is no K^+ in $H^2(Y, \mathbb{R})$ so the analogy isn't absolutely direct).

But the nontriviality of the restriction

$$\widetilde{\tau}(G_{g_0}, \mathcal{J}_{g_0}, \Omega_{g_0})|_{i_J(X)}$$

depends just on the class $[i_J(X)] \in H_4(Y, \mathbb{Z})$ which is the same for all almost complex structures on X with the same canonical class, since

$$[i_J(X)] = (\pi^*(K_J))^*. \tag{3.5}$$

More rigouresly, we have the following equality

$$\int_Y (\widetilde{\tau}(G_{g_0}, \mathcal{J}_{g_0}, \Omega_{g_0}))^2 . \pi^* K_J = \int_X (\tau(g_0, J, \omega))^2 \tag{3.6}$$

therefore nontriviality of the left side integral implies nontriviality of the image for any triple from $\mathcal{M}_X^{K_J}$.

So the proof of the Proposition 1 is completed.

§4. Semitwistor spaces

Now we know that nontriviality of τ is stable with respect to deformations of the second element of triples. The next step is to establish the same stability with respect to deformations of the first element in triples.

For this we'll study a new object: an anolog of the twistor space, which depends on a fixed *almost complex structure* (instead of a riemannian metric). Above we got twistor space as total space for all (pointwise) almost complex structures compatible with a given riemannian metric. If one fixed viceversa an almost complex structure

J one can consider all riemannian metrics compatible with J over a fixed point. Let us input some linear algebra to establish what is a fiber in this case.

For $\mathbb{R}^4$ with given standard complex structure

$$J = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$$

we are interested in such positive defined bilinear forms which are preserved by J. Every form can be represented by a symmetric matrix A_Q. Direct calculations give us the following answer

$$A_Q = \begin{pmatrix} a & b & 0 & 0 \\ b & c & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & b & c \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$ and

$$ac - b^2 > 0, \qquad a + c > 0, \tag{4.1}$$

for positivity.

The set $\mathcal{L}^+$ of points satisfied the two conditions above is the inner half part of quadratic cone $ac - b^2 = 0$ which belongs to the condition $a + c > 0$. Points on the boundary cone $ac - b^2 = 0$ correspond to *degenerated* metrics.

In turn to the conformal classes (of compatible riemannian metrics) we get a bundle

$$w : W \to X \tag{4.2}$$

where $w^{-1}(pt)$ is topologically *a two - dimensional disc*. Let us call the total space of this bundle W *semitwistor space* corresponding to the given J. A reason to call it so is as follows: let us take together with $\mathcal{L}^+$ the other part

$$\mathcal{L}^- = \{(a, b, c)| \quad ac - b^2 > 0 \quad \text{b}ut \quad a + c < 0\}.$$

This part corresponds to *negative defined* metrics, but nevertheless after "projectivization" we get a sphere S^2_p with marked equator $S^1_p \subset S^2_p$ containing conformal classes of degenerated metrics. Globalizing this picture over whole X one gets a bundle on 2- spheres

$$t : T \to X, \tag{4.3}$$

where $t^{-1}(p) = S^2_p$, and this bundle, at least topologically, corresponds to twistor bundle $\pi : \mathbb{P}^1 \to X$ above.

But now we'd like to consider the following construction. Let us realise the disc, which belongs to the set of all conformal classes, compatible with a given almost complex structure J as *the Lobachevsky plane*. Really, on $\mathcal{L}^+$ we have the natural norm

$$v = (a, b, c) \mapsto |v| = ac - b^2 \tag{4.4}$$

and one can consider all vectors with the unit norm. They form hyperbolic surface inside $\mathcal{L}^+$ which we'll denote as $\mathcal{L}ob$. It's clear that $\mathcal{L}ob$ is naturally isomorphic to the Lobachevsky plane. So one can regard

$$w : W \to X$$

as fibration on the Lobachevsky planes. So as on $\mathcal{L}ob$ there exists special riemannian metric one can define the universal conformal class $*_G$ on the whole W in the same way as it was defined for the universal almost complex structure on twistor space. For each $s \in W$ one has

$$TW_s = TX_{w(s)} \times T(\mathcal{L}ob)_s,$$

and on the second component we have standard riemannian metric and on the first component one has tautological conformal class on X corresponding to this point s. So we have the universal conformal class $*_J$ on our semitwistor space W and hence one can define an almost complex structure on W. Namely, this universal conformal class defines over each point the orthogonal decomposition of the tangent space so one can define the direct product of two complex structure - the given on TX and the standard on $T(\mathcal{L}ob)$.

