

q -LEIBNIZ ALGEBRAS

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ABSTRACT. An algebra (A, \circ) is called Leibniz if $a \circ (b \circ c) = (a \circ b) \circ c - (a \circ c) \circ b$ for all $a, b, c \in A$. We study identities for the algebras $A^{(q)} = (A, \circ_q)$, where $a \circ_q b = a \circ b + qb \circ a$ is the q -commutator. We show that the class of q -Leibniz algebras is defined by one identity of degree 3 if $q^2 \neq 1$, $q \neq -2$, by two identities of degree 3 if $q = -2$, and by the commutativity identity and one identity of degree 4 if $q = 1$. In the case of $q = -1$ we construct two identities of degree 5 that form a base of identities of degree 5 for -1 -Leibniz algebras. Any identity of degree < 5 for -1 -Leibniz algebras follows from the anti-commutativity identity.

1. INTRODUCTION

Denote by $A = (A, \circ)$ an algebra with vector space A over a field K of characteristic $\neq 2, 3$ and multiplication $(a, b) \mapsto a \circ b$. Let $(a, b, c) = a \circ (b \circ c) - (a \circ b) \circ c$ be the associator and $a \circ_q b = a \circ b + qb \circ a$ be the q -commutator, where $q \in K$. Denote by $A^{(q)} = (A, \circ_q)$ the algebra with the q -commutator. Notice that $a \circ_{-1} b = a \circ b - b \circ a$ is a commutator (Lie bracket, usually denoted by $[a, b]$) and $a \circ_1 b = a \circ b + b \circ a$ is an anti-commutator (Jordan bracket, sometimes denoted by $\{a, b\}$).

Example. If A is an associative algebra, then $A^{(-1)} = (A, [,]) is a Lie algebra,$

$$[a, b] = -[b, a],$$

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0,$$

and $A^{(+1)} = (A, \{ , \})$ is a Jordan algebra,

$$\{a, b\} = \{b, a\},$$

$$\{\{a, a\}, \{b, a\}\} = \{\{\{a, a\}, b\}, a\}.$$

Usually, q -commutators are studied in a frame of quantum groups. It seems that studying of q -identities has their own interest. We try to demonstrate it in a class of Leibniz algebras. We call an algebra A *Leibniz* (more exactly *right-Leibniz*) if for all $a, b, c \in A$

$$a \circ (b \circ c) = (a \circ b) \circ c - (a \circ c) \circ b.$$

Leibniz algebras were introduced in [2], [3]. In other words, Leibniz algebras are algebras with the identity $lei = 0$, where

$$lei = lei(t_1, t_2, t_3) = t_1(t_2t_3) - (t_1t_2)t_3 + (t_1t_3)t_2.$$

Example. Let (L, \star) be a Lie algebra under multiplication \star and let M be an L -module under the right action $(M, L) \rightarrow M, (m, a) \mapsto ma$. Make M a trivial left L -module: $am = 0, a \in L, m \in M$. Then the vector space $L \oplus M$ under multiplication

$$(a + m) \circ (b + n) = a \star b + mb$$

becomes right-Leibniz. Indeed,

$$\begin{aligned} (a + m) \circ ((b + n) \circ (c + s)) &= (a + m) \circ (b \star c + nc) \\ &= a \star (b \star c) + m(b \star c) = (a \star b) \star c - (a \star c) \star b + (mb)c - (mc)b \\ &= ((a + m) \circ (b + n)) \circ (c + s) - ((a + m) \circ (c + s)) \circ (b + n). \end{aligned}$$

We call the so-obtained algebra $L + M$ (a semi-direct sum of Leibniz algebras) *standard* Leibniz.

Endow a standard Leibniz algebra $(L + M, \circ)$ with the commutator $[,]$. Then

$$\begin{aligned} [a + m, b + n] &= (a + m) \circ (b + n) - (b + n) \circ (a + m) = (a \star b) + mb - (b \star a) - na \\ &= 2[a, b] + mb - na, \end{aligned}$$

where $[a, b] = a \star b - b \star a$. The algebra $(L + M, [,])$ (more exactly, $L + M$ under multiplication $[a, b] + (mb - na)/2$) is called *Omni-Lie* [4], [5].

Given non-associative polynomials f_1, \dots, f_s , we let $Var(f_1, \dots, f_s)$ denote the variety of algebras with identities $f_1 = 0, \dots, f_s = 0$.

In this paper we construct identities for q -(right)-Leibniz algebras. In particular, we describe identities for Omni-Lie algebras.

We prove that the category of q -Leibniz algebras is equivalent to the category of Leibniz algebras if $q^2 \neq 1, q \neq -2$. This means that, for $q \neq \pm 1, -2$, every algebra with identity $lei^{(q)} = 0$ can be obtained as $A^{(q)}$ from some Leibniz algebra A and, conversely, if B is an algebra with identity $lei^{(q)} = 0$, then $B^{(-q)}$ is right-Leibniz. In the case of $q = -2$ we should add to the identity $lei^{(q)} = 0$ the identity $lei_1^{(q)} = 0$ in order to obtain equivalent categories.

Theorem 1.1. ($q \neq -1, 1, -2$) *The class of q -Leibniz algebras $\mathfrak{Lei}^{(q)}$ satisfies the identity $lei^{(q)} = 0$, where*

$$lei^{(q)} = lei^{(q)}(t_1, t_2, t_3) =$$

$$(q^2 - 1)(t_1(t_2t_3) - t_2(t_1t_3)) + (q^2 + q - 1)(t_2t_1)t_3 + (t_2t_3)t_1 - t_1(t_3t_2) - qt_3(t_1t_2).$$

The varieties \mathfrak{Lei} , $\mathfrak{Lei}^{(q)}$ and $Var(lei^{(q)})$ are equivalent.

In particular, $Var(lei^{(q)})$ has no special identity for $\mathfrak{L}^{(q)}$ if $q \neq -2, q^2 \neq 1$. The identity $lei_1^{(q)} = 0$ is a consequence of the identity $lei^{(q)} = 0$ if $q \neq -2, q^2 \neq 1$.

Theorem 1.2. ($q^2 \neq 1$) *Let $q = -2$. The class of q -Leibniz algebras $\mathfrak{L}ei^{(-2)}$ satisfies the identities $lei^{(-2)} = 0$ and $lei_1^{(-2)} = 0$, where $lei^{(q)}$ is given above and*

$$lei_1^{(q)} = lei_1^{(q)}(t_1, t_2, t_3) = -t_1(t_2t_3 + t_3t_2) + q(t_2t_3 + t_3t_2)t_1.$$

The varieties $\mathfrak{L}ei$, $\mathfrak{L}ei^{(-2)}$ and $Var(lei^{(-2)}, lei_1^{(-2)})$ are equivalent.

So the identity $lei_1^{(-2)} = 0$ is a special identity for $Var(lei^{(-2)})$ which does not follow from the identity $lei^{(-2)} = 0$, and there are no other special identities for $\mathfrak{L}ei^{(-2)}$.

Define non-commutative non-associative polynomials $leilie_1, leilie_2$ of degree five by

$$leilie_1(t_1, t_2, t_3, t_4, t_5) = 2ljac(ljac(t_1, t_2, t_3), t_4, t_5) - [ljac(t_1, t_2, t_3), [t_4, t_5]],$$

$$\begin{aligned} & leilie_2(t_1, t_2, t_3, t_4, t_5) \\ &= -\frac{1}{2} \sum_{\sigma \in Sym(2,3,4,5)} sign \sigma (-4(((t_{\sigma(2)}t_{\sigma(3)})t_{\sigma(4)})t_{\sigma(5)})t_1 + 2(((t_{\sigma(2)}t_{\sigma(3)})t_1)t_{\sigma(4)})t_{\sigma(5)} \\ &+ 2(((t_{\sigma(2)}t_{\sigma(3)})t_{\sigma(4)})t_1)t_{\sigma(5)} + ((t_1t_{\sigma(2)}t_{\sigma(3)})(t_{\sigma(4)}t_{\sigma(5)}) + ((t_1t_{\sigma(2)})(t_{\sigma(4)}t_{\sigma(5)}))t_{\sigma(3)}). \end{aligned}$$

For a non-commutative non-associative polynomial $f(t_1, \dots, t_k)$, denote by $Alt(f)$ its skew-symmetrization

$$Alt f(t_1, \dots, t_k) = \sum_{\sigma \in Sym_k} sign \sigma f(t_{\sigma(1)}, \dots, t_{\sigma(k)}).$$

Let

$$leilie(t_1, t_2, t_3, t_4, t_5) = Alt(4(((t_1t_2)t_3)t_4)t_5 - ((t_1t_2)t_3)(t_4t_5)).$$

Theorem 1.3. ($q = -1$) *Let A be a right-Leibniz algebra. Then $A^{(-1)}$ satisfies the identities $acom = 0$, $leilie_1 = 0$ and $leilie_2 = 0$. Any polylinear identity of $\mathfrak{L}ei^{(-1)}$ of degree no more than 4 follows from the anti-commutativity identity. Any polylinear identity of $\mathfrak{L}ei^{(-1)}$ of degree 5 follows from the identities $acom = 0$, $leilie_1 = 0$ and $leilie_2 = 0$.*

Corollary 1.4. *Let A be a right-Leibniz algebra. Then $A^{(-1)}$ satisfies the identity $leilie = 0$.*

Corollary 1.5. *Every Omni-Lie algebra satisfies the polynomial identities $acom = 0$, $leilie_1 = 0$, $leilie_2 = 0$ and $leilie = 0$. The identities $acom = 0$, $leilie_1 = 0$ and $leilie_2 = 0$ form a base of identities in the space of polylinear identities of degree no more than 5 for the class of Omni-Lie algebras.*

Note that the polynomials $leilie_1$, $leilie_2$ and $leilie$ have 9, 60 and 90 terms respectively.

Let

$$leijor(t_1, t_2, t_3, t_4) = (t_1 t_2)(t_3 t_4).$$

Theorem 1.6. ($q = 1$) *Let A be a right-Leibniz algebra. Then $A^{(1)}$ satisfies the identities $com = 0$ and $leijor = 0$. Every polylinear identity which is true for any Leibniz-Jordan algebra follows from the identities $com = 0$ and $leijor = 0$.*

In other words, there are no special identities for the class of Leibniz-Jordan algebras.

2. NON-COMMUTATIVE NON-ASSOCIATIVE POLYNOMIALS

Let $K\langle t_1, t_2, \dots \rangle$ be the space of non-commutative non-associative polynomials in variables t_1, t_2, \dots (free magma). For a polynomial $f = f(t_1, \dots, t_k) \in K\langle t_1, t_2, \dots \rangle$, we say that $f = 0$ is an *identity* on an algebra (A, \circ) if $f(a_1, \dots, a_k) = 0$ for all $a_1, \dots, a_k \in A$.

Recall that there are exists $\frac{1}{k} \binom{2(k-1)}{k-1}$ types of bracketing for the string $t_1 \dots t_k$. For example, there are 5 types of bracketing for 4 elements:

$$((t_1 t_2) t_3) t_4, (t_1 t_2)(t_3 t_4), t_1(t_2(t_3 t_4)), t_1((t_2 t_3) t_4), (t_1(t_2 t_3)) t_4.$$

Order the types of bracketing somehow. If σ is a type of bracketing, denote by $\sigma(t_{i_1}, \dots, t_{i_k})$ the string $t_{i_1} \dots t_{i_k}$ with bracketing type σ . For example, if $k = 4$ and σ is the bracketing type $(t_1(t_2 t_3)) t_4$ then $\sigma(t_1, t_2, t_1, t_3) = (t_1(t_2 t_1)) t_3$.

Let α be some bracketing type of t_1, \dots, t_n . We say that a monomial of the form $\alpha(t_{i_1}, \dots, t_{i_n})$ has *polydegree* (r_1, \dots, r_k) if $\{i_1, \dots, i_n\} \subseteq \{1, \dots, k\}$ and $r_m = |\{s : i_s = m, s = 1, \dots, n\}|$ is the number of indexes i_s equal to m for any $m = 1, \dots, k$. Call $f = f(x_1, \dots, x_k)$ *homogeneous of degree* (r_1, \dots, r_k) if f is a linear combination of monomials of polydegree (r_1, \dots, r_k) . Say that a homogeneous polynomial f has *degree* l if $r_1 + \dots + r_k = l$.

A homogeneous polynomial $f = f(t_1, \dots, t_k)$ of polydegree $(1, \dots, 1)$ is called *polylinear*. Notice that the degree of a polylinear polynomial $f \in K\langle t_1, \dots, t_k \rangle$ is equal to the number of variables, k . In other

words a polynomial f is polylinear if f is a linear combination of monomials of the form $\alpha(t_{i_1}, \dots, t_{i_k})$, where $\binom{1 \dots k}{i_1 \dots i_k} \in \text{Sym}_k$ is a permutation of the set $\{1, \dots, k\}$ and α is a bracketing.

Given polynomials $f_1, \dots, f_s, g \in K\langle t_1, \dots, t_k \rangle$, we say that the identity $g = 0$ follows from the identities $f_1 = 0, \dots, f_s = 0$, and write $\{f_1 = 0, \dots, f_s = 0\} \Rightarrow g = 0$, if $g = 0$ is an identity for any algebra $A \in \mathcal{L}$ with the identities $f_1 = 0, \dots, f_s = 0$.

Suppose that polynomials f_1, \dots, f_s are homogeneous and have degrees $n_1 \leq \dots \leq n_s$. Suppose that $(A, \circ_q) \in \mathcal{L}^{(q)}$ has identities $f_1 = 0, \dots, f_s = 0$. We say that these identities are $\mathcal{L}^{(q)}$ -minimal if

- for any $r = 1, \dots, s$ the identity $f_r = 0$ does not follow from the identities $f_1 = 0, \dots, f_{r-1} = 0, f_{r+1} = 0, \dots, f_s = 0$
- if $\{f_1 = 0, \dots, f_{r-1} = 0, g = 0, f_{r+1} = 0, \dots, f_r = 0\} \Rightarrow f_r = 0$ then $\{f_1 = 0, \dots, f_{r-1} = 0, f_r = 0, f_{r+1} = 0, \dots, f_s = 0\} \Rightarrow g = 0$.

The space $K\langle t_1, t_2, \dots \rangle$ has the natural multiplication $(f, g) \mapsto f \cdot g = fg$. Let us endow it with the multiplication $(f, g) \mapsto f \cdot_q g$ given by $f \cdot_q g = f \cdot g + qg \cdot f$. For example,

$$(t_1 + 3t_1t_2) \cdot ((t_2t_3)t_1) = t_1((t_2t_3)t_1) + 3(t_1t_2)((t_2t_3)t_1),$$

$$\begin{aligned} & (t_1 + 3t_1t_2) \cdot_q ((t_2t_3)t_1) \\ &= t_1((t_2t_3)t_1) + 3(t_1t_2)((t_2t_3)t_1) + q((t_2t_3)t_1)t_1 + 3q((t_2t_3)t_1)(t_1t_2). \end{aligned}$$

Let

$$\tau_q : K\langle t_1, t_2, \dots \rangle \rightarrow K\langle t_1, t_2, \dots \rangle$$

be a linear map defined by

$$\tau_q(t_i) = t_i,$$

$$\tau_q(f \cdot g) = \tau_q(f) \cdot \tau_q(g) + q\tau(g) \cdot \tau_q(f),$$

for any $f, g \in K\langle t_1, t_2, \dots \rangle$. Then

$$\tau_q : (K\langle t_1, t_2, \dots \rangle, \cdot) \rightarrow (K\langle t_1, t_2, \dots \rangle, \cdot_q)$$

is the homomorphism

$$\tau_q(f \cdot g) = \tau_q(f) \cdot_q \tau_q(g).$$

Given a bracketing type σ , we set

$$\sigma_q = \tau_q\sigma.$$

In other words, $\sigma_q(t_1, \dots, t_k)$ is a polynomial obtained from $\sigma(t_1, \dots, t_k)$ by multiplication \circ_q . For example, if σ is the bracketing type $(t_1 t_2) t_3$, then

$$\sigma_q(t_3, t_1, t_2) = (t_3 t_1) t_2 + q((t_1 t_3) t_2 + t_2(t_3 t_1)) + q^2 t_2(t_1 t_3).$$

Lemma 2.1. *For any bracketing type σ*

$$\sigma_{-q} \sigma_q(t_{i_1}, \dots, t_{i_k}) = (1 - q^2)^{k-1} \sigma_0(t_{i_1}, \dots, t_{i_k}).$$

Proof. We use the induction on k . For $k = 2$ the statement is true:

$$\sigma_q(t_{i_1}, t_{i_2}) = t_{i_1} t_{i_2} + q t_{i_2} t_{i_1},$$

and

$$\begin{aligned} & \sigma_{-q} \sigma_q(t_{i_1}, t_{i_2}) \\ &= t_{i_1} t_{i_2} - q t_{i_2} t_{i_1} + q t_{i_2} \cdot t_{i_1} - q^2 t_{i_1} t_{i_2} \\ &= (1 - q^2) t_{i_1} t_{i_2} = (1 - q^2) \sigma_0(t_{i_1}, t_{i_2}). \end{aligned}$$

