THE HOMOTOPY CLASSIFICATION OF (n–1)–CONNECTED (n+3)–DIMENSIONAL POLYHEDRA, n$\geq$4

by

Hans–Joachim Baues and Matthias Hennes

Max–Planck–Institut für Mathematik
Gottfried–Claren–Str. 26
5300 Bonn 3
Federal Republic of Germany

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The classification of homotopy types of finite polyhedra is a classical and fundamental task of topology. Here we mean a classification by minimal algebraic data which for example allows the explicit computation of the number of homotopy types with prescribed homology. The main result on this problem in the literature is due to J.H.C. Whitehead (1949) who classified (n–1)–connected (n+2)–dimensional polyhedra. In this paper we consider the next step concerning finite (n–1)–connected (n+3)–dimensional polyhedra. In the stable range, $n \geq 4$, they are classified by the following decomposition theorem, see (3.9).

Theorem: *Each finite* (n–1)*–connected* (n+3)*–dimensional polyhedron* X, $n \geq 4$, *admits a homotopy equivalence*

$$X \simeq X_1 \vee \ldots \vee X_r$$

where the right hand side is a one point union of indecomposable complexes which is unique up to permutation. Moreover a complete list of these indecomposable complexes is given by the corresponding spheres and Moore-spaces of cyclic groups $\mathbb{Z}/p^k$, p *prime, and by the complexes* X(w), X(w,φ) *which are in* 1–1*–correspondence to special words, see* (3.1). *Such words as well can be described by graphs as in Figure* 2.

For example the real projective 3–space $\mathbb{R}P_3$ has the stabilization $\Sigma^{n-1}(\mathbb{R}P_3) \simeq S^{n+2} \vee M(\mathbb{Z}/2,n)$ where $M(\mathbb{Z}/2,n)$ is a Moore space while $\Sigma^{n-1}(\mathbb{R}P_4) \simeq X({}_1\xi^1)$ is an indecomposable complex given by the special word ${}_1\xi^1$.

Considering the homology of indecomposable complexes one for example gets the

Corollary: *Let* $n \geq 4$ *and let* X *be an* $(n-1)$*–connected* $(n+3)$*–dimensional finite polyhedron with Betti numbers* $\beta_i(X)$. *If*
$2 < \beta_n(X) + \beta_{n+1}(X) + \beta_{n+2}(X) + \beta_{n+3}(X)$ *or if* $H_{n+1}(X)$ *contains the direct sum of two cyclic groups then* X *is decomposable.*

A further application is the following

Example: There are exactly 4732 simply connected homotopy types X which have the reduced homology groups $(n \geq 4)$

$$\tilde{H}_i(X) = \begin{cases} \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z} & , i = n \\ \mathbb{Z}/8 \oplus \mathbb{Z} & , i = n+1 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z} & , i = n+2 \\ \mathbb{Z} & , i = n+3 \\ 0 & \text{otherwise} \end{cases}$$

The example indicates that a homotopy type is not nearly determined by its integral homology. Still the Whitehead theorem shows that the homology is the basic homotopy invariant of a simply connected polyhedron X. Therefore one wants to represent the homotopy type of X directly in terms of a suitable natural algebraic structure on the homology of X. Such a structure was obtained in 1949 by J.H.C. Whitehead for 1–connected 4–dimensional polyhedra [39], [41] and later also for $(n-1)$–connected $(n+2)$–dimensional polyhedra, $n \geq 3$, see [40]; it was used by Chang [6], [7] for the computation of the corresponding indecomposable polyhedra. Since then various authors studied the classification of $(n-1)$–connected $(n+3)$–dimensional polyhedra, $n \geq 4$, in terms of primary and secondary cohomology operations, see [8],[9],[10],[11],[12],[13],[14],[31],[36]. The classifying data still remained intricate. The proof of the decomposition theorem above is based on a new kind of invariant which simplifies the algebraic representation considerably:

<u>Classification theorem</u> (see (4.10)): *Let* $n \geq 4$. *Then the homotopy types of* $(n-1)$*–connected* $(n+3)$*–dimensional polyhedra are in* 1–1 *correspondence to the isomorphism classes of stable* A_n^3*–systems, see* (4.7).

The restriction of this result to $(n+2)$–dimensional complexes is literally Whitehead's classification in [42] which is the easy part of the theorem; the main new feature in A_n^3–systems is the 'boundary invariant' β.

Finally we remark implications of our results for manifolds. Each $(n-1)$–connected $(2n+3)$–dimensional compact manifold M, $n \geq 4$, admits a homotopy equivalence $M \simeq C_g$ where C_g is the mapping cone of a map

$$g : S^{2n+2} \longrightarrow X_1 \vee \ldots \vee X_m = X.$$

Here X is a one point union of indecomposable complexes X_i as in the decomposition theorem. For $n \geq 5$ this one point union has the additional property that the space X is self dual with respect to Spanier–Whitehead duality, see [33]. Since the dual of X is given by $DX_1 \vee \ldots \vee DX_m$, self duality of X means that $DX_1, \ldots, DX_m$ is a permutation of $X_1, \ldots, X_m$. This can readily be checked by the determination of dual complexes in (3.10).

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§ 1 The decomposition problem in representation theory and topology

Let $\underline{\underline{C}}$ be a category with an initial object $*$ and assume sums, denoted by $A \vee B$, exist in $\underline{\underline{C}}$. An object X in $\underline{\underline{C}}$ is decomposable if there exists an isomorphism $X \cong A \vee B$ in $\underline{\underline{C}}$ where A and B are not isomorphic to $*$. Whence an object X is indecomposable if $X \cong A \vee B$ implies $A \cong *$ or $B \cong *$. A decomposition of X is an isomorphism

$$X \cong A_1 \vee \ldots \vee A_n \,, \; n < \infty, \tag{1.1}$$

in $\underline{\underline{C}}$ where A_i is indecomposable for all $i \in \{1,\ldots,n\}$. The decomposition of X is unique if $B_1 \vee \ldots \vee B_m \cong X \cong A_1 \vee \ldots \vee A_n$ implies that $m = n$ and that there is a permutation σ with $B_{\sigma_i} \cong A_i$ for all i. A morphism f in $\underline{\underline{C}}$ is indecomposable if the object f is indecomposable in the category $\underline{\underline{Pair}}(\underline{\underline{C}})$. The objects of $\underline{\underline{Pair}}(\underline{\underline{C}})$ are the morphisms of $\underline{\underline{C}}$ and the morphisms $f \longrightarrow g$ in $\underline{\underline{Pair}}(\underline{\underline{C}})$ are the pairs (α,β) of morphism in $\underline{\underline{C}}$ with $g\alpha = \beta f$. The sum of f and g is the morphism $f \vee g = (i_1 f, i_2 g)$. The decomposition problem in $\underline{\underline{C}}$ can be described by the following task: Find a complete list of indecomposable isomorphism types in $\underline{\underline{C}}$ and describe the possible decompositions of objects in $\underline{\underline{C}}$! We now consider various examples and solutions of such decomposition problems. These examples originated in representation theory and topology.

First let R be a ring and let $\underline{\underline{C}}$ be a full category of R–modules (satisfying some finiteness restraint). The initial object in $\underline{\underline{C}}$ is the trivial module 0 and the sum in $\underline{\underline{C}}$ is the direct sum of modules, denoted by $M \oplus N$. With respect to the decomposition problem for modules in $\underline{\underline{C}}$ Gabriel states in the introduction of [21]:

"The main and perhaps hopeless purpose of representation theory is to find an efficient general method for constructing the indecomposable objects by means of simple objects, which are supposed to be given". Various results on such decomposition problems are outlined in [21]. In this paper we shall use only the following examples.

(1.2) Example: For $R = \mathbb{Z}$ let $\underset{=}{C}$ be the category of finitely generated abelian groups. In this case the indecomposable objects are well known; they are given by the cyclic groups $\mathbb{Z}$ and $\mathbb{Z}/p^i$ where p is a prime and $i \geq 1$.

(1.3) Example: Let k be a field and let R be the quotient ring $R = k\langle X,Y\rangle/(X^2,Y^2)$. Here (X^2,Y^2) stands for the ideal generated by X^2 and Y^2 in the free associative algebra $k\langle X,Y\rangle$ in the variables X and Y. Let $\underset{=}{C}$ be the full category of R–modules which are finite dimensional as k–vector spaces. C.M. Ringel [30] gave a complete list of indecomposable objects in $\underset{=}{C}$. These objects are characterized by certain words which are partially of a similar nature as the words used in §2 below.

(1.4) Example: In topology we also consider graded rings like the Steenrod algebra and graded modules like the homology or cohomology of a space. Let $R = \mathfrak{A}_p$ be the mod p Steenrod algebra and let $k \geq 0$. We consider the category $\underset{=}{C}$ of all graded R–modules H for which H_i is a finite $\mathbb{Z}/p$–vector space and for which $H_i = 0$ for $i < 0$ and $i > k$. It is a hard problem to compute the indecomposable objects of $\underset{=}{C}$; only for $k \leq 4p-5$ the answer is known by the work of Henn [22]. In fact, Henn's result is highly related to the result of Ringel in (1.3) above; to see this we consider the case $p = 2$. The restriction $k \leq 3$ then implies that the $\mathfrak{A}_2$–module structure of H is completely determined by Sq_1 and Sq_2 with $Sq_1Sq_1 = 0$ and $Sq_2Sq_2 = 0$. Whence, forgetting degrees, the module H is actually a module over the ring $\mathbb{Z}/2\langle X,Y\rangle/(X^2,Y^2)$ with $X = Sq_1$, $Y = Sq_2$ and such modules were classified by Ringel.

Next we describe the decomposition problem of homotopy theory. Let $\underline{\underline{\mathrm{Top}}}^*/\simeq$ be the homotopy category of pointed topological spaces. The set of morphisms $X \longrightarrow Y$ in $\underline{\underline{\mathrm{Top}}}^*/\simeq$ is the set of homotopy classes $[X,Y]$. Isomorphisms in $\underline{\underline{\mathrm{Top}}}^*/\simeq$ are called homotopy equivalences and isomorphism types in $\underline{\underline{\mathrm{Top}}}^*/\simeq$ are homotopy types. Let $\underline{\underline{A}}_n^k$ be the full subcategory of $\underline{\underline{\mathrm{Top}}}^*/\simeq$ consisting of (n–1)–connected (n+k)–dimensional CW–complexes, the objects of $\underline{\underline{A}}_n^k$ are also called A_n^k–polyhedra, see [40]. The suspension Σ gives us the sequence of functors

$$\underline{\underline{A}}_1^k \xrightarrow{\Sigma} \underline{\underline{A}}_2^k \longrightarrow \dots \longrightarrow \underline{\underline{A}}_n^k \xrightarrow{\Sigma} \underline{\underline{A}}_{n+1}^k \longrightarrow \dots \tag{1.5}$$

which is the <u>k–stem of homotopy categories</u>. The Freudenthal suspension theorem shows that for $k + 1 < n$ the functor $\Sigma : \underline{\underline{A}}_n^k \longrightarrow \underline{\underline{A}}_{n+1}^k$ is an equivalence of categories; moreover for $k + 1 = n$ this functor is full and a $1 - 1$ correspondence of homotopy types. We say that the homotopy types of $\underline{\underline{A}}_n^k$ are <u>stable</u> if $k + 1 \leq n$, the morphisms of $\underline{\underline{A}}_n^k$, however, are stable if $k + 1 < n$. The computation of the k–stem is a classical and principal problem of homotopy theory which, in particular, was studied for $k \leq 2$ by J.H.C. Whitehead [39], [40], [42]. The k–stem of homotopy groups of spheres, denoted by $\pi_{n+k}(S^n)$, $n \geq 2$, now is known for fairly large k; for example one can find a complete list for $k \leq 19$ in Toda's book [35]. The k–stem of homotopy types, however, is still mysterious even for very small k. The initial object of the category $\underline{\underline{A}}_n^k$ is the point $*$ and the sum in $\underline{\underline{A}}_n^k$ is the one point union of spaces. The suspension Σ in (1.5) carries a sum to a sum and $\Sigma : \underline{\underline{A}}_n^k \longrightarrow \underline{\underline{A}}_{n+1}^k$ yields a 1–1 correspondence of indecomposable homotopy types for $k + 1 \leq n$. As in the case of modules we use a finiteness restraint, we consider the decomposition problem only for finite (or equivalently compact) CW–complexes. Therefore we introduce the full subcategory $\underline{\underline{FA}}_n^k$,

$$\underline{\underline{FA}}_n^k \subset \underline{\underline{A}}_n^k \subset \underline{\underline{\mathrm{Top}}}^*/\simeq, \tag{1.6}$$

consisting of (n–1)–connected (n+k)–dimensional CW–complexes with only fintely many cells. The following results on the decomposition problem in $\underline{\underline{FA}}_n^k$ are known. Recall that a Moore space $M(A,m)$ is a simply connected CW–complex with homology groups $H_m M(A,m) \cong A$ and $\tilde{H}_i M(A,m) = 0$ for $i \neq m$. The sphere S^m is a Moore space $M(\mathbb{Z},m)$ and $M(\mathbb{Z}/k,m)$ is the mapping cone of the map $k\iota : S^m \longrightarrow S^m$ of degree k. The elementary Moore spaces of $\underline{\underline{FA}}_n^k$ are the spheres S^m, $n \leq m \leq n+k$, and the Moore spaces $M(\mathbb{Z}/p^i,m)$ where p is a prime, $i \geq 1$, $n \leq m < n+k$. These are indecomposable objects of $\underline{\underline{FA}}_n^k$. The next result essentially follows from (1.2) by use of the Hurewicz theorem.

(1.7) Proposition: (A) For $n \geq 1$ the sphere S^n is the only indecomposable homotopy type of $\underline{\underline{FA}}_n^0$, and each object in $\underline{\underline{FA}}_n^0$ has a unique decomposition.
(B) Let $n \geq 2$. The elementary Moore spaces of $\underline{\underline{FA}}_n^1$ are the only indecomposable homotopy types in $\underline{\underline{FA}}_n^1$ and each object in $\underline{\underline{FA}}_n^1$ has a unique decomposition.