So for every almost complex structure J on X we defined (absolutely canonically) a 6- dimensional open manifold W with boundary S with universal pair $*_J$ and $\mathcal{J}_J$. But another problem is that $*_J$ doesn't admit a nondegenerated extension to the boundary S and the same happens with our universal almost complex structure $\mathcal{J}_J$.

Hence we have an analogy with the twistor case but now the situation is much more complicated. First of all while a twistor space is compact (and is smooth 6-dimensional manifold), a semitwistor space is not compact and is represented by an open almost complex 6- dimensional riemannian manifold. The bounded manifold can be described as S^1- bundle over X and is denoted as

$$\partial W = S \to X.$$

Now the description of all *globally defined* conformal classes compatible with our given J is the following. For each $*_g$ on based manifold X one has the corresponding *smooth inclusion*

$$i_{*_g}(X) \to W$$

such that $i_{*_g}(X)$ is *a pseudoholomorphic* with respect to $\mathcal{J}_J$ 4- submanifold which intersects the fibers transversally. Again we have

$$\begin{aligned} *_J|_{i_{*_g}(X)} &= *_g \\ \mathcal{J}_J|_{i_{*_g}(X)} &= J. \end{aligned} \tag{4.5}$$

Now we want to repeat the argument, which took place in the previous section to establish nontriviality of our canonical map τ for all metrics compatible with our given almost complex structure J. But in the present case one has to use two additional arguments.

First of all the construction of semitwistor spaces above deals with *conformal classes* instead of riemannian metrics. So we have to fix a nondegenerated 4- form on X, denoted as $d\mu$ (the volume form) and then get a one- to- one correspondence between conformal classes and riemannian metrics with the same volume form (so this fixing defines a section of fibration

$$\widetilde{W} \to W, \quad i_{d\mu}(W) \hookrightarrow \widetilde{W} \tag{4.6}$$

where $\widetilde{W}$ is the space of all riemannian metrics pointwise compatible with J).

Then one gets on W *the universal riemannian metric* G instead of the universal conformal class $*_G$ and universal 2- form Ω.

Further, the second point is that W isn't compact and one has to impose the following argument to establish stability of our canonical map τ with respect to changing of the first element in hermitian triple. Namely instead of ordinary cohomology one has to work with cohomology with compact support. Then since our universal riemannian metric G descends to zero on the boundary we can repeat all arguments on the Hodge decomposition of Ω in terms of the cohomology with compact support and get the following statement:

Proposition 2. *Let one fix a hermitian triple (g, J, ω) with nontrivial image in $K^+ \subset H^2(X, \mathbb{R})$. Then for every metric g_1 compatible with J and with the same volume form $d\mu$ as for the original one the image of (g_1, J, ω_1) is nontrivial.*

§5. From metric to conformal class

As we've seen in the previous section the nontriviality condition for τ is stable with respect to changing of the second element in hermitian triples. Now we'd like to consider the next step.

Proposition 3. *Let for (g, J, ω) the image $\tau(g, J, \omega) \in K^+_X$ is nontrivial. Then the same is true for any triple of the shape $(e^f \cdot g, J, e^f \cdot \omega)$ where $f : X \to \mathbb{R}$ is a smooth real function*

$$\tau(e^f \cdot g, J, e^f \cdot \omega) \neq [0].$$

Proof. Let (g, J, ω) is a triple with nontrivial image

$$\tau(g, J, \omega) = [\omega_H] \in H^2(X, \mathbb{R}) \quad | \quad [\omega_H] \neq [0]$$

where ω_H is the corresponding harmonic 2- form with respect to $*_g$ and the orientation. Consider the following integral

$$\int_X e^f \omega \wedge \omega_H. \tag{5.1}$$

Since both forms appearing in this integral are self dual this expression equals to projection of the first form $e^f \omega$ to the harmonic ray $\mathbb{R} < \omega_H >$ modulo the norm of ω_H. So to prove the Proposition 3 it's sufficient to show that this integral isn't equal to zero.