Suppose that our statement is true for $k - 1$. Let

$$\sigma(t_{i_1}, \dots, t_{i_k}) = \sigma'(t_{i_1}, \dots, t_{i_{k'}}) \sigma''(t_{i_{k'+1}}, \dots, t_{i_k})$$

for some $1 \leq k' \leq k$ and for some bracketings σ' , σ'' . Then

$$\begin{aligned} & \sigma_q(t_{i_1}, \dots, t_{i_k}) \\ &= \sigma'_q(t_{i_1}, \dots, t_{i_{k'}}) \sigma''_q(t_{i_{k'+1}}, \dots, t_{i_k}) \\ &+ q \sigma''_q(t_{i_{k'+1}}, \dots, t_{i_k}) \sigma'_q(t_{i_1}, \dots, t_{i_{k'}}) \end{aligned}$$

and

$$\begin{aligned} & \sigma_{-q} \sigma_q(t_{i_1}, \dots, t_{i_k}) \\ &= \sigma'_{-q} \sigma'_q(t_{i_1}, \dots, t_{i_{k'}}) \sigma''_{-q} \sigma''_q(t_{i_{k'+1}}, \dots, t_{i_k}) \\ &- q \sigma''_{-q} \sigma''_q(t_{i_{k'+1}}, \dots, t_{i_k}) \sigma'_{-q} \sigma'_q(t_{i_1}, \dots, t_{i_{k'}}) \\ &+ q \sigma''_{-q} \sigma''_q(t_{i_{k'+1}}, \dots, t_{i_k}) \sigma'_{-q} \sigma'_q(t_{i_1}, \dots, t_{i_{k'}}) \\ &- q^2 \sigma'_{-q} \sigma'_q(t_{i_1}, \dots, t_{i_{k'}}) \sigma''_{-q} \sigma''_q(t_{i_{k'+1}}, \dots, t_{i_k}) \\ &= \sigma'_{-q} \sigma'_q(t_{i_1}, \dots, t_{i_{k'}}) \sigma''_{-q} \sigma''_q(t_{i_{k'+1}}, \dots, t_{i_k}) \\ &- q^2 \sigma'_{-q} \sigma'_q(t_{i_1}, \dots, t_{i_{k'}}) \sigma''_{-q} \sigma''_q(t_{i_{k'+1}}, \dots, t_{i_k}). \end{aligned}$$

By the induction hypothesis

$$\begin{aligned} & \sigma'_{-q} \sigma'_q(t_{i_1}, \dots, t_{i_{k'}}) \\ &= (1 - q^2)^{k'-1} \sigma'_0(t_{i_1}, \dots, t_{i_{k'}}), \\ & \sigma''_{-q} \sigma''_q(t_{i_{k'+1}}, \dots, t_{i_k}) \end{aligned}$$

$$= (1 - q^2)^{k-k'-1} \sigma_0''(t_{i_{k'+1}}, \dots, t_{i_k}).$$

Therefore,

$$\begin{aligned} & \sigma'_{-q} \sigma'_q(t_{i_1}, \dots, t_{i_{k'}}) \sigma''_{-q} \sigma''_q(t_{i_{k'+1}}, \dots, t_{i_k}) \\ &= (1 - q^2)^{k-2} \sigma_0(t_{i_1}, \dots, t_{i_k}), \end{aligned}$$

$$\begin{aligned} & -q^2 \sigma'_{-q} \sigma'_q(t_{i_1}, \dots, t_{i_{k'}}) \sigma''_{-q} \sigma''_q(t_{i_{k'+1}}, \dots, t_{i_k}) \\ &= -q^2 (1 - q^2)^{k-2} \sigma_0(t_{i_1}, \dots, t_{i_k}) \end{aligned}$$

and

$$\begin{aligned} & \sigma_{-q} \sigma_q(t_{i_1}, \dots, t_{i_k}) \\ &= \sigma'_{-q} \sigma'_q(t_{i_1}, \dots, t_{i_{k'}}) \sigma''_{-q} \sigma''_q(t_{i_{k'+1}}, \dots, t_{i_k}) \\ & - q^2 \sigma'_{-q} \sigma'_q(t_{i_1}, \dots, t_{i_{k'}}) \sigma''_{-q} \sigma''_q(t_{i_{k'+1}}, \dots, t_{i_k}) \\ &= (1 - q^2)^{k-1} \sigma_0(t_{i_1}, \dots, t_{i_k}). \end{aligned}$$

From Lemma 2.1 we infer the following

Theorem 2.2. ($q^2 \neq 1$) *Let f_1, \dots, f_s be homogeneous polynomials of degree k . Then the class of q -algebras $\text{Var}(f_1, \dots, f_s)^{(q)}$ forms a variety generated by the polynomial identities $\sigma_{-q} f_1 = 0, \dots, \sigma_{-q} f_s = 0$. This variety is equivalent to $\text{Var}(f_1, \dots, f_s)$ and the equivalence can be given by $A = (A, \star) \mapsto A^{(-q)} = (A, \star_{-q})$.*

The equivalence of varieties means the following. There exist functors

$$\begin{aligned} F : \text{Var}(f_1, \dots, f_s) &\rightarrow \text{Var}(\sigma_{-q} f_1, \dots, \sigma_{-q} f_s), & (A, \circ) &\rightarrow (A, \circ_q), \\ G : \text{Var}(\sigma_{-q} f_1, \dots, \sigma_{-q} f_s) &\rightarrow \text{Var}(f_1, \dots, f_s), & (A, \star) &\rightarrow (A, \star'_q) \end{aligned}$$

such that

$$GF(A, \circ) = (A, \circ), \quad FG(A, \star) = (A, \star).$$

Here

$$a \star'_q b = \frac{1}{(1 - q^2)^{k-1}} a \star_q b.$$

Recall that all polynomials f_1, \dots, f_s are supposed homogeneous. Notice that, for any $(A, \circ), (B, \cdot) \in \text{Var}(f_1, \dots, f_s)$ and a morphism between them, i.e., a homomorphism $\psi : (A, \circ) \rightarrow (B, \cdot)$, there corresponds a morphism of algebras $\psi : F(A, \circ) \rightarrow F(B, \cdot)$ in the category $\text{Var}(\sigma_{-q} f_1, \dots, \sigma_{-q} f_s)$, i.e., a homomorphism $\psi : (A, \circ_q) \rightarrow (B, \cdot_q)$. Indeed,

$$\begin{aligned} & \psi(a_1 \circ_q a_2) \\ &= \psi(a_1 \circ a_2 + q a_2 \circ a_1) \\ &= \psi(a_1 \circ a_2) + q \psi(a_2 \circ a_1) \end{aligned}$$

$$\begin{aligned}
&= \psi(a_1) \cdot \psi(a_2) + q \psi(a_2) \cdot \psi(a_1) \\
&= \psi(a_1) \cdot_q \psi(a_2).
\end{aligned}$$

If I is an ideal of (A, \circ) then I is an ideal of (A, \circ_q) . Therefore, simplicity, nilpotency and solvability properties of algebras in the category $Var(f_1, \dots, f_s)$ remain the same for the corresponding algebras in $Var(\sigma_{-q}f_1, \dots, \sigma_{-q}f_s)$. If (A, \circ) is free in the variety $Var(f_1, \dots, f_s)$, then (A, \circ_q) is free in the variety $Var(\sigma_{-q}f_1, \dots, \sigma_{-q}f_s)$. We pay attention to the fact that the categories $Var(f_1, \dots, f_s)$ and $Var(\sigma_{-q}f_1, \dots, \sigma_{-q}f_s)$ are equivalent only in the case of $q^2 \neq 1$.

Let g_1, \dots, g_s, h be non-commutative non-associative polynomials. Suppose that, for a class \mathfrak{L} of algebras, the corresponding class $\mathfrak{L}^{(q)}$ of q -algebras satisfies the identities $g_1 = 0, \dots, g_s = 0$ and $h = 0$. In this case we say that $h = 0$ is a *special* identity or an s -identity for $Var(g_1, \dots, g_s)$.

We give another application of Lemma 2.1.

Theorem 2.3. *If $q \neq \pm 1$, then the map*

$$\tau_q : (K\langle t_1, t_2, \dots \rangle, \cdot) \rightarrow (K\langle t_1, t_2, \dots \rangle, \cdot_q)$$

is an isomorphism.

Let \mathfrak{L} be some class of algebras. For a polynomial $f \in K\langle t_1, t_2, \dots \rangle$, we say that $f = 0$ is an identity for \mathfrak{L} if every algebra $A \in \mathfrak{L}$ satisfies the identity $f = 0$. Recall that each class of algebras satisfying polynomial identities forms a variety.

Let \mathfrak{Lei} be the class of Leibniz algebras, i.e., the variety of algebras generated by the (right)-Leibniz identity $lei = 0$. Denote by $\mathfrak{Lei}^{(q)}$ the class of q -Leibniz algebras, i.e., algebras of the form $A^{(q)} = (A, \circ_q)$, where $A \in \mathfrak{Lei}$.

Define non-commutative polynomials *com* (commutativity), *acom* (anti-commutativity), *ljac* (left-Jacobian), *rjac* (right-Jacobian), *lalia* (left-anti-Lie-admissible), *ralia* (right-Anti-Lie-admissible), *lia* (Lie-admissible), s_k^l (standard left-skew-symmetric), s_k^r (standard right-skew-symmetric) and $s_k^{[r]}$ (s_k -Lie-admissible) by

$$com(t_1, t_2) = t_1 t_2 - t_2 t_1,$$

$$acom(t_1, t_2) = t_1 t_2 + t_2 t_1,$$

$$ljac(t_1, t_2, t_3) = (t_1 t_2) t_3 + (t_2 t_3) t_1 + (t_3 t_1) t_2,$$

$$rjac(t_1, t_2, t_3) = t_1 (t_2 t_3) + t_2 (t_3 t_1) + t_3 (t_1 t_2),$$

$$lalia(t_1, t_2, t_3) = [t_1, t_2] t_3 + [t_2, t_3] t_1 + [t_3, t_1] t_2,$$

$$\begin{aligned} ralia(t_1, t_2, t_3) &= t_1[t_2, t_3] + t_2[t_3, t_1] + t_3[t_1, t_2], \\ lia(t_1, t_2, t_3) &= [[t_1, t_2], t_3] + [[t_2, t_3], t_1] + [[t_3, t_1], t_2], \\ alia^{(q)} &= lalia + qralia, \quad q \in K. \end{aligned}$$

Recall that for a non-commutative non-associative polynomial $f(t_1, \dots, t_k)$, we denote by $Alt(f)$ its skew-symmetrization

$$Alt f(t_1, \dots, t_k) = \sum_{\sigma \in Sym_k} sign \sigma f(t_{\sigma(1)}, \dots, t_{\sigma(k)}).$$

Let

$$\begin{aligned} s_k^r(t_1, \dots, t_k) &= Alt(t_1(t_2(\dots(t_{k-1}t_k))), \\ s_k^l(t_1, \dots, t_k) &= Alt((\dots(t_1t_2)\dots t_{k-1})t_k, \\ s_k^{[r]}(t_1, \dots, t_k) &= Alt([t_1, [t_2, \dots, [t_{l-1}, t_k]]]). \end{aligned}$$

Notice that

$$\begin{aligned} com &= s_2, \\ lalia &= s_3^l, \quad ralia = s_3^r, \quad lia = s_3^l - s_3^r = lalia - ralia. \end{aligned}$$

If polynomials are anti-commutative, i.e., satisfy the identity $acom = 0$, then

$$\begin{aligned} ljac &= -rjac, \\ lia &= 4ljac. \end{aligned}$$

3. RIGHT-CENTER AND LIE ELEMENTS

Let $F = F(V)$ be a free right-Leibniz algebra generated by a space V . Let $(F^{lie}, [,])$ be the subspace of F generated by V under the commutator $[,]$. We say that $a \in F$ is a *Lie-element* if $a \in (F^{lie}, [,])$. Homomorphic images of Lie elements of any Leibniz algebras are called Lie elements as well.

Let (A, \circ) be a right-Leibniz algebra. An element $z \in A$ is called *right-central* if

$$a \circ z = 0$$

for all $a \in A$. Let A^{rann} be the set of right-central elements of A . It was noticed in [3] that A^{rann} is an ideal with trivial left action, $a \circ z = 0, z \in A^{rann}, a \in A$, such that

$$\{a, b\} = a \circ b + b \circ a \in A^{rann}$$

for all $a, b \in A$. We construct new right-central elements.

Observe that

$$s_{k+1}^l(a_1, \dots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^i s_k^l(a_1, \dots, \hat{a}_i, \dots, a_{k+1}) \circ a_i. \quad (1)$$

Lemma 3.1. *Let (A, \circ) be a right-Leibniz algebra. Then A^{rann} is an ideal such that*

$$a \circ z = 0.$$

For any $q \in K$,

$$(a \circ_q b) \circ c = (a \circ c) \circ_q b + a \circ_q (b \circ c). \quad (2)$$

In particular,

$$\{a, b\} \circ c = \{a \circ c, b\} + \{a, b \circ c\}.$$

For any $k \geq 3$

$$s_k^l(a_1, \dots, a_k), \quad s_k^r(a_1, \dots, a_k) \in A^{rann}.$$

Moreover,

$$s_k^l(a_1, \dots, a_k) = s_k^{[r]}(a_1, \dots, a_k)$$

are Lie elements,

$$s_k^r(a_1, \dots, a_k) = 0, k \geq 4,$$

and

$$s_3^r(a, b, c) = 2s_3^l(a, b, c).$$

In other words, any right-Leibniz algebra A is $-1/2$ -Alia,

$$alia^{(-1/2)}(a, b, c) = 0$$

for all $a, b, c \in A$.

Proof. We have

$$\begin{aligned} & (a \circ_q b) \circ c \\ &= (a \circ b + qb \circ a) \circ c \\ &= a \circ (b \circ c) + (a \circ c) \circ b + qb \circ (a \circ c) + q(b \circ c) \circ a \\ &= (a \circ c) \circ_q b + a \circ_q (b \circ c). \end{aligned}$$

So, (2) is established. Thus, in the case $q = 0$ we obtain the right-Leibniz identity

$$(a \circ b) \circ c = (a \circ c) \circ b + a \circ (b \circ c).$$

Let $k = 3$. Notice that

$$s_3^r(a, b, c) = ralia(a, b, c).$$

We have

$$\begin{aligned} & ralia(a, b, c) \\ &= a \circ [b, c] + b \circ [c, a] + c \circ [a, b] \\ &= 2(a \circ (b \circ c) + b \circ (c \circ a) + c \circ (a \circ b)) - a \circ \{b, c\} - b \circ \{c, a\} - c \circ \{a, b\} \\ &= 2rjac(a, b, c). \end{aligned}$$

By the right-Leibniz identity

$$ljac(a, b, c) = [a, b] \circ c + [b, c] \circ a + [c, a] \circ b = lalia(a, b, c),$$

and

$$lia[a, b, c] = lalia(a, b, c) - ralia(a, b, c) = rjac(a, b, c) - 2rjac(a, b, c) = -rjac(a, b, c).$$

So, $s_3^r(a, b, c) = 2rjac(a, b, c) = -2lia(a, b, c)$ is a Lie-element.

By the right-Leibniz identity

$$\begin{aligned} & u \circ rjac(a, b, c) \\ &= ((u \circ a) \circ (b \circ c) - ((u \circ (b \circ c)) \circ a + ((u \circ b) \circ (c \circ a) \\ &\quad - ((u \circ (c \circ a)) \circ b + ((u \circ c) \circ (a \circ b) - ((u \circ (a \circ b)) \circ c \\ &= ((u \circ a) \circ b) \circ c - ((u \circ a) \circ c) \circ b - ((u \circ b) \circ c) \circ a + ((u \circ c) \circ b) \circ a \\ &\quad + ((u \circ b) \circ c) \circ a - ((u \circ b) \circ a) \circ c - ((u \circ c) \circ a) \circ b + ((u \circ a) \circ c) \circ b \\ &\quad + ((u \circ c) \circ a) \circ b - ((u \circ c) \circ b) \circ a - ((u \circ a) \circ b) \circ c + ((u \circ b) \circ a) \circ c \\ &= 0. \end{aligned}$$

So, the element $s_3^l(a, b, c)$ is right-central.