It is known that there are 2–dimensional complexes in $\underline{\underline{FA}}_1^1$ which admit different decompositions, see [18]. For the next result we define the elementary complexes of Chang which we denote by

$$(1.8) \qquad X(\eta),\ X(\eta q),\ X({}_p\eta),\ X({}_p\eta q)$$

where $p,q, \in \mathbb{N} = \{1,2,\dots\}$. They are given by the mapping cones of the maps $f_1 : S^{n+1} \longrightarrow S^n$, $f_2 : S^{n+1} \longrightarrow S^{n+1} \vee S^n$, $f_3 : S^{n+1} \vee S^n \longrightarrow S^n$ and $f_4 : S^{n+1} \vee S^n \longrightarrow S^{n+1} \vee S^n$ respectively; here $f_1 = \eta$ is the Hopf map, moreover $f_2 = i_1(2^q\iota) + i_2\eta$, $f_3 = (\eta, 2^p\iota)$, $f_4 = (i_1(2^q\iota) + i_2\eta,\ i_2(2^p\iota))$ where i_1, resp. i_2, denotes the inclusions of S^{n+1}, resp. S^n, into $S^{n+1} \vee S^n$. These complexes are also discussed in the books of Hilton [24], [25]. We visualize the words η, ηq, ${}_p\eta$ and ${}_p\eta q$ by the corresponding subgraphs of the graph in Figure 1 where the edge in this graph, connecting the levels 0 and 2, is denoted by η.

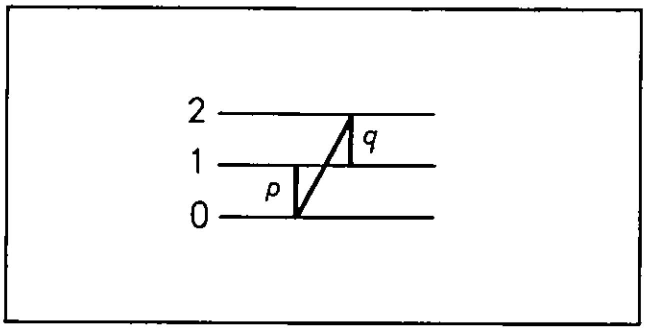


Figure 1

(1.9) Theorem of Chang [6]: Let $n \geq 3$. The elementary Moore spaces and the elementary complexes of Chang above are the only indecomposable homotopy types in $\underline{\underline{FA}}_n^2$ and each object in $\underline{\underline{FA}}_n^2$ has a unique decomposition.

This result is based on Whitehead's algebraic classification of A_n^2–polyhedra [40]. Our main result (3.9) below gives a complete solution of the decomposition problem in $\underline{\underline{FA}}_n^3$. The solution involves two main steps. First we obtain an algebraic classification of all A_n^3–polyhedra, $n \geq 4$, and then we solve the decomposition problem by use of the algebraic invariants. The second step is purely algebraic and can be considered as a kind of generalized decomposition problem of representation theory; at this point we also use the results of Ringel and Henn described in (1.3) and (1.4) above. In addition Spanier–Whitehead duality turns out to be an important tool.

An algebraic classification of all A_n^4–polyhedra, $n \geq 5$, is not yet known, though Unsöld [37] gave an algebraic classification of such polyhedra if they have torsion free homology. It would be very interesting to use Unsöld's result for the classification of all indecomposable stable A_n^4–polyhedra with fintitely generated torsion free homology. Since the primes 2 and 3 appear decomposition is not unique, see [20], [26], [27]. This is avoidable by localization. There are many rings R which are wild in the sense that there seems to be no hope for a complete classification of indecomposable R–modules, see for example [29]. It is not at all clear whether a similar kind of "wildness" appears in the decomposition problem of stable homotopy types. In fact, it might be true that the Steenrod algebra itself is wild in the sense of representation theory, nevertheless the collection of those indecomposable modules over the Steenrod algebra which are actually realizable might not be wild.

§2 Spanier–Whitehead duality and homotopy groups of Moore spaces

We here introduce certain generators of homotopy groups of Moore spaces which play an essential role for the construction of the indecomposable A_n^3–polyhedra in the next section. The generators chosen are compatible with Spanier–Whitehead duality. With respect to Spanier–Whitehead duality we refer the reader to [32] and [34], we here only recall a few facts needed in this paper.

In the stable range $m < 2n - 1$ the Spanier–Whitehead (n+m)–duality is a contravariant isomorphism of categories

$$D : \underline{\underline{FA}}_n^{m-n} \xrightarrow{\cong} \underline{\underline{FA}}_n^{m-n}. \tag{2.1}$$

This isomorphism carries X to $DX = X^*$ and carries the homotopy class $f \in [X,Y]$ to the homotopy class $Df = f^* \in [Y^*,X^*]$. The isomorphism D satisfies $DD =$ identity that is $X^{**} = X$ and $f^{**} = f$. The functor D depends on the choice of (n+m)–duality maps $D_X : X^* \wedge X \longrightarrow S^{n+m}$ which satisfy certain properties, see [32] or [34]. The homotopy type of X^*, however, is well defined and does not depend on this choice. As an example we have the dual $D(S^{n+q}) = S^{m-q}$ for $q \leq m - n$, then the dual of $f : S^{n+q} \longrightarrow S^{n+q'}$ is $f^* = \Sigma^k f : S^{m-q'} \longrightarrow S^{m-q}$ with $k = m-n-q'-q$. This shows that Moore spaces satisfy

$$M(\mathbb{Z}/r,n+q)^* = M(\mathbb{Z}/r,m-q-1). \tag{2.2}$$

In fact, for a mapping cone C_f we can choose $DC_f = C_g$ where g represents f^*.

We now consider maps between Moore spaces of cyclic groups. For the pseudo projective plane $P_r = S^1 \cup_r e^2$ we have $\Sigma^{n-1} P_r = M(\mathbb{Z}/r,n)$. This yields the function $\Sigma^{n-1}: [P_r,P_t] \longrightarrow [M(\mathbb{Z}/r,n),M(\mathbb{Z}/t,n)]$ between sets of homotopy classes, see (1.5).

(2.3) Proposition [3]: Let $n \geq 3$. For $\varphi \in \mathrm{Hom}(\mathbb{Z}/r,\mathbb{Z}/t)$ there exists a unique element $B\varphi \in [M(\mathbb{Z}/r,n), M(\mathbb{Z}/t,n)]$ which induces φ in homology and which is in the image of the function Σ^{n-1} above.

Clearly B in (2.3) satisfies $B(\mathrm{id}) = \mathrm{id}$ and $B(\varphi\Psi) = (B\varphi)(B\Psi)$ for compositions $\varphi\Psi$; the function B, however, is not additive. Let $\chi : \mathbb{Z}/p^r \longrightarrow \mathbb{Z}/p^t$ be the canonical generator of $\mathrm{Hom}(\mathbb{Z}/p^r,\mathbb{Z}/p^t) = \mathbb{Z}/p^{\min(r,t)}$ given by $\chi(1) = 1$ if $r \geq t$ and by $\chi(1) = p^{t-r}\cdot 1$ for $r < t$. Using (2.3) we get for $n \geq 3$ and a prime p the well known result

$$(2.4)\quad [M(\mathbb{Z}/p^r,n),M(\mathbb{Z}/p^t,n)] = \begin{cases} \mathbb{Z}/p^{\min(r,t)} B(\chi) & \text{for } p \neq 2 \\ \mathbb{Z}/4\ \ B(\chi) & \text{for } p^r = p^t = 2 \\ \mathbb{Z}/2^{\min(r,t)} B(\chi) \oplus \mathbb{Z}/2\, i\eta q & \text{otherwise} \end{cases}$$

Here we write $A = \mathbb{Z}/k\, B$ if A is a cyclic group of order k with generator B. The generator $i\eta q$ is given by the inclusion $i : S^n \subset M(\mathbb{Z}/p^t,n)$, the pinch map $q : M(\mathbb{Z}/p^r,n) \longrightarrow S^{n+1}$, and the Hopf map η with $[S^{n+1},S^n] = \mathbb{Z}/2\ \eta$. Moreover we get

$$(2.5)\qquad [S^{n+1},M(\mathbb{Z}/2^t,n)] = \mathbb{Z}/2\, i\eta \text{ and } [M(\mathbb{Z}/2^t,n),S^n] = \mathbb{Z}/2\ \eta q$$

which are (2n+1)–dual groups with $(i\eta)^* = \eta q$. On the other hand we get the (2n+2)–dual groups, $n \geq 4$,

$$[S^{n+2},M(\mathbb{Z}/2^t,n)] = \begin{cases} \mathbb{Z}/4\ \ \xi_1 & \text{for } t = 1 \\ \mathbb{Z}/2\ \ \xi_t \oplus \mathbb{Z}/2\ \ i\eta\eta & \text{for } t > 1 \end{cases}$$

(2.6)

$$[M(\mathbb{Z}/2^t,n+1),S^n] = \begin{cases} \mathbb{Z}/4\ \ \eta^1 & \text{for } t = 1 \\ \mathbb{Z}/2\ \ \eta^t \oplus \mathbb{Z}/2\ \ \eta\eta q & \text{for } t > 1 \end{cases}$$

Here we choose a generator ξ_1 and we set $\xi_t = B(\chi)\xi_1$, moreover we set $\eta^1 = (\xi_1)^*$ and $\eta^t = (\xi_t)^* = \eta^1 B(\chi)$. The map $\eta\eta$ is the double Hopf map with $[S^{n+2},S^n] = \mathbb{Z}/2\eta\eta$. Finally we get for $n \geq 4$

$$(2.7) \quad [M(\mathbb{Z}/2^s,n+1),M(\mathbb{Z}/2^r,n)] = \begin{cases} \mathbb{Z}/2\ \xi_1^1 \oplus \mathbb{Z}/2\ \eta_1^1 & \text{for } s = r = 1 \\ \mathbb{Z}/4\ \xi_1^s \oplus \mathbb{Z}/2\ \eta_1^s & \text{for } s > 1 = r \\ \mathbb{Z}/2\ \xi_r^1 \oplus \mathbb{Z}/4\ \eta_r^1 & \text{for } s = 1 < r \\ \mathbb{Z}/2\ \xi_r^s \oplus \mathbb{Z}/2\ \eta_r^s \oplus \mathbb{Z}/2\epsilon_r^s & \text{otherwise} \end{cases} .$$

Here we set $\xi_r^s = B(\chi)\xi_1 q$, $\eta_r^s = i\eta^1 B(\chi)$ and $\epsilon_r^s = i\eta\eta q$. We have the (2n+2)–dualities $(\xi_r^s)^* = \eta_s^r$ and $(\epsilon_r^s)^* = \epsilon_s^r$.

(2.8) <u>Proposition</u>: The groups (2.4), (2.5), (2.6) and (2.7) determine exactly all non trivial groups [X,Y] where X and Y are elementary Moore spaces of $\underline{\underline{FA}}_n^2$, $n \geq 4$.

This essentially is proved in [5], a complete proof is also given by Jäschke [28]. The explicit definition of generators above as well determines all compositions of maps between elementary Moore spaces in $\underline{\underline{FA}}_n^2$.

§3 The indecomposable (n–1)–connected (n+3)–dimensional polyhedra, $n \geq 4$

In this section we describe the main result of this paper which solves the decomposition problem in the category $\underline{FA}_n^3$, $n \geq 4$, see (1.6).

For the description of the indecomposable objects we use certain words. Let L be a set, the elements of which are called 'letters'. A word with letters in L is an element in the free monoid generated by L. Such a word a is written $a = a_1a_2...a_n$ with $a_i \in L$, $n \geq 0$; for $n = 0$ this is the empty word ϕ. Let $b = b_1...b_k$ be a word. We write $w = ... b$ if there is a word a with $w = ab$, similarly we write $w = b...$ if there is a word c with $w = bc$ and we write $w = ...b...$ if there exist words a and c with $w = abc$. A subword of an infinite sequence $... a_{-2}a_{-1}a_0a_1a_2 ...$ with $a_i \in L$, $i \in \mathbb{Z}$, is a finite connected subsequence $a_na_{n+1}...a_{n+k}$, $n \in \mathbb{Z}$. For the word $a = a_1...a_n$ we define the word $-a = a_na_{n-1}...a_1$ by reversing the order in a.

(3.1) Definition: We define a collection of finite words $w = w_1w_2 ... w_k$. The letters w_i of w are the symbols ξ,η,ϵ or natural numbers t,s_i,r_i, $i \in \mathbb{Z}$. We write the letters s_i as upper indices, the letters r_i as lower indices, and the letter t in the middle of the line since we have to distinguish between these numbers. For example $\eta 5\xi^2\eta_3$ is such a word with $t = 5$, $r_1 = 3$, $s_1 = 2$. A basic sequence is defined by

$$(1) \qquad \xi^{s_1}\eta_{r_1}\xi^{s_2}\eta_{r_2}...$$

This is the infinite product $a(1)a(2)...$ of words $a(i) = \xi^{s_i}\eta_{r_i}$, $i \geq 1$. A basic word is any subword of (1). A central sequence is defined by

$$(2) \qquad \ldots {}^{s_{-2}}\xi_{r_{-2}} \eta^{s_{-1}} \xi_{r_{-1}} \eta t \xi^{s_1} \eta_{r_1} \xi^{s_2} \eta_{r_2} \ldots$$

A central word w is any subword of (2) containing the number t. Whence a central word w is of the form w = atb where –a and b are basic words. An ϵ–sequence is defined by

$$(3) \qquad \ldots {}^{s_{-2}}\xi_{r_{-2}} \eta^{s_{-1}} \xi_{r_{-1}} \epsilon^{s_1} \eta_{r_1} \xi^{s_2} \eta_{r_2} \ldots$$

An ϵ–word w is any subword of (3) containg the letter ϵ; again we can write $w = a\epsilon b$ where –a and b are basic words.