For this one can observe that if

$$(\omega, \omega_H) \in C^\infty(X \to \mathbb{R}) \tag{5.2}$$

is non negative real function then the integral above has to be strictly positive. So now our claim is to establish that the inner product (ω, ω_H), induced by our original riemannian metric g, is non negative everywhere. (And let's recall that $\omega \wedge \omega_H = (\omega \wedge \omega_H)$ everywhere because of the self duality.)

It's clear that since

$$\int_X \omega \wedge \omega_H = \int_X \omega_H \wedge \omega_H > 0 \tag{5.3}$$

(by the defintion), the function

$$s = (\omega, \omega_H)$$

has positive integral over X with respect to the volume form $d\mu_g$. So if there exist points with negative volumes of s then it should impose the following picture. There is smooth 3- dimensional submanifold $B \subset X$ which is

$$B = s^{-1}(0)$$

and such that $X \setminus B = U^+ \cup U^-$ where

$$U^{\pm} = s^{-1}(\mathbb{R}^{\pm}).$$

First of all, it's easy to see that ω_H is equal to zero form on B

$$\omega_H|_B = 0.$$

Really, let us consider smooth inclusion

$$i : B \hookrightarrow X$$

which gives us riemannian metric i^*g, nondegenerated everywhere 2- form $i^*\omega$ and inherited orientation on B. So one can use again the Hodge theorem over B and it's clear that the harmonic part of $i^*\omega$ should be equal to $i^*\omega_H$:

$$i^*\omega = i^*\omega_H + d\rho_1 + d^*\rho_2 \tag{5.4}$$

over B where $\rho_1 \in \Omega^1_B, \rho_2 \in \Omega^3_B$. Since we supposed that $(\omega, \omega_H) = 0$ over B, we immediately get

$$\int_B (i^*\omega_H, i^*\omega_H) d\mu_{i^*g} = \int_B (i^*\omega, i^*\omega_H) d\mu_{i^*g} = 0 \tag{5.5}$$

so B *has to be* a subset of zeroset for ω_H

$$\omega_H|_B = 0. \tag{5.6}$$

These arguments can be used to establish what we need. Namely, let B_1 is a smooth 3- dimensional submanifold of X such that

$$B_1 \subset U^-$$

and

$$i_1 : B_1 \hookrightarrow X$$

is the corresponding smooth inclusion. Then with respect to $i_1^* g$ and the inherited orientation over B_1 one has the same decomposition as (5.4)

$$i_1^* \omega = i_1^* \omega_H + d\rho_3 + d^* \rho_4. \tag{5.4'}$$

And the point is that

$$\int_{B_1} (i_1^* \omega_H, i_1^* \omega_H) d\mu_{i_1^* g} = \int_{B_1} (i_1^* \omega, i_1^* \omega_H) d\mu_{i_1^* g} \tag{5.5'}$$

has to be negative so we get the contradiction.

From all above one can conlcude that:

1. $(\omega, \omega_H) \geq 0$ everywhere

and moreover

2. zerosets of (ω, ω_H) and (ω_H, ω_H) coincide.

Because of this for any smooth function $f \in C^\infty(X \to \mathbb{R})$ we get

$$\begin{gathered} \int_X e^f \omega \wedge \omega_H = \int_X e^f (\omega, \omega_H) d\mu_g \geq \\ \int_X e^t (\omega, \omega_H) d\mu_g = e^t [\omega_H]^2 > 0 \end{gathered} \tag{5.7}$$

where t is the minimum of our smooth function f on our compact smooth manifold X.

Hence we get the statement of the Proposition 3.