Suppose that $s_k^l(a_1, \dots, a_k) = s^{[r]}(a_1, \dots, a_k)$ is a Lie element and is right-central. Prove that $s_{k+1}^l(a_1, \dots, a_{k+1})$ is also a Lie element which is right-central. Since $s_k^l(a_1, \dots, \hat{a}_i, \dots, a_{k+1}) \in A^{rann}$ for every $i = 1, \dots, k+1$ and since A^{rann} is an ideal, we have

$$s_{k+1}^l(a_1, \dots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+k+1} s_k^l(a_1, \dots, \hat{a}_i, \dots, a_{k+1}) \circ a_i \in A^{rann}.$$

Further,

$$s_{k+1}^{[r]}(a_1, \dots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} [a_i, s_k^{[r]}(a_1, \dots, \hat{a}_i, \dots, a_{k+1})]$$

(by the induction hypothesis)

$$= \sum_{i=1}^{k+1} (-1)^{i+1} [a_i, s_k^l(a_1, \dots, \hat{a}_i, \dots, a_{k+1})]$$

(since $s_k^l(a_1, \dots, \hat{a}_i, \dots, a_{k+1}) \in A^{rann}$)

$$= \sum_{i=1}^{k+1} (-1)^i s_k^l(a_1, \dots, \hat{a}_i, \dots, a_{k+1}) \circ a_i = s_{k+1}^l(a_1, \dots, a_{k+1}).$$

4. q -COMMUTATORS OF LEIBNIZ ALGEBRAS IN CASE $q^2 \neq 1$.

Lemma 4.1. *For any Leibniz algebra A its q -algebra $A^{(q)}$ satisfies the identities $lei^{(q)} = 0$ and $lei_1^{(q)} = 0$.*

Proof. We have

$$\begin{aligned}
& lei^{(q)}(a, b, c) \\
&= (q^2 - 1) a \circ_q (b \circ_q c) - a \circ_q (c \circ_q b) - (q^2 - 1) b \circ_q (a \circ_q c) - q c \circ_q (a \circ_q b) \\
&\quad + (q^2 + q - 1) (b \circ_q a) \circ_q c + (b \circ_q c) \circ_q a \\
&= (q^2 - 1)(a \circ (b \circ c) + (1 + q)a \circ (c \circ b) - b \circ (a \circ c) - qb \circ (c \circ a) + (q + q^2)c \circ (a \circ b) \\
&\quad + qc \circ (b \circ a) + q(a \circ b) \circ c - q(a \circ c) \circ b + (1 - q)(b \circ a) \circ c \\
&\quad + (q - 1)(b \circ c) \circ a - q^2(c \circ a) \circ b + q^2(c \circ b) \circ a) \\
&= (q^2 - 1)(a \circ (b \circ c) + (1 + q)a \circ (c \circ b) - b \circ (a \circ c) + qb \circ (a \circ c) \\
&\quad + q^2c \circ (a \circ b) + q(a \circ b) \circ c - q(a \circ c) \circ b + (1 - q)(b \circ a) \circ c \\
&\quad + (q - 1)(b \circ c) \circ a - q^2(c \circ a) \circ b + q^2(c \circ b) \circ a) \\
&= (q^2 - 1)(qa \circ (c \circ b) + (q - 1)b \circ (a \circ c) + q^2c \circ (a \circ b) + q((a \circ b) \circ c - (a \circ c) \circ b) \\
&\quad + (1 - q)((b \circ a) \circ c - (b \circ c) \circ a) - q^2((c \circ a) \circ b - (c \circ b) \circ a)) \\
&= (q^2 - 1)(q(a \circ (c \circ b) + (a \circ b) \circ c - (a \circ c) \circ b) + (1 - q)(-b \circ (a \circ c) + (b \circ a) \circ c - (b \circ c) \circ a) \\
&\quad - q^2(-c \circ (a \circ b) + (c \circ a) \circ b - (c \circ b) \circ a)) \\
&\text{(by the right-Leibniz identity)} \\
&= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& lei_1^{(q)}(a, b, c) = -a \circ_q (b \circ_q c) - a \circ_q (c \circ_q b) + q(b \circ_q c) \circ_q a + q(c \circ_q b) \circ_q a \\
&= -a \circ (b \circ c) - qa \circ (c \circ b) - q(b \circ c) \circ a - q^2(c \circ b) \circ a \\
&\quad - a \circ (c \circ b) - qa \circ (b \circ c) - q(c \circ b) \circ a - q^2(b \circ c) \circ a \\
&\quad + q(b \circ c) \circ a + q^2(c \circ b) \circ a + q^2a \circ (b \circ c) + q^3a \circ (c \circ b) \\
&\quad + q(c \circ b) \circ a + q^2(b \circ c) \circ a + q^2a \circ (c \circ b) + q^3a \circ (b \circ c) \\
&= -(a \circ b) \circ c + (a \circ c) \circ b - q(a \circ c) \circ b + q(a \circ b) \circ c
\end{aligned}$$

$$\begin{aligned}
 & -q(b \circ c) \circ a - q^2(c \circ b) \circ a \\
 & -(a \circ c) \circ b + (a \circ b) \circ c - q(a \circ b) \circ c + q(a \circ c) \circ b \\
 & -q(c \circ b) \circ a - q^2(b \circ c) \circ a \\
 & +q(b \circ c) \circ a + q^2(c \circ b) \circ a \\
 & +q^2(a \circ b) \circ c - q^2(a \circ c) \circ b + q^3(a \circ c) \circ b - q^3(a \circ b) \circ c \\
 & +q(c \circ b) \circ a + q^2(b \circ c) \circ a \\
 & +q^2(a \circ c) \circ b - q^2(a \circ b) \circ c + q^3(a \circ b) \circ c - q^3(a \circ c) \circ b \\
 \\
 & = (-1 + q + 1 - q + q^2 - q^3 - q^2 + q^3)(a \circ b) \circ c \\
 & + (1 - q - 1 + q - q^2 + q^3 + q^2 - q^3)(a \circ c) \circ b \\
 & + (-q - q^2 + q + q^2)(b \circ c) \circ a \\
 & + (-q^2 - q + q^2 + q)(c \circ b) \circ a \\
 \\
 & = 0.
 \end{aligned}$$

Lemma 4.2. *If $q \neq -2$, then*

$$Alt(lei^{(q)}) = -(q+2)(q-1)alia^{\left(\frac{-(2q+1)}{q+2}\right)}.$$

If $q = -2$, then

$$Alt(lei^{(-2)}) = 9ralia.$$

Proof. Consider the case $q \neq -2$. We have

$$\begin{aligned}
 & lei^{(q)}(t_1, t_2, t_3) + lei^{(q)}(t_2, t_3, t_1) + lei^{(q)}(t_3, t_1, t_2) \\
 & - lei^{(q)}(t_2, t_1, t_3) - lei^{(q)}(t_3, t_2, t_1) - lei^{(q)}(t_1, t_3, t_2) \\
 & = (q-1)\{(2q+1)(t_1[t_2, t_3] + t_2[t_3, t_1]) + t_3[t_1, t_2]\} \\
 & \quad - (q+2)([t_1, t_2]t_3 + [t_3, t_1]t_2 + [t_2, t_3]t_1) \\
 & = (2-q-q^2)ralia^{\left(\frac{-(2q+1)}{q+2}\right)}.
 \end{aligned}$$

The case $q = -2$ is considered in a similar manner.

Lemma 4.3. *Let L be a free Leibniz algebra with 3 generators, $q \in K, q \neq 0, \pm 1$. Then any polylinear identity of $L^{(q)}$ of degree 3 follows from the identities $lei^{(q)} = 0$ and $lei_1^{(q)} = 0$. If $q \neq -2$ then $lei_1^{(q)} = 0$ is a consequence of the identity $lei^{(q)} = 0$. If $q = -2$, then $lei^{(q)} = 0$ and $lei_1^{(q)} = 0$ are independent identities.*

Proof. Let $L = (L, \circ)$ be a free Leibniz algebra generated by three elements a, b, c . Write the q -commutator in $L^{(q)}$ by $uv = u \circ v + qv \circ u$.

The polylinear part of the free magma algebra (the algebra of non-commutative non-associative polynomials) in degree 3 has dimension 12. It is generated by the following 12 monomials:

$$\begin{aligned} e_1 &= e_1(t_1, t_2, t_3) = t_1(t_2t_3), & e_2 &= e_2(t_1, t_2, t_3) = t_2(t_3t_1), \\ e_3 &= e_3(t_1, t_2, t_3) = t_3(t_1t_2), & e_4 &= e_4(t_1, t_2, t_3) = t_2(t_1t_3), \\ e_5 &= e_5(t_1, t_2, t_3) = t_3(t_2t_1), & e_6 &= e_6(t_1, t_2, t_3) = t_1(t_3t_2), \\ e_7 &= e_7(t_1, t_2, t_3) = (t_1t_2)t_3, & e_8 &= e_8(t_1, t_2, t_3) = (t_2t_3)t_1, \\ e_9 &= e_9(t_1, t_2, t_3) = (t_3t_1)t_2, & e_{10} &= e_{10}(t_1, t_2, t_3) = (t_2t_1)t_3, \\ e_{11} &= e_{11}(t_1, t_2, t_3) = (t_3t_2)t_1, & e_{12} &= e_{12}(t_1, t_2, t_3) = (t_1t_3)t_2. \end{aligned}$$

Let $X = X(t_1, t_2, t_3) = \sum_{i=1}^{12} \lambda_i e_i(t_1, t_2, t_3)$ be a polynomial such that $X(a, b, c) = 0$ is an identity on $L^{(q)}$.

Substitute the generator elements $a, b, c \in L$ for the parameters t_1, t_2, t_3 . Write e_i instead of $e_i(a, b, c)$. We have

$$\begin{aligned} e_1 &= a \circ (b \circ c) + qa \circ (c \circ b) + q(b \circ c) \circ a + q^2(c \circ b) \circ a \\ &= (a \circ b) \circ c - (a \circ c) \circ b + q(a \circ c) \circ b - q(a \circ b) \circ c \\ &\quad + q(b \circ c) \circ a + q^2(c \circ b) \circ a. \end{aligned}$$

Similar calculations show that

$$\begin{aligned} e_2 &= (b \circ c) \circ a - (b \circ a) \circ c + q(b \circ a) \circ c - q(b \circ c) \circ a \\ &\quad + q(c \circ a) \circ b + q^2(a \circ c) \circ b, \end{aligned}$$

$$\begin{aligned} e_3 &= (c \circ a) \circ b - (c \circ b) \circ a + q(c \circ b) \circ a - q(c \circ a) \circ b \\ &\quad + q(a \circ b) \circ c + q^2(b \circ a) \circ c, \end{aligned}$$

$$\begin{aligned} e_4 &= (b \circ a) \circ c - (b \circ c) \circ a + q(b \circ c) \circ a - q(b \circ a) \circ c \\ &\quad + q(a \circ c) \circ b + q^2(c \circ a) \circ b, \end{aligned}$$

$$\begin{aligned} e_5 &= (c \circ b) \circ a - (c \circ a) \circ b + q(c \circ a) \circ b - q(c \circ b) \circ a \\ &\quad + q(b \circ a) \circ c + q^2(a \circ b) \circ c, \end{aligned}$$

$$\begin{aligned} e_6 &= (a \circ c) \circ b - (a \circ b) \circ c + q(a \circ b) \circ c - q(a \circ c) \circ b \\ &\quad + q(c \circ b) \circ a + q^2(b \circ c) \circ a, \end{aligned}$$

$$\begin{aligned} e_7 &= (a \circ b) \circ c + q(b \circ a) \circ c + q(c \circ a) \circ b - q(c \circ b) \circ a \\ &\quad + q^2(c \circ b) \circ a - q^2(c \circ a) \circ b, \end{aligned}$$

$$e_8 = (b \circ c) \circ a + q(c \circ b) \circ a + q(a \circ b) \circ c - q(a \circ c) \circ b \\ + q^2(a \circ c) \circ b - q^2(a \circ b) \circ c,$$

$$e_9 = (c \circ a) \circ b + q(a \circ c) \circ b + q(b \circ c) \circ a - q(b \circ a) \circ c \\ + q^2(b \circ a) \circ c - q^2(b \circ c) \circ a,$$

$$e_{10} = (b \circ a) \circ c + q(a \circ b) \circ c + q(c \circ b) \circ a - q(c \circ a) \circ b \\ + q^2(c \circ a) \circ b - q^2(c \circ b) \circ a,$$

$$e_{11} = (c \circ b) \circ a + q(b \circ c) \circ a + q(a \circ c) \circ b - q(a \circ b) \circ c \\ + q^2(a \circ b) \circ c - q^2(a \circ c) \circ b,$$

$$e_{12} = (a \circ c) \circ b + q(c \circ a) \circ b + q(b \circ a) \circ c - q(b \circ c) \circ a \\ + q^2(b \circ c) \circ a - q^2(b \circ a) \circ c.$$

So,

$$X =$$

$$\begin{aligned} & \lambda_1(a \circ b) \circ c - \lambda_1(a \circ c) \circ b + q\lambda_1(a \circ c) \circ b - q\lambda_1(a \circ b) \circ c \\ & \quad + q\lambda_1(b \circ c) \circ a + q^2\lambda_1(c \circ b) \circ a \\ & + \lambda_2(b \circ c) \circ a - \lambda_2(b \circ a) \circ c + q\lambda_2(b \circ a) \circ c - q\lambda_2(b \circ c) \circ a \\ & \quad + q\lambda_2(c \circ a) \circ b + q^2\lambda_2(a \circ c) \circ b \\ & + \lambda_3(c \circ a) \circ b - \lambda_3(c \circ b) \circ a + q\lambda_3(c \circ b) \circ a - q\lambda_3(c \circ a) \circ b \\ & \quad + q\lambda_3(a \circ b) \circ c + q^2\lambda_3(b \circ a) \circ c \\ & \lambda_4(b \circ a) \circ c - \lambda_4(b \circ c) \circ a + q\lambda_4(b \circ c) \circ a - q\lambda_4(b \circ a) \circ c \\ & \quad + q\lambda_4(a \circ c) \circ b + q^2\lambda_4(c \circ a) \circ b \\ & \lambda_5(c \circ b) \circ a - \lambda_5(c \circ a) \circ b + q\lambda_5(c \circ a) \circ b - q\lambda_5(c \circ b) \circ a \\ & \quad + q\lambda_5(b \circ a) \circ c + q^2\lambda_5(a \circ b) \circ c \\ & \lambda_6(a \circ c) \circ b - \lambda_6(a \circ b) \circ c + q\lambda_6(a \circ b) \circ c - q\lambda_6(a \circ c) \circ b \\ & \quad + q\lambda_6(c \circ b) \circ a + q^2\lambda_6(b \circ c) \circ a \\ & + \lambda_7(a \circ b) \circ c + q\lambda_7(b \circ a) \circ c + q\lambda_7(c \circ a) \circ b - q\lambda_7(c \circ b) \circ a \\ & \quad + q^2\lambda_7(c \circ b) \circ a - q^2\lambda_7(c \circ a) \circ b \\ & + \lambda_8(b \circ c) \circ a + q\lambda_8(c \circ b) \circ a + q\lambda_8(a \circ b) \circ c - q\lambda_8(a \circ c) \circ b \\ & \quad + q^2\lambda_8(a \circ c) \circ b - q^2\lambda_8(a \circ b) \circ c \\ & + \lambda_9(c \circ a) \circ b + q\lambda_9(a \circ c) \circ b + q\lambda_9(b \circ c) \circ a - q\lambda_9(b \circ a) \circ c \\ & \quad + q^2\lambda_9(b \circ a) \circ c - q^2\lambda_9(b \circ c) \circ a \end{aligned}$$

$$\begin{aligned}
& +\lambda_{10}(b \circ a) \circ c + q\lambda_{10}(a \circ b) \circ c + q\lambda_{10}(c \circ b) \circ a - q\lambda_{10}(c \circ a) \circ b \\
& \quad + q^2\lambda_{10}(c \circ a) \circ b - q^2\lambda_{10}(c \circ b) \circ a \\
& +\lambda_{11}(c \circ b) \circ a + q\lambda_{11}(b \circ c) \circ a + q\lambda_{11}(a \circ c) \circ b - q\lambda_{11}(a \circ b) \circ c \\
& \quad + q^2\lambda_{11}(a \circ b) \circ c - q^2\lambda_{11}(a \circ c) \circ b \\
& +\lambda_{12}(a \circ c) \circ b + q\lambda_{12}(c \circ a) \circ b + q\lambda_{12}(b \circ a) \circ c - q\lambda_{12}(b \circ c) \circ a \\
& \quad + q^2\lambda_{12}(b \circ c) \circ a - q^2\lambda_{12}(b \circ a) \circ c \\
& = (\lambda_1 - q\lambda_1 + q\lambda_3 + q^2\lambda_5 - \lambda_6 + q\lambda_6 + \lambda_7 + q\lambda_8 - q^2\lambda_8 + q\lambda_{10} - q\lambda_{11} + q^2\lambda_{11})(a \circ b) \circ c \\
& + (-\lambda_1 + q\lambda_1 + q^2\lambda_2 + q\lambda_4 + \lambda_6 - q\lambda_6 - q\lambda_8 + q^2\lambda_8 + q\lambda_9 + q\lambda_{11} - q^2\lambda_{11} + \lambda_{12})(a \circ c) \circ b \\
& + (-\lambda_2 + q\lambda_2 + q^2\lambda_3 + \lambda_4 - q\lambda_4 + q\lambda_5 + q\lambda_7 - q\lambda_9 + q^2\lambda_9 + \lambda_{10} + q\lambda_{12} - q^2\lambda_{12})(b \circ a) \circ c \\
& + (q\lambda_1 + \lambda_2 - q\lambda_2 - \lambda_4 + q\lambda_4 + q^2\lambda_6 + \lambda_8 + q\lambda_9 - q^2\lambda_9 + q\lambda_{11} - q\lambda_{12} + q^2\lambda_{12})(b \circ c) \circ a \\
& + (q\lambda_2 + \lambda_3 - q\lambda_3 + q^2\lambda_4 - \lambda_5 + q\lambda_5 + q\lambda_7 - q^2\lambda_7 + \lambda_9 - q\lambda_{10} + q^2\lambda_{10} + q\lambda_{12})(c \circ a) \circ b \\
& + (q^2\lambda_1 - \lambda_3 + q\lambda_3 + \lambda_5 - q\lambda_5 + q\lambda_6 - q\lambda_7 + q^2\lambda_7 + q\lambda_8 + q\lambda_{10} - q^2\lambda_{10} + \lambda_{11})(c \circ b) \circ a
\end{aligned}$$

