On the degree and the birationality of the second adjunction mapping*

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Abstract

Let $\mathcal{L}$ be a very ample line bundle on $\mathcal{M}$, a projective manifold of dimension $n \geq 3$. Under the assumption that $K_{\mathcal{M}} + (n-2)\mathcal{L}$ has Kodaira dimension n, we study the degree of the map ϕ associated to the complete linear system $|2(K_{\mathcal{M}} + (n-2)\mathcal{L})|$. In particular we show that under a number of conditions, e.g., $n \geq 5$ or $K_{\mathcal{M}} + (n-3)\mathcal{L}$ having nonnegative Kodaira dimension, the degree ϕ is one, i.e., ϕ is birational. We also show that under a mild condition on the linear system $|K_{\mathcal{M}} + (n-2)\mathcal{L}|$ satisfied for all known examples, ϕ is birational unless $(\mathcal{M}, \mathcal{L})$ is a three dimensional variety with very restricted invariants. Moreover there is an example with these invariants such that $\deg \phi = 2$.

Introduction

Let $\mathcal{L}$ be a very ample line bundle on an n-dimensional projective manifold $\mathcal{M}$. We assume throughout this introduction that $n \geq 3$ and also that $\kappa(K_{\mathcal{M}}+(n-2)\mathcal{L}) = n$, i.e., that the Kodaira dimension of $K_{\mathcal{M}}+(n-2)\mathcal{L}$ is n. This condition on the Kodaira dimension is satisfied except for a short list of special varieties $(\mathcal{M}, \mathcal{L})$ (see e.g., [2]).

The condition $\kappa(K_{\mathcal{M}} + (n-2)\mathcal{L}) = n$ implies that $\mathcal{M}$ is the blowing up of a projective manifold M, $\pi : \mathcal{M} \to M$, at a finite set such that $K_{\mathcal{M}} + (n-2)\mathcal{L} \cong \pi^*(K_M + (n-2)L)$, where $L := (\pi_*\mathcal{L})^{**}$ is an ample line bundle, $K_M + (n-1)L$ is very ample, and $\mathcal{K}_M := K_M + (n-2)L$ is nef. Thus the a priori meromorphic map associated to $|2(K_{\mathcal{M}} + (n-2)\mathcal{L})|$ factors as π composed with the mapping associated to $|2\mathcal{K}_M|$. It is a theorem of the second author (see [2, (13.2.5)]) that in the situation of this paper $2\mathcal{K}_M$ is spanned. Thus we are reduced to describing the structure of the morphism $\phi : M \to \mathbb{P}_{\mathbb{C}}$ associated to $|2\mathcal{K}_M|$. The morphism ϕ factors as $r \circ s$, where $r : M \to Y$ is a birational morphism with connected fibers onto a normal projective variety Y and s is finite. The structure of r is completely understood; cf., [2, 12] for a description of this map.

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In Theorem (2.1) we show that under the added assumption that there is a smooth divisor $A \in |\mathcal{K}_M|$ the map s (and hence ϕ) is birational unless $n = 3$, $\mathcal{K}_M^3 = 1$, $\mathcal{K}_M^2 \cdot L = 3$, $\mathcal{K}_M \cdot L^2 = 9$, and A is a Del Pezzo surface with $K_A^2 = 1$. There is moreover a degree 27 threefold $M \subset \mathbb{P}^{13}$ with these invariants and with $\deg \phi = 2$ (see (2.2)). Though we expect such a smooth A to exist always, this is not known. We show that, without any assumption on the existence of a smooth $A \in |\mathcal{K}_M|$,

1) the same conclusion holds under the added hypothesis that $n = \dim M \geq 5$ (see Corollary (2.4));

2) the morphism ϕ is birational if $\kappa(K_{\mathcal{M}} + (n-3)\mathcal{L}) \geq 0$ (see Theorem (2.3)); and

3) $\deg \phi \leq 7$ if $n = 3$ (see Theorem (3.1)).

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1 Background material

We work over the complex field $\mathbb{C}$. Throughout the paper we deal with projective varieties V. We denote by $\mathcal{O}_V$ the structure sheaf of V and by K_V the canonical bundle. For any coherent sheaf $\mathcal{F}$ on V, $h^i(\mathcal{F})$ denotes the complex dimension of $H^i(V, \mathcal{F})$.

Let L be a line bundle on V. The line bundle L is said to be *numerically effective* (*nef*, for short) if $L \cdot C \geq 0$ for all effective curves C on V. L is said to be *big* if $\kappa(L) = \dim V$, where $\kappa(L)$ denotes the Kodaira dimension of L. If L is nef then this is equivalent to $c_1(L)^n > 0$, where $c_1(L)$ is the first Chern class of L and $n = \dim V$.

1.1 Notation. The notation used in this paper are standard from algebraic geometry. Let us only fix the following.

$\approx$, the linear equivalence of line bundles;

$\chi(L) = \sum_i (-1)^i h^i(L)$, the Euler characteristic of a line bundle L;

$|L|$, the complete linear system associated to a line bundle L;

$\Gamma(L)$, the space of the global sections of a line bundle, L, on a variety V; we say that L is *spanned* if it is spanned at all points of V by $\Gamma(L)$;

$q(V) = h^1(\mathcal{O}_V)$, the irregularity, for V smooth;

$\kappa(D)$, the Kodaira dimension of the line bundle associated to a Cartier divisor D on V;

$\kappa(V) := \kappa(K_V)$, the Kodaira dimension of V, for V smooth.

Line bundles and divisors are used with little (or no) distinction. We almost always use the additive notation.

1.2 Genus formula. For a line bundle L on an irreducible normal variety V of dimension n the sectional genus, $g(L) = g(V, L)$, of (V, L) is defined by $2g(L) - 2 = (K_V + (n-1)L) \cdot L^{n-1}$. Note that if $|L|$ contains $n-1$ elements $H_1, \dots, H_{n-1}$ meeting in a reduced irreducible curve C contained in the smooth points of V, then $g(L) = g(C) = 1 - \chi(\mathcal{O}_C)$, the arithmetic genus of C.