§6. Good path, joining two triples

Only one problem remains — namely, how one can join two triples from a component $\mathcal{M}_X^{K_i} \subset \mathcal{M}_X$ by an appropriate path, covered by our Propositions 1, 2, 3 which would give us the statement of the Main Theorem. Good path, joining two triples (g_0, J_0, ω_0) and (g_1, J_1, ω_1) can be constructed using the following

"Junction" Lemma. *Let (g_0, J_0, ω_0) and (g_1, J_1, ω_1) be a pair of hermitian triples over X with the same canonical class $K_{J_i} = K \in H^2(X, \mathbb{Z})$. Then there exist an almost complex structure J_{junct} with the same canonical class together with two riemannian metrics g_p, g_q such that*

1) g_p is compatible similtaneously with J_0 and J_{junct}

and

2) g_q is compatible similtaniously with J_{junct} and J_1.

So we'll construct a good path in $\mathcal{M}_X^K$ joining two given triple such that this path consists of five "linear" pathes — the first one is induced by changing of riemannian metric with the fixed almost complex structure J_0, the second is induced by changing of almost complex structures with the fixed riemannian metric g_p, the third is induced by changing of riemannian metric with the fixed almost complex structure J_{junct}, the fourth is induced by changing of almost complex structures with the fixed riemannian metric g_q. Of course, one has to impose the last "line" — changing of riemannian metrics from g_q to g_1 with fixed almost complex structure J_1.

The main step in this procedure is to construct an appropriate almost complex structure J_{junct} so we begin with the explanation of this point.

We work locally over a point $x \in X$. So we have in $\mathbb{CP}^3 = \mathbb{P}(T_x^{\mathbb{C}} X)$ two pairs of projective lines $l_i, \Theta_x(l_i), i = 0, 1$, which correspond to our given almost complex structures J_0, J_1. It's clear that in general there is no a riemannian metric (i.e.: a nondegenerated real quadric) in general which contains all four projective lines. Let's consider what we have in $\mathbb{CP}^5 = \mathbb{P}(\Lambda^2 T_x^{*\mathbb{C}} X)$. There are two pairs of points on the Grassmanian $Gr \subset \mathbb{CP}^5$ and our four projective lines lie on the same quadric in $\mathbb{CP}^3$ if and only if the corresponding four points in $\mathbb{CP}^5$ lie on the same real 2-plane (which is $\mathbb{P}(\Lambda_x^+)$ of course).

So given these four points $l_0, \Theta_x(l_0), l_1, \Theta_x(l_1)$ our aim is to construct such pair of points $l_{junct}, \Theta_x(l_{junct})$ that:

the span $< l_i, \Theta_x(l_i), l_{junct}, \Theta_x(l_{junct}) >$ is equal to projective 2- plane $\mathbb{P}_i^2, i = 0, 1,$. The point is that these two planes are automatically real (so $\Theta_x(\mathbb{P}_i^2) = \mathbb{P}_i^2$) and one can choose g_p and g_q such that $\mathbb{P}_0^2 \subset \mathbb{CP}^5$ is the projectivization of $\Lambda_{g_p}^+$ and $\mathbb{P}_1^2$ is the projectivization of $\Lambda_{g_q}^+$.

So the remaining part of the proof is just an exercise in classical projective geometry. To construct two conjugated points $l_{junct}, \Theta_x(l_{junct})$ one can use the following procedure. Let us take the set of all projective 2-planes which contain the projective line $< l_0, l_1 >$. Then if we find such a 2- plane containing this line that it has *two real points* r_0, r_1 (such points that

$$\Theta_x(r_i) = r_i)$$

our problem will be solved. Really, let's suppose that we've found such projective 2- plane π that

$$l_0, l_1, r_0, r_1 \in \pi$$

where l_0, l_1 our given points and r_0, r_1 some distinct real points. Then there exists conjugated 2- plane $\Theta_x(\pi)$ which contains the points $\Theta_x(l_0), \Theta_x(l_1), r_0, r_1$. So one can construct the following two points

$$\begin{aligned} l'_{junct} &= < l_0, r_0 > \cap < l_1, r_1 > \in \pi \\ \Theta_x(l'_{junct}) &= < \Theta_x(l_0), r_0 > \cap < \Theta_x(l_1), r_1 > \in \Theta_x(\pi). \end{aligned} \tag{6.1}$$

It's easy to see that projective lines $< l_0, \Theta_x(l_0) >$ and $< l'_{junct}, \Theta_x(l'_{junct}) >$ lie in the same projective 2- plane (which *isn't our* π!) and the same is true for projective lines $< l_1, \Theta_x(l_1) >$ and $< l'_{junct}, \Theta_x(l'_{junct}) >$.