We thus we obtain the following system of equations

$$\begin{aligned}
(1-q)\lambda_1 + q\lambda_3 + q^2\lambda_5 + (q-1)\lambda_6 + \lambda_7 + (q-q^2)\lambda_8 + q\lambda_{10} + (q^2-q)\lambda_{11} &= 0, \\
(q-1)\lambda_1 + q^2\lambda_2 + q\lambda_4 + (1-q)\lambda_6 + (q^2-q)\lambda_8 + q\lambda_9 + (q-q^2)\lambda_{11} + \lambda_{12} &= 0, \\
(q-1)\lambda_2 + q^2\lambda_3 + (1-q)\lambda_4 + q\lambda_5 + q\lambda_7 + (q^2-q)\lambda_9 + \lambda_{10} + (q-q^2)\lambda_{12} &= 0, \\
q\lambda_1 + (1-q)\lambda_2 + (q-1)\lambda_4 + q^2\lambda_6 + \lambda_8 + (q-q^2)\lambda_9 + q\lambda_{11} + (q^2-q)\lambda_{12} &= 0, \\
q\lambda_2 + (1-q)\lambda_3 + q^2\lambda_4 + (q-1)\lambda_5 + (q-q^2)\lambda_7 + \lambda_9 + (q^2-q)\lambda_{10} + q\lambda_{12} &= 0, \\
q^2\lambda_1 + (q-1)\lambda_3 + (1-q)\lambda_5 + q\lambda_6 + (q^2-q)\lambda_7 + q\lambda_8 + (q-q^2)\lambda_{10} + \lambda_{11} &= 0.
\end{aligned}$$

The transposed matrix of this system is

$$\begin{array}{cccccc}
1-q & q & 0 & 0 & q^2 & q-1 \\
0 & 1-q & q & q-1 & 0 & q^2 \\
q & 0 & 1-q & q^2 & q-1 & 0 \\
0 & q-1 & q^2 & 1-q & 0 & q \\
q^2 & 0 & q-1 & q & 1-q & 0 \\
q-1 & q^2 & 0 & 0 & q & 1-q \\
1 & 0 & q-q^2 & q & q^2-q & 0 \\
q-q^2 & 1 & 0 & 0 & q & q^2-q \\
0 & q-q^2 & 1 & q^2-q & 0 & q \\
q & 0 & q^2-q & 1 & q-q^2 & 0 \\
q^2-q & q & 0 & 0 & 1 & q-q^2 \\
0 & q^2-q & q & q-q^2 & 0 & 1
\end{array}$$

The determinant of the 6×6 -matrix composed of the first 6 rows is $(1-q)^5q^3(1+q)^3(q+2)$. So, this system has rank 6 if $q^2 \neq 1, q \neq 0, -2$.

One can choose $\lambda_i, 7 \leq i \leq 12$ as free parameters. Now, we consider two cases.

Suppose that $q \neq -2$. In this case the system has the following solution

$$\lambda_1 = -\frac{-1+q+q^2}{(q+2)q}(\lambda_7 + \lambda_8 + \lambda_9 + (1-q-q^2)\lambda_{10} + (1+q)\lambda_{11} - \lambda_{12}),$$

$$\lambda_2 = -\frac{1}{(q+2)q}(\lambda_7 + (q^2+q-1)\lambda_8 + \lambda_9 - \lambda_{10} + (1-q-q^2)\lambda_{11} + (q+1)\lambda_{12}),$$

$$\lambda_3 = -\frac{1}{(q+2)q}(\lambda_7 + \lambda_8 + (q^2+q-1)\lambda_9 + (q+1)\lambda_{10} - \lambda_{11} - (q^2+q-1)\lambda_{12}),$$

$$\lambda_4 = -\frac{1}{(q+2)q}((1-q-q^2)\lambda_7 - \lambda_8 + (q+1)\lambda_9 + (q^2+q-1)\lambda_{10} + \lambda_{11} + \lambda_{12}),$$

$$\lambda_5 = -\frac{1}{(q+2)q}((1+q)\lambda_7 - (q^2+q-1)\lambda_8 - \lambda_9 + \lambda_{10} + (q^2+q-1)\lambda_{11} + \lambda_{12}),$$

$$\lambda_6 = -\frac{1}{(q+2)q}(-\lambda_7 + (q+1)\lambda_8 + (1-q-q^2)\lambda_9 + \lambda_{10} + \lambda_{11} + (q^2+q-1)\lambda_{12}).$$

Substitute these expressions for $\lambda_i, 1 \leq i \leq 6$, in $X(t_1, t_2, t_3)$ and collect the coefficients of $\lambda_j, 7 \leq j \leq 12$. We obtain a presentation of the polynomial $X(t_1, t_2, t_3)$ as a linear combination of the following 6 polynomials

$$\begin{aligned} f_1 = & (q-1)t_1(t_2t_3) - (q^3-q+1)t_1(t_3t_2) - (q-1)t_2(t_1t_3) - (q^2+q-1)t_2(t_3t_1) \\ & + (q^3-q)t_3(t_1t_2) + (q^3+q^2-q)(t_1t_3)t_2 + q(t_2t_3)t_1 \end{aligned}$$

$$\begin{aligned} f_2 = & (-1+q^2)t_1(t_2t_3) - t_1(t_3t_2) - (q^2-1)t_2(t_1t_3) - qt_3(t_1t_2) + (q^2+q-1)(t_2t_1)t_3 + (t_2t_3)t_1 \end{aligned}$$

$$\begin{aligned} f_3 = & (-q^3+q-1)t_1(t_2t_3) - t_1(t_3t_2) + (q^3-q+1)t_2(t_1t_3) - (q^2+q-1)t_2(t_3t_1) \\ & - qt_3(t_1t_2) + (q^3+q^2-q)(t_1t_2)t_3 + (q^2+q)(t_2t_3)t_1 \end{aligned}$$

$$\begin{aligned} f_4 = & -t_1(t_2t_3) - (1+q)t_1(t_3t_2) + t_2(t_1t_3) - (q^2+q-1)t_2(t_3t_1) - t_3(t_1t_2) \\ & + (q^2+q-1)t_3(t_2t_1) + (q^2+2q)(t_2t_3)t_1 \end{aligned}$$

$$f_5 = (1-q)t_1(t_2t_3) + (q^3 - q + 1)t_1(t_3t_2) - q^2t_2(t_1t_3) - (q^3 - q)t_3(t_1t_2) - q(t_2t_3)t_1 + (q^3 + q^2 - q)(t_3t_1)t_2$$

$$f_6 = -t_1(t_2t_3) - t_1(t_3t_2) + q(t_2t_3)t_1 + q((t_3t_2)t_1).$$

We see that if $q^2 \neq 1, q \neq -2$, then

$$f_1 = \frac{1}{(q-1)(q+1)(q+2)}(-lei^{(q)}(t_1, t_2, t_3) - (-1+q+q^2)lei^{(q)}(t_2, t_1, t_3) + (-1+q+q^2)^2lei^{(q)}(t_3, t_1, t_2) + (-1+q+q^2)lei^{(q)}(t_3, t_2, t_1)),$$

$$f_2 = lei^{(q)},$$

$$f_3 = \frac{1}{(q-1)(q+1)(q+2)}(- (1+q)lei^{(q)}(t_1, t_2, t_3) + (-1+q+q^2)^2lei^{(q)}(t_2, t_1, t_3) - (-1+q+q^2)lei^{(q)}(t_3, t_1, t_2) + (1+q)(-1+q+q^2)lei^{(q)}(t_3, t_2, t_1)),$$

$$f_4 = \frac{1}{(q+1)(q-1)}(-lei^{(q)}(t_1, t_2, t_3) + (-1+q+q^2)lei^{(q)}(t_3, t_2, t_1)),$$

$$f_5 = \frac{1}{(q-1)(q+1)(q+2)}(lei^{(q)}(t_1, t_2, t_3) + (-1+q+q^2)^2 lei^{(q)}(t_1, t_3, t_2) - (-1+q+q^2)lei^{(q)}(t_2, t_3, t_1) - (-1+q+q^2)lei^{(q)}(t_3, t_2, t_1)),$$

$$f_6 = \frac{1}{(q-1)(q+1)(q+2)}(-lei^{(q)}(t_1, t_2, t_3) - lei^{(q)}(t_1, t_3, t_2) + (-1+q+q^2)lei^{(q)}(t_2, t_3, t_1) + (-1+q+q^2)lei^{(q)}(t_3, t_2, t_1)).$$

Now, we consider the case $q = -2$. In this case, similar arguments show that X is a linear combination of the following polynomials

$$g_1 = t_3(t_2t_1) + 2/3(t_1t_2)t_3 + 4/3(t_1t_3)t_2 + 4/3(t_2t_1)t_3 - 4/3(t_2t_3)t_1 + 5/3(t_3t_1)t_2 - 5/3(t_3t_2)t_1,$$

$$g_2 = t_2(t_3t_1) + 4/3(t_1t_2)t_3 + 2/3(t_1t_3)t_2 + 5/3(t_2t_1)t_3 - 5/3(t_2t_3)t_1 + 4/3(t_3t_1)t_2 - 4/3(t_3t_2)t_1,$$

$$g_3 = t_1(t_2t_3) - 5/3(t_1t_2)t_3 + 5/3(t_1t_3)t_2 - 4/3(t_2t_1)t_3 + 4/3(t_2t_3)t_1$$

$$+4/3(t_3t_1)t_2 + 2/3(t_3t_2)t_1,$$

$$g_4 = t_1(t_3t_2) + 5/3(t_1t_2)t_3 - 5/3(t_1t_3)t_2 + 4/3(t_2t_1)t_3 + 2/3(t_2t_3)t_1 \\ -4/3(t_3t_1)t_2 + 4/3(t_3t_2)t_1,$$

$$g_5 = t_2(t_1t_3) - 4/3(t_1t_2)t_3 + 4/3(t_1t_3)t_2 - 5/3(t_2t_1)t_3 + 5/3(t_2t_3)t_1 \\ +2/3(t_3t_1)t_2 + 4/3(t_3t_2)t_1,$$

$$g_6 = t_3(t_1t_2 + 4/3(t_1t_2)t_3 - 4/3(t_1t_3)t_2 + 2/3(t_2t_1)t_3 + 4/3(t_2t_3)t_1 \\ -5/3(t_3t_1)t_2 + 5/3(t_3t_2)t_1.$$

We have

$$g_1 - 1/3(4lei^{(-2)}(t_1, t_2, t_3) + 3lei^{(-2)}(t_1, t_3, t_2) + 2lei^{(-2)}(t_2, t_1, t_3) \\ +4lei_1^{(-2)}(t_1, t_2, t_3) - lei_1^{(-2)}(t_2, t_1, t_3)) = 1/3ralia(t_1, t_2, t_3),$$

$$g_2 - 1/3(5lei^{(-2)}(t_1, t_2, t_3) + 6lei^{(-2)}(t_1, t_3, t_2) + 4lei^{(-2)}(t_2, t_1, t_3) \\ +5lei_1^{(-2)}(t_1, t_2, t_3) + lei_1^{(-2)}(t_2, t_1, t_3)) = 8/3ralia(t_1, t_2, t_3),$$

$$g_3 - 1/3(-4lei^{(-2)}(t_1, t_2, t_3) - 6lei^{(-2)}(t_1, t_3, t_2) - 5lei^{(-2)}(t_2, t_1, t_3) \\ -4lei_1^{(-2)}(t_1, t_2, t_3) - 5lei_1^{(-2)}(t_2, t_1, t_3)) = -10/3ralia(t_1, t_2, t_3),$$

$$g_4 - 1/3(4lei^{(-2)}(t_1, t_2, t_3) + 6lei^{(-2)}(t_1, t_3, t_2) + 5lei^{(-2)}(t_2, t_1, t_3) \\ +lei_1^{(-2)}(t_1, t_2, t_3) + 5lei_1^{(-2)}(t_2, t_1, t_3)) = 10/3ralia(t_1, t_2, t_3),$$

$$g_5 - 1/3(-5lei^{(-2)}(t_1, t_2, t_3) - 6lei^{(-2)}(t_1, t_3, t_2) - 4lei^{(-2)}(t_2, t_1, t_3) \\ -5lei_1^{(-2)}(t_1, t_2, t_3) - 4lei_1^{(-2)}(t_2, t_1, t_3)) = -8/3ralia(t_1, t_2, t_3),$$

$$g_6 - 1/3(2lei^{(-2)}(t_1, t_2, t_3) + 3lei^{(-2)}(t_1, t_3, t_2) + 4lei^{(-2)}(t_2, t_1, t_3) \\ -lei_1^{(-2)}(t_1, t_2, t_3) + 4lei_1^{(-2)}(t_2, t_1, t_3)) = 8/3ralia(t_1, t_2, t_3).$$

By Lemma 4.2 $ralia = 0$ is a consequence of the identity $lei^{(q)} = 0$. Therefore, all the identities $g_i = 0$ are consequences of the identities $lei^{(q)} = 0$ and $lei_1^{(q)} = 0$.

We have proved that any identity of degree 3 of $L^{(q)}$ for $q = -2$ follows from the identities $lei^{(q)} = 0$ and $lei_1^{(q)} = 0$. Notice that the equation

$$\begin{aligned} lei_1^{(q)}(a, b, c) &= \mu_1 lei^{(q)}(a, b, c) + \mu_2 lei^{(q)}(b, c, a) + \mu_3 lei^{(q)}(c, a, b) \\ &\quad + \mu_4 lei^{(q)}(b, a, c) + \mu_5 lei^{(q)}(c, b, a) + \mu_6 lei^{(q)}(a, c, b) \end{aligned}$$

in $L^{(q)}$ with unknowns $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6$ is not solvable. Therefore, this system of identities $lei^{(q)} = 0, lei_1^{(q)} = 0$ is $\mathfrak{L}\mathfrak{e}\mathfrak{i}^{(q)}$ -minimal if $q = -2$.

Lemma 4.4. *Suppose that $q \neq 0, \pm 1$ and an algebra (A, \star) satisfies the identities $lei^{(q)} = 0$ and $lei_1^{(q)} = 0$. Then the algebra (A, \circ) , where $a \circ b = (1 - q^2)^{-1}(a \star b - q b \star a)$, is a (right)-Leibniz algebra, and the algebras (A, \star) and (A, \circ_q) are isomorphic.*

Proof. One checks that

$$\begin{aligned} lei(t_1, t_2, t_3) &= \\ -2 lei^{(q)}(t_1, t_2, t_3) - 2/3 lei^{(q)}(t_1, t_3, t_2) - lei^{(q)}(t_2, t_1, t_3) + 2/3 (lei^{(q)}(t_2, t_3, t_1) \\ &\quad - 2 lei_1^{(q)}(t_1, t_2, t_3) - lei_1^{(q)}(t_2, t_1, t_3)) \end{aligned}$$

for $q^2 \neq 1, q = -2$, and

$$\begin{aligned} lei(t_1, t_2, t_3) &= \\ \frac{1}{(q^2 - 1)(q + 2)} &(q(q + 1) lei^{(q)}(t_1, t_2, t_3) - (-1 + 2q + q^2) lei^{(q)}(t_1, t_3, t_2) \\ &\quad - (q + 1) lei^{(q)}(t_2, t_1, t_3) + (1 - q + q^2 + q^3) lei^{(q)}(t_2, t_3, t_1) \\ &\quad + (q + 1) lei^{(q)}(t_3, t_1, t_2) - (q + q^2) lei^{(q)}(t_3, t_2, t_1)) \end{aligned}$$

for $q^2 \neq 1, q \neq -2$.