A general word is a basic word, a central word or an ϵ–word. A general word w is called special if w contains at least one of the letters ξ, η or ϵ and if the following conditions (i), D(i), (ii) and D(ii) are satisfied in case $w = a\epsilon b$ is an ϵ–word. We associate with b the tuple

$$s(b) = (s_1^b, s_2^b, \ldots) = \begin{cases} (s_1, \ldots, s_m, \infty, 0, 0, \ldots) & \text{if } b = \ldots \xi \\ (s_1, \ldots, s_m, 0, 0, 0, \ldots) & \text{otherwise} \end{cases}$$

$$r(b) = (r_1^b, r_2^b, \ldots) = \begin{cases} (r_1, \ldots, r_\ell, \infty, 0, 0, \ldots) & \text{if } b = \ldots \eta \\ (r_1, \ldots, r_\ell, 0, 0, 0, \ldots) & \text{otherwise} \end{cases}$$

where $s_1 \ldots s_m$ and $r_1 \ldots r_\ell$ are the words of upper indices and lower indices respectively given by b. In the same way we get $s(-a) = (s_1^{-a}, s_2^{-a}, \ldots)$ and $r(-a) = (r_1^{-a}, r_2^{-a}, \ldots)$ with $s_i^{-a} \in \{s_{-i}, \infty, 0\}$ and $r_i^{-a} \in \{r_{-i}, \infty, 0\}$, $i \in \mathbb{N}$. The conditions in question on the ϵ–word $w = a\epsilon b$ are:

(i) $b = \phi \Rightarrow a \neq \xi_1$

D(i) $a = \phi \Rightarrow b \neq {}^1\eta$

(ii) $s_1 = 1 \Rightarrow r_{-1} \geq 2$ and

$$(r_1^b, -s_2^b, r_2^b, -s_3^b, r_3^b, \ldots, -s_i^b, r_i^b, \ldots) < (r_1^{-a} - 1, -s_1^{-a}, r_2^{-a}, -s_2^{-a}, r_3^{-a}, -s^{-a}, -s_3^{-a}, \ldots, r_i^{-a}, -s_i^{-a}, \ldots)$$

D(ii) $r_{-1} = 1 \Rightarrow s_1 \geq 2$ and

$$(-s_1^b + 1, r_1^b, -s_2^b, r_2^b, -s_3^b, r_3^b, \ldots, -s_i^b, r_i^b, \ldots) < (-s_1^{-a}, r_2^{-a}, -s_2^{-a}, r_3^{-a}, -s_3^{-a}, \ldots, r_i^{-a}, -s_i^{-a}, \ldots)$$

In (ii) and D(ii) we use the lexicographical ordering $<$ from the left and the index i runs through $i = 2,3,\ldots$ as indicated.

Finally we define a <u>cyclic</u> <u>word</u> by a pair (w,φ) where w is a basic word of the form $(p \geq 1)$

(4) $$w = \xi^{s_1}\eta_{r_1}\xi^{s_2}\eta_{r_2}\cdots\xi^{s_p}\eta_{r_p}$$

and where φ is an automorphism of a finite dimensional $\mathbb{Z}/2$–vector space $V = V(\varphi)$. Two cyclic words (w,φ) and (w',φ') are <u>equivalent</u> if w' is a cyclic permutation of w, that is

$$w' = \xi^{s_i}\eta_{r_i}\cdots\xi^{s_p}\eta_{r_p}\xi^{s_1}\eta_{r_1}\cdots\xi^{s_{i-1}}\eta_{r_{i-1}},$$

and if there is an isomorphism $\Psi : V(\varphi) \cong V(\varphi')$ with $\varphi = \Psi^{-1}\varphi'\Psi$. A cyclic word (w,φ) is a <u>special</u> <u>cyclic</u> <u>word</u> if φ is an indecomposable automorphism and if w is not of the the form $w = w'w'\ldots w'$ where the right hand side is a j–fold power of a word w' with $j > 1$.

The sequences in (2.1) can be visualized by the infinite graphs in Figure 2. The letters s_i, resp. r_i, correspond to vertical edges connecting the levels 2 and 3, resp. the levels 0,1. The letters ξ, resp. η, correspond to diagonal edges connecting the levels 0 and 2, resp. the levels 1 and 3. Moreover ϵ connects the levels 0 and 3 and t the levels 1 and 2. We identify a general word in (3.1) with the corresponding subgraph of the graphs in Figure 2. Therefore the <u>vertices of level</u> i of a general word are defined by the vertices of level i of the corresponding graph, $i \in \{0,1,2,3\}$. We also write $|x| = i$ if x is a vertex of level i.

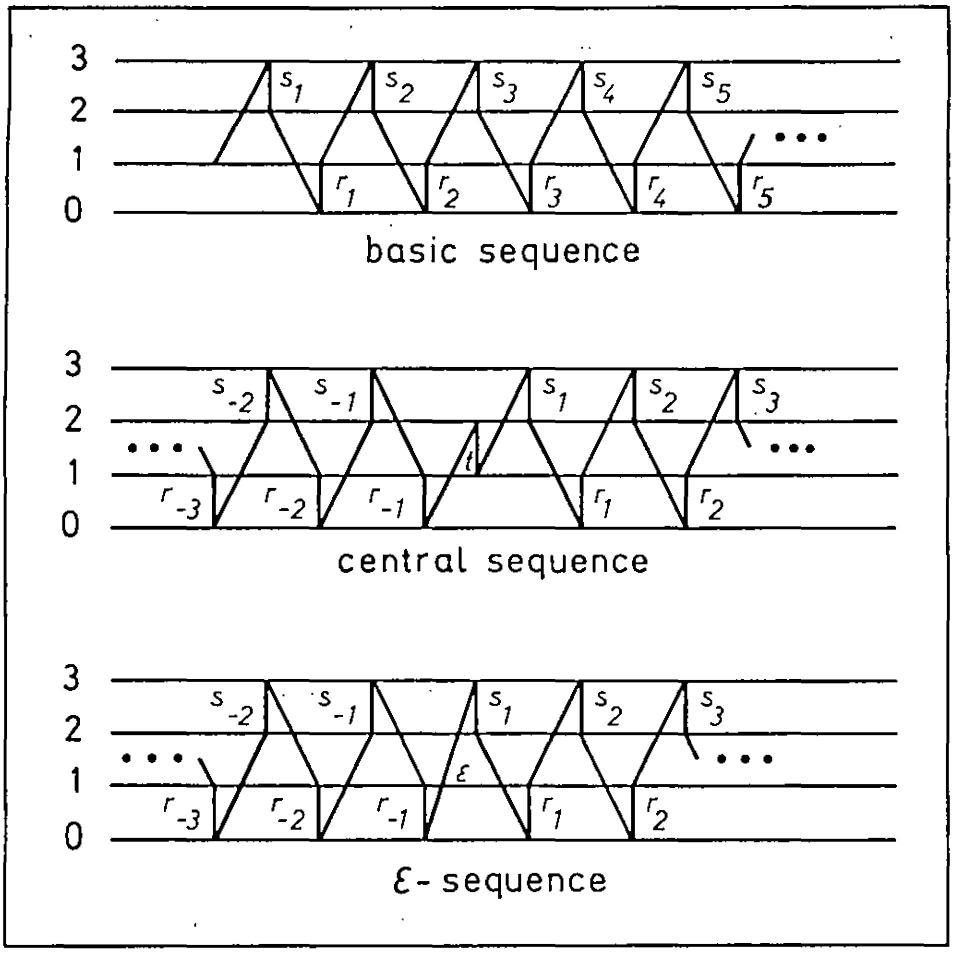


Figure 2

(3.2) Remark: There is a simple rule which creates exactly all graphs corresponding to general words. Draw in the plane $\mathbb{R}^2$ a connected finite graph of total height at most 3 that alternatingly consists of vertical edges of height one and diagonal edges of height 2 or 3. Moreover endow each vertical edge with a natural number. An equivalence relation on such graphs is generated by reflection at a vertical line. One readily checks that the equivalence classes of such graphs are in 1–1 correspondence to all general words.

(3.3) Definition: Let w be a basic word, a central word or an ϵ–word. We obtain the dual word D(w) by reflection of the graph w at a horizontal line and by using the equivalence defined in (3.2). Then D(w) is again a basic word, a central word, or an ϵ–word respectively. Clearly the reflection replaces each letter ξ in w by the letter η and vice versa, moreover it turns a lower index into an upper index and vice versa. We define the dual cyclic word $D(w,\varphi)$ as follows. For the cyclic word (w,φ) in (3.1)(4) let $D(w,\varphi) = (w',(\varphi^*)^{-1})$. Here we set

$$w' = \xi^{r_1}\eta_{s_2}\xi^{r_2} \dots \eta_{s_p}\xi^{r_p}\eta_{s_1}$$

and we set $\varphi^* = \mathrm{Hom}(\varphi,\mathbb{Z}/2)$ with $V(\varphi^*) = \mathrm{Hom}\,(V(\varphi),\mathbb{Z}/2)$. Up to a cyclic permutation w' is just D(w) defined above. We point out that the dual words D(w) and $D(w,\varphi)$ are special if w and (w,φ) respectively are special.

We are going to construct certain A^3_n–polyhedra, $n \geq 4$, associated to the words in (3.1). To this end we first define the homology of a word.

(3.4) Definition: Let w be a general word and let $r_\alpha \dots r_\beta$ and $s_\mu \dots s_\nu$ be the words of lower indices and of upper indices respectively given by w. We define the torsion groups of w by

(1) $T_0(w) = \mathbb{Z}/2^{r_\alpha} \oplus \ldots \oplus \mathbb{Z}/2^{r_\beta}$,

(2) $T_1(w) = \mathbb{Z}/2^t$ if w is a central word,

(3) $T_2(w) = \mathbb{Z}/2^{s_\mu} \oplus \ldots \oplus \mathbb{Z}/2^{s_\nu}$,

and we set $T_i(w) = 0$ otherwise. We define the <u>integral homology</u> of w by

(4) $H_i(w) = \mathbb{Z}^{L_i(w)} \oplus T_i(w) \oplus \mathbb{Z}^{R_i(w)}$.

Here $\beta_i(w) = L_i(w) + R_i(w)$ is the <u>Betti number</u> of w; this is the number of end points of the graph w which are vertices of level i and which are not vertices of vertical edges; we call such vertices x <u>spherical vertices</u> of w. Let $L(w)$, resp. $R(w)$, be the <u>left</u>, resp. <u>right</u>, spherical vertex of w in case they occur. Now we set $L_i(w) = 1$ if $|L(w)| = i$ and $R_i(w) = 1$ if $|R(w)| = i$, moreover $L_i(w) = 0$ and $R_i(w) = 0$ otherwise.

Using the equation (4) we have specified an <u>ordered basis</u> B_i of $H_i(w)$. We point out that

(5) $\beta_0(w) + \beta_1(w) + \beta_2(w) + \beta_3(w) \leq 2$.

For a cyclic word (w,φ) we set

(6) $H_i(w,\varphi) = \bigoplus_v T_i(w)$

where $v = \dim V(\varphi)$ and where the right hand side is the v–fold direct sum of $T_i(w)$.

(3.5) <u>Definition</u>: Let $n \geq 4$ and let w be a general word. We define the A_n^3–polyhedron $X(w) = C_f$ by the mapping cone C_f of a map $f(w) : A \longrightarrow B$ where

(1) $$\begin{cases} A = M(H_3,n+2) \vee M(H_2,n+1) \vee S_c^{n+1} \\ B = M(H_0,n) \vee S_c^{n+1} \vee S_b^{n+1} \end{cases}$$

Here $H_i = H_i(w)$ is the homology group in (3.4) above. We set $S_c^{n+1} = S^{n+1}$ if w is a central word and we set $S_c^{n+1} = *$ otherwise, moreover we set $S_b^{n+1} = S^{n+1}$ if w is a basic word of the form $w = \xi \ldots$ and we set $S_b^{n+1} = *$ otherwise. For the following short words w we can describe $f(w)$ directly in terms of the generators defined in §2:

(2) $$\begin{aligned}&f(\eta) = \eta = \eta_n : S^{n+1} \longrightarrow S^n ,\\ &f(\xi) = \eta = \eta_{n+1} : S^{n+2} \longrightarrow S^{n+1} ,\\ &f(\epsilon) = \eta\eta : S^{n+2} \longrightarrow S^n ,\\ &f(t) = t\iota : S^{n+1} \longrightarrow S^{n+1} ,\end{aligned}$$

$$f({}_r\xi^s) = \xi_r^s, \qquad f({}^s\eta_r) = \eta_r^s, \qquad f({}_r\epsilon^s) = \epsilon_r^s = i\eta\eta q,$$

$$f({}_r\xi) = \xi_r, \qquad f({}^s\eta) = \eta^s, \qquad f({}_r\epsilon) = \epsilon_r = i\eta\eta ,$$

$$f(\xi^s) = \eta_{n+1}q, \qquad f(\eta_r) = i\eta_n, \qquad f(\epsilon^s) = \epsilon^s = \eta\eta q .$$

In general we obtain $f(w) : A \longrightarrow B$ as follows. For this we first describe B and A in (1) as one point unions of elementary Moore spaces. For each letter r_δ of $r_\alpha \ldots r_\beta$ (see 3.4) we have the inclusion

(3) $$j(r_\delta) : M(\mathbb{Z}/2^{r_\delta}, n) \subset B$$

Moreover for each spherical vertex x of w with $|x| \leq 1$ we have the inclusion

(4) $$j(x) : S^{n+|x|} \subset B$$

This is the inclusion of S_b^{n+1} if $|x| = 1$. The space B is exactly the one point union of the subspaces (3), (4) and of $j_c : S_c^{n+1} \subset B$. Next we consider the space A in (1). For each letter s_τ of $s_\mu \ldots s_\nu$ (see (3.4)) we have the inclusion

(5) $$j(s_\tau) : M(\mathbb{Z}/2^{s_\tau}, n+1) \subset A$$

Moreover for each spherical vertex x of w with $|x| \geq 2$ we have the inclusion

(6) $$j(x) : S^{n+|x|-1} \subset A$$

The space A is exactly the one point union of the subspaces (5), (6) and of $j_c : S_c^{n+1} \subset A$.

We now define $f = f(w)$ by the following equations. For a letter s_τ as above and for $\delta = \tau - 1$ we set

$$(7)\quad fj(s_\tau) = \begin{cases} j(r_\delta)\xi_{r_\delta}^{s_\tau} + j(r_\tau)\eta_{r_\tau}^{s_\tau} & \text{if } w = \dots {}_{r_\delta}\xi^{s_\tau}\eta_{r_\tau}\dots \\ j(r_\delta)\eta_{r_\delta}^{s_\tau} + j(r_\tau)\xi_{r_\tau}^{s_\tau} & \text{if } w = \dots {}_{r_\delta}\eta^{s_\tau}\xi_{r_\tau}\dots \\ j_c\eta_{n+1}q + j(r_1)\eta_{r_1}^{s_1} & \text{if } w = \dots t\xi^{s_1}\eta_{r_1}\dots \text{ and } \tau = 1 \\ j(r_{-1})\epsilon_{r_{-1}}^{s_1} + j(r_1)\eta_{r_1}^{s_1} & \text{if } w = \dots {}_{r_{-1}}\epsilon^{s_1}\eta_{r_1}\dots \text{ and } \tau = 1 \end{cases}$$

The first equation also holds if the letters r_δ or r_τ are empty that is if $w = \xi^{s_\tau}\eta\dots$ or if $w = \dots\xi^{s_\tau}\eta$ respectively. In this case we set $j(r_\delta) = j(x)$, if $x = L(w)$, resp. $j(r_\tau) = j(y)$, if $y = R(w)$, see (3.4). We use a similar convention for the other equations in (7). Using (2) and (7) we see that $fj(s_\tau)$ is well defined for all general words w. Next we define $fj(x)$ where x is a spherical vertex of w with $|x| \geq 2$.

$$(8)\quad fj(x) = \begin{cases} j(r_\alpha)\xi_{r_\alpha} & \text{if } w = \xi_{r_\alpha}\dots,\ |x| = 3,\ x = L(w) \\ j(r_\alpha)i\eta & \text{if } w = \eta_{r_\alpha}\dots,\ |x| = 2,\ x = L(w) \\ j(r_\beta)\xi_{r_\beta} & \text{if } w = \dots {}_{r_\beta}\xi,\ |x| = 3,\ x = R(w) \\ j(r_{-1})i\eta\eta & \text{if } w = \dots {}_{r_{-1}}\epsilon,\ |x| = 3,\ x = R(w) \end{cases}$$

Using (8) and (2) the element $fj(x)$ is well defined for all general words w. Finally we define fj_c by

$$(9)\qquad fj_c = \begin{cases} j(r_{-1})i\eta + j_c(t\iota) & \text{if} \quad w = \dots {}_{r_{-1}}\eta t\dots \\ j(x)\eta + j_c(t\iota) & \text{if} \quad w = \eta t\dots,\ x = L(w) \\ j_c(t\iota) & \text{if} \quad w = t\dots \end{cases}$$

This completes the definition of $f = f(w)$ and whence the definition of $X(w) = C_f$.