1.3 Castelnuovo's bound. Let C be a reduced irreducible projective curve. Assume that $\psi : C \to \mathbb{P}^N$ is a generically one-to-one morphism, and that $\psi(C)$ does not lie in any hyperplane. Let d denote the degree of $\psi(C)$ in $\mathbb{P}^N$. Let $g(C)$ be the arithmetic genus of C. Then Castelnuovo's bound (see, e.g., [6, Theorem 3.7]) reads

$$g(C) \le \mathrm{Castel}(d, N) := \left[\frac{d-2}{N-1}\right]\left(d - N - \left(\left[\frac{d-2}{N-1}\right] - 1\right)\frac{N-1}{2}\right), \qquad (1)$$

where $[x]$ means the greatest integer $\le x$.

1.4 Reductions. (See e.g., [2, Chapters 7, 12]). Let $(\mathcal{M}, \mathcal{L})$ be a smooth variety of dimension $n \ge 2$ polarized with a very ample line bundle $\mathcal{L}$. A smooth polarized variety (M, L) is called a *reduction* of $(\mathcal{M}, \mathcal{L})$ if there is a morphism $\pi : \mathcal{M} \to M$ expressing $\mathcal{M}$ as the blowing up of X at a finite set of points, B, such that $L := (\pi_* \mathcal{L})^{**}$ is ample and $\mathcal{L} \approx \pi^* L - [\pi^{-1}(B)]$ or, equivalently, $K_{\mathcal{M}} + (n-1)\mathcal{L} \approx \pi^*(K_M + (n-1)L)$.

Note that there is a one-to-one correspondence between smooth divisors of $|L|$ which contain the set B and smooth divisors of $|\mathcal{L}|$.

Except for an explicit list of well understood pairs $(\mathcal{M}, \mathcal{L})$ (see in particular [2, §§7.2, 7.3, 7.4]) we can assume:

a) $K_{\mathcal{M}} + (n-1)\mathcal{L}$ is spanned and big, and $K_M + (n-1)L$ is very ample. Note that this reduction, (M, L), is unique up to isomorphism. We will refer to it as *the first reduction* of $(\mathcal{M}, \mathcal{L})$;

b) $K_M + (n-2)L$ is nef and big, for $n \ge 3$.

Since by the above we can assume that $K_M + (n-2)L$ is nef and big, from the Kawamata-Shokurov base point free theorem (see [9, §3]) we know that $|m(K_M + (n-2)L)|$, for $m \gg 0$, gives rise to a morphism $\varphi : M \to X$ with connected fibers and normal image. Thus there is an ample line bundle $\mathcal{K}$ on X such that

$K_M + (n-2)L \approx \varphi^*\mathcal{K}$. The morphism φ is very well behaved (see e.g., [2, §§7.5, 7.6, 7.7 and Chapter 12]). Furthermore X has terminal, 2-Gorenstein (i.e., $2K_X$ is a line bundle) isolated singularities and $\mathcal{K} \approx K_X + (n-2)\mathcal{D}$, where $\mathcal{D} := (\varphi_* L)^{**}$ is a 2-Cartier divisor such that $2L \approx \varphi^*(2\mathcal{D}) - \Delta$ for some effective divisor Δ on X which is φ-exceptional and $\dim \varphi(\Delta) \leq 1$ (see [2, (7.5.7)]). The pair $(X, \mathcal{D})$ is known as the *second reduction* of $(\mathcal{M}, \mathcal{L})$. For definition and properties of terminal singularities we also refer to [9].

We say that $(\mathcal{M}, \mathcal{L})$ is of *log-general type* if $\kappa(K_{\mathcal{M}} + (n-2)\mathcal{L}) = n$. Notice that this is equivalent to saying that $K_M + (n-2)L$ is nef and big (see [2, (7.6.9)]). Let $\widehat{S}$ be the smooth surface obtained as transversal intersection of $n-2$ general members of $|\mathcal{L}|$ and let $S := \pi(\widehat{S})$ be the corresponding smooth surface in M. Since $K_M + (n-2)L$ is nef and big the canonical bundle K_S of S is nef and big, so that S is a minimal surface of general type (see also [2, (7.6.10)]). We have

$$K_S \cdot K_S < 9\chi(\mathcal{O}_S). \tag{2}$$

Indeed, Miyaoka's inequality yields $K_S \cdot K_S \leq 9\chi(\mathcal{O}_S)$. Note that the equality cannot happen. Otherwise S is ball quotient and hence a $K(\pi, 1)$, which contradicts [2, (1.3)].

For further properties of log-general type polarized pairs see, e.g., [2, §13.2] and [3, (0.10)].

1.5 Pluridegrees. Let $(\mathcal{M}, \mathcal{L})$, (M, L) be as in (1.4). Define the *pluridegrees*, for $j = 0, \ldots, n = \dim \mathcal{M}$, by

$$\widehat{d_j} := (K_{\mathcal{M}} + (n-2)\mathcal{L})^j \cdot \mathcal{L}^{n-j} \ \text{ and } \ d_j := (K_M + (n-2)L)^j \cdot L^{n-j}.$$

If γ denotes the number of points blown up under $\pi : \mathcal{M} \to M$, then because $K_{\mathcal{M}} + (n-2)\mathcal{L} \approx K_M + (n-2)L + \sum_i E_i$, E_i the exceptional divisors, the invariants $\widehat{d_j}$, d_j are related by

$$\widehat{d_j} = d_j - (-1)^j \gamma.$$

We put $\hat{d} := \widehat{d_0}$, $d := d_0$. If $K_M + (n-2)L$ is nef, by the generalized Hodge index theorem (see e.g., [2, (2.5.1), (13.1)]) one has

$$d_j^2 \geq d_{j-1} d_{j+1}, \ \ j = 1, \ldots, n-1, \tag{3}$$

and the parity Lemma (13.1.1) of [2] says that

$$d_j \equiv d_{j+1} \bmod(2) \ \text{ for } \ j = 0, \ldots, n-1 \ \text{ and } \ j \text{ even}. \tag{4}$$

Moreover if $K_M + (n-2)L$ is nef and big, i.e., if $(\mathcal{M}, \mathcal{L})$ is of log-general type, the d_j's are positive.