But two constructed points aren't what we need because a priori these points don't lie on our Grassmanian Gr so don't represent any lines in our $\mathbb{CP}^3$. So one has to construct the intersection

$$< l'_{junct}, \Theta_x(l'_{junct}) > \cap Gr = \{l_{junct}, \Theta_x(l_{junct})\} \tag{6.2}$$

and these two points would be our l_{junct} and $\Theta_x(l_{junct})$. It's not hard to see that this intersection is a pair of points indeed because our quadric $Gr \subset \mathbb{CP}^5$ is real with respect to Θ_x and our projective line $< l'_{junct}, \Theta_x(l'_{junct}) >$ is real but without real points (by the construction).

So it remains to explain why there exists such appropriate projective 2- plane π which contains given two points l_0, l_1 and a pair of distinct real points.

We construct π as follows:

1. consider the span

$$< l_0, l_1, \Theta_x(l_0), \Theta_x(l_1) >= \mathbb{CP}^3 \subset \mathbb{CP}^5; \tag{6.3}$$

2. our standard real structure Θ_x restricted on $\mathbb{CP}^3$ gives us *a standard real structure* (which we denote as Θ'_x) on $\mathbb{CP}^3$;

3. since Θ'_x is a standard real structure there exists invariant with respect to Θ'_x subset $\mathbb{RP}^3 \subset \mathbb{CP}^3$;

4. consider a 2- plane π, containing the projective line $< l_0, l_1 >$ and lieing in our $\mathbb{CP}^3$;

5. since every two projective planes in $\mathbb{CP}^3$ have non trivial intersection (which is a projective line) we get the projective line

$$\rho = \pi \cap \Theta'_x(\pi); \tag{6.4}$$

6. this projective line ρ is invariant with respect to the Θ'_x- action (i.e. is a real projective line)

$$\Theta'_x(\rho) = \rho;$$

7. every real projective line in $\mathbb{CP}^3$ has at least two real points.

One can see that really *every* 2- plane, contained in the span (6.3), satisfies the required property (i.e. has two distinct real points).

We have to check only step 2 and step 7 from the list above. The other steps are rather obvious.

First of all, let us recall that there exist two types of real structure on projective spaces. The first one is standard real structure and in appropriate homogeneous coordinates in $\mathbb{CP}^n$ it has the form

$$\Theta_s(z_0 : ... : z_n) = (\overline{z}_0 : ... : \overline{z}_n).$$

Evedentely, it has real points. The second type exists only on $\mathbb{CP}^n$ with n odd. In this odd dimensional case we have quaternionic real structure:

$$\Theta_q(z_0 : z_1 : ... : z_{n-1} : z_n) = (-\overline{z}_1 : \overline{z}_0 : ... : -\overline{z}_n : \overline{z}_{n-1}).$$

This quaternionic structure has no real point (it can be checked directly).

Step 2. Since our $\mathbb{CP}^3$ is preserved by Θ_x (the four points are transformed ones to the others) we get well defined real structure Θ'_x on $\mathbb{CP}^3$. We have to establish that this restricted Θ'_x is a standard real structure. For this it is sufficient to show that there exists a real point (because there are only two possibilities — standard and quaternionic). So consider the projective line $< l_0, \Theta_x(l_0) >\subset \mathbb{CP}^3 \subset \mathbb{CP}^5$. This line is preserved by the real structure Θ_x. Hence this reduction of the step 2 together with the step 7 will be proved if we prove the following

Statement. *Let a projective line $\rho \in \mathbb{CP}^n$ is preserved by a fixed standard real structure Θ_s. Then there exist at least two real points on this projective line.*