We point out that $X(w)$ in (3.5) coincides with the corresponding elementary complex in (1.8) if w is one of the words η, ηq, ${}_p\eta$, ${}_p\eta q$. Moreover the suspension of such complexes are given by

(3.6) $\Sigma X(\eta) = X(\xi)$, $\Sigma X(\eta q) = X(\xi^q)$, $\Sigma X({}_p\eta) = X(p\xi)$, $\Sigma X({}_p\eta q) = X(p\xi^q)$.

The words ${}_p\eta q$ and $p\xi^q$ correspond to the two possible subgraphs in a central sequence which both look like the graph in Figure 1. This precisely describes the embedding of indecomposable A_m^2–polyhedra $(m = n,\ n + 1)$ into the set of indecomposable A_n^3–polyhedra. In a similar way indecomposable A_m^3–polyhedra $(m = n,\ n + 1)$ are embedded in the set of indecomposable A_n^4–polyhedra; this already signifies the complexity of the decomposition problem in $\underline{\underline{FA}}_n^4$.

(3.7) <u>Definition</u>: Let $n \geq 4$ and let (w,φ) be a cyclic word. We define the A_n^3–polyhedron $X(w,\varphi) = C_f$ by the mapping cone of a map $f = f(w,\varphi)$ where

(1) $$f : M(H_2,n+1) \longrightarrow M(H_0,n)$$

with $H_i = H_i(w,\varphi)$, see (3.4)(6). For $u \in \{1,\dots,v\}$ we have the inclusion $(m = n,n + 1$ and $i = 0,2)$

(2) $$j_u : M(T_i(w),m) \subset M(H_i,m)$$

by the direct sum decomposition in (3.4)(6). Moreover we have for each letter r_δ and s_τ of $r_1 \dots r_p$ and $s_1 \dots s_p$ (see (3.1)(4)) the inclusions

(3) $$j(r_\delta) : M(\mathbb{Z}/2^{r_\delta},n) \subset M(T_0(w),n)\ ,$$

(4) $$j(s_\tau) : M(\mathbb{Z}/2^{s_\tau}, n+1) \subset M(T_2(w), n+1) .$$

Compare (3.5)(3) and (3.5)(5). We choose a basis $\{b_1,\dots,b_v\}$ of the vector space $V(\varphi)$ and we define $\varphi_u^e \in \{0,1\}$ by $\varphi(b_u) = \sum_{e=1}^{v} \varphi_u^e b_e$. This yields a definition of f by the following formulas (5) and (6).

(5) $$f j_u j(s_\tau) = j_u[j(r_\delta)\xi_{r_\delta}^{s_\tau} + j(r_\tau)\eta_{r_\tau}^{s_\tau}]$$

if $w = \dots{}_{r_\delta}\xi^{s_\tau}\eta_\tau\dots$, $\tau \in \{2,\dots,p\}$ and $\delta = \tau - 1$, see (3.1)(4). Moreover we set

(6) $$f\, j_u\, j(s_1) = j_u\, j(r_1)\eta_{r_1}^{s_1} + \sum_{e=1}^{v} \varphi_u^e\, j_e j(r_p)\, \xi_{r_p}^{s_1} .$$

The spaces $X(w)$ and $X(w,\varphi)$ are constructed in such a way that the integral homology is given by

(3.8) $$H_{n+i}X(w) = H_i(w),\ \ H_{n+i}X(w,\varphi) = H_i(w,\varphi)$$

where we use the homology of the words w and (w,φ) in (3.4). For an elementary Moore space $M(\mathbb{Z}/2^k, n+j)$ in $\underline{\underline{FA}}_n^3$ we get $X(w) = M(\mathbb{Z}/2^k, n+j)$ if the graph w consists only of the edge k connecting the levels j and $j+1$, moreover $X(w) = S^{n+j}$ is a sphere if the graph w consists only of a vertex at level j.

The next result solves the decomposition problem in $\underline{\underline{FA}}_n^3$, see (1.6), we prove this result in §6 below.

(3.9) <u>Decomposition theorem:</u> Let $n \geq 4$. The elementary Moore spaces in $\underline{\underline{FA}}_n^3$, the complexes $X(w)$ where w is a special word, and the complexes $X(w,\varphi)$ where (w,φ) is a special cyclic word furnish a complete list of all indecomposable homotopy types in $\underline{\underline{FA}}_n^3$.

For two complexes X,X' in this list there is a homotopy equivalence $X \simeq X'$ if and only if there are equivalent special cyclic words $(w,\varphi) \sim (w',\varphi')$ with $X = X(w,\varphi)$ and $X' = X(w',\varphi')$. Moreover each homotopy type in $\underline{\underline{FA}}_n^3$ has a unique decomposition.

Spanier–Whitehead duality of indecomposable complexes in $\underline{\underline{FA}}_n^3$ is completely clarified by the next result.

(3.10) Theorem: Let $n \geq 5$. For a general word w and for a cyclic word (w,φ) let Dw and $D(w,\varphi)$ be the dual words defined in (3.3). Then X(Dw) is the Spanier–Whitehead (2n+3)–dual of X(w) and $X(D(w,\varphi))$ is the Spanier–Whitehead (2n+3)–dual of $X(w,\varphi)$.

Proof of (3.10): The result essentially follows from the careful choice of generators in §2 which is compatible with Spanier–Whitehead duality. This implies that there are (2n+2)–dualities $f(w)^* = f(Dw)$ and $f(w,\varphi)^* = f(D(w,\varphi))$. Whence (3.10) is a consequence of the remark on mapping cones following (2.2).

§4 Algebraic invariants

We describe algebraic stable A_n^3–systems which classify the homotopy types of A_n^3–polyhedra, $n \geq 4$. To this end we introduce the following notation.

Let $F : \underline{\underline{C}} \longrightarrow \underline{\underline{K}}$ be a functor. We say that an object X in $\underline{\underline{K}}$ is F–realizable if there is an object Y in $\underline{\underline{C}}$ together with an isomorphism $\alpha : FY \cong X$ in $\underline{\underline{K}}$. We call Y or the pair (α, Y) an F–realization of X. We say that F is a detecting functor if F is full, if each object in $\underline{\underline{K}}$ is F–realizable, and if F satisfies the following sufficiency condition: A morphism β in $\underline{\underline{C}}$ is an isomorphism if and only if the morphism $F\beta$ is an isomorphism in $\underline{\underline{K}}$. One readily observes that a detecting functor F induces a 1–1 correspondence between isomorphism classes of objects; here a 1–1 correspondence is a function which is injective and surjective. Moreover a detecting functor F induces a 1–1 correspondence between isomorphism classes of indecomposable objects if F preserves sums.

We shall use graded abelian groups H with

(4.1) $$H_i = 0 \text{ for } i < 0,\ i > 3 \text{ and } H_3 \text{ free abelian.}$$

For example the reduced integral homology H of an A_n^3–polyhedron X has this property; here we set

(4.2) $$H_i = H_{n+i}(X) \text{ for } i \in \mathbb{Z}.$$

We now consider the following commutative diagram of additive functors, $n \geq 4$, where $\underline{\underline{A}}_n^3$ is the homotopy category of A_n^3–polyhedra.

(4.3)
$$\begin{array}{ccc} \underline{\underline{A}}_n^3 & \xrightarrow{\;\underset{\sim}{U}\;} & \underline{\underline{H}} \\ {\scriptstyle \underset{\sim}{Q}}\downarrow & & \downarrow{\scriptstyle \underset{\sim}{R}} \\ \underline{\underline{S}} & \xrightarrow[\;\underset{\sim}{V}\;]{} & \underline{\underline{G}} \end{array}$$

The categories $\underline{\underline{H}}$, $\underline{\underline{S}}$, and $\underline{\underline{G}}$ are purely algebraic and the functors $\underset{\sim}{Q}$ and $\underset{\sim}{R}$ which we define below are detecting functors. The objects of the categories $\underline{\underline{H}}$, $\underline{\underline{S}}$, and $\underline{\underline{G}}$ are given by specifying additonal structure on "homology" groups H as in (4.1). Let $\underline{\underline{FH}}$, $\underline{\underline{FS}}$, and $\underline{\underline{FG}}$ be the full subcategories of $\underline{\underline{H}}$, $\underline{\underline{S}}$, and $\underline{\underline{G}}$ respectively for which all objects have <u>finitely generated</u> homology.

(4.4) <u>Definition</u> <u>of</u> $\underline{\underline{H}}$: Objects are triples $H_S = (H,H(2),S)$ where H satisfies (4.1) and where $H(2)$ is a graded R–module with $R = \mathbb{Z}/2<Sq_1,Sq_2>/(Sq_1^2,Sq_2^2)$ and $|Sq_1| = -1$, $|Sq_2| = -2$, compare (1.4). Moreover S is a short exact sequence

(1) $$H \otimes \mathbb{Z}/2 \overset{\bar{r}}{\rightarrowtail} H(2) \overset{\bar{b}}{\twoheadrightarrow} H*\mathbb{Z}/2$$

with degree $|\bar{b}| = -1$ such that Sq_1 is the composition

(2) $$H(2) \overset{\bar{b}}{\twoheadrightarrow} H*\mathbb{Z}/2 \overset{i}{\rightarrowtail} H \overset{q}{\twoheadrightarrow} H \otimes \mathbb{Z}/2 \overset{\bar{r}}{\rightarrowtail} H(2).$$

Here i is the inclusion of the 2–torsion and q is the quotient map. A morphism $H_S \longrightarrow H'_S$ in $\underline{\underline{H}}$ is a pair (F,G) of degree 0 homomorphisms $F : H \longrightarrow H'$, $G : H(2) \longrightarrow H'(2)$, such that G is R–linear and such that F and G are compatible with respect to the sequences S and S′.

The <u>functor</u> $\underset{\sim}{U} : \underline{\underline{A}}_n^3 \longrightarrow \underline{\underline{H}}$ carries an object X to $\underset{\sim}{U}(X) = (H,H(2),S)$ where H is the homology in (4.2), where $H(2)_i = H_{n+i}(X,\mathbb{Z}/2)$ is endowed with the action of the Steenrod operations Sq_1,Sq_2, and where S is the universal coefficient sequence. The functor $\underset{\sim}{U}$ was considered by Henn [22], who showed that an analogous functor for the

Steenrod algebra $\mathfrak{A}_p$, p odd, is a detecting functor. In our case, however, we use the prime 2 so that $\underset{\approx}{U}$ is not a detecting functor since in general there are non trivial higher order cohomology operations on spaces X in $\underline{\underline{A}}_n^3$, for example Adem operations, which are not detected by $\underset{\approx}{U}(X)$. We therefore need the better algebraic invariants of X obtained by the functor $\underset{\approx}{Q}$ in (4.3), see (4.11) below.

In the definition of the category $\underline{\underline{S}}$ we use the following notation on abelian groups K and L. We have the natural isomorphism $\Psi : \mathrm{Ext}(K,L \otimes \mathbb{Z}/2) \cong \mathrm{Hom}(K*\mathbb{Z}/2,L\otimes\mathbb{Z}/2)$ which we use as an identification. Here Ψ is defined as follows. Let $\{E\}$ be the class of the extension $L \otimes \mathbb{Z}/2 \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} K$ and let $x \in K$ with $2x = 0$. Then we set $\Psi\{E\}(x) = i^{-1}(2p^{-1}x)$. The element $qi \in \mathrm{Hom}(K*\mathbb{Z}/2,K\otimes\mathbb{Z}/2) = \mathrm{Ext}(K*\mathbb{Z}/2,K\otimes\mathbb{Z}/2)$, defined as in (4.4)(2), yields an extension of abelian groups

(4.5) $$K\otimes\mathbb{Z}/2 \overset{\mu}{\rightarrowtail} G(K) \overset{\Delta}{\twoheadrightarrow} K*\mathbb{Z}/2 .$$

For each homomorphism $\varphi : K \longrightarrow L$ there is a homomorphism $\overline{\varphi} : G(K) \longrightarrow G(L)$ with $(\varphi*1)\Delta = \Delta\overline{\varphi}$ and $\overline{\varphi}\mu = \mu(\varphi \otimes 1)$. Moreover we obtain for $\overline{G}(K) = \mathrm{Hom}(G(K),\mathbb{Z}/4)$ the extension

(4.6) $$\mathrm{Ext}(K,\mathbb{Z}/2) \overset{\overline{\Delta}}{\rightarrowtail} \overline{G}(K) \overset{\overline{\mu}}{\twoheadrightarrow} \mathrm{Hom}(K,\mathbb{Z}/2)$$

where $\overline{\Delta}$ and $\overline{\mu}$ can be identified with $\mathrm{Hom}(\Delta,\mathbb{Z}/4)$ and $\mathrm{Hom}(\mu,\mathbb{Z}/4)$ respectively.