If the second reduction, $(X, \mathcal{D})$, $\varphi : M \to X$, with $\mathcal{D} = (\varphi_* L)^{**}$, $\mathcal{K} \approx K_X + (n-2)\mathcal{D}$, of $(\mathcal{M}, \mathcal{L})$ exists, then we define

$$d'_j := \mathcal{K}^j \cdot \mathcal{D}^{n-j}, \ \ j = 0, \ldots, n, \ d' := d'_0.$$

Note that $d_j = d'_j$ for $j \geq 2$. To see this recall that $2L \approx \varphi^*(2\mathcal{D}) - \Delta$ for some effective Cartier divisor Δ which is φ-exceptional (see (1.4)) and compute, for $j \geq 2$,

$$\begin{aligned} 2^{n-j}d_j &= (K_M + (n-2)L)^j \cdot (2L)^{n-j} \\ &= (\varphi^*\mathcal{K})^j \cdot (\varphi^*(2\mathcal{D}) - \Delta)^{n-j} = 2^{n-j}\mathcal{K}^{n-j} \cdot \mathcal{D}^j = 2^{n-j}d'_j, \end{aligned}$$

where the last but one equality follows from the fact that $\dim \varphi(\Delta) \leq 1$.

The following is a consequence of the Tsuji inequality (see [13, (5.2)]), the log version of the usual Yau inequality (see also [2, (13.1.7), (13.1.8)]).

Proposition 1.6 (Tsuji inequality) *Let $\mathcal{M}$ be a smooth 3-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that the first reduction, (M, L), of $(\mathcal{M}, \mathcal{L})$ exists. Let d_j, $0 \leq j \leq 3$, be the pluridegrees of (M, L). Assume that $K_M + L$ is nef. Then we have*

$$(K_M + L)^3 + \frac{8}{3}K_S \cdot L_S \leq 32(2h^0(K_M + L) - \chi(\mathcal{O}_S)), \text{ or}$$

$$h^0(K_M + L) \geq \frac{\chi(\mathcal{O}_S)}{2} + \frac{d_1}{24} + \frac{d_3}{64}.$$

The following lower bound for the degree is not optimal, but it is sufficient for our purposes.

Lemma 1.7 *Let $\mathcal{M}$ be a smooth n-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let (M, L) be the first reduction of $(\mathcal{M}, \mathcal{L})$. Then either $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^{n+1}$ or $d \geq \widehat{d} := \mathcal{L}^n \geq 8$.*

Proof. We can assume that $\Gamma(\mathcal{L})$ embeds $\mathcal{M}$ in $\mathbb{P}^N$ with $N \geq n+2$. Let $\widehat{S}$ be the smooth surface obtained as transversal intersection of $n-2$ general members of $|\mathcal{L}|$. Since $\widehat{S}$ is of general type we have by a result of Castelnuovo (see [2, (8.1)], [7, (0.6)]) that $\widehat{d} > 2(N-n)+2 \geq 6$. Thus $d \geq \widehat{d} \geq 7$. Assume $\widehat{d} = 7$. Then $N = n+2$ since otherwise $\widehat{d} > 2(N-n)+2 \geq 8$. Thus by Castelnuovo's bound we conclude that $g(C) \leq 6$ for any smooth curve section C of $\mathcal{M} \subset \mathbb{P}^{n+2}$. Therefore, by the genus formula, $7 + d_1 \leq 7 + \widehat{d}_1 = 2g(C) - 2 \leq 10$, or $d_1 \leq 3$. But then $d_2 \leq 1$ by the Hodge index relation $d_1^2 \geq dd_2$. Since $(\mathcal{M}, \mathcal{L})$ is of log-general type we know that $d_j \geq 1$ for each $j = 1, \ldots, n$. Therefore we conclude that $d_2 = 1$ and $d_2^2 \geq d_1d_3$ implies $d_1 = 1$. This gives the absurdity that $d = 1$. Q.E.D.

2 Two birationality results for the second adjunction mapping

Let $\mathcal{M}$ be a smooth connected n-fold polarized by a very ample line bundle $\mathcal{L}$, $n \geq 3$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + (n-2)\mathcal{D}$, be the first and the second reductions of $(\mathcal{M}, \mathcal{L})$ as in (1.4).

Set $\mathcal{K}_M := K_M + (n-2)L$. In this section we study the birationality of the map given by the complete linear system $|2\mathcal{K}_M|$ under the assumption that $|K_M+(n-2)L|$ contains a smooth element (see (2.1)). We consider this result to be one of the most important results in this paper since it is likely optimal. We also prove the birationality of $|2\mathcal{K}_M|$ in the case when $\kappa(K_{\mathcal{M}} + (n-3)\mathcal{L}) \geq 0$ (see (2.3)).

Note that, since $\mathcal{K}_M \approx \varphi^*\mathcal{K}$, the map associated to $|2\mathcal{K}|$ is a birational morphism if and only if the map associated to $|2\mathcal{K}_M|$ is a birational morphism.

Let us recall the following basic result of the second author (see e.g., [2, (13.2.5)]).

- With the notations as above, assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Then $2\mathcal{K}_M$ is spanned by its global sections.

The first result we have is the following.