Therefore, for any algebra (A, \star) with identities $lei^{(q)} = 0$ and $lei_1^{(q)} = 0$ the algebra (A, \circ) , where $a \circ b = (1 - q^2)^{-1}(a \star b - q b \star a)$, satisfies the identity $lei = 0$. It is evident that

$$\begin{aligned} &a \circ_q b \\ &= (1 - q^2)^{-1}(a \circ b + q b \circ a) = (1 - q^2)^{-1}(a \star b - q b \star a + q b \star a - q^2 a \star b) \\ &= a \star b \end{aligned}$$

Proof of Theorems 1.1 and 1.2. By Lemmas 4.1, 4.3, 4.4 our theorems are true.

5. LEIBNIZ-LIE ALGEBRAS

In this section we study identities for Leibniz-Lie algebras, i.e., algebras $(A, [\ , \])$ under -1 -commutator for Leibniz algebras (A, \circ) . Recall that

$$\begin{aligned} & leilie_1(t_1, t_2, t_3, t_4, t_5) \\ &= -ljac(t_1, t_2, t_3)(t_4t_5) + 2((ljac(t_1, t_2, t_3))t_4)t_5 - 2((ljac(t_1, t_2, t_3)t_5)t_4 \\ &= -rjac(t_1, t_2, t_3)(t_4t_5) + 2((rjac(t_1, t_2, t_3))t_4)t_5 - 2((rjac(t_1, t_2, t_3)t_5)t_4 \\ &\quad = ((t_1t_2)t_3)(t_4t_5) - ((t_1t_3)t_2)(t_4t_5) + ((t_2t_3)t_1)(t_4t_5) \\ &\quad - 2(((t_1t_2)t_3)t_4)t_5 + 2(((t_1t_2)t_3)t_5)t_4 + 2(((t_1t_3)t_2)t_4)t_5 \\ &\quad - 2(((t_1t_3)t_2)t_5)t_4 - 2(((t_2t_3)t_1)t_4)t_5 + 2(((t_2t_3)t_1)t_5)t_4, \end{aligned}$$

$$\begin{aligned} & leilie_2(t_1, t_2, t_3, t_4, t_5) \\ &= -((t_1t_2)t_3)(t_4t_5) + ((t_1t_2)t_4)(t_3t_5) - ((t_1t_2)t_5)(t_3t_4) - ((t_1t_2)(t_3t_4))t_5 \\ &\quad + ((t_1t_2)(t_3t_5))t_4 - ((t_1t_2)(t_4t_5))t_3 + ((t_1t_3)t_2)(t_4t_5) - ((t_1t_3)t_4)(t_2t_5) \\ &\quad + ((t_1t_3)t_5)(t_2t_4) + ((t_1t_3)(t_2t_4))t_5 - ((t_1t_3)(t_2t_5))t_4 + ((t_1t_3)(t_4t_5))t_2 \\ &\quad - ((t_1t_4)t_2)(t_3t_5) + ((t_1t_4)t_3)(t_2t_5) - ((t_1t_4)t_5)(t_2t_3) - ((t_1t_4)(t_2t_3))t_5 \\ &\quad + ((t_1t_4)(t_2t_5))t_3 - ((t_1t_4)(t_3t_5))t_2 + ((t_1t_5)t_2)(t_3t_4) - ((t_1t_5)t_3)(t_2t_4) \\ &\quad + ((t_1t_5)t_4)(t_2t_3) + ((t_1t_5)(t_2t_3))t_4 - ((t_1t_5)(t_2t_4))t_3 + ((t_1t_5)(t_3t_4))t_2 \\ &\quad - 2(((t_2t_3)t_1)t_4)t_5 + 2(((t_2t_3)t_1)t_5)t_4 - 2(((t_2t_3)t_4)t_1)t_5 + 4(((t_2t_3)t_4)t_5)t_1 \\ &\quad + 2(((t_2t_3)t_5)t_1)t_4 - 4(((t_2t_3)t_5)t_4)t_1 + 2(((t_2t_4)t_1)t_3)t_5 - 2(((t_2t_4)t_1)t_5)t_3 \\ &\quad + 2(((t_2t_4)t_3)t_1)t_5 - 4(((t_2t_4)t_3)t_5)t_1 - 2(((t_2t_4)t_5)t_1)t_3 + 4(((t_2t_4)t_5)t_3)t_1 \\ &\quad - 2(((t_2t_5)t_1)t_3)t_4 + 2(((t_2t_5)t_1)t_4)t_3 - 2(((t_2t_5)t_3)t_1)t_4 + 4(((t_2t_5)t_3)t_4)t_1 \\ &\quad + 2(((t_2t_5)t_4)t_1)t_3 - 4(((t_2t_5)t_4)t_3)t_1 - 2(((t_3t_4)t_1)t_2)t_5 + 2(((t_3t_4)t_1)t_5)t_2 \\ &\quad - 2(((t_3t_4)t_2)t_1)t_5 + 4(((t_3t_4)t_2)t_5)t_1 + 2(((t_3t_4)t_5)t_1)t_2 - 4(((t_3t_4)t_5)t_2)t_1 \\ &\quad + 2(((t_3t_5)t_1)t_2)t_4 - 2(((t_3t_5)t_1)t_4)t_2 + 2(((t_3t_5)t_2)t_1)t_4 - 4(((t_3t_5)t_2)t_4)t_1 \\ &\quad - 2(((t_3t_5)t_4)t_1)t_2 + 4(((t_3t_5)t_4)t_2)t_1 - 2(((t_4t_5)t_1)t_2)t_3 + 2(((t_4t_5)t_1)t_3)t_2 \\ &\quad - 2(((t_4t_5)t_2)t_1)t_3 + 4(((t_4t_5)t_2)t_3)t_1 + 2(((t_4t_5)t_3)t_1)t_2 - 4(((t_4t_5)t_3)t_2)t_1. \end{aligned}$$

We see that $leilie_1(t_1, t_2, t_3, t_4, t_5)$ has type $(3, 2)$, i.e., it is skew-symmetric in t_1, t_2, t_3 and in t_4, t_5 , and $leilie_2(t_1, t_2, t_3, t_4, t_5)$ has type $(1, 4)$, is skew-symmetric in t_2, t_3, t_4, t_5 .

Let

$$lei_3^{(-1)}(t_1, t_2, t_3, t_4, t_5) =$$

$$\begin{aligned} & ((t_1t_2)t_3)(t_4t_5) + ((t_1t_2)t_5)(t_3t_4) + ((t_1t_2)(t_3t_4))t_5 - 2((t_1t_2)(t_3t_5))t_4 \\ & + ((t_1t_2)(t_4t_5))t_3 - ((t_1t_3)t_5)(t_2t_4) + ((t_1t_3)(t_2t_4))t_5 - ((t_1t_4)t_3)(t_2t_5) \end{aligned}$$

$$\begin{aligned}
& +((t_1 t_4) t_5)(t_2 t_3) + ((t_1 t_4)(t_2 t_3)) t_5 - ((t_1 t_4)(t_2 t_5)) t_3 + 2((t_1 t_4)(t_3 t_5)) t_2 \\
& + ((t_1 t_5) t_3)(t_2 t_4) - ((t_1 t_5)(t_2 t_4)) t_3 + ((t_2 t_3) t_5)(t_1 t_4) + ((t_2 t_4) t_3)(t_1 t_5) \\
& - ((t_2 t_4) t_5)(t_1 t_3) - 2((t_2 t_4)(t_3 t_5)) t_1 - ((t_2 t_5) t_3)(t_1 t_4) + ((t_3 t_5) t_4)(t_1 t_2) \\
& + 2(((t_1 t_2) t_3) t_5) t_4 - 6(((t_1 t_2) t_4) t_3) t_5 + 6(((t_1 t_2) t_4) t_5) t_3 - 2(((t_1 t_2) t_5) t_3) t_4 \\
& - 2(((t_1 t_3) t_2) t_4) t_5 + 2(((t_1 t_3) t_4) t_2) t_5 + 6(((t_1 t_3) t_5) t_2) t_4 - 6(((t_1 t_3) t_5) t_4) t_2 \\
& + 6(((t_1 t_4) t_2) t_3) t_5 - 6(((t_1 t_4) t_2) t_5) t_3 - 2(((t_1 t_4) t_3) t_5) t_2 + 2(((t_1 t_4) t_5) t_3) t_2 \\
& + 2(((t_1 t_5) t_2) t_4) t_3 - 6(((t_1 t_5) t_3) t_2) t_4 + 6(((t_1 t_5) t_3) t_4) t_2 - 2(((t_1 t_5) t_4) t_2) t_3 \\
& + 2(((t_2 t_3) t_1) t_4) t_5 - 2(((t_2 t_3) t_4) t_1) t_5 - 6(((t_2 t_3) t_5) t_1) t_4 + 6(((t_2 t_3) t_5) t_4) t_1 \\
& - 6(((t_2 t_4) t_1) t_3) t_5 + 6(((t_2 t_4) t_1) t_5) t_3 + 2(((t_2 t_4) t_3) t_5) t_1 - 2(((t_2 t_4) t_5) t_3) t_1 \\
& - 2(((t_2 t_5) t_1) t_4) t_3 + 6(((t_2 t_5) t_3) t_1) t_4 - 6(((t_2 t_5) t_3) t_4) t_1 + 2(((t_2 t_5) t_4) t_1) t_3 \\
& + 2(((t_3 t_4) t_1) t_2) t_5 - 2(((t_3 t_4) t_2) t_1) t_5 - 4(((t_3 t_4) t_5) t_1) t_2 + 4(((t_3 t_4) t_5) t_2) t_1 \\
& + 4(((t_3 t_5) t_1) t_2) t_4 - 4(((t_3 t_5) t_1) t_4) t_2 - 4(((t_3 t_5) t_2) t_1) t_4 + 4(((t_3 t_5) t_2) t_4) t_1 \\
& + 2(((t_3 t_5) t_4) t_1) t_2 - 2(((t_3 t_5) t_4) t_2) t_1 + 2(((t_4 t_5) t_1) t_2) t_3 - 2(((t_4 t_5) t_2) t_1) t_3 \\
& - 4(((t_4 t_5) t_3) t_1) t_2 + 4(((t_4 t_5) t_3) t_2) t_1.
\end{aligned}$$

Lemma 5.1. *The identity $lei_3^{(-1)} = 0$ is a consequence of the identities $leilie_1 = 0$, $leilie_2 = 0$ and the anti-commutativity identity.*

Proof. Let

$$\begin{aligned}
R = & lei_3^{(-1)}(t_1, t_2, t_3, t_4, t_5) + leilie_1(t_1, t_2, t_3, t_4, t_5) - 2leilie_1(t_1, t_2, t_4, t_3, t_5) \\
& + leilie_1(t_1, t_2, t_5, t_3, t_4) + leilie_1(t_1, t_3, t_4, t_2, t_5) + leilie_1(t_1, t_4, t_5, t_2, t_3) \\
& - leilie_1(t_2, t_3, t_4, t_1, t_5) - leilie_1(t_2, t_4, t_5, t_1, t_3) + leilie_1(t_3, t_4, t_5, t_1, t_2) \\
& + leilie_2(t_1, t_2, t_3, t_4, t_5) - leilie_2(t_2, t_1, t_3, t_4, t_5) - leilie_2(t_4, t_1, t_2, t_3, t_5).
\end{aligned}$$

Direct calculations show that

$$\begin{aligned}
R = & +2((t_1 t_3)(t_2 t_4)) t_5 + ((t_2 t_4)(t_1 t_3)) t_5 - ((t_4 t_2)(t_1 t_3)) t_5 \\
& + ((t_2 t_1)(t_3 t_4)) t_5 + ((t_4 t_3)(t_1 t_2)) t_5 \\
& - ((t_2 t_3)(t_1 t_4)) t_5 + ((t_4 t_1)(t_2 t_3)) t_5 \\
& - ((t_1 t_2)(t_3 t_5)) t_4 - ((t_2 t_1)(t_3 t_5)) t_4 \\
& - ((t_2 t_5)(t_1 t_3)) t_4 - ((t_1 t_3)(t_2 t_5)) t_4 \\
& + ((t_1 t_5)(t_2 t_3)) t_4 + ((t_2 t_3)(t_1 t_5)) t_4 \\
& - 2((t_1 t_5)(t_2 t_4)) t_3 - ((t_2 t_4)(t_1 t_5)) t_3 + ((t_4 t_2)(t_1 t_5)) t_3 \\
& + ((t_2 t_5)(t_1 t_4)) t_3 - ((t_4 t_1)(t_2 t_5)) t_3 \\
& + ((t_2 t_1)(t_4 t_5)) t_3 - ((t_4 t_5)(t_1 t_2)) t_3 \\
& + ((t_1 t_3)(t_4 t_5)) t_2 + ((t_4 t_5)(t_1 t_3)) t_2 \\
& + ((t_1 t_4)(t_3 t_5)) t_2 + ((t_4 t_1)(t_3 t_5)) t_2
\end{aligned}$$

$$\begin{aligned}
 & +((t_1t_5)(t_3t_4))t_2 - ((t_4t_3)(t_1t_5))t_2 \\
 & -((t_2t_3)(t_4t_5))t_1 - ((t_4t_5)(t_2t_3))t_1 \\
 & -((t_2t_4)(t_3t_5))t_1 - ((t_4t_2)(t_3t_5))t_1 \\
 & +((t_4t_3)(t_2t_5))t_1 - ((t_2t_5)(t_3t_4))t_1 \\
 & +((t_1t_2)t_3)(t_4t_5) + ((t_2t_1)t_3)(t_4t_5) \\
 & +((t_1t_4)t_2)(t_3t_5) + ((t_4t_1)t_2)(t_3t_5) \\
 & -((t_2t_4)t_1)(t_3t_5) - ((t_4t_2)t_1)(t_3t_5) \\
 & -((t_1t_2)t_4)(t_3t_5) - ((t_2t_1)t_4)(t_3t_5) \\
 & +((t_1t_2)t_5)(t_3t_4) + ((t_2t_1)t_5)(t_3t_4) \\
 & -((t_1t_4)t_3)(t_2t_5) - ((t_4t_1)t_3)(t_2t_5) \\
 & +((t_3t_4)t_1)(t_2t_5) + ((t_4t_3)t_1)(t_2t_5) \\
 & +((t_1t_4)t_5)(t_2t_3) + ((t_4t_1)t_5)(t_2t_3) \\
 & +((t_2t_4)t_3)(t_1t_5) + ((t_4t_2)t_3)(t_1t_5) \\
 & -(((t_3t_4)t_2)(t_1t_5) - ((t_4t_3)t_2)(t_1t_5) \\
 & -((t_2t_4)t_5)(t_1t_3) - ((t_4t_2)t_5)(t_1t_3) \\
 & +((t_3t_4)t_5)(t_1t_2) + ((t_4t_3)t_5)(t_1t_2) \\
 & = 0,
 \end{aligned}$$

if $t_1t_2 - t_2t_1 = 0$ is identity.

Lemma 5.2. *Let (A, \circ) be a Leibniz algebra. Then the Leibniz-Lie algebra $(A, [,])$ satisfies the identities $lei_1^{(-1)} = 0$, $leilie_2 = 0$.*

Proof. By Lemma 3.1 $leilie_1 = 0$ is an identity on $A^{(-1)}$.