(4.7) Definition of $\underline{\underline{S}}$: Objects in $\underline{\underline{S}}$ are stable A_n^3–systems $A = (H,\pi_1,D,\beta) = (H^A,\pi_1^A,D^A,\beta^A)$. Here H satisfies (4.1) and π_1 is an abelian group. Moreover D is a diagram of unbroken arrows in $\underline{\underline{Ab}}$ as follows:

$$
\text{(1)} \qquad
\begin{array}{ccccccccccccc}
 & & \pi_1\otimes\mathbb{Z}/2 & & & & & & & & & & \\
 & & \big\downarrow{\scriptstyle\mu} & & & & & & & & & & \\
H_3 & \xrightarrow{b_3} & \Gamma & \dashrightarrow & \pi_2 & \dashrightarrow & H_2 & \xrightarrow{b_2} & H_0\otimes\mathbb{Z}/2 & \xrightarrow{i} & \pi_1 & \xrightarrow{h} & H_1 . \\
 & & \big\downarrow{\scriptstyle\Delta} & & & & & & & & & & \\
 & & H_0*\mathbb{Z}/2 & & & & & & & & & &
\end{array}
$$

The row is an exact sequence; (the group π_2 in this row is considered in (4.8) below). Moreover the column is the short exact sequence defined by the following push out in $\underline{\underline{\mathrm{Ab}}}$:

$$
\text{(2)} \qquad
\begin{array}{ccccc}
H_0\otimes\mathbb{Z}/2 & \xrightarrow{i\,\otimes\,1} & \pi_1\otimes\mathbb{Z}/2 & & \\
{\scriptstyle\mu}\big\downarrow & \text{push} & \big\downarrow{\scriptstyle\mu} & & \\
G(H_0) & \xrightarrow[\alpha_*]{} & \Gamma & \xrightarrow[q]{} & \mathrm{cok}(b_3) .
\end{array}
$$

Here we use (4.5) and the homomorphism i in (1). The map Δ in (1) is induced by Δ in (4.5) and q is the projection for the cokernel of b_3 in (1). We use the composition $v = q\mu(i\otimes 1)$ in (2) for the definition of the push out $\Gamma(K;A)$ in $\underline{\underline{\mathrm{Ab}}}$ defined by the following diagram (3); here we use the exact sequence (4.6).

$$
\text{(3)} \qquad
\begin{array}{ccccc}
\mathrm{Ext}(K,\mathbb{Z}/2)\otimes H_0 & \xrightarrow{\overline{\Delta}\otimes 1} & \overline{G}(K)\otimes H_0 & \xrightarrow{\overline{\mu}\otimes 1} & \mathrm{Hom}(K,\mathbb{Z}/2)\otimes H_0 \\
\| & & \big\downarrow & & \| \\
\mathrm{Ext}(K,H_0\otimes\mathbb{Z}/2) \ \text{push} & & & & \\
{\scriptstyle v_*}\big\downarrow & & & & \\
\mathrm{Ext}(K,\mathrm{cok}\ b_3) & \xrightarrow{\Delta} & \Gamma(K;A) & \xrightarrow{\mu} & \mathrm{Hom}(K,H_0\otimes\mathbb{Z}/2)
\end{array}
$$

The map μ is induced by $\overline{\mu}\otimes 1$. Finally β in the object $A = (H,\pi_1,D,\beta)$ is an element

$$
\text{(4)} \qquad \beta \in \Gamma(H_2;A) \ \text{ with } \ \mu(\beta) = b_2
$$

where b_2 is given by (1). A <u>morphism</u> $\varphi : A \longrightarrow B$ in $\underline{S}$ is a tuple of homomorphisms

$$\varphi_i : H_i^A \longrightarrow H_i^B,\ \varphi_\pi : \pi_1^A \longrightarrow \pi_1^B,\ \varphi_\Gamma : \Gamma^A \longrightarrow \Gamma^B,$$

such that φ is compatible with all unbroken arrows in (1) and such that

(5) $$\varphi_*(\beta^A) = \overline{\varphi}_2^*(\beta^B)$$

Here $\varphi_* : \Gamma(H_2^A;A) \longrightarrow \Gamma(H_2^A;B)$ is induced on the push out (3) by $\mathrm{Ext}(H_2^A,\tilde{\varphi}_\Gamma) \oplus \overline{G}(H_2^A) \otimes \varphi_0$ where $\tilde{\varphi}_\Gamma : \mathrm{cok}\ b_2^A \longrightarrow \mathrm{cok}\ b_2^B$ is induced by φ_Γ. Moreover $\overline{\varphi}_2$ is a map $\overline{\varphi}_2 : G(H_2^A) \longrightarrow G(H_2^B)$ as in (4.5) and $\overline{\varphi}_2^* : \Gamma(H_2^B;B) \longrightarrow \Gamma(H_2^A;B)$ is given on the push out (3) by $\mathrm{Ext}(\varphi_2,\mathrm{cok}\ b_3^B) \oplus \mathrm{Hom}(\overline{\varphi}_2,\mathbb{Z}/4) \otimes H_0^B$, see (4.6).

(4.8) <u>Remark</u>: The "homotopy group" $\pi_2 = \pi_2^A$ in the exact sequence (4.7)(1) is determined by the element $\beta = \beta^A$ as follows. The inclusion $\Psi : \ker b_2 \subset H_2$ induces a map $\overline{\Psi}^* : \Gamma(H_2;A) \longrightarrow \Gamma(\ker b_2;A)$ as in (4.7)(5). Now we get $\mu\overline{\Psi}^*\beta = \Psi^*\mu\beta = \Psi^* b_2 = 0$ by (4.7)(4). Whence an element

(1) $$\{\pi_2\} = \Delta^{-1}\overline{\Psi}^*\beta \in \mathrm{Ext}(\ker b_2,\mathrm{cok}\ b_3)$$

is well defined by the exact sequence in the bottom row of (4.7)(3) where we set $K = \ker(b_2)$. This element $\{\pi_2\}$ determines the extension

(2) $$\mathrm{cok}\ b_3 \rightarrowtail \pi_2 \twoheadrightarrow \ker b_2$$

in the exact sequence (4.7)(1).

Recall that J.H.C. Whitehead [42] introduced for an (n–1)–connected space X, $n \geq 3$, the exact sequence

(4.9)
$$H_{n+3}X \xrightarrow{b} \Gamma_{n+2}X \xrightarrow{i} \pi_{n+2}X \xrightarrow{h} H_{n+2}X \xrightarrow{b} H_nX\otimes\mathbb{Z}/2 \xrightarrow{i} \pi_{n+1} \xrightarrow{h} H_{n+1}X$$

where h is the Hurewicz homomorphism and where b is the secondary boundary operator with $\Gamma_m X = \mathrm{im}\{\pi_m X^{m-1} \longrightarrow \pi_m X^m\}$, X^m = m–skeleton of X. Compare also (XII.3) in [38]. We are now ready to state the following result which we prove in (5.18) below.

(4.10) <u>Classification theorem</u>: Let $n \geq 4$, then there exists a detecting functor $\underset{\sim}{Q} : \underline{\underline{A}}_n^3 \longrightarrow \underline{\underline{S}}$ such that $\underset{\sim}{Q}(X) = (H,\pi_1,D,\beta)$ has the following properties. The homology H is given as in (4.2), the groups π_1 and π_2 in D are the homotopy groups $\pi_1 = \pi_{n+1}(X)$ and $\pi_2 \cong \pi_{n+2}(X)$. Moreover there is a natural isomorphism $\Gamma \cong \Gamma_{n+2}(X)$ such that the row of D in (4.7)(1) is naturally isomorphic to Whitehead's exact sequence (4.9).

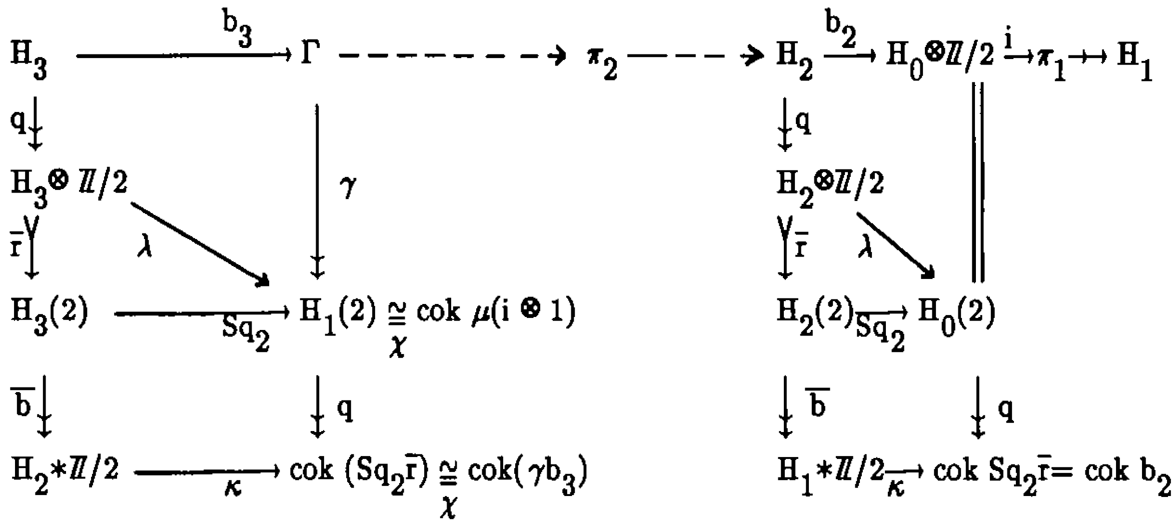
A complete definition of the functor $\underset{\sim}{Q}$ is given in (5.13) below. The main new feature of the functor $\underset{\sim}{Q}$ is the invariant β in $\underset{\sim}{Q}(X)$ which we call the <u>boundary</u> <u>invariant</u> of X. As pointed out in [4] the boundary invariants are the true Eckmann Hilton dual's of the Postnikov invariants; a further discussion of these invariants will appear elesewhere. In Part II of [4] the objects of $\underline{\underline{S}}$ are called A_n^3–systems, $n \geq 4$, here we use the convention that we omit n in the description of objects in $\underline{\underline{S}}$ since we are in the stable range.

The functors $\underset{\sim}{Q}$ and $\underset{\sim}{U}$ in (4.3) have the following connection which we prove in (5.19) below.

(4.11) <u>Proposition:</u> Let X be an object in $\underline{\underline{A}}_n^3$ and let

$$\underset{\sim}{U}(X) = (H,H(2),S) \text{ and } \underset{\sim}{Q}(X) = (H,\pi_1,D,\beta).$$

Then there is a natural isomorphism χ such that the following diagram commutes.



We shall use χ for the identification $H_1(2) = \text{cok}\ \mu(i \otimes 1)$. The map γ and the arrows q denote quotient maps. The top row is given by D and coincides up to isomorphism with (4.9). The maps Sq_2, $\bar{r}$ and $\bar{b}$ are given by $\underline{U}(X)$. Moreover we obtain the maps κ in the diagram by the elements

(1) $\kappa = \{\pi_1\} \in \text{Ext}(H_1, \text{cok}\ b_2) = \text{Hom}(H_1 * \mathbb{Z}/2, \text{cok}\ b_2)$ and

(2) $\kappa = \{\bar{\beta}\} \in \text{Ext}(H_2, \text{cok}\ \gamma b_3) = \text{Hom}(H_2 * \mathbb{Z}/2, \text{cok}\ \gamma b_3)$ respectively.

Here $\{\pi_1\}$ is given by the extension $\text{cok}\ b_2 \rightarrowtail \pi_1 \twoheadrightarrow H_1$ which is part of the top row of the diagram. Moreover let $\gamma_* : \text{cok}\ b_3 \to \text{cok}\ \gamma b_3$ be induced by γ above, then we observe that $\gamma_* v = \gamma_* q \mu(i \otimes 1) = 0$ is trivial since $\gamma\mu(i \otimes 1) = 0$ by definition of γ. Whence we obtain the map $(H_2 = H_2^A)$

(3) $$\gamma_{\#} = (\text{Ext}(H_2, \gamma_*), 0) : \Gamma(H_2; A) \longrightarrow \text{Ext}(H_2, \text{cok}\ \gamma\, b_3)$$

on the push out in (4.7)(3). Using this map we define $\{\bar{\beta}\} = \gamma_{\#}\beta$ where β is the boundary invariant given by $\underline{Q}(X)$. The diagram in (4.11) shows us exactly the connection between the Steenrod squaring operations and Whitehead's certain exact sequence (4.9). The commutativity of the right hand side of the diagram is actually an old result of J.H.C. Whitehead, compare [41] and (XII. 4) in [38].

We are now ready for the definition of the category $\underline{\underline{G}}$ and of the functors $\underset{\sim}{R}$ and $\underset{\sim}{V}$.

(4.12) <u>Definition of</u> $\underline{\underline{G}}$: Objects are triples $W = (H,D_1,D_2)$ where H satisfies (4.1) and where D_1 and D_2 are the following diagrams (q denotes the quotient map).

$$H_2\otimes\mathbb{Z}/2 \xrightarrow{\lambda} H_0\otimes\mathbb{Z}/2 \xrightarrow{q} \operatorname{cok}\lambda \xleftarrow{\kappa} H_1*\mathbb{Z}/2\ ,$$

$$\begin{array}{ccccccc} & & H_1\otimes\mathbb{Z}/2 & & & & \\ & & \downarrow{\scriptstyle \bar{r}} & & & & \\ H_3\otimes\mathbb{Z}/2 & \xrightarrow{\lambda} & H_1(2) & \xrightarrow{q} & \operatorname{cok}\lambda & \xleftarrow{\kappa} & H_1*\mathbb{Z}/2 \\ & & \downarrow{\scriptstyle \bar{b}} & & & & \\ & & H_0*\mathbb{Z}/2 & & & & \end{array}$$

A morphism $\varphi : W \longrightarrow W'$ in $\underline{\underline{G}}$ is a homomorphism $\varphi : H \longrightarrow H'$ of degree 0 for which there exists a morphism $\Psi : H_1(2) \longrightarrow H_1'(2)$ such that φ and Ψ are compatible with the diagrams D_1 and D_2. Sometimes we write H^W and $H(W)$ for H in $W = (H,D_1,D_2)$.

Using diagram (4.11) we obtain obvious functors $\underset{\sim}{R} : \underline{\underline{H}} \longrightarrow \underline{\underline{G}}$ and $\underset{\sim}{V} : \underline{\underline{S}} \longrightarrow \underline{\underline{G}}$. Namely for an object $H_S = (H,H(2),S)$ in $\underline{\underline{H}}$ we get $W = \underset{\sim}{R}(H_S)$ by $\lambda = Sq_2\bar{r}$ and by $\kappa\bar{b} = qSq_2$. For an object A in $\underline{\underline{S}}$ we get $W = \underset{\sim}{V}(A)$ as follows. In D_1 we define λ and κ by $\lambda q = b_2$, $\kappa = \{\pi_1\}$ and in D_2 we define λ and κ by $\lambda q = \gamma b_3$, $\kappa = \{\overline{\beta}\}$. Now Proposition (4.11) exactly shows that the diagram of functors (4.3) commutes.