Theorem 2.1 *Let $\mathcal{M}$ be a smooth connected n-fold polarized by a very ample line bundle $\mathcal{L}$, $n \geq 3$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$ be the first reduction of $(\mathcal{M}, \mathcal{L})$. Let $\mathcal{K}_M := K_M + (n-2)L$. Assume that there exists a smooth element A in the complete linear system $|\mathcal{K}_M|$. Then the map associated to $|2\mathcal{K}_M|$ is a birational morphism unless $n = 3$, $d_3 = 1$, $d_2 = 3$, $d_1 = 9$ and A is a Del Pezzo surface with $K_A^2 = 1$.*

Proof. Let L_A be the restriction of L to A. Consider on M the exact sequence

$$0 \to K_M + (n-2)L \to 2\mathcal{K}_M \to K_A + (\dim A - 1)L_A \to 0.$$

Since $2\mathcal{K}_M$ is spanned by $\bullet$) and $h^1(K_M + (n-2)L) = 0$ we see that $K_A + (\dim A - 1)L_A$ is spanned and defines a morphism ψ_A which coincides with the restriction, ϕ_A, of the morphism ϕ associated to $|2\mathcal{K}_M|$. Note that the connected part of the morphism ψ_A is the first reduction map, $\pi_A : A \to A'$, of the pair (A, L_A) and that the positive dimensional fibers of π_A are contained in the exceptional set of ϕ. We have $\psi_A = s_A \circ \pi_A$, where $s_A : A' \to \mathbb{P}_{\mathbb{C}}$ is a finite-to-one morphism.

Assume first $n \geq 4$. Then by a well known result due to the second author and Van de Ven (see e.g., [2, (11.3.1)]), s_A is an embedding. Let $\mathfrak{t}$ be the sheet number of ϕ, i.e., ϕ is a generically $\mathfrak{t}$-to-one morphism. Choose a general point $x \in M$. Then $\phi^{-1}(\phi(x))$ consists of $\mathfrak{t}$ distinct points. Note that, since $2A$ belongs to the linear system defining ϕ, a smooth element $A \in |\mathcal{K}_M|$ containing x contains all the $\mathfrak{t}$ points. Therefore $\psi_A^{-1}(\psi_A(x))$ consists of $\mathfrak{t}$ distinct points and hence, since s_A is an embedding, $\pi_A^{-1}(\pi_A(x))$ consists of $\mathfrak{t}$ distinct points. This contradicts the fact that π_A has connected fibers. Thus we conclude that $\mathfrak{t} = 1$.

It remains to consider the case when $n = 3$. In this case exactly the same argument works unless the polarized surface (A, L_A) is one of the four exceptional pairs described e.g., in [2, (10.3.1)].

The first possibility is that A is a degree 9 Del Pezzo surface, $K_A^2 = 1$, $L_A \approx -3K_A$. Then $L_A \approx -3(2K_M + L)_A$, or $(6K_M + 4L)_A \approx \mathcal{O}_A$. It thus follows that $(3\mathcal{K}_M - L)_A \approx \mathcal{O}_A$ and hence, since $A \in |\mathcal{K}_M|$, we have $(3\mathcal{K}_M - L) \cdot \mathcal{K}_M \cdot D = 0$ for any effective divisor D on M. Therefore, by taking $D = A$ and $D = L$, we find

$3d_3 = d_2$ and $3d_2 = d_1$ respectively. Thus $d_1 = 9d_3$. If $d_3 = 1$, we have $d_1 = 9$, $d_2 = 3$ and we find the 3-dimensional exceptional case as in the statement. Note that the case $d_3 \geq 2$ cannot occur, since in this case $d_1 \geq 18$ and we get the numerical contradiction $9 = L_A \cdot L_A = L \cdot L \cdot \mathcal{K}_M = d_1 \geq 18$.

The second possibility is that A is a degree 8 Del Pezzo surface, $K_A^2 = 2$, $L_A \approx -2K_A$. The same argument as in the case considered above gives now $(4\mathcal{K}_M - L)_A \approx \mathcal{O}_A$ and hence $d_1 = 4d_2$, $d_2 = 4d_3$. Thus we find the numerical contradiction $8 = L_A \cdot L_A = L \cdot L \cdot \mathcal{K}_M = d_1 = 16d_3$.

The third possibility is when A is the blowing up at one point, $r : A \to A'$, of a degree 8 Del Pezzo surface (A', L') with $L' \approx -2K_{A'}$, $K_{A'}^2 = 2$. In this case $2K_A + L_A \approx r^*(2K_{A'} + L') + \ell \approx \ell$, where ℓ is the exceptional line of r. Therefore $2K_A + L_A$ has a section. Since $A \in |\mathcal{K}_M|$ it follows from $K_A \approx (2\mathcal{K}_M - L)_A$ that $(4K_M + 3L)_A \approx (4\mathcal{K}_M - L)_A$ has a section. Then from

$$(4\mathcal{K}_M - L) \cdot \mathcal{K}_M^2 = (4\mathcal{K}_M - L)_A \cdot \mathcal{K}_{M|A} = (2K_A + L_A) \cdot \mathcal{K}_{M|A} = \ell \cdot \mathcal{K}_{M|A} = 0$$

and

$$(4\mathcal{K}_M - L) \cdot \mathcal{K}_M \cdot L = (2K_A + L_A) \cdot L_A = \ell \cdot L_A = 1$$

we get $4d_3 = d_2$ and $4d_2 = d_1 + 1$ respectively. But $d_1 = \mathcal{K}_M \cdot L \cdot L = L_A \cdot L_A = 7$, so that we find the numerical contradiction $7 = d_1 = 16d_3 - 1 \geq 15$.