Let us check that $leilie_2 = 0$. Present

$$h = h(a, b, c, d, e)$$

$$\begin{aligned}
 \stackrel{def}{=} & -4[[[[b, c], d], e], a] + 2[[[[b, c], a], d], e] + 2[[[[b, c], d], a], e] \\
 & + [[a, b], c], [d, e]] + [[a, b], [d, e]], c]
 \end{aligned}$$

in the form $h_1 + h_2$, where

$$h_1 = h_1(a, b, c, d, e) = -4[[[[b, c], d], e], a] + 2[[[[b, c], a], d], e] + 2[[[[b, c], d], a], e],$$

$$h_2 = h_2(a, b, c, d, e) = [[a, b], c], [d, e]] + [[a, b], [d, e]], c].$$

Notice that

$$\begin{aligned}
 & -2[[[[[[u, b], c], a] + [[[[[[u, a], d], e] + [[[[[[u, d], a], e] \\
 = & -8(((a \circ b) \circ u) \circ c) + 8(((a \circ c) \circ b) \circ u) - 8(((a \circ c) \circ u) \circ b) \\
 & + 2(((a \circ d) \circ u) \circ e) + 8(((a \circ u) \circ b) \circ c) - 3(((a \circ u) \circ d) \circ e) \\
 & + 2(((b \circ u) \circ c) \circ a) - 4(((c \circ b) \circ u) \circ a) + 4(((c \circ u) \circ b) \circ a) \\
 & + 2(((d \circ a) \circ u) \circ e) - 3(((d \circ u) \circ a) \circ e) - 4(((e \circ a) \circ d) \circ u)
 \end{aligned}$$

$$\begin{aligned}
& +8(((e \circ a) \circ u) \circ d) - 4(((e \circ d) \circ a) \circ u) + 8(((e \circ d) \circ u) \circ a) \\
& -4(((e \circ u) \circ a) \circ d) - 4(((e \circ u) \circ d) \circ a) + (((u \circ a) \circ d) \circ e) \\
& -2(((u \circ b) \circ c) \circ a) + (((u \circ d) \circ a) \circ e).
\end{aligned}$$

Therefore,

$$\begin{aligned}
h_1 &= -4[[[b, c], d], e], a] + 2[[[b, c], a], d], e] + 2[[[b, c], d], a], e] \\
&= 20((((a \circ b) \circ c) \circ d) \circ e) - 20((((a \circ c) \circ b) \circ d) \circ e) - 24((((a \circ d) \circ b) \circ c) \circ e) \\
&+ 24((((a \circ d) \circ c) \circ b) \circ e) - 32((((a \circ e) \circ b) \circ c) \circ d) + 32((((a \circ e) \circ c) \circ b) \circ d) \\
&+ 32((((a \circ e) \circ d) \circ b) \circ c) - 32((((a \circ e) \circ d) \circ c) \circ b) + 2((((b \circ c) \circ a) \circ d) \circ e) \\
&+ 2((((b \circ c) \circ d) \circ a) \circ e) - 4((((b \circ c) \circ d) \circ e) \circ a) - 2((((c \circ b) \circ a) \circ d) \circ e) \\
&- 2((((c \circ b) \circ d) \circ a) \circ e) + 4((((c \circ b) \circ d) \circ e) \circ a) + 8((((d \circ a) \circ b) \circ c) \circ e) \\
&- 8((((d \circ a) \circ c) \circ b) \circ e) - 12((((d \circ b) \circ c) \circ a) \circ e) + 8((((d \circ b) \circ c) \circ e) \circ a) \\
&+ 12((((d \circ c) \circ b) \circ a) \circ e) - 8((((d \circ c) \circ b) \circ e) \circ a) + 32((((e \circ a) \circ b) \circ c) \circ d) \\
&- 32((((e \circ a) \circ c) \circ b) \circ d) - 16((((e \circ a) \circ d) \circ b) \circ c) + 16((((e \circ a) \circ d) \circ c) \circ b) \\
&- 16((((e \circ b) \circ c) \circ a) \circ d) + 16((((e \circ c) \circ b) \circ a) \circ d) - 16((((e \circ d) \circ a) \circ b) \circ c) \\
&+ 16((((e \circ d) \circ a) \circ c) \circ b) + 16((((e \circ d) \circ b) \circ c) \circ a) - 16((((e \circ d) \circ c) \circ b) \circ a).
\end{aligned}$$

Further,

$$\begin{aligned}
& [[u, b], c] + [[u, c], b] \\
&= 2((b \circ c) \circ u) - 3((b \circ u) \circ c) + 2((c \circ b) \circ u) - 3((c \circ u) \circ b) + ((u \circ b) \circ c) + ((u \circ c) \circ b).
\end{aligned}$$

Thus,

$$\begin{aligned}
h_2 &= [[[a, b], c], [d, e]] + [[[a, b], [d, e]], c] \\
&= 2((((a \circ b) \circ c) \circ d) \circ e) - 2((((a \circ b) \circ c) \circ e) \circ d) + 2((((a \circ b) \circ d) \circ e) \circ c) \\
&- 2((((a \circ b) \circ e) \circ d) \circ c) - 2((((b \circ a) \circ c) \circ d) \circ e) + 2((((b \circ a) \circ c) \circ e) \circ d) \\
&- 2((((b \circ a) \circ d) \circ e) \circ c) + 2((((b \circ a) \circ e) \circ d) \circ c) - 12((((c \circ a) \circ b) \circ d) \circ e) \\
&+ 12((((c \circ a) \circ b) \circ e) \circ d) + 12((((c \circ b) \circ a) \circ d) \circ e) - 12((((c \circ b) \circ a) \circ e) \circ d) \\
&+ 8((((c \circ d) \circ e) \circ a) \circ b) - 8((((c \circ d) \circ e) \circ b) \circ a) - 8((((c \circ e) \circ d) \circ a) \circ b) \\
&+ 8((((c \circ e) \circ d) \circ b) \circ a) - 6((((d \circ e) \circ a) \circ b) \circ c) + 6((((d \circ e) \circ b) \circ a) \circ c) \\
&+ 4((((d \circ e) \circ c) \circ a) \circ b) - 4((((d \circ e) \circ c) \circ b) \circ a) + 6((((e \circ d) \circ a) \circ b) \circ c) \\
&- 6((((e \circ d) \circ b) \circ a) \circ c) - 4((((e \circ d) \circ c) \circ a) \circ b) + 4((((e \circ d) \circ c) \circ b) \circ a).
\end{aligned}$$

Hence, $h = h_1 + h_2$ can be presented in the form

$$h = r_1 + r_2 + r_3 + r_4 + r_5,$$

where $r_i = r_i(a, b, c, d, e)$, $i = 1, 2, 3, 4, 5$, and

$$\begin{aligned}
r_1 &= 22((((a \circ b) \circ c) \circ d) \circ e) - 2((((a \circ b) \circ c) \circ e) \circ d) + 2((((a \circ b) \circ d) \circ e) \circ c) \\
&- 2((((a \circ b) \circ e) \circ d) \circ c) - 20((((a \circ c) \circ b) \circ d) \circ e) - 24((((a \circ d) \circ b) \circ c) \circ e) \\
&+ 24((((a \circ d) \circ c) \circ b) \circ e) - 32((((a \circ e) \circ b) \circ c) \circ d) + 32((((a \circ e) \circ c) \circ b) \circ d) \\
&+ 32((((a \circ e) \circ d) \circ b) \circ c) - 32((((a \circ e) \circ d) \circ c) \circ b),
\end{aligned}$$

$$\begin{aligned} r_2 = & -2((((b \circ a) \circ c) \circ d) \circ e) + 2((((b \circ a) \circ c) \circ e) \circ d) - 2((((b \circ a) \circ d) \circ e) \circ c) \\ & + 2((((b \circ a) \circ e) \circ d) \circ c) - 12((((c \circ a) \circ b) \circ d) \circ e) + 12((((c \circ a) \circ b) \circ e) \circ d) \\ & + 8((((d \circ a) \circ b) \circ c) \circ e) - 8((((d \circ a) \circ c) \circ b) \circ e) + 32((((e \circ a) \circ b) \circ c) \circ d) \\ & - 32((((e \circ a) \circ c) \circ b) \circ d) - 16((((e \circ a) \circ d) \circ b) \circ c) + 16((((e \circ a) \circ d) \circ c) \circ b), \end{aligned}$$

$$\begin{aligned} r_3 = & 2((((b \circ c) \circ a) \circ d) \circ e) + 10((((c \circ b) \circ a) \circ d) \circ e) - 12((((c \circ b) \circ a) \circ e) \circ d) \\ & - 6((((d \circ e) \circ a) \circ b) \circ c) - 10((((e \circ d) \circ a) \circ b) \circ c) + 16((((e \circ d) \circ a) \circ c) \circ b), \end{aligned}$$

$$\begin{aligned} r_4 = & 2((((b \circ c) \circ d) \circ a) \circ e) - 2((((c \circ b) \circ d) \circ a) \circ e) + 8((((c \circ d) \circ e) \circ a) \circ b) \\ & - 8((((c \circ e) \circ d) \circ a) \circ b) - 12((((d \circ b) \circ c) \circ a) \circ e) + 12((((d \circ c) \circ b) \circ a) \circ e) \\ & + 6((((d \circ e) \circ b) \circ a) \circ c) + 4((((d \circ e) \circ c) \circ a) \circ b) - 16((((e \circ b) \circ c) \circ a) \circ d) \\ & + 16((((e \circ c) \circ b) \circ a) \circ d) - 6((((e \circ d) \circ b) \circ a) \circ c) - 4((((e \circ d) \circ c) \circ a) \circ b), \end{aligned}$$

$$\begin{aligned} r_5 = & -4((((b \circ c) \circ d) \circ e) \circ a) + 4((((c \circ b) \circ d) \circ e) \circ a) - 8((((c \circ d) \circ e) \circ b) \circ a) \\ & + 8((((c \circ e) \circ d) \circ b) \circ a) + 8((((d \circ b) \circ c) \circ e) \circ a) - 8((((d \circ c) \circ b) \circ e) \circ a) \\ & - 4((((d \circ e) \circ c) \circ b) \circ a) + 16((((e \circ d) \circ b) \circ c) \circ a) - 12((((e \circ d) \circ c) \circ b) \circ a). \end{aligned}$$

Show that $Alt(r_1) = 0$. Let us collect the coefficients of $(((a \circ b) \circ c) \circ d) \circ e$ in r_1 . We see that this yields

$$\begin{aligned} & 22 \operatorname{sign} \begin{pmatrix} a & b & c & d & e \\ a & b & c & d & e \end{pmatrix} - 2 \operatorname{sign} \begin{pmatrix} a & b & c & d & e \\ a & b & c & e & d \end{pmatrix} \\ & + 2 \operatorname{sign} \begin{pmatrix} a & b & c & d & e \\ a & b & d & e & c \end{pmatrix} - 2 \operatorname{sign} \begin{pmatrix} a & b & c & d & e \\ a & b & e & d & c \end{pmatrix} \\ & - 20 \operatorname{sign} \begin{pmatrix} a & b & c & d & e \\ a & c & b & d & e \end{pmatrix} - 24 \operatorname{sign} \begin{pmatrix} a & b & c & d & e \\ a & d & b & c & e \end{pmatrix} \\ & + 24 \operatorname{sign} \begin{pmatrix} a & b & c & d & e \\ a & d & c & b & e \end{pmatrix} - 32 \operatorname{sign} \begin{pmatrix} a & b & c & d & e \\ a & e & b & c & d \end{pmatrix} \\ & + 32 \operatorname{sign} \begin{pmatrix} a & b & c & d & e \\ a & e & c & b & d \end{pmatrix} + 32 \operatorname{sign} \begin{pmatrix} a & b & c & d & e \\ a & e & d & b & c \end{pmatrix} \\ & \quad - 32 \operatorname{sign} \begin{pmatrix} a & b & c & d & e \\ a & e & d & c & b \end{pmatrix} \\ & = 22 + 2 + 2 + 2 + 20 - 24 - 24 + 32 + 32 - 32 - 32 = 0. \end{aligned}$$

So, by skew-symmetry, $r_1 = 0$. Analogous calculations show that

$$Alt(r_i) = 0$$

for all $i = 1, 2, 3, 4, 5$. Thus,

$$leilie_2 = Alt(r_1) + \cdots + Alt(r_5) = 0.$$

Lemma 5.3. *Any identity of degree 4 for $\mathfrak{L}\mathfrak{e}\mathfrak{i}^{(-1)}$ follows from the identity $acom = 0$.*

Proof. Let

$$\begin{aligned} X_4(t_1, t_2, t_3, t_4) = & \\ & \lambda_1(t_1 t_2)(t_3 t_4) + \lambda_2(t_1 t_3)(t_2 t_4) + \lambda_3(t_2 t_3)(t_1 t_4) + \lambda_4((t_1 t_2) t_3) t_4 + \lambda_{10}((t_1 t_2) t_4) t_3 \\ & + \lambda_5((t_1 t_3) t_2) t_4 + \lambda_{11}((t_1 t_3) t_4) t_2 + \lambda_6((t_1 t_4) t_2) t_3 + \lambda_{12}((t_1 t_4) t_3) t_2 + \lambda_7((t_2 t_3) t_1) t_4 \\ & + \lambda_{13}((t_2 t_3) t_4) t_1 + \lambda_8((t_2 t_4) t_1) t_3 + \lambda_{14}((t_2 t_4) t_3) t_1 + \lambda_9((t_3 t_4) t_1) t_2 + \lambda_{15}((t_3 t_4) t_2) t_1 \end{aligned}$$

be a generic skew-symmetric polynomial of degree 4. For t_1, t_2, t_3, t_4 , we substitute the elements a, b, c, d of the free Leibniz algebra, and calculate $X_4(a, b, c, d)$ under the commutator $[u, v] = u \circ v - v \circ u$. We obtain

$$\begin{aligned} X_4(a, b, c, d) = & \\ & (2\lambda_1 + \lambda_4 - 2\lambda_7 - 4\lambda_{13} + 4\lambda_{15})(((a \circ b) \circ c) \circ d) \\ & + (-2\lambda_1 - 2\lambda_8 + \lambda_{10} - 4\lambda_{14} - 4\lambda_{15})(((a \circ b) \circ d) \circ c) \\ & + (2\lambda_2 + \lambda_5 + 2\lambda_7 + 4\lambda_{13} + 4\lambda_{14})(((a \circ c) \circ b) \circ d) \\ & + (-2\lambda_2 - 2\lambda_9 + \lambda_{11} - 4\lambda_{14} - 4\lambda_{15})(((a \circ c) \circ d) \circ b) \\ & + (-2\lambda_3 + \lambda_6 + 2\lambda_8 + 4\lambda_{13} + 4\lambda_{14})(((a \circ d) \circ b) \circ c) \\ & + (2\lambda_3 + 2\lambda_9 + \lambda_{12} - 4\lambda_{13} + 4\lambda_{15})(((a \circ d) \circ c) \circ b) \\ & + (-2\lambda_1 - \lambda_4 - 2\lambda_5 + 4\lambda_9 - 4\lambda_{11})(((b \circ a) \circ c) \circ d) \\ & + (2\lambda_1 - 2\lambda_6 - 4\lambda_9 - \lambda_{10} - 4\lambda_{12})(((b \circ a) \circ d) \circ c) \\ & + (2\lambda_3 + 2\lambda_5 + \lambda_7 + 4\lambda_{11} + 4\lambda_{12})(((b \circ c) \circ a) \circ d) \\ & + (-2\lambda_3 - 4\lambda_9 - 4\lambda_{12} + \lambda_{13} - 2\lambda_{15})(((b \circ c) \circ d) \circ a) \\ & + (-2\lambda_2 + 2\lambda_6 + \lambda_8 + 4\lambda_{11} + 4\lambda_{12})(((b \circ d) \circ a) \circ c) \\ & + (2\lambda_2 + 4\lambda_9 - 4\lambda_{11} + \lambda_{14} + 2\lambda_{15})(((b \circ d) \circ c) \circ a) \\ & + (-2\lambda_2 - 2\lambda_4 - \lambda_5 + 4\lambda_8 - 4\lambda_{10})(((c \circ a) \circ b) \circ d) \\ & + (2\lambda_2 - 4\lambda_6 - 4\lambda_8 - \lambda_{11} - 2\lambda_{12})(((c \circ a) \circ d) \circ b) \\ & + (-2\lambda_3 + 2\lambda_4 + 4\lambda_6 - \lambda_7 + 4\lambda_{10})(((c \circ b) \circ a) \circ d) \\ & + (2\lambda_3 - 4\lambda_6 - 4\lambda_8 - \lambda_{13} - 2\lambda_{14})(((c \circ b) \circ d) \circ a) \\ & + (-2\lambda_1 + 4\lambda_6 + \lambda_9 + 4\lambda_{10} + 2\lambda_{12})(((c \circ d) \circ a) \circ b) \\ & + (2\lambda_1 + 4\lambda_8 - 4\lambda_{10} + 2\lambda_{14} + \lambda_{15})(((c \circ d) \circ b) \circ a) \\ & + (2\lambda_3 - 4\lambda_4 - \lambda_6 + 4\lambda_7 - 2\lambda_{10})(((d \circ a) \circ b) \circ c) \\ & + (-2\lambda_3 - 4\lambda_5 - 4\lambda_7 - 2\lambda_{11} - \lambda_{12})(((d \circ a) \circ c) \circ b) \\ & + (2\lambda_2 + 4\lambda_4 + 4\lambda_5 - \lambda_8 + 2\lambda_{10})(((d \circ b) \circ a) \circ c) \\ & + (-2\lambda_2 - 4\lambda_5 - 4\lambda_7 - 2\lambda_{13} - \lambda_{14})(((d \circ b) \circ c) \circ a) \end{aligned}$$

$$\begin{aligned}
 &+(2\lambda_1 + 4\lambda_4 + 4\lambda_5 - \lambda_9 + 2\lambda_{11})(((d \circ c) \circ a) \circ b) \\
 &+(-2\lambda_1 - 4\lambda_4 + 4\lambda_7 + 2\lambda_{13} - \lambda_{15})(((d \circ c) \circ b) \circ a).
 \end{aligned}$$

Since all 24 left-bracketed elements like $((a \circ b) \circ c) \circ d$ are basis elements, the condition $X_4(a, b, c, d) = 0$ gives us the system of 24 linear equations in 15 unknowns $\lambda_i, i = 1, \dots, 15$. We see that the rank of this system is 15 and our system has the trivial solution only: $\lambda_i = 0$ for all $i = 1, 2, \dots, 15$. In other words, any polylinear identity of degree 4 for $\mathfrak{L}\mathfrak{e}\mathfrak{i}^{(-1)}$ follows from the identity $acom = 0$.