All categories in (4.3) are in an obvious way additive categories and all functors are additive. The direct sum in $\underline{\underline{H}}$,$\underline{\underline{S}}$ and $\underline{\underline{G}}$ respectively is defined via direct sums of abelian groups.

(4.13) <u>Lemma</u>: $\underset{\sim}{R}$ is a detecting functor.

Proof: Let $W = (H,D_1,D_2)$ be an object in $\underline{\underline{SH}}$. We get an $\underset{\sim}{R}$–realization $H_S = (H,H(2),S)$ as follows. $H_0(2) = H_0 \otimes \mathbb{Z}/2$. $H_1(2)$ and the corresponding part of S is given by D_2. Let $H_i(2) = H_i \otimes \mathbb{Z}/2 \oplus H_{i-1} * \mathbb{Z}/2$ for $i = 2,3$. Then S is completely defined. Now Sq_2 can be chosen such that $\underset{\sim}{R}(H_S) = W$. If $\varphi : \underset{\sim}{R}(H_S) \longrightarrow \underset{\sim}{R}(H_S')$ is a morphism in $\underline{\underline{G}}$ we can choose $\Psi : H_S \longrightarrow H_S'$ with $\underset{\sim}{R}(\Psi) = \varphi$, since the involved exact sequences are split.

(4.14) Lemma: Each object in $\underline{\underline{G}}$ has a $\underset{\sim}{V}$–realization.

(4.15) Corollary: Each object in $\underline{\underline{H}}$ has a $\underset{\sim}{U}$–realization.

The corollary follows from (4.14) and from the fact that $\underset{\sim}{Q}$ and $\underset{\sim}{R}$ are detecting functors and that (4.3) commutes.

Proof of (4.14): Let $W = (H,D_1,D_2)$ be an object in $\underline{\underline{G}}$. We get a $\underset{\sim}{V}$–realization $A = (H,\pi_1,D,\beta)$ by $b_2 = \lambda q$; we choose the extension π_1 such that $\{\pi_1\} = \kappa$ as in (4.11)(1). This gives us Γ by (4.7)(2), such that there is an isomorphism $\chi : H_1(2) \cong \operatorname{cok} \mu(i \otimes 1)$. We choose b_3 such that $\gamma b_3 = \chi\lambda q$, see (4.11). $\gamma_\#$ in (4.11)(3) is surjective. We can choose β with $\gamma_\#\beta = \kappa$. This completes the definition of A. The isomorphism $\varphi : W \cong \underset{\sim}{V}(A)$ is given by the identity on H.

§5 Proof of the classification theorem

In this section we first define the functor $\underset{\sim}{Q} : \underline{\underline{A}}_n^3 \longrightarrow \underline{\underline{S}}$, $n \geq 4$, and we show that $\underset{\sim}{Q}$ has the properties in (4.11). Then we show that $\underset{\sim}{Q}$ is a detecting functor. The definition of $\underset{\sim}{Q}(X)$ uses Whitehead's exact sequence (4.9) and a new 'boundary invariant'.

We first consider Whitehead's group $\Gamma_{n+2}(X)$. Let X be an (n–1)–connected CW–space with $n \geq 4$. Then we have the following natural short exact sequence which we denote by $S_\Gamma(X)$:

(5.1) $$\pi_{n+1}X \otimes \mathbb{Z}/2 \overset{\mu}{\rightarrowtail} \Gamma_{n+2}X \overset{\Delta}{\twoheadrightarrow} H_nX * \mathbb{Z}/2.$$

We define μ by $\mu(x \otimes 1) = \eta^* x$, $x \in \pi_{n+1}X$, where $\eta : S^{n+2} \longrightarrow S^{n+1}$ is the Hopf map. Moreover we define Δ by the first k–invariant $\beta : X \longrightarrow K(H_0,n) = K_0$ of X with $H_0 = H_nX = \pi_nX$. Now Δ is the composition of $\Gamma_{n+2}(\beta)$ and of the isomorphism $\Gamma_{n+2}K_0 \cong H_{n+3}K_0 \cong H_0 * \mathbb{Z}/2$, compare [19]. These definitions show that μ and Δ are natural. For the exactness it is enough to consider the case when X is (n+1)–dimensional and whence of the form $X \simeq M(H_0,n) \vee M(F,n+1)$; here F is a free abelian group. Therefore (5.1) follows from the special case (5.2) below. For a Moore space $X = M(K,n)$ of an abelian group K we have the exact sequence G_K:

(5.2) $$K \otimes \mathbb{Z}/2 \overset{\mu}{\rightarrowtail} \pi_{n+2}M(K,n) \overset{\Delta}{\twoheadrightarrow} K * \mathbb{Z}/2$$

where (4.9) shows that $K \otimes \mathbb{Z}/2 = \pi_{n+1}M(K,n)$, compare [1].
The extension (5.2) conicides with the extension (4.5) so that we may set

(5.3) $$G(K) = \pi_{n+2}M(K,n)\,,$$

compare [1]. The group $G(K)$ can be used for the following algebraic characterization of maps between Moore spaces

(1) $$[M(K,n),M(L,n)] \cong \mathrm{Hom}(G_K,G_L),\ \ n \geq 3.$$

Here $\mathrm{Hom}(G_K,G_L)$ is the set of all pairs $(\overline{\varphi},\varphi)$ where $\overline{\varphi} : G(K) \longrightarrow G(L)$ and $\varphi : K \longrightarrow L$ are homomorphisms with $\Delta\overline{\varphi} = (\varphi * 1)\Delta$ and $\mu(\varphi \otimes 1) = \overline{\varphi}\mu$. The natural isomorphism (1) carries a map $\overline{\varphi} : M(K,n) \longrightarrow M(L,n)$ to the pair $(\pi_{n+2}\overline{\varphi},H_n\overline{\varphi})$, we use this isomorphism as an identification; compare (V. 3a. 8) in [2].

For $H_0 = H_n(X)$ we choose a map $\alpha : M(H_0,n) \longrightarrow X$ which induces the identity $H_n(\alpha) = 1$. Using (5.1), (5.2) and (5.3) one gets the following commutative diagram.

(5.4) $$\begin{array}{ccccc}
H_0\otimes\mathbb{Z}/2 & \xrightarrow{\ i\,\otimes\,1\ } & \pi_{n+1}X\otimes\mathbb{Z}/2 & \xrightarrow{\ h\ } & H_{n+1}(X)\otimes\mathbb{Z}/2 \\
\downarrow \mu & \text{push} & \downarrow \mu & & \downarrow \overline{\Gamma} \\
G(H_0) & \xrightarrow{\ \alpha_*\ } & \Gamma_{n+2}X & \longrightarrow & H_{n+1}(X,\mathbb{Z}/2) \\
\downarrow \Delta & & \downarrow \Delta & & \downarrow \overline{b} \\
H_0*\mathbb{Z}/2 & = & H_nX*\mathbb{Z}/2 & = & H_nX*\mathbb{Z}/2
\end{array}$$

The left hand side of the diagram is given by $\alpha_* : S_\Gamma(M(K,n)) \longrightarrow S_\Gamma(X)$; since the columns are exact we see that the subdiagram push is actually a push out of abelian groups. One gets the right hand side of the diagram by the map $j_* : S_\Gamma(X) \longrightarrow S_\Gamma(SP_\infty X)$ induced by the inclusion $j : X \longrightarrow SP_\infty X$ where $SP_\infty X$ is the infinite symmetric product of X. We use the theorem of Dold–Thom [17] for the identification $\chi_0 : S_\Gamma(SP_\infty X) \cong S$ where S is the right hand column of (5.4) which is also part of $\underline{\mathrm{U}}(X)$, see (4.4). This way we define the natural map γ in (5.4) by the composition $\gamma = \chi_0\Gamma_{n+2}(j)$. The top row of

(5.4) is obtained by the exact sequence (4.9); therefore diagram (5.4) yields the isomorphism

(5.5) $$\chi : \mathrm{cok}\, \mu(i \otimes 1) \cong H_{n+1}(X, \mathbb{Z}/2) .$$

Next we need homotopy groups with coefficients defined by $\pi_n(K;X) = [M(K,n),X]$. One has the universal coefficient sequence

(5.6) $$\mathrm{Ext}(K, \pi_{n+1}X) \rightarrowtail^{\Delta} \pi_n(K;X) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(K, \pi_n X),$$

compare [25]. As a special case one gets for $\pi_{n+1}(K;S^n)$ the sequence

(5.7) $$\mathrm{Ext}(K, \mathbb{Z}/2) \rightarrowtail^{\Delta} \pi_{n+1}(K;S^n) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(K, \mathbb{Z}/2)$$

which is naturally isomorphic to the sequence (4.6), in particular, there is an isomorphism

(5.8) $$\overline{G}(K) = \pi_{n+1}(K;S^n) \cong \mathrm{Hom}(G(K), \mathbb{Z}/4)$$

which we use as an identification. For the definition of this isomorphism we first observe that the maps ξ_1 and η^1 in (2.6) induce isomorphisms

(1) $$\xi_1^* : [M(\mathbb{Z}/2,n), M(K,n)] \cong [S^{n+2}, M(K,n)] = G(K)$$

(2) $$\eta^1_* : [M(K,n+1), M(\mathbb{Z}/2,n+1)] \cong [M(K,n+1), S^n] = \overline{G}(K).$$

This is readily seen by comparing the corresponding short exact sequences for these groups. Now the isomorphism (5.8) carries $x \in \pi_{n+1}(K;S^n)$ to the homomorphism

(3) $$\begin{cases}\Psi_x : \pi_{n+2}M(K,n) \longrightarrow \mathbb{Z}/4 = \pi_{n+1}(\mathbb{Z}/2;M(\mathbb{Z}/2,n+1)),\\ \text{with } \Psi_x(y) = (\eta^1_*)^{-1}(x) \circ \Sigma(\xi_1^*)^{-1}(y) .\end{cases}$$

Obviously (5.8) is natural in the sense that $\overline{\varphi} : M(K',n) \longrightarrow M(K,n)$ induces $(\Sigma\overline{\varphi})^* = \mathrm{Hom}(\pi_{n+2}(\overline{\varphi}), \mathbb{Z}/4)$ where $(\Sigma\overline{\varphi})^*(x) = x \circ (\Sigma\overline{\varphi})$, compare (5.3)(1).

We now use the groups $\overline{G}(K)$ and $G(L)$ above for the computation of the group $\pi_{n+1}(K;M(L,n))$. This group is embedded in the following commutative diagram in which the rows are exact sequences

(5.9) $$\begin{array}{ccccc}
\mathrm{Ext}(K,\mathbb{Z}/2)\otimes L & \xrightarrow{\Delta\otimes 1} & \overline{G}(K)\otimes L & \xrightarrow{\mu\otimes 1} & \mathrm{Hom}(K,\mathbb{Z}/2)\otimes L \\
\| & & & & \| \\
\mathrm{Ext}(K,L\otimes\mathbb{Z}/2) & \text{push} & \Big\downarrow t & & \| \\
\Big\downarrow \mathrm{Ext}(K,\mu) & & & & \| \\
\mathrm{Ext}(K,G(L)) & \stackrel{\Delta}{\rightarrowtail} & \pi_{n+1}(K;M(L,n)) & \stackrel{\mu}{\twoheadrightarrow} & \mathrm{Hom}(K,L\otimes\mathbb{Z}/2)
\end{array}$$

The bottom row and the top row are given by (5.6), (5.7) and μ is induced by μ in (5.2). Moreover the homomorphism t is defined for $x \in \overline{G}(K)$, $y \in L = \pi_n M(L,n)$ by the composition $t(x \otimes y) = y \circ x$, see (5.8). One readily checks that the diagram commutes. Whence exactness of the rows implies that the subdiagram push in (5.9) is a push out diagram of abelian groups. All maps in (5.9) are natural in the obvious fashion.

Recall that Whitehead's Γ–groups $\Gamma_m Y$ of a CW–complex Y are given by $\Gamma_m Y = \mathrm{image}\,\{i_* : \pi_m Y^{m-1} \longrightarrow \pi_m Y^m\}$. Whence we have the inclusion $i : \Gamma_m Y \subset \pi_m Y^m$ and the projection $p : \pi_m Y^{m-1} \twoheadrightarrow \Gamma_m Y$. We now introduce the <u>Γ–groups</u> $\Gamma_m(K;Y)$ <u>with coefficients</u> in the abelian group K by the commutative diagram

(5.10)
$$\begin{array}{ccccc}
\operatorname{Ext}(K,\Gamma_{m+1}Y) & \xleftarrow{\;p_*\;} & \operatorname{Ext}(K,\pi_{m+1}Y^m) & \overset{\Delta}{\searrow} & \\
\Delta\downarrow & \text{push} & \downarrow & & \\
\Gamma_m(K;Y) & \xleftarrow{\;p_\#\;} & \Gamma_m^\# & \rightarrowtail & \pi_m(K;Y^m) \\
\mu\downarrow & & \downarrow & \text{pull} & i_*\downarrow\;\downarrow\mu \\
\operatorname{Hom}(K,\Gamma_m Y) & = & \operatorname{Hom}(K,\Gamma_m Y) & \rightarrowtail & \operatorname{Hom}(K,\pi_m Y^m)
\end{array}$$

Here push and pull denote a push out diagram and a pull back diagram respectively. The right hand side is given by the m–skeleton Y^m of Y and by (5.6). By definition the left hand column is a short exact sequence, which is the universal coefficient sequence for the group $\Gamma_m(K;Y)$. This sequence is clearly natural with respect to cellular maps $f: Y' \longrightarrow Y$; in case Y and Y' are simply connected the induced map $f_*: \Gamma_m(K;Y') \longrightarrow \Gamma_m(K;Y)$ depends only on the homotopy class of f. This shows that the group $\Gamma_m(K;Y)$ is actually a new homotopy invariant of the space Y. We shall discuss the properties of these groups elsewhere; here we are only interested in the group $\Gamma_{n+1}(K;X)$ where X is an (n–1)–connected space. In this case the map α in (5.4) induces the commutative diagram

(5.11)
$$\begin{array}{ccccc}
\operatorname{Ext}(K,G(H_0)) & \overset{\Delta}{\rightarrowtail} & \pi_{n+1}(K;M(H_0,n)) & \overset{\mu}{\twoheadrightarrow} & \operatorname{Hom}(K,H_0\otimes\mathbb{Z}/2) \\
\downarrow \operatorname{Ext}(K,\alpha_*) & \text{push} & \downarrow \alpha_* & & \| \\
\operatorname{Ext}(K,\Gamma_{n+2}X) & \overset{\Delta}{\rightarrowtail} & \Gamma_{n+1}(K;X) & \overset{\mu}{\twoheadrightarrow} & \operatorname{Hom}(K,H_0\otimes\mathbb{Z}/2)
\end{array}$$

with short exact rows. This follows from the naturality of (5.10) since for an (n+1)–dimensional complex M we have $\pi_{n+1}(K;M) = \Gamma_{n+1}(K;M)$. Since the rows are short exact sequences we see that (5.11) again is a push out of abelian groups. This push out can be combined with the push out in (5.9) so that one gets an explicit formula for the groups $\Gamma_{n+1}(K;X)$ in terms of $\overline{G}(K)$ and $\Gamma_{n+2}X$.