The fourth possibility is when A is a $\mathbb{P}^1$-bundle over a smooth elliptic curve of invariant $e = -1$, $L_A \approx 3E$, E a section of minimal self-intersection. Let $f \cong \mathbb{P}^1$ be a fiber of the $\mathbb{P}^1$-bundle. Then $K_A \approx -2E + f$ and $(K_A + L_A) \cdot L_A = (E + f) \cdot 3E = 6$. Thus from $2L_A \cdot \mathcal{K}_{M|A} = (K_A + L_A) \cdot L_A = 6$ we get $L_A \cdot \mathcal{K}_{M|A} = L \cdot \mathcal{K}_M \cdot \mathcal{K}_M = d_2 = 3$. Moreover $d_1 = L_A \cdot L_A = 9$, so that the Hodge inequality $d_1 \cdot d_3 \leq d_2^2$ yields $d_3 = 1$. This gives the contradiction $q(\mathcal{M}) = 0$ ([4, (3.2)]). Q.E.D.

Example 2.2 Notation as in (2.1). Note that the exceptional case of a polarized 3-fold (M, L) with the morphism associated to $|2(K_M + L)|$ not birational and with invariants $d_3 = 1$, $d_2 = 3$, $d_1 = 9$ as in (2.1) really occurs. The example in [10] has $\mathcal{L} = 3H$ very ample with $\mathcal{K}_{\mathcal{M}} \cong -2H$. Thus all reduction maps are isomorphisms with $\mathcal{K}_M = K_M + L \approx H$ having $h^0(\mathcal{K}_M) = 3$, $d_3 = \mathcal{K}_M^3 = 1$, $d_2 = \mathcal{K}_M^2 \cdot \mathcal{L} = 3H^3 = 3$, $d_1 = \mathcal{K}_M^2 \cdot \mathcal{L} = 3H^3 = 9$. The morphism ϕ associated to $|2\mathcal{K}_M|$ has degree 2 (see also [11, (5.3)]). Indeed, from the exact sequence

$$0 \to \mathcal{K}_M \cong H \to 2H \to 2H_H \to 0,$$

we obtain $h^0(2\mathcal{K}_M) = 7$ and $\phi : M \to \mathbb{P}^6$ maps M surjectively onto the cone $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ over $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$. Since $(2\mathcal{K}_M)^3 = 8$ and the tautological bundle of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ satisfies $\xi^3 = 4$ we conclude that ϕ has degree 2.

A second result we have is the following.

Theorem 2.3 *Let $\mathcal{M}$ be a smooth connected n-fold polarized by a very ample line bundle $\mathcal{L}$, $n \geq 3$. Assume that $\kappa(K_{\mathcal{M}} + (n-3)\mathcal{L}) \geq 0$. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$ be the first reduction of $(\mathcal{M}, \mathcal{L})$. Let $\mathcal{K}_M := K_M + (n-2)L$. Then the map associated to $|2\mathcal{K}_M|$ is a birational morphism.*

Proof. We claim that $d_2 := K_S^2 \geq 10$, where $S = \pi(\widehat{S})$ and $\widehat{S}$ is the transversal intersection of $n-2$ general members of $|\mathcal{L}|$. To see this consider the 3-fold section $\mathcal{V} = \mathcal{V}_3$ of $\mathcal{M}$ obtained as transversal intersection of $n-3$ general members of $|\mathcal{L}|$. Let $\mathcal{L}_{\mathcal{V}}$ be the restriction of $\mathcal{L}$ to $\mathcal{V}$. Let $V := \pi(\mathcal{V})$ and let L_V be the restriction of L to V. Then the reduction is compatible with the restriction, i.e., (V, L_V) is the first reduction of $(\mathcal{V}, \mathcal{L}_{\mathcal{V}})$. Let $d = d_0$, d_1, d_2, d_3 be the pluridegrees of (V, L_V) as in (1.5), so that $d_2 = K_S^2$. Note that the assumption $\kappa(K_{\mathcal{M}} + (n-3)\mathcal{L}) \geq 0$ implies $\kappa(\mathcal{V}) \geq 0$. Then from [12, (1.5), (3.1)] we know that

$$d_3 \geq d_2 \geq d_1 \geq d. \tag{5}$$

From Lemma (1.7) we have $d \geq 8$. First consider the case when $|\mathcal{L}_{\mathcal{V}}|$ embeds $\mathcal{V}$ in $\mathbb{P}^N$, $N \geq 6$. If $d = 8, 9$, Castelnuovo's bound (1.3) gives $g := g(\mathcal{L}_{\mathcal{V}}) \leq 5, 7$ respectively. Since $d_1 \geq d$ by (5) the genus formula $d + d_1 = 2g - 2$ gives a numerical contradiction. Content-Length: 11321 X-Lines: 228 Status: RO

Thus we can assume that $|\mathcal{L}_{\mathcal{V}}|$ embeds $\mathcal{V}$ in $\mathbb{P}^5$. By looking over the list of small degree 3-folds in $\mathbb{P}^5$ (see e.g., [5, Chapter 6]) we see that the only possible cases with $d = 8, 9$ are when $\mathcal{V}$ is the complete intersection of either a quadric and a quartic or two cubics in $\mathbb{P}^5$ and $\mathcal{L}_{\mathcal{V}} = \mathcal{O}_{\mathcal{V}}(1)$. Accordingly, $\mathcal{M}$ is the complete intersection either of type $(2,4)$ or of type $(3,3)$ in $\mathbb{P}^{n+2}$ with $\mathcal{L} \approx \mathcal{O}_{\mathcal{M}}(1)$. In both cases $K_{\mathcal{M}} + (n-2)\mathcal{L}$ is very ample. Thus from now on we can assume that $d_2 \geq 10$.