Lemma 5.4. *Any identity of a free Leibniz algebra of degree 5 follows from the identities $leilie_1 = 0$, $leilie_2 = 0$, $lei_3^{(-1)} = 0$.*

Proof. Let $f = f(t_1, \dots, t_5)$ be a non-commutative non-associative polynomial such that $f = 0$ is an identity for any right-Leibniz algebra. Notice that there exist 105 anti-commutative non-associative polynomials. Present f as a linear combination of these 105 elements.

Insert in f the elements of the free Leibniz algebra generated by 5 elements u_1, u_2, u_3, u_4, u_5 and calculate the polynomial f under the commutator $[u, v] = u \circ v - v \circ u$, where $(u, v) \mapsto u \circ v$ is the multiplication in a free (right)-Leibniz algebra. Expand this expression in terms of the multiplication \circ by using the Leibniz rule

$$u \circ (v \circ w) = (u \circ v) \circ w - (u \circ w) \circ v.$$

We obtain the element that is a linear combination of 120 elements of the form $((u_{\sigma(1)} \circ u_{\sigma(2)}) \circ u_{\sigma(3)}) \circ u_{\sigma(4)} \circ u_{\sigma(5)}$, where $\sigma \in Sym_5$. The identity condition $f = 0$ on $L^{(-1)}$ gives us 120 linear equations in 105 unknowns λ_i . Solve this system of equations. We do this using the computer system Mathematica. We find out that the system has 14 free parameters. It shows that f can be presented as a linear combination of the following 14 polynomials

$$f_1 = leilie_1,$$

$$\begin{aligned}
 f_2(t_1, t_2, t_3, t_4, t_5) = & \\
 &((t_1 t_2) t_4)(t_3 t_5) - ((t_1 t_4) t_2)(t_3 t_5) + ((t_2 t_4) t_1)(t_3 t_5) \\
 &- 2(((t_1 t_2) t_4) t_3) t_5 + 2(((t_1 t_2) t_4) t_5) t_3 + 2(((t_1 t_4) t_2) t_3) t_5 \\
 &- 2(((t_1 t_4) t_2) t_5) t_3 - 2(((t_2 t_4) t_1) t_3) t_5 + 2(((t_2 t_4) t_1) t_5) t_3,
 \end{aligned}$$

$$\begin{aligned}
 f_3(t_1, t_2, t_3, t_4, t_5) = & \\
 &((t_1 t_2) t_5)(t_3 t_4) - ((t_1 t_5) t_2)(t_3 t_4) + ((t_2 t_5) t_1)(t_3 t_4) \\
 &- 2(((t_1 t_2) t_5) t_3) t_4 + 2(((t_1 t_2) t_5) t_4) t_3 + 2(((t_1 t_5) t_2) t_3) t_4
 \end{aligned}$$

$$-2(((t_1 t_5) t_2) t_4) t_3 - 2(((t_2 t_5) t_1) t_3) t_4 + 2(((t_2 t_5) t_1) t_4) t_3,$$

$$\begin{aligned} f_4(t_1, t_2, t_3, t_4, t_5) = & \\ & ((t_1 t_3) t_4) (t_2 t_5) - ((t_1 t_4) t_3) (t_2 t_5) + ((t_3 t_4) t_1) (t_2 t_5) \\ & - 2(((t_1 t_3) t_4) t_2) t_5 + 2(((t_1 t_3) t_4) t_5) t_2 + 2(((t_1 t_4) t_3) t_2) t_5 \\ & - 2(((t_1 t_4) t_3) t_5) t_2 - 2(((t_3 t_4) t_1) t_2) t_5 + 2(((t_3 t_4) t_1) t_5) t_2, \end{aligned}$$

$$\begin{aligned} f_5(t_1, t_2, t_3, t_4, t_5) = & \\ & ((t_2 t_3) t_4) (t_1 t_5) - ((t_2 t_4) t_3) (t_1 t_5) + ((t_3 t_4) t_2) (t_1 t_5) \\ & - 2(((t_2 t_3) t_4) t_1) t_5 + 2(((t_2 t_3) t_4) t_5) t_1 + 2(((t_2 t_4) t_3) t_1) t_5 \\ & - 2(((t_2 t_4) t_3) t_5) t_1 - 2(((t_3 t_4) t_2) t_1) t_5 + 2(((t_3 t_4) t_2) t_5) t_1, \end{aligned}$$

$$\begin{aligned} f_6(t_1, t_2, t_3, t_4, t_5) = & \\ & ((t_1 t_3) t_5) (t_2 t_4) - ((t_1 t_5) t_3) (t_2 t_4) + ((t_3 t_5) t_1) (t_2 t_4) \\ & - 2(((t_1 t_3) t_5) t_2) t_4 + 2(((t_1 t_3) t_5) t_4) t_2 + 2(((t_1 t_5) t_3) t_2) t_4 \\ & - 2(((t_1 t_5) t_3) t_4) t_2 - 2(((t_3 t_5) t_1) t_2) t_4 + 2(((t_3 t_5) t_1) t_4) t_2, \end{aligned}$$

$$\begin{aligned} f_7(t_1, t_2, t_3, t_4, t_5) = & \\ & ((t_2 t_3) t_5) (t_1 t_4) - ((t_2 t_5) t_3) (t_1 t_4) + ((t_3 t_5) t_2) (t_1 t_4) \\ & - 2(((t_2 t_3) t_5) t_1) t_4 + 2(((t_2 t_3) t_5) t_4) t_1 + 2(((t_2 t_5) t_3) t_1) t_4 \\ & - 2(((t_2 t_5) t_3) t_4) t_1 - 2(((t_3 t_5) t_2) t_1) t_4 + 2(((t_3 t_5) t_2) t_4) t_1, \end{aligned}$$

$$f_8 = leilie_2,$$

$$\begin{aligned} f_9(t_1, t_2, t_3, t_4, t_5) = & \\ & -((t_1 t_2) t_3) (t_4 t_5) + ((t_1 t_2) t_4) (t_3 t_5) - ((t_1 t_2) t_5) (t_3 t_4) - ((t_1 t_2) (t_3 t_4)) t_5 \\ & + ((t_1 t_2) (t_3 t_5)) t_4 - ((t_1 t_2) (t_4 t_5)) t_3 + ((t_1 t_3) t_2) (t_4 t_5) - ((t_1 t_3) t_4) (t_2 t_5) \\ & + ((t_1 t_3) t_5) (t_2 t_4) + ((t_1 t_3) (t_2 t_4)) t_5 - ((t_1 t_3) (t_2 t_5)) t_4 + ((t_1 t_3) (t_4 t_5)) t_2 \\ & - ((t_1 t_4) t_2) (t_3 t_5) + ((t_1 t_4) t_3) (t_2 t_5) - ((t_1 t_4) (t_2 t_3)) t_5 + ((t_1 t_4) (t_2 t_5)) t_3 \\ & - ((t_1 t_4) (t_3 t_5)) t_2 + ((t_1 t_5) t_2) (t_3 t_4) - ((t_1 t_5) t_3) (t_2 t_4) + ((t_1 t_5) (t_2 t_3)) t_4 \\ & - ((t_1 t_5) (t_2 t_4)) t_3 + ((t_1 t_5) (t_3 t_4)) t_2 + ((t_4 t_5) t_1) (t_2 t_3) - 2(((t_1 t_4) t_5) t_2) t_3 \\ & + 2(((t_1 t_4) t_5) t_3) t_2 + 2(((t_1 t_5) t_4) t_2) t_3 - 2(((t_1 t_5) t_4) t_3) t_2 \\ & - 2(((t_2 t_3) t_1) t_4) t_5 + 2(((t_2 t_3) t_1) t_5) t_4 - 2(((t_2 t_3) t_4) t_1) t_5 + 4(((t_2 t_3) t_4) t_5) t_1 \\ & + 2(((t_2 t_3) t_5) t_1) t_4 - 4(((t_2 t_3) t_5) t_4) t_1 + 2(((t_2 t_4) t_1) t_3) t_5 - 2(((t_2 t_4) t_1) t_5) t_3 \\ & + 2(((t_2 t_4) t_3) t_1) t_5 - 4(((t_2 t_4) t_3) t_5) t_1 - 2(((t_2 t_4) t_5) t_1) t_3 + 4(((t_2 t_4) t_5) t_3) t_1 \\ & - 2(((t_2 t_5) t_1) t_3) t_4 + 2(((t_2 t_5) t_1) t_4) t_3 - 2(((t_2 t_5) t_3) t_1) t_4 + 4(((t_2 t_5) t_3) t_4) t_1 \end{aligned}$$

$$\begin{aligned}
 &+2(((t_2t_5)t_4)t_1)t_3 - 4(((t_2t_5)t_4)t_3)t_1 - 2(((t_3t_4)t_1)t_2)t_5 + 2(((t_3t_4)t_1)t_5)t_2 \\
 &- 2(((t_3t_4)t_2)t_1)t_5 + 4(((t_3t_4)t_2)t_5)t_1 + 2(((t_3t_4)t_5)t_1)t_2 - 4(((t_3t_4)t_5)t_2)t_1 \\
 &+ 2(((t_3t_5)t_1)t_2)t_4 - 2(((t_3t_5)t_1)t_4)t_2 + 2(((t_3t_5)t_2)t_1)t_4 - 4(((t_3t_5)t_2)t_4)t_1 \\
 &- 2(((t_3t_5)t_4)t_1)t_2 + 4(((t_3t_5)t_4)t_2)t_1 - 4(((t_4t_5)t_1)t_2)t_3 + 4(((t_4t_5)t_1)t_3)t_2 \\
 &- 2(((t_4t_5)t_2)t_1)t_3 + 4(((t_4t_5)t_2)t_3)t_1 + 2(((t_4t_5)t_3)t_1)t_2 - 4(((t_4t_5)t_3)t_2)t_1,
 \end{aligned}$$

$$\begin{aligned}
 &f_{10}(t_1, t_2, t_3, t_4, t_5) = \\
 &((t_1t_2)(t_3t_4))t_5 - ((t_1t_2)(t_3t_5))t_4 + ((t_1t_2)(t_4t_5))t_3 + ((t_1t_3)t_2)(t_4t_5) \\
 &+ ((t_1t_3)(t_2t_4))t_5 - ((t_1t_3)(t_2t_5))t_4 - ((t_1t_4)t_2)(t_3t_5) - ((t_1t_4)(t_2t_3))t_5 \\
 &+ ((t_1t_4)(t_2t_5))t_3 + ((t_1t_5)t_2)(t_3t_4) + ((t_1t_5)(t_2t_3))t_4 - ((t_1t_5)(t_2t_4))t_3 \\
 &- ((t_2t_3)t_4)(t_1t_5) + ((t_2t_3)t_5)(t_1t_4) + ((t_2t_3)(t_4t_5))t_1 + ((t_2t_4)t_3)(t_1t_5) \\
 &- ((t_2t_4)t_5)(t_1t_3) - ((t_2t_4)(t_3t_5))t_1 - ((t_2t_5)t_3)(t_1t_4) + ((t_2t_5)t_4)(t_1t_3) \\
 &\quad + ((t_2t_5)(t_3t_4))t_1 + 2(((t_1t_2)t_3)t_4)t_5 - 2(((t_1t_2)t_3)t_5)t_4 \\
 &- 2(((t_1t_2)t_4)t_3)t_5 + 2(((t_1t_2)t_4)t_5)t_3 + 2(((t_1t_2)t_5)t_3)t_4 - 2(((t_1t_2)t_5)t_4)t_3 \\
 &- 4(((t_1t_3)t_2)t_4)t_5 + 4(((t_1t_3)t_2)t_5)t_4 - 2(((t_1t_3)t_4)t_2)t_5 + 4(((t_1t_3)t_4)t_5)t_2 \\
 &+ 2(((t_1t_3)t_5)t_2)t_4 - 4(((t_1t_3)t_5)t_4)t_2 + 4(((t_1t_4)t_2)t_3)t_5 - 4(((t_1t_4)t_2)t_5)t_3 \\
 &+ 2(((t_1t_4)t_3)t_2)t_5 - 4(((t_1t_4)t_3)t_5)t_2 - 2(((t_1t_4)t_5)t_2)t_3 + 4(((t_1t_4)t_5)t_3)t_2 \\
 &- 4(((t_1t_5)t_2)t_3)t_4 + 4(((t_1t_5)t_2)t_4)t_3 - 2(((t_1t_5)t_3)t_2)t_4 + 4(((t_1t_5)t_3)t_4)t_2 \\
 &+ 2(((t_1t_5)t_4)t_2)t_3 - 4(((t_1t_5)t_4)t_3)t_2 + 2(((t_2t_3)t_1)t_4)t_5 - 2(((t_2t_3)t_1)t_5)t_4 \\
 &- 2(((t_2t_4)t_1)t_3)t_5 + 2(((t_2t_4)t_1)t_5)t_3 + 2(((t_2t_5)t_1)t_3)t_4 - 2(((t_2t_5)t_1)t_4)t_3 \\
 &- 2(((t_3t_4)t_1)t_2)t_5 + 4(((t_3t_4)t_1)t_5)t_2 - 2(((t_3t_4)t_2)t_1)t_5 + 2(((t_3t_4)t_2)t_5)t_1 \\
 &- 4(((t_3t_4)t_5)t_1)t_2 + 2(((t_3t_4)t_5)t_2)t_1 + 2(((t_3t_5)t_1)t_2)t_4 - 4(((t_3t_5)t_1)t_4)t_2 \\
 &+ 2(((t_3t_5)t_2)t_1)t_4 - 2(((t_3t_5)t_2)t_4)t_1 + 4(((t_3t_5)t_4)t_1)t_2 - 2(((t_3t_5)t_4)t_2)t_1 \\
 &- 2(((t_4t_5)t_1)t_2)t_3 + 4(((t_4t_5)t_1)t_3)t_2 - 2(((t_4t_5)t_2)t_1)t_3 + 2(((t_4t_5)t_2)t_3)t_1 \\
 &\quad - 4(((t_4t_5)t_3)t_1)t_2 + 2(((t_4t_5)t_3)t_2)t_1,
 \end{aligned}$$

$$\begin{aligned}
 &f_{11}(t_1, t_2, t_3, t_4, t_5) = \\
 &((t_1t_2)(t_3t_4))t_5 - ((t_1t_2)(t_3t_5))t_4 + ((t_1t_2)(t_4t_5))t_3 + ((t_1t_3)t_2)(t_4t_5) \\
 &+ ((t_1t_3)(t_2t_4))t_5 - ((t_1t_3)(t_2t_5))t_4 - ((t_1t_4)t_2)(t_3t_5) - ((t_1t_4)(t_2t_3))t_5 \\
 &+ ((t_1t_4)(t_2t_5))t_3 + ((t_1t_5)t_2)(t_3t_4) + ((t_1t_5)(t_2t_3))t_4 - ((t_1t_5)(t_2t_4))t_3 \\
 &- ((t_2t_3)t_4)(t_1t_5) + ((t_2t_3)t_5)(t_1t_4) + ((t_2t_3)(t_4t_5))t_1 + ((t_2t_4)t_3)(t_1t_5) \\
 &- ((t_2t_4)t_5)(t_1t_3) - ((t_2t_5)t_3)(t_1t_4) + ((t_2t_5)(t_3t_4))t_1 + ((t_4t_5)t_2)(t_1t_3) \\
 &+ 2(((t_1t_2)t_3)t_4)t_5 - 2(((t_1t_2)t_3)t_5)t_4 - 2(((t_1t_2)t_4)t_3)t_5 + 2(((t_1t_2)t_4)t_5)t_3 \\
 &+ 2(((t_1t_2)t_5)t_3)t_4 - 2(((t_1t_2)t_5)t_4)t_3 - 4(((t_1t_3)t_2)t_4)t_5 + 4(((t_1t_3)t_2)t_5)t_4 \\
 &- 2(((t_1t_3)t_4)t_2)t_5 + 4(((t_1t_3)t_4)t_5)t_2 + 2(((t_1t_3)t_5)t_2)t_4 - 4(((t_1t_3)t_5)t_4)t_2 \\
 &+ 4(((t_1t_4)t_2)t_3)t_5 - 4(((t_1t_4)t_2)t_5)t_3 + 2(((t_1t_4)t_3)t_2)t_5 - 4(((t_1t_4)t_3)t_5)t_2
 \end{aligned}$$