The natural quotient map $q : \Gamma_{n+2}(X) \to \operatorname{cok} b_{n+3}$, see (4.9), yields the push out group $\Gamma^b_{n+1}(K;X)$ given by the diagram

$$
(5.12)\qquad
\begin{array}{ccccc}
\operatorname{Ext}(K,\Gamma_{n+2}X) & \rightarrowtail & \Gamma_{n+1}(K;X) & \twoheadrightarrow & \operatorname{Hom}(K,H_0\otimes\mathbb{Z}/2) \\
\downarrow \operatorname{Ext}(K,q) & \text{push} & \downarrow q_{\#} & & \| \\
\operatorname{Ext}(K,\operatorname{cok} b_{n+3}) & \underset{\Delta}{\rightarrowtail} & \Gamma^b_{n+1}(K;X) & \underset{\mu}{\twoheadrightarrow} & \operatorname{Hom}(K,H_0\otimes\mathbb{Z}/2)
\end{array}
$$

We are now ready for the definition of the functor $\underset{\approx}{Q}$ in the classification theorem.

(5.13) <u>Definition</u>: Let X be a CW–complex in $\underline{\underline{A}}^3_n$. Then we define the object $\underset{\approx}{Q}(X) = (H,\pi_1,D,\beta)$ in $\underline{\underline{S}}$, see (4.7), as follows. The graded group H is the homology (4.2); the group π_1 is the homotopy group $\pi_{n+1}(X)$. Moreover, diagram D is defined by the exact sequence (4.9) and by (5.1); here we use the identification $\Gamma = \Gamma_{n+2}X$ given by (4.7)(2) and (5.4). Combining the push out diagrams (5.9), (5.11) and (5.12) we get the identification $\Gamma(K;\underset{\approx}{Q}(X)) = \Gamma^b_{n+1}(K;X)$, compare (4.7)(3) where $v = q\mu(i\otimes 1) = q\alpha_*\mu$ by (5.4). Finally the boundary invariant $\beta \in \Gamma^b_{n+1}(K;X)$ is obtained by the next lemma, see (5.14)(3) below.

Let X be a CW–complex in $\underline{\underline{A}}^3_n$. The homology $H = H^X$ with $H^X_i = H_{n+i}(X)$ is given by the cellular chain complex $C_* = C^X_*$ with $C^X_i = H_{n+i}(X^{n+i},X^{n+i-1})$. Let $B_i \subset Z_i \subset C_i$ be the subgroup of boundaries and cycles respectively. Then we have the one point unions of Moore spaces:

$$
\begin{cases}
X' = M(H_0,n) \vee M(Z_1,n+1) \\
X'' = M(H_3,n+2) \vee M(H_2,n+1) \vee M(B_1,n+1)
\end{cases}
$$

(5.14) <u>Lemma</u>: There is a map $f : X'' \to X'$ such that the mapping cone C_f is homotopy equivalent to X.

We point out that the complexes $X(w)$ and $X(w,\varphi)$ in (3.9) are special examples of such mapping cones C_f.

Proof: We first observe that we have the homotopy equivalences $X' \simeq X^{n+1}$ and $\Sigma X'' \simeq X/X^{n+1}$. Now f is the desuspension of the boundary map $X/X^{n+1} \longrightarrow \Sigma X^{n+1}$ in the cofiber sequence of the inclusion $X^{n+1} \subset X$. Since we assume $n \geq 4$ the desuspension is well defined up to homotopy.

The map f constructed in the proof of (5.14) has the additional property that the induced homology homomorphism $H_*(f)$ is given by the inclusion $B_1 \subset Z_1$. Moreover the inclusion $i_2 : M(H_2,n+1) \subset X''$ yields the element

(1) $\quad f\, i_2 \in [M(H_2,n+1),X^{n+1}]$ with

(2) $\quad \mu(f\, i_2) = i_* b_2 \in \mathrm{Hom}(H_2,\pi_{n+1}X^{n+1})$

Here $i : \Gamma_{n+1}X = H_0 \otimes \mathbb{Z}/2 \subset \pi_{n+1}X^{n+1}$ is the inclusion as in (5.10) and b_2 is given by (4.9). Equation (2) shows that $f\, i_2$ is an element in $\Gamma^{\#}_{n+1}$, see (5.10) where we set $m = n+1$, $Y = X^{n+1}$ and $K = H_2$. Therefore the boundary invariant

(3) $$\beta = \beta^X = q_{\#} p_{\#}(f\, i_2) \in \Gamma^b_{n+1}(H_2;X) = \Gamma(H_2;\underset{\approx}{Q}(X))$$

is defined with $\mu(\beta) = b_2$. Here we use the maps $q_{\#}$ and $p_{\#}$ in (5.11) and (5.10) respectively. Though the map f in (5.14) is not canonically given by X the element β is a homotopy invariant in the following sense. Let $F : X \longrightarrow Y$ be a map in $\underline{\underline{A}}^3_n$ and let $\varphi = H_{n+2}(F) : H^X_2 \longrightarrow H^Y_2$. Then any map $\overline{\varphi} : M(H^X_2,n+1) \longrightarrow M(H^Y_2,n+1)$ which induces φ satisfies the equation

(4) $$F_*(\beta^X) = \overline{\varphi}^*(\beta^Y)\,.$$

This, in fact, shows that $\underset{\approx}{Q}$ in (5.13) is a well defined functor $\underline{\underline{A}}^3_n \longrightarrow \underline{\underline{S}}$.

(5.15) Proposition: Each object in $\underline{\underline{S}}$ is $\underset{\approx}{Q}$–realizable.

Proof: Let $A = (H,\pi_1,D,\beta)$ be an object in $\underline{\underline{S}}$. We define $X = C_f$ with $\mathcal{Q}(X) \cong A$ as follows. First we choose a free resolution $B_1 \rightarrowtail Z_1 \twoheadrightarrow H_1$ of H_1 and we define X' and X'' as in (5.14). Let $i_0 : M(H_0,n) \subset X'$ and $i_Z : M(Z_1,n+1) \subset X'$ be the inclusions. Then we have the obvious coordinates of f,

$$\text{(1)} \qquad \begin{cases} f = (f_3, f_2, f_B) \quad \text{with} \\ f_3 = i_0 f_3^0 + i_Z f_3^Z \quad \text{and } f_B = i_0 f_B^0 + i_Z f_B^Z . \end{cases}$$

The map f_B^Z is given by the inclusion $B_1 \rightarrowtail Z_1$. Next we choose a commutative diagram

$$\text{(2)} \qquad \begin{array}{ccccccc} H_2 & \xrightarrow{b_2} & H_0 \otimes \mathbb{Z}/2 & \xrightarrow{i} & \pi_1 & \twoheadrightarrow & H_1 \\ & & \uparrow f_B^0 & & \uparrow g & & \| \\ & & B_1 & \rightarrowtail & Z_1 & \twoheadrightarrow & H_1 \end{array}$$

where the top row is given by diagram D in A. The homomorphism f_B^0 determines the map f_B^0 in (1). Next we choose for $b_3 : H_3 \longrightarrow \Gamma$ in D homomorphisms $f_3^0 : H_3 \longrightarrow G(H_0)$ and $f_3^Z : H_3 \longrightarrow Z_1 \otimes \mathbb{Z}/2$ such that

$$\text{(3)} \qquad b_3 = \alpha_* f_3^0 + \mu(g \otimes 1) f_3^Z .$$

Then f_3^0 and f_3^Z determine the coordinates of the map f_3. Finally we choose for β an element $f_2 \in \Gamma_{n+1}^{\#}$ with $q_{\#} p_{\#} f_2 = \beta$, see (5.14)(3). Then f_2 gives us the coordinate f_2 in (1). One readily checks that $\mathcal{Q}(C_f)$ in (5.13) is isomorphic to A.

(5.16) Proposition: The functor $\mathcal{Q} : \underline{\underline{A}}_n^3 \longrightarrow \underline{\underline{S}}$ is full.

Proof: Let $X = C_f$, $Y = C_g$ (with $f : X'' \longrightarrow X'$ and $g : Y'' \longrightarrow Y'$) be objects in $\underline{\underline{A}}_n^3$, see (5.14). Moreover, let

$$\text{(1)} \qquad (\varphi,\varphi_\pi,\varphi_\Gamma) : A = \mathcal{Q}(X) \longrightarrow B = \mathcal{Q}(Y)$$

be a morphism in $\underline{\underline{S}}$. We have to show that there is a cellular map $F : X \longrightarrow Y$ with $\mathcal{Q}(F) = (\varphi,\varphi_\pi,\varphi_\Gamma)$. We first construct the restriction $F' : X' \longrightarrow Y'$ of the map F. The map F' has the coordinates

(2) $$F' = (i_0\overline{\varphi}_0 + i_Z F_1, F_2) \text{ with}$$
$$\overline{\varphi}_0 \in [M(H_0^X,n), M(H_0^Y,n)],\ H_n(\overline{\varphi}_0) = \varphi_0,$$
$$F_1 \in [M(H_0^X,n), M(Z_1^Y,n+1)] = \mathrm{Ext}(H_0^X, Z_1^Y),$$
$$F_2 \in [M(Z_1^X,n+1), Y'].$$

The inclusion $i_Z : M(Z_1^X,n+1) \subset X'$ induces a homomorphism $g : Z_1^X \longrightarrow \pi_{n+1}X' \twoheadrightarrow \pi_{n+1}X = \pi_1^X$ compare (5.15)(2). We now choose a homomorphism F_2 such that the diagram

(3) $$\begin{array}{ccc} \pi_1^X & \xleftarrow{\quad g \quad} & Z_1^X \\ \downarrow{\scriptstyle \varphi_\pi} & & \downarrow{\scriptstyle F_2} \\ \pi_1^Y & \xleftarrow{\quad j \quad} & \pi_{n+1}Y' = H_0^Y \otimes \mathbb{Z}/2 \oplus Z_1^Y \end{array}$$

commutes. Here $j = (i,g)$ is induced by the inclusion $Y' \subset Y$. The homomorphism F_2 in (3) determines the map F_2 in (1). Next we consider the following commutative diagram for φ_Γ.

(4) $$\begin{array}{ccccc} & & (\alpha_*,\mu) & & \\ \Gamma^X = \Gamma_{n+2}(X) & \xleftarrow{\ p\ } & \pi_{n+2}X' & \xrightarrow{\ g_*\ } & G(H_0^X) \oplus \pi_1^X \otimes \mathbb{Z}/2 \\ \varphi_\Gamma \downarrow & & \downarrow \pi_{n+2}(F') & & \downarrow \begin{bmatrix} \overline{\varphi}_0 & 0 \\ \Psi\Delta & \varphi_\pi \otimes 1 \end{bmatrix} \\ \Gamma^Y = \Gamma_{n+2}(Y) & \xleftarrow{\ p\ } & \pi_{n+2}Y' & \xrightarrow{\ g_*\ } & G(H_0^Y) \oplus \pi_1^Y \otimes \mathbb{Z}/2 \\ & & (\alpha_*,\mu) & & \end{array}$$

Here p is defined as in (5.10) and $\alpha = j\,i_0$ is the inclusion. The map (α_*,μ) is surjective by (5.4). For $\pi_{n+2}X' = G(H_0^X) \oplus Z_1^X \otimes \mathbb{Z}/2$ we obtain g_* by $1 \oplus g \otimes 1$ where g is defined as in (3). The homomorphism $\overline{\varphi}_0$ corresponds to a map as in (1) by (5.3)(1). Moreover Ψ in (4) is a homomorphism

(5) $$\Psi : H_0^X * \mathbb{Z}/2 \longrightarrow \pi_1^Y \otimes \mathbb{Z}/2$$

We claim that there exist $\overline{\varphi}_0$, Ψ and F_1 in (2) such that diagram (4) is commutative. To see this we first choose a map $\varphi_0' : G(H_0^X) \longrightarrow G(H_0^Y)$ compatible with φ_0. Then

$\varphi_0' \oplus \varphi_\pi \otimes 1$ induces by (5.4) a homomorphism $\varphi_\Gamma' : \Gamma_{n+2}X \longrightarrow \Gamma_{n+2}Y$ which as well is compatible with μ and Δ in (5.4). Whence there is a homomorphism Ψ' with $\varphi_\Gamma = \varphi_\Gamma' + \mu\Psi'\Delta$. Next we choose for Ψ' a lifting as in the diagram

$$\begin{array}{ccc} & & Z_1^Y\otimes\mathbb{Z}/2 \oplus H_0^Y\otimes\mathbb{Z}/2 \\ & \overset{(\Psi_0,\Psi_1)}{\nearrow} & \big\downarrow (g\otimes 1\,,\, i\otimes 1) \\ H_0^X * \mathbb{Z}/2 & \xrightarrow[\Psi']{} & \pi_1^Y \otimes \mathbb{Z}/2 \end{array} \tag{6}$$

and we set

$$\begin{cases} \overline{\varphi}_0 = \varphi_0' + \mu\Psi_1\Delta, \\ \Psi \ = (g\otimes 1)\Psi_0 \,. \end{cases} \tag{7}$$

Moreover we choose for Ψ_0 an element F_1 with $q_*(F_1) = \Psi_0$. Here

$$q_* : \mathrm{Ext}(H_0^X, Z_1^Y) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Ext}(H_0^X, Z_1^Y \otimes \mathbb{Z}/2) = \mathrm{Hom}(H_0^X * \mathbb{Z}/2, Z_1^Y \otimes \mathbb{Z}/2)$$

is induced by the quotient map $q : Z_1^Y \longrightarrow Z_1^Y \otimes \mathbb{Z}/2$. One can check that for these choices diagram (4) commutes.