First assume $n = 3$, so that $\kappa(\mathcal{M}) \geq 0$. Recalling $\bullet$), consider the morphism $\phi : M \to Y$ associated to $|2\mathcal{K}_M|$. Assume that ϕ is not birational, i.e., is not generically one-to-one. Thus there exists a dense open set $U \subset Y$ such that for any point $y \in Y$, the fiber $\phi^{-1}(y)$ contains at least two points $x_1 := x_{1,y}$, $x_2 := x_{2,y}$ (depending on y) such that $\phi(x_1) = \phi(x_2) = y$. From a Bertini's type theorem (see e.g., [2, (1.7.9)]) we know that there is a smooth surface $\widehat{S} \in |\mathcal{L}|$ passing through $\pi^{-1}(x_1)$, $\pi^{-1}(x_2)$. Then the image $S = \pi(\widehat{S})$ is a smooth surface in $|L|$ passing through x_1, x_2.

The exact sequence

$$0 \to K_M + \mathcal{K}_M \to 2\mathcal{K}_M \to 2K_S \to 0$$

gives a surjection $\Gamma(2\mathcal{K}_M) \to \Gamma(2K_S) \to 0$. Thus we see that ϕ restricts to the bicanonical map ϕ_S associated to $|2K_S|$. Hence in particular ϕ_S is not an embedding. Recall that S is a minimal surface of general type under our assumptions (see (1.4)). Then we know by Reider's theorem (see e.g., [2, (8.5.1)]) that there exists an effective curve $C_y \subset S$, C_y depending on y and containing x_1, x_2, such that

$$K_S \cdot C_y - 2 \leq C_y \cdot C_y < \frac{K_S \cdot C_y}{2} < 2.$$

Therefore, since K_S is nef, we have $0 \leq K_S \cdot C_y \leq 3$. The case $K_S \cdot C_y = 3$ is excluded. Indeed in this case $C_y^2 = 1$ and the Hodge index relation $d_2 \leq K_S^2 C_y^2 \leq (K_S \cdot C_y)^2$ contradicts the assumption $d_2 \geq 10$.

Note that when y varies in U, the curves $\{C_y\}_{y \in U}$ cover a dense open set $\phi^{-1}(U)$ of M. This is clear since the points $x_{1,y}$, $x_{2,y}$ cover $\phi^{-1}(U)$.

Assume $K_S \cdot C_y = 0$. Then, by the genus formula, $C_y^2 = -2$ and C_y is the union of rational curves. Thus by the above, there is a dense open set $U \subset M$ covered by rational curves. This contradicts the assumption $\kappa(M) \geq 0$ (see e.g., [8, Part I, (5.8)]). Since $\kappa(M) \geq 0$, a multiple tK_M of the canonical bundle K_M is effective for some $t \gg 0$. Since the curve C_y moves it thus follows that $K_M \cdot C_y \geq 0$.

If $K_S \cdot C_y = 1, 2$, from $K_S \cdot C_y = (K_M + L) \cdot C_y \geq L \cdot C_y$ we obtain $L \cdot C_y \leq 2$. Since L is very ample outside of a finite set of points, this implies that C_y is a rational curve. Thus we conclude as in the previous case.

It remains to consider the general case when $n \geq 4$. Let $\mathcal{V}_t$ be the t-fold section of $\mathcal{M}$ obtained as transversal intersection of $n - t$ general members of $|\mathcal{L}|$, $t = 2, \ldots, n$, $\mathcal{V}_n := \mathcal{M}$, $\mathcal{V}_2 := \widehat{S}$. Let $\mathcal{L}_t$ be the restriction of $\mathcal{L}$ to $\mathcal{V}_t$. let $V_t := \pi(\mathcal{V}_t)$ and let L_t be the restriction of L to V_t. Reductions are compatible with restrictions, i.e., (V_t, L_t) is the first reduction of $(\mathcal{V}_t, \mathcal{L}_t)$. Set $\mathcal{K}_t := K_{V_t} + (t-2)L_t$. Consider the exact sequence, for $t = 3, \ldots, n-1$,

$$0 \to K_{V_t} + \mathcal{K}_t + (n-3)L_t \to 2\mathcal{K}_t \to 2\mathcal{K}_{t-1} \to 0,$$

which gives a surjective map $\Gamma(2\mathcal{K}_t) \to \Gamma(2\mathcal{K}_{t-1})$. Assuming ϕ is not birational and using induction on t, we conclude as above that the bicanonical map of the surface section $V_2 := S$ of M with respect to L is not an embedding. Recalling that the assumption $\kappa(K_{\mathcal{M}} + (n-3)\mathcal{L}) \geq 0$ implies $\kappa(\mathcal{V}_3) = \kappa(V_3) \geq 0$, the above argument shows that we can reduce to the 3-fold case. Q.E.D.

By using results from [1] we obtain from Theorem (2.3) the following rather strong consequence.

Corollary 2.4 *Let $\mathcal{M}$ be a smooth connected n-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + (n-2)\mathcal{D}$, be the first and the second reductions of $(\mathcal{M}, \mathcal{L})$. Let $\mathcal{K}_M := K_M + (n-2)L$. If $n \geq 5$, then the map associated to $|2\mathcal{K}_M|$ is a birational morphism.*

Proof. First assume $n \geq 7$. Then by [1, (3.1)] we know that $K_X + (n-3)\mathcal{K}$ is nef. Thus $t(K_X + (n-3)\mathcal{K})$ is effective for some integer $t \in \mathbb{Z}$, i.e., $K_X + (n-3)\mathcal{K}$ is $\mathbb{Q}$-effective. Therefore $\kappa(K_X + (n-3)\mathcal{K}) \geq 0$. Hence from [2, (7.6.1), (7.6.2)] we have

$$\kappa(K_X + (n-3)\mathcal{D}) = \kappa(K_{\mathcal{M}} + (n-3)\mathcal{L}) = \kappa(K_M + (n-3)L) \geq 0.$$

Thus Theorem (2.3) applies to give the result.