$$\begin{aligned}
& -2((t_1 t_4) t_5) t_2 t_3 + 4((t_1 t_4) t_5) t_3 t_2 - 4((t_1 t_5) t_2) t_3 t_4 + 4((t_1 t_5) t_2) t_4 t_3 \\
& -2((t_1 t_5) t_3) t_2 t_4 + 4((t_1 t_5) t_3) t_4 t_2 + 2((t_1 t_5) t_4) t_2 t_3 - 4((t_1 t_5) t_4) t_3 t_2 \\
& + 2((t_2 t_3) t_1) t_4 t_5 - 2((t_2 t_3) t_1) t_5 t_4 - 2((t_2 t_4) t_1) t_3 t_5 + 2((t_2 t_4) t_1) t_5 t_3 \\
& -2((t_2 t_4) t_5) t_1 t_3 + 2((t_2 t_4) t_5) t_3 t_1 + 2((t_2 t_5) t_1) t_3 t_4 - 2((t_2 t_5) t_1) t_4 t_3 \\
& + 2((t_2 t_5) t_4) t_1 t_3 - 2((t_2 t_5) t_4) t_3 t_1 - 2((t_3 t_4) t_1) t_2 t_5 + 4((t_3 t_4) t_1) t_5 t_2 \\
& -2((t_3 t_4) t_2) t_1 t_5 + 2((t_3 t_4) t_2) t_5 t_1 - 4((t_3 t_4) t_5) t_1 t_2 + 2((t_3 t_4) t_5) t_2 t_1 \\
& + 2((t_3 t_5) t_1) t_2 t_4 - 4((t_3 t_5) t_1) t_4 t_2 + 2((t_3 t_5) t_2) t_1 t_4 - 2((t_3 t_5) t_2) t_4 t_1 \\
& + 4((t_3 t_5) t_4) t_1 t_2 - 2((t_3 t_5) t_4) t_2 t_1 - 2((t_4 t_5) t_1) t_2 t_3 + 4((t_4 t_5) t_1) t_3 t_2 \\
& -4((t_4 t_5) t_2) t_1 t_3 + 4((t_4 t_5) t_2) t_3 t_1 - 4((t_4 t_5) t_3) t_1 t_2 + 2((t_4 t_5) t_3) t_2 t_1,
\end{aligned}$$

$$f_{12}(t_1, t_2, t_3, t_4, t_5) =$$

$$\begin{aligned}
& ((t_1 t_2) t_5) (t_3 t_4) - ((t_1 t_2) (t_3 t_5)) t_4 + ((t_1 t_2) (t_4 t_5)) t_3 - ((t_1 t_3) t_5) (t_2 t_4) \\
& + ((t_1 t_3) (t_2 t_5)) t_4 - ((t_1 t_3) (t_4 t_5)) t_2 + ((t_1 t_4) t_5) (t_2 t_3) - ((t_1 t_4) (t_2 t_5)) t_3 \\
& + ((t_1 t_4) (t_3 t_5)) t_2 + ((t_1 t_5) (t_2 t_3)) t_4 - ((t_1 t_5) (t_2 t_4)) t_3 + ((t_1 t_5) (t_3 t_4)) t_2 \\
& + ((t_2 t_3) t_5) (t_1 t_4) + ((t_2 t_3) (t_4 t_5)) t_1 - ((t_2 t_4) t_5) (t_1 t_3) - ((t_2 t_4) (t_3 t_5)) t_1 \\
& - ((t_2 t_5) (t_3 t_4)) t_1 + ((t_3 t_4) t_5) (t_1 t_2) + 4(((t_1 t_2) t_3) t_4) t_5 - 2(((t_1 t_2) t_3) t_5) t_4 \\
& -4(((t_1 t_2) t_4) t_3) t_5 + 2(((t_1 t_2) t_4) t_5) t_3 - 4(((t_1 t_2) t_5) t_3) t_4 + 4(((t_1 t_2) t_5) t_4) t_3 \\
& -4(((t_1 t_3) t_2) t_4) t_5 + 2(((t_1 t_3) t_2) t_5) t_4 + 4(((t_1 t_3) t_4) t_2) t_5 - 2(((t_1 t_3) t_4) t_5) t_2 \\
& +4(((t_1 t_3) t_5) t_2) t_4 - 4(((t_1 t_3) t_5) t_4) t_2 + 4(((t_1 t_4) t_2) t_3) t_5 - 2(((t_1 t_4) t_2) t_5) t_3 \\
& -4(((t_1 t_4) t_3) t_2) t_5 + 2(((t_1 t_4) t_3) t_5) t_2 - 4(((t_1 t_4) t_5) t_2) t_3 + 4(((t_1 t_4) t_5) t_3) t_2 \\
& +2(((t_1 t_5) t_2) t_3) t_4 - 2(((t_1 t_5) t_2) t_4) t_3 - 2(((t_1 t_5) t_3) t_2) t_4 + 2(((t_1 t_5) t_3) t_4) t_2 \\
& +2(((t_1 t_5) t_4) t_2) t_3 - 2(((t_1 t_5) t_4) t_3) t_2 + 4(((t_2 t_3) t_1) t_4) t_5 - 2(((t_2 t_3) t_1) t_5) t_4 \\
& -4(((t_2 t_3) t_4) t_1) t_5 + 2(((t_2 t_3) t_4) t_5) t_1 - 4(((t_2 t_3) t_5) t_1) t_4 + 4(((t_2 t_3) t_5) t_4) t_1 \\
& -4(((t_2 t_4) t_1) t_3) t_5 + 2(((t_2 t_4) t_1) t_5) t_3 + 4(((t_2 t_4) t_3) t_1) t_5 - 2(((t_2 t_4) t_3) t_5) t_1 \\
& +4(((t_2 t_4) t_5) t_1) t_3 - 4(((t_2 t_4) t_5) t_3) t_1 - 2(((t_2 t_5) t_1) t_3) t_4 + 2(((t_2 t_5) t_1) t_4) t_3 \\
& +2(((t_2 t_5) t_3) t_1) t_4 - 2(((t_2 t_5) t_3) t_4) t_1 - 2(((t_2 t_5) t_4) t_1) t_3 + 2(((t_2 t_5) t_4) t_3) t_1 \\
& +4(((t_3 t_4) t_1) t_2) t_5 - 2(((t_3 t_4) t_1) t_5) t_2 - 4(((t_3 t_4) t_2) t_1) t_5 + 2(((t_3 t_4) t_2) t_5) t_1 \\
& -4(((t_3 t_4) t_5) t_1) t_2 + 4(((t_3 t_4) t_5) t_2) t_1 + 2(((t_3 t_5) t_1) t_2) t_4 - 2(((t_3 t_5) t_1) t_4) t_2 \\
& -2(((t_3 t_5) t_2) t_1) t_4 + 2(((t_3 t_5) t_2) t_4) t_1 + 2(((t_3 t_5) t_4) t_1) t_2 - 2(((t_3 t_5) t_4) t_2) t_1 \\
& -2(((t_4 t_5) t_1) t_2) t_3 + 2(((t_4 t_5) t_1) t_3) t_2 + 2(((t_4 t_5) t_2) t_1) t_3 - 2(((t_4 t_5) t_2) t_3) t_1 \\
& -2(((t_4 t_5) t_3) t_1) t_2 + 2(((t_4 t_5) t_3) t_2) t_1,
\end{aligned}$$

$$f_{13} = lei_3^{(-1)},$$

$$f_{14}(t_1, t_2, t_3, t_4, t_5) =$$

$$((t_1 t_2) t_3) (t_4 t_5) + ((t_1 t_2) (t_3 t_4)) t_5 - ((t_1 t_2) (t_3 t_5)) t_4 + ((t_1 t_3) (t_2 t_4)) t_5$$

$$\begin{aligned}
& -((t_1t_3)(t_2t_5))t_4 + ((t_1t_3)(t_4t_5))t_2 - ((t_1t_4)t_3)(t_2t_5) + ((t_1t_4)(t_2t_3))t_5 \\
& + ((t_1t_4)(t_3t_5))t_2 + ((t_1t_5)t_3)(t_2t_4) - ((t_1t_5)(t_2t_3))t_4 - ((t_1t_5)(t_3t_4))t_2 \\
& - ((t_2t_3)(t_4t_5))t_1 + ((t_2t_4)t_3)(t_1t_5) - ((t_2t_4)(t_3t_5))t_1 - ((t_2t_5)t_3)(t_1t_4) \\
& + ((t_2t_5)(t_3t_4))t_1 + ((t_4t_5)t_3)(t_1t_2) - 4(((t_1t_2)t_3)t_4)t_5 + 4(((t_1t_2)t_3)t_5)t_4 \\
& - 2(((t_1t_2)t_4)t_3)t_5 + 4(((t_1t_2)t_4)t_5)t_3 + 2(((t_1t_2)t_5)t_3)t_4 - 4(((t_1t_2)t_5)t_4)t_3 \\
& + 2(((t_1t_3)t_2)t_4)t_5 - 2(((t_1t_3)t_2)t_5)t_4 - 2(((t_1t_3)t_4)t_2)t_5 + 2(((t_1t_3)t_4)t_5)t_2 \\
& + 2(((t_1t_3)t_5)t_2)t_4 - 2(((t_1t_3)t_5)t_4)t_2 + 2(((t_1t_4)t_2)t_3)t_5 - 4(((t_1t_4)t_2)t_5)t_3 \\
& + 4(((t_1t_4)t_3)t_2)t_5 - 4(((t_1t_4)t_3)t_5)t_2 + 4(((t_1t_4)t_5)t_2)t_3 - 2(((t_1t_4)t_5)t_3)t_2 \\
& - 2(((t_1t_5)t_2)t_3)t_4 + 4(((t_1t_5)t_2)t_4)t_3 - 4(((t_1t_5)t_3)t_2)t_4 + 4(((t_1t_5)t_3)t_4)t_2 \\
& - 4(((t_1t_5)t_4)t_2)t_3 + 2(((t_1t_5)t_4)t_3)t_2 - 2(((t_2t_3)t_1)t_4)t_5 + 2(((t_2t_3)t_1)t_5)t_4 \\
& + 2(((t_2t_3)t_4)t_1)t_5 - 2(((t_2t_3)t_4)t_5)t_1 - 2(((t_2t_3)t_5)t_1)t_4 + 2(((t_2t_3)t_5)t_4)t_1 \\
& - 2(((t_2t_4)t_1)t_3)t_5 + 4(((t_2t_4)t_1)t_5)t_3 - 4(((t_2t_4)t_3)t_1)t_5 + 4(((t_2t_4)t_3)t_5)t_1 \\
& - 4(((t_2t_4)t_5)t_1)t_3 + 2(((t_2t_4)t_5)t_3)t_1 + 2(((t_2t_5)t_1)t_3)t_4 - 4(((t_2t_5)t_1)t_4)t_3 \\
& + 4(((t_2t_5)t_3)t_1)t_4 - 4(((t_2t_5)t_3)t_4)t_1 + 4(((t_2t_5)t_4)t_1)t_3 - 2(((t_2t_5)t_4)t_3)t_1 \\
& - 2(((t_3t_4)t_1)t_2)t_5 + 2(((t_3t_4)t_1)t_5)t_2 + 2(((t_3t_4)t_2)t_1)t_5 - 2(((t_3t_4)t_2)t_5)t_1 \\
& - 2(((t_3t_4)t_5)t_1)t_2 + 2(((t_3t_4)t_5)t_2)t_1 + 2(((t_3t_5)t_1)t_2)t_4 - 2(((t_3t_5)t_1)t_4)t_2 \\
& - 2(((t_3t_5)t_2)t_1)t_4 + 2(((t_3t_5)t_2)t_4)t_1 + 2(((t_3t_5)t_4)t_1)t_2 - 2(((t_3t_5)t_4)t_2)t_1 \\
& + 4(((t_4t_5)t_1)t_2)t_3 - 2(((t_4t_5)t_1)t_3)t_2 - 4(((t_4t_5)t_2)t_1)t_3 + 2(((t_4t_5)t_2)t_3)t_1 \\
& - 4(((t_4t_5)t_3)t_1)t_2 + 4(((t_4t_5)t_3)t_2)t_1.
\end{aligned}$$

We see that

$$f_2 = leilie_1(t_1, t_2, t_4, t_3, t_5),$$

$$f_3 = leilie_1(t_1, t_2, t_5, t_3, t_4),$$

$$f_4 = leilie_1(t_1, t_3, t_4, t_2, t_5),$$

$$f_5 = leilie_1(t_2, t_3, t_4, t_1, t_5),$$

$$f_6 = leilie_1(t_1, t_3, t_5, t_2, t_4),$$

$$f_7 = leilie_1(t_2, t_3, t_5, t_1, t_4),$$

$$f_9 = leilie_1(t_1, t_4, t_5, t_2, t_3) + leilie_2(t_1, t_2, t_3, t_4, t_5),$$

$$\begin{aligned}
f_{10} = & (leilie_1(t_3, t_4, t_5, t_1, t_2) + 2leilie_2(t_1, t_2, t_3, t_4, t_5) + lei_3^{(-1)}(t_1, t_2, t_3, t_4, t_5) \\
& - lei_3^{(-1)}(t_1, t_2, t_3, t_5, t_4) - lei_3^{(-1)}(t_1, t_2, t_4, t_3, t_5))/2,
\end{aligned}$$

$$f_{11} = (2leilie_1(t_2, t_4, t_5, t_1, t_3) + leilie_1(t_3, t_4, t_5, t_1, t_2) + 2leilie_2(t_1, t_2, t_3, t_4, t_5) + lei_3^{(-1)}(t_1, t_2, t_3, t_4, t_5) - lei_3^{(-1)}(t_1, t_2, t_3, t_5, t_4) - lei_3^{(-1)}(t_1, t_2, t_4, t_3, t_5))/2,$$

$$f_{12} = (leilie_1(t_3, t_4, t_5, t_1, t_2) + lei_3^{(-1)}(t_1, t_2, t_3, t_4, t_5) + lei_3^{(-1)}(t_1, t_2, t_3, t_5, t_4) - lei_3^{(-1)}(t_1, t_2, t_4, t_3, t_5))/2,$$

$$f_{14} = (leilie_1(t_3, t_4, t_5, t_1, t_2) + lei_3^{(-1)}(t_1, t_2, t_3, t_4, t_5) - lei_3^{(-1)}(t_1, t_2, t_3, t_5, t_4) + lei_3^{(-1)}(t_1, t_2, t_4, t_3, t_5))/2.$$

So, by Lemma 5.1 the 9-term polynomial $leilie_1$ and the 60-term polynomial $leilie_2$ form a base of polylinear identities of degree 5.

Proof of Theorem 1.4. Follows from Lemmas 5.1, 5.2, 5.3 and 5.4.

6. LEIBNIZ-JORDAN ALGEBRAS

Proof of Theorem 1.6. It is easy to check that $leijor = 0$ is identity for any algebra of a form $A^{(1)}$, where A is a Leibniz algebra.

Let A be an associative algebra and M a right-module over A . Then $A^{(-1)}$ is a Lie algebra and M can be made into an antisymmetric $A^{(-1)}$ -module. Let $L = A + M$ be the standard Leibniz algebra corresponding to these Lie and antisymmetric module structures. If we denote by \star the multiplication in the Leibniz algebra L , then

$$(a + m) \star (b + n) = [a, b] + mb,$$

and

$$\{a + m, b + n\} = [a, b] + mb + [b, a] + na = na + mb.$$

In particular,

$$\{a, m\} = ma, \quad \{a, b\} = 0, \quad \{m, n\} = 0 \quad (3)$$

for all $a, b \in A, m, n \in M$. Recall that

$$\{t_1, t_2\} = t_1 t_2 + t_2 t_1$$

is the Jordan commutator.

Suppose that $f = 0$ is a minimal identity on the Leibniz-Jordan algebra $(L, \{, \})$ which does not follow from the identity $leijor = 0$. We can assume that f is polylinear and $f = f(t_1, \dots, t_k)$ is a linear combination of left-bracketed monomials of the form $((t_{i_1} t_{i_2}) \dots) t_{i_k}$. So,

$$f(t_1, \dots, t_k) = \sum_{\sigma \in Sym_k} \lambda_{\sigma} ((t_{\sigma(1)} t_{\sigma(2)}) \dots) t_{\sigma(k)}$$

for some $\lambda_\sigma \in K$. Write the condition $f(a_1, \dots, a_{k-1}, m) = 0$ by using the multiplication rules (3) for Leibniz-Jordan algebras. We have

$$f(a_1, \dots, a_{k-1}, m) = \sum_{\sigma \in \text{Sym}_{k-1}} \lambda_\sigma((ma_{\sigma(1)}) \cdots) a_{\sigma(k-1)} \quad (4)$$

for any $a_1, \dots, a_{k-1} \in A, m \in M$.

Take $A = \text{Mat}_n$ to be the matrix algebra and $M = K^n$ the n -dimensional natural module. Then conditions (4) imply that

$$\sum_{\sigma \in \text{Sym}_{k-1}} \lambda_\sigma((a_{\sigma(1)} a_{\sigma(2)}) \cdots) a_{\sigma(k-1)} = 0$$

is an identity on Mat_n . By the Amitsur-Levitsky theorem [1], matrix algebras have no identity of degree k if $k < 2n + 1$. So, $f = 0$ is not an identity for Leibniz-Jordan algebras of the form $\text{Mat}_n + K^n$ if $n > (k - 1)/2$. In other words, any s -identity for Leibniz-Jordan algebras follows from the identities $leijor = 0, com = 0$.

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