Proof. Let us fix appropriate homogeneous coordinates on $\mathbb{CP}^n$ such that our given real structure Θ_s has the form (??) in this coordinate system. Then consider a system of homogeneous equations in the same coordinates, corresponding to our projective line ρ:

$$\sum_{i=0}^{n} \alpha_j^i z_i = 0, \tag{6.5}$$

$\alpha_j^i \in \mathbb{C}, j = 1, ..., n-1$. Because of the Θ_s- invariance we get that the points of ρ satisfy the following system

$$\sum_{i=0}^{n} \alpha_j^i \overline{z}_i = 0 \tag{6.5'}$$

as well as the original one. Then it is easy to see that the points

$$r_0 = (Rez_0 : ... : Rez_0), \quad r_1 = (Imz_0 : ... : Imz_0)$$

should satisfy both systems. So we've found two real points.

Of course, there exist not only two real points on ρ. There is a real projective line

$$\rho_{\mathbb{R}} \cong \mathbb{RP}^1, \quad \rho_{\mathbb{R}} \subset \rho$$

containing real points, but for our aim it is sufficient to find only two distinct real points.

With the Statement in hand we prove the step 2 and the step 7, applying this result to the case $n = 5$ and the case $n = 3$ respectively.

It remains to note that one can globalize this local picture over whole X (using putching arguments) and then deduce that such good path always exists.

So now we have a good "partially linear" path joinig two given hermitian triples, which were choosen arbitrary, and then applying Propositions 1, 2 and 3 along this path one gets the statement of our Main Theorem.

§7. Additional remarks

First of all it's necessory to say that the Main Theorem above gives us new invariants of smooth structures in 4- geometry.

Really, it's clear that for two diffeomorphic 4- manifolds X_1, X_2 with some diffeomorphism

$$\phi : X_1 \to X_2$$

if $K_2 \in H^2(X_2, \mathbb{Z})$ has *nontrivial generalized image* in $H^2(X_2, \mathbb{R})$ (so if for every triple (g, J, ω) with $K_J = K_2$ the image $\tau(g, J, \omega) \neq [0]$) the corresponding class $\phi^* K_2 \in H^2(X_1, \mathbb{Z})$ should be "nontrivial" in this sense too.

Let us define the following subset in $H^2(X, \mathbb{Z})$ for a smooth compact orientable 4- manifold X. A class $K_i \in H^2(X, \mathbb{Z})$ is called *PS*- class if the generalized image is nontrivial. The subset $N_{PS} \subset H^2(X, \mathbb{Z})$ *is preserved by the group* $Diff^+X$ (as the basic classes in the gauge theories). Of course, the generalization of the Taubes result ([9]) gives us the following inclusion

$$N_{PS} \subset \{the \quad set \quad of \quad basic \quad classes\}$$

but we'd like to emphercise that our definition of the PS- classes works in the case $b_2^+ = 1$ as well as in the "classical" for the gauge theories case $b_2^+ > 1$ (of course, if $b_2^+ = 0$ the subset N_{PS} is empty).

But one could try to derive some more think invariants of smooth structures from the construction of the canonical map τ. Let us consider the complete image $Im\tau \subset K^+ \subset H^2(X, \mathbb{R})$. It's clear again that

1. $Im\tau$ is a "subcone" in K^+. The point is that if some point $k \in K^+$ is realized as the image of a triple (g, J, ω) then for every positive real number $r > 0$ the point $r \cdot K \in K^+$ is realized too;

2. "subcone" $Im\tau$ is a connected subset and the canonical map τ is a contineous map;

3. it's quite natural to projectivize this "subcone" and then the topology of the porjective "manifold" reflects properties of the underling smooth structure. Namely if there are two smooth structures on the same topological based manifold then two corresponding "projectivized subcones" should be topologically ismorphic one to other.

But now it's the time to finish this article hoping that the new notion of pseudo symplectic manifolds inspired by the considerations in the framework of Seiberg — Witten theory will lead to new results in the smooth classification of 4- manifolds.

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MPI, Bonn
E-mail address: tyurin@tyurin.mccme.ru jtyurin@mpim-bonn.mpg.de