Assume $n = 6$. Then from [1, (4.1)] we know that $K_X + 3\mathcal{K} = K_X + (n-3)\mathcal{K}$ is nef unless $(X, \mathcal{K}) \overset{\sigma}{\cong} (\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1))$. In this case $\mathcal{K}$ is very ample, so $|2\mathcal{K}_M| = |\sigma^*\mathcal{O}_{\mathbb{P}^6}(2)|$ defines a birational morphism. Thus the same argument as above lets us conclude that $\kappa(K_M + 3L) \geq 0$ and hence that $|2\mathcal{K}_M|$ defines a birational morphism by (2.3).

Assume $n = 5$. Then from [1, (4.1)] we know that $K_X + 3\mathcal{K}$ is nef unless either $(X, \mathcal{K}) \overset{\sigma}{\cong} (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(1))$, $\mathcal{Q}$ hyperquadric in $\mathbb{P}^6$, or $L^5 = 121 (\neq \frac{3^5-1}{2})$ and X is a singular 2-Gorenstein Fano 5-fold with $2K_X \approx -7\mathcal{K}$.

In the first case, $\mathcal{K}$ is very ample, so that $|2\mathcal{K}_M| = |\sigma^*\mathcal{O}_{\mathcal{Q}}(2)|$ gives a birational morphism again by (2.3).

In the second case, note that X is not Gorenstein since otherwise $K_X \approx -7A$ for some ample line bundle A on X. This contradicts the well known fact that X has index $\leq \dim X + 1 = 6$ (see also [2, (3.3.2)]). Thus, since X is not Gorenstein, [2, (0.3.3)] applies to say that $h^0(K_{\mathcal{M}} + 2\mathcal{L}) = h^0(K_M + 2L) > 0$.

Write $2\mathcal{K}_M = \mathcal{K}_M + (K_M + 2L) + L$. Since $\mathcal{K}_M \approx \varphi^*(\mathcal{K})$, $|L|$ gives a birational map (L is very ample off a finite set of points) and $K_M + 2L$ is effective, we see that $2\mathcal{K}_M$ is the sum of a nef and big and an effective line bundle. Thus we conclude that $|2\mathcal{K}_M|$ gives a birational morphism. Q.E.D.

The following two lemmas are not essentially used in the paper, but have some interest in themselves.

Lemma 2.5 *Let $\mathcal{M}$ be a smooth connected n-fold polarized by a very ample line bundle $\mathcal{L}$, $n \geq 3$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$ be the first reduction of $(\mathcal{M}, \mathcal{L})$. Let $\mathcal{K}_M := K_M + (n-2)L$. If $h^0(K_M + \mathcal{K}_M + tL) \neq 0$ for $t \leq n-3$, then the morphism given by $|2\mathcal{K}_M|$ is birational.*

Proof. Write $2\mathcal{K}_M$ as $2\mathcal{K}_M = K_M + \mathcal{K}_M + tL + (n-2-t)L$. Since $K_M + \mathcal{K}_M + tL$ is effective, $|L|$ gives a birational map and $n-2-t > 0$ we are done. Q.E.D.

Lemma 2.6 *Let $\mathcal{M}$ be a smooth connected n-fold polarized by a very ample line bundle $\mathcal{L}$, $n \geq 3$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$ be the first reduction of $(\mathcal{M}, \mathcal{L})$. Let $\mathcal{K}_M := K_M + (n-2)L$. Let V be the t-fold section of (M, L) obtained as transversal intersection of $n-t$ general members of $|L|$. Let L_V be the restriction of L to V and let $\mathcal{K}_V := K_V + (t-2)L_V$. If $t \geq 3$ and the map given by $|2\mathcal{K}_M|$ is not birational, then*

$$h^0(K_V + \mathcal{K}_V + jL_V) = 0 \ \ for \ \ 0 \leq j \leq t-3.$$

Proof. Consider on M the Koszul complex

$$0 \to -(n-t)L \to \cdots \to \oplus^{n-t-1}(-2L) \to \oplus^{n-t}(-L) \to \mathcal{O}_M \to \mathcal{O}_V \to 0.$$

By tensoring with $\mathcal{K}_M + K_M + (n-t+j)L$ we get the exact sequence

$$0 \to K_M + \mathcal{K}_M + jL \to \cdots \to K_M + \mathcal{K}_M + (n-t+j)L \to K_V + \mathcal{K}_V + jL_V \to 0.$$

Note that $h^1(K_M + \mathcal{K}_M + jL) = 0$ by Kodaira vanishing and $h^0(K_M + \mathcal{K}_M + (n-t+j)L) = 0$ by Lemma (2.5) since $n-t+j \leq n-3$. Thus $h^0(K_V + \mathcal{K}_V + jL_V) = 0$.Q.E.D.

As a consequence of (2.5) and (2.6) we have the following general fact.

Remark 2.7 Let $\mathcal{M}$ be a smooth connected n-fold polarized by a very ample line bundle $\mathcal{L}$, $n \geq 3$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$ be the first reduction of $(\mathcal{M}, \mathcal{L})$. Let $\mathcal{K}_M := K_M + (n-2)L$. If the morphism associated to $|2\mathcal{K}_M|$ is not birational, then $h^i(\mathcal{O}_M) = 0$, $i \geq 3$.

To see this, assume first $n = 3$. Then $h^3(\mathcal{O}_M) = h^0(K_M)$. If $h^0(K_M) > 0$, we would have an inclusion $0 \to L \to K_M + L$, so that the morphism associated to $|\mathcal{K}_M| = |K_M + L|$ would be birational.

To show the statement in the general case, assume e.g., $n = 4$. Let V be the general smooth element of $|L|$ and consider the exact sequence

$$0 \to -L \to \mathcal{O}_M \to \mathcal{O}_V \to 0.$$

Look at the cohomology associated to this sequence. Note that $h^i(-L) = 0$, $i = 0, 1, 2, 3$. Moreover $h^4(\mathcal{O}_M) = h^0(K_M) = 0$ since otherwise we would have an inclusion

$$0 \to (n-2)L = 2L \to K_M + 2L = \mathcal{K}_M,$$

and hence the same contradiction as above. Let L_V be the restriction of L to V. Set $\mathcal{K}_V := K_V + 2L_V$. By the assumption, it follows that also the morphism associated to $|2\mathcal{K}_M|$ is not birational. Therefore Lemma (2.6) yields $h^0(K_V + \mathcal{K}_V) = 0$ and thus $h^0(K_V) = h^3(\mathcal{O}_V) = 0$. Then $h^3(\mathcal{O}_M) = 0$.

In the general case the same argument as above gives the result, starting from the Koszul complex

$$0 \to -(n-3)L \cdots \to \oplus^{n-4}(-2L) \to \oplus^{n-3}(-L) \to \mathcal{O}_M \to \mathcal{O}_V \to 0.$$

3 A bound for the degree of the second adjunction mapping in the case $n = 3$

We keep the notation and the assumptions as at the beginning of §2. This section is devoted to the proof of the following result.

Theorem 3.1 *Let $\mathcal{M}$ be a smooth connected threefold polarized by a very ample line bundle $\mathcal{L}$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$ be the first reduction of $(\mathcal{M}, \mathcal{L})$. Let $\mathcal{K}_M := K_M + L$. Let $\mathfrak{t}$ be the degree of the morphism, ϕ, associated to $|2\mathcal{K}_M|$. Then $\mathfrak{t} \leq 7$.*

Proof. Let d_j, $j = 0, 1, 2, 3$, $d_0 = d$, be the pluridegrees of (M, L). Let S be a smooth general member of $|L|$.

Note that we can assume $h^0(K_M + \mathcal{K}_M) = h^0(2\mathcal{K}_M - L) = 0$. Indeed otherwise $\Gamma(2\mathcal{K}_M - L)$ gives an embedding $\Gamma(L) \hookrightarrow \Gamma(2\mathcal{K}_M)$ and hence $\Gamma(2\mathcal{K}_M)$ gives a birational morphism, given on a Zariski open set by sections of $\Gamma(L)$. Therefore a Riemann-Roch computation (see [4, §6]) gives the general relation

$$4\chi(\mathcal{O}_S) + d_2 = d_3 + h, \tag{6}$$

where $h := h^0(K_M + L)$. Note also that $\mathcal{K}_M$ is nef and big. Then the exact sequence

$$0 \to K_M + \mathcal{K}_M \to 2\mathcal{K}_M \to 2K_S \to 0$$

yields $h^0(2\mathcal{K}_M) = d_2 + \chi(\mathcal{O}_S)$. Thus we obtain

$$(2\mathcal{K}_M)^3 = 8d_3 = \mathfrak{t} \deg \phi(M) \geq \mathfrak{t}(d_2 + \chi(\mathcal{O}_S) - 3). \tag{7}$$

Reasoning by contradiction, assume $\mathfrak{t} \geq 8$. Then (7) yields $d_3 \geq d_2 + \chi(\mathcal{O}_S) - 3$, which, combined with (6), reads $\chi(\mathcal{O}_S) \geq 2h - 1$.

From this and Tsuji's inequality (1.6) we infer that $\frac{d_1}{12} + \frac{d_3}{32} \leq 1$, i.e.,

$$\frac{d_1}{12} + \frac{d_3}{32} = 1. \tag{8}$$

Hence in particular $d_3 \geq 2$ since d_1 is integer. Content-Length: 4233 X-Lines: 126 Status: RO

Recall that $h \geq 2$ from [3, (1.2)]. Thus (6) and the above inequality $\chi(\mathcal{O}_S) \geq 2h - 1$ give

$$\frac{d_3 - d_2}{2} = 2\chi(\mathcal{O}_S) - 3h \geq h - 2 \geq 0,$$

so that $d_3 \geq d - 2$.

Recall that $d \geq 8$ by Lemma (1.7) and $d_3 \geq 2$ by the above. Then simply starting from $d \geq 8$, $d_1 \geq 1$, $d_2 \geq 1$ and $d_3 \geq 2$ and by using repeatedly the Hodge index inequalities (3) we obtain $d \geq 8$, $d_1 \geq 8$, $d_2 \geq 8$, $d_3 \geq 8$.

Assume $\widehat{d} := \mathcal{L}^3 = 8$. Then $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^5$. Indeed otherwise Castelnuovo's bound (1) gives $g(\mathcal{L}) \leq \mathrm{Castel}(8, 4) \leq 5$ and hence we would have the numerical contradiction $16 \leq \widehat{d} + \widehat{d}_1 = d + d_1 = 2g(\mathcal{L}) - 2 \leq 8$. By looking over the list of 3-folds in $\mathbb{P}^5$ of degree ≤ 12 (see [5, Chapter 6]) we see that the only degree 8 log-general type case is the complete intersection of a quadric and a quartic. In this case $\mathcal{K}_M \approx \mathcal{O}_M(1)$ and therefore $\mathfrak{t} = 1$, contradicting the present assumption $\mathfrak{t} \geq 8$.

Thus we can assume $\widehat{d} \geq 9$, so that $d \geq 9$. Then starting from $d \geq 9$, $d_1 \geq 1$, $d_2 \geq 1$, $d_3 \geq 2$ and using repeatedly the Hodge index inequalities (3) we obtain $d \geq 9$, $d_1 \geq 9$, $d_2 \geq 9$, $d_3 \geq 9$. Now use relation (8) to get the numerical contradiction

$$1 = \frac{d_1}{12} + \frac{d_3}{32} \geq \frac{9}{12} + \frac{9}{32} > \frac{9}{12} + \frac{8}{32} = 1.$$

Thus we conclude that $\mathfrak{t} \leq 7$. Q.E.D.

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