Any extension F of F' with $H_*(F) = \varphi$ induces the map $\underset{\sim}{Q}(F) = (\varphi, \varphi_\pi, \varphi_\Gamma)$. One obtains the existence of such a map F by the construction of a map $F'' : X'' \longrightarrow Y''$ such that $\underset{\sim}{g}F'' = F'\underset{\sim}{f}$ and such that $H_*(F'')$ is compatible with φ_3, φ_2. This is a direct but somewhat tedious procedure. A more elegant proof for the existence of the extension F is obtained as follows. First we choose a chain map

$$\xi : C_*(X) \longrightarrow C_*(Y) \tag{8}$$

between cellular chain complexes such that ξ coincides with $C_*(F')$ in degree $\le n+1$ and such that $H_*(\xi) = \varphi$. Since (φ, φ_π) is compatible with diagram D we can find an extension $F^{n+2} : X^{n+2} \longrightarrow Y^{n+2}$ of F' compatible with ξ. Therefore the obstruction

$$\mathcal{O}(\xi, F') \in H^{n+3}(X, \Gamma^Y) \tag{9}$$

is defined, see (V. 5.14)(2) in [2]. This obstruction is trivial if and only if there exists an extension F of F' with $C_*F = \xi$. Now we have the short exact sequence

(10) $$\mathrm{Ext}(H_2^X,\Gamma^Y) \rightarrowtail^{b} H^{n+3}(X,\Gamma^Y) \overset{r}{\twoheadrightarrow} \mathrm{Hom}(H_3^X,\Gamma^Y)$$

which satisfies

(11) $$r\mathcal{O}(\xi,F') = -b_3^Y\varphi_3 + \varphi_\Gamma b_3^X = 0.$$

Moreover for the projection $q : \Gamma^Y \twoheadrightarrow \mathrm{cok}\, b_3^Y$ we have the equation, $q_* = \mathrm{Ext}\,(H_2^X,q)$,

(12) $$\Delta q_* b^{-1}\mathcal{O}(\xi,F') = -\overline{\varphi}_2^*(\beta^Y) + F'_*(\beta^X) = 0.$$

Here the right hand side is an element of $\Gamma^b_{n+1}(H_2^X;Y)$ which is trivial by assumption, see (4.7)(5). Next we can alter the chain map ξ by an element $\alpha \in \mathrm{Ext}(H_2^X,H_3^Y)$ so that we get the chain map $\xi + \alpha$, see (VI 10.13) in [16]. Then the obstruction (9) satisfies the formula

(13) $$\mathcal{O}(\xi + \alpha,F') = \mathcal{O}(\xi) + b(b_3^Y)_*(\alpha)$$

with $(b_3^Y)_* = \mathrm{Ext}(H_2^X,b_3^Y)$ and with b in (10). Since image $(b_3^Y)_* = \mathrm{kernel}\ q_*$ we see by (13), (12) and (11) that there is an α such that $\mathcal{O}(\xi+\alpha,F') = 0$. This shows that F' has an extension F with $\mathcal{Q}(F) = (\varphi,\varphi_\pi,\varphi_\Gamma)$ and the proof of (5.16) is complete.

(5.17) Remark: The boundary invariant β defined in (5.14)(3) is a special element in a sequence of boundary invariant $\beta_i(X)$, $i \geq 4$, defined for any simply connected space X, see [4]. All these boundary invariants satisfy a formula as in (12) above; we shall discuss this fact elsewhere. Moreover one has formulas as (11) and (13) above also in the case of non simply connected spaces, see [3]. Special cases of such formulas were also used in (IX, 4.13)[2].

(5.18) Proof of the classification theorem:

Using the Whitehead theorem the classification theorem (4.10) is a consequence of (5.15) and (5.16).

(5.19) Proof of (4.11): We already pointed out that the commutativity of the right hand side in diagram (4.11) was proved by J.H.C. Whitehead [41], see also (XII.4) in [38]. This as well yields the commutativity of the left hand side of diagram (4.11) in case the n–skeleton of X is trivial. Now we can use naturality arguments for the quotient map $X \longrightarrow X/X^n$ to obtain the commutativity in general, for this we also use the explicit formula for $\mathcal{Q}(X)$ in terms of f with $X = C_f$, see (5.13) and (5.14)(3).

§6 Proof of the decomposition theorem

The detecting functor $\underset{\sim}{Q} : \underline{\underline{FA}}^3_n \longrightarrow \underline{\underline{FS}}$ in (4.11) induces a 1–1 correspondence between indecomposable A^3_n–polyhedra in $\underline{\underline{FA}}^3_n$ and indecomposable A^3_n–systems in $\underline{\underline{FS}}$. Whence we can solve the decomposition problem in $\underline{\underline{FA}}^3_n$ by the classification of indecomposable objects in $\underline{\underline{FS}}$. For this we use a result of Henn [22] from which we derive the following solution of the decomposition problem in the category $\underline{\underline{FH}}$.

(6.1) Theorem: The objects $\underset{\sim}{U}X$, $\underset{\sim}{U}X(w)$, $\underset{\sim}{U}X(w,\varphi)$ of $\underline{\underline{FH}}$, where X is an elementary Moore space in $\underline{\underline{FA}}^3_n$, where w is a special word which is basic or central, and where (w,φ) is a special cyclic word furnish a complete list of indecomposable objects of $\underline{\underline{FH}}$. Two objects H_S and H'_S in this list are isomorphic in $\underline{\underline{FH}}$ if and only if there are equivalent special cyclic words $(w,\varphi) \sim (w',\varphi')$ with $H_S = UX(w,\varphi)$ and $H'_S = \underset{\sim}{U}X(w',\varphi')$. Moreover decomposition is unique in $\underline{\underline{FH}}$.

This result is very similar to the decomposition theorem (3.9); the crucial difference is described by the special ϵ–words $w = a\epsilon b$ which as well yield indecomposable homotopy types $X(w)$ in $\underline{\underline{FA}}^3_n$ but for which $\underset{\sim}{U}X(w)$ is decomposable, namely

$$\underset{\sim}{U}X(a\epsilon b) = \underset{\sim}{U}X(-a) \oplus \underset{\sim}{U}X(b). \tag{6.2}$$

Here $\underset{\sim}{U}X(-a)$ and $\underset{\sim}{U}X(b)$ are again indecomposable objects in $\underline{\underline{FH}}$.

For the proof of the decomposition theorem (3.9) we first observe an easy fact.

(6.3) Lemma: Let X be a finite A^3_n–polyhedron, $n \geq 4$. Then there exists a homotopy equivalence $X \simeq X' \vee M$ where the integral homology H_*X' of X' has no odd torsion and where M is a one point union of elementary Moore spaces $M(p^i,j)$ with p an odd prime, $i \geq 1$, $n \leq j \leq n+2$.

The lemma easily follows from the classification theorem (4.10).

We now describe the indecomposable objects in $\underline{\underline{FH}}$ and $\underline{\underline{FS}}$ in terms of the graphs associated to the general words w.

(6.4) Definition: Let w be a basic word or a central word. Then we obtain the object $\underset{\sim}{Y}(w) = (H,H(2),S)$ in $\underline{\underline{FH}}$ with $\underset{\sim}{Y}(w) = \underset{\sim}{Y}X(w)$ as follows. The homology $H = H(w)$ is given by the formula (3.4)(4). The generators of H are given by the elements g(I) where I is a spherical vertex of w or where I is one of the vertical edges of w denoted by s_τ, r_δ or t, see Fig. 2. Let $\ell(I)$ be the lower vertex of I and let u(I) be the upper vertex of I. If I is a spherical vertex we set $\ell(I) = I$ and $u(I) = \phi$. The degree $|g(I)|$ of the generator g(I) is the level of $\ell(I)$. Now let H(2) be the free $\mathbb{Z}/2$–module with generators h(x) where x is a vertex of w. The degree $|h(x)|$ of h(x) is the level of x. We define the exact sequence S:

$$(1) \qquad H \otimes \mathbb{Z}/2 \overset{\bar{r}}{\rightarrowtail} H(2) \overset{\bar{b}}{\twoheadrightarrow} H * \mathbb{Z}/2$$

by $\bar{r}(g(I) \otimes 1) = h(\ell(I))$ and $\bar{b}(h(x)) = g(I)*1$ for $x = u(I)$ and $\bar{b}(h(x)) = 0$ otherwise. Moreover we define

$$(2) \qquad Sq_2 : H(2) \longrightarrow H(2)$$

by $Sq_2h(x) = h(y)$ if there exists a diagonal edge connecting the vertices x and y with $|y| = |x| - 2$ and by $Sq_2h(x) = 0$ otherwise. This completes the definition of $\underset{\sim}{Y}(w)$. In a similar way we obtain the object $\underset{\sim}{Y}(w,\varphi)$ in $\underline{\underline{FH}}$ with $\underset{\sim}{Y}(w,y) = \underset{\sim}{Y}X(w,\varphi)$. Henn [22] describes the objects $\underset{\sim}{Y}(w)$ by using words W of a different form which are 1–1 corresponded to the words w used here. For example Henn's word

$W = (1,9,0;2,3,3;1,4,0;2,2,1;3,1,2;0,0,0)$ corresponds to $w = {}_9\eta^3\xi_4\eta 2\xi^1\eta$. Using this correspondence we derive (6.1) from [22], compare also [23]. Finally we define the Spanier Whitehead dual of $\underset{\approx}{U}(w)$ and $\underset{\approx}{U}(w,\varphi)$ by

(4) $$D\underset{\approx}{U}(w) = \underset{\approx}{U}(Dw),\ D\underset{\approx}{U}(w,\varphi) = \underset{\approx}{U}(D(w,\varphi)) .$$

Here the word Dw and $D(w,\varphi)$ are given by (3.3).

(6.5) <u>Definition</u>: Let w be a general word. Then we obtain the object $\underset{\approx}{Q}(w) = (H,\pi_1,D,\beta)$ in $\underline{\underline{FS}}$ with $\underset{\approx}{Q}(w) = \underset{\approx}{Q}X(w)$ as follows. We choose the generators of the groups in $\underset{\approx}{Q}(w)$ compatible with the generators in $\underset{\approx}{U}X(w)$ above, see (4.12) and (6.2). The generators $g(I)$ of $H = H(w)$ are defined in (6.4). Now let w be basic or central. We define b_2 by

(1) $$b_2(g(I)) = g(J) \otimes 1,\quad |g(I)| = 2,$$

if there is a diagonal η connecting $\ell(I)$ and $\ell(J)$; and we set $b_2(g(I)) = 0$ otherwise. This shows that π_1 is a cyclic group with generator g_π satisfying

(2) $$\pi_1 = \begin{cases} H_1 & \text{if } \operatorname{cok} b_2 = 0 \\ \mathbb{Z}/2^{t+1} & \text{if } \operatorname{cok} b_2 \neq 0 \text{ and } H_1 = \mathbb{Z}/2^t \\ \mathbb{Z}/2 & \text{if } \operatorname{cok} b_2 \neq 0 \text{ and } H_1 = 0 \end{cases}$$

and satisfying $h(g_\pi) = g(I)$, $|g(I)| = 1$, for $H_1 \neq 0$. This as well determines the homomorphism i since $\operatorname{cok}(b_2) = \mathbb{Z}/2$ or $\operatorname{cok}(b_2) = 0$. We obtain $\Gamma = \Gamma(i)$ by

(3) $$\Gamma = \begin{cases} (\mathbb{Z}/2)^d & \text{if } w \neq {}_{r_1}\cdots \\ \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{d-1} & \text{if } w = {}_{r_1}\cdots, \quad r_1 = 1, \\ \mathbb{Z}/2 \oplus (\mathbb{Z}/2)^d & \text{if } w = {}_{r_1}\cdots, \quad r_1 \geq 2, \end{cases}$$

where d is given by $H_1(2) = (\mathbb{Z}/2)^d$. We choose generators $h_\Gamma(x)$ with $\gamma h_\Gamma(x) = h(x)$ for all vertices x of level 1 where we use γ in (4.12). The generator of $\mathbb{Z}/4$ in (3) is $h_\Gamma(u(r_1))$, the generator of the first summand $\mathbb{Z}/2$ in the bottom row of (3) is denoted by h_Γ with $\gamma(h_\Gamma) = 0$. In terms of these generators we obtain Δ,μ and b_3 in (4.8)(1) as follows. First we get $\Delta = \overline{b}\gamma$ by γ above and $\overline{b}$ in (6.4) and we get μ by

$$(4) \quad \mu(g_\pi \otimes 1) = \begin{cases} h_\Gamma(\ell(I)) & \text{if } g(I) \text{ generates } H_1 \neq 0, \\ 2h_\Gamma(u(r_1)) & \text{if } w = {}_{r_1}\cdots, \ r_1 = 1, \\ h_\Gamma & \text{if } w = {}_{r_1}\cdots, \ r_1 \geq 2. \end{cases}$$

Moreover, b_3 is defined by

$$(5) \qquad b_3(g(x)) = h_\Gamma(y), \quad |g(x)| = 3,$$

if there is a diagonal ξ connecting x and y.

It remains to define $\beta \in \Gamma(H_2,v)$. We describe β as an equivalence class $\beta = \{(\beta_E, b(w))\}$ where

$$(6) \qquad \begin{cases} b(w) \in \overline{G}(H_2) \otimes H_0 \ , \\ \beta_E \ \in \mathrm{Ext}(H_2, \mathrm{cok}\ b_3). \end{cases}$$

We obtain a basis of $\overline{G}(H_2)$ by choosing for each cyclic summand C of H_2 the generators in

$$(7) \quad \overline{G}(C) = \begin{cases} \mathbb{Z}/4\,\overline{g}(s_\tau) & \text{if } C = \mathbb{Z}/2^{s_\tau} g(s_\tau),\ s_\tau = 1, \\ \mathbb{Z}/2\,\overline{g}(s_\tau) \oplus \mathbb{Z}/2\,\overline{\overline{g}}(s_\tau) & \text{if } C = \mathbb{Z}/2^{s_\tau} g(s_\tau),\ s_\tau > 1, \\ \mathbb{Z}/2\,\overline{g}(x) & \text{if } C = \mathbb{Z}g(x), \end{cases}$$

for which $\overline{\mu}\,\overline{g}(I)$ is the generator of $\mathrm{Hom}(C,\mathbb{Z}/2)$. Now we define $b(w)$ by

$$(8) \qquad b(w) = \sum b(I,J)\,\overline{g}(I) \otimes g(J)$$

where $b(I,J) = 1$ if $b_2(g(I)) = g(J) \otimes 1$ and where $b(I,J) = 0$ otherwise. The sum runs through all I,J for which $g(I)$ and $g(J)$ are generators of H_2 and H_0 respectively. Whence we get $(\overline{\mu} \otimes 1)(b(w)) = b_2$. Using the basis elements $g(s_\tau) \in H_2$ and a cyclic subgroup $C \subset \mathrm{cok}(b_3)$ with generator $c \in C$ we obtain the cyclic subgroup $\mathrm{Ext}(\mathbb{Z}\cdot g(s_\tau),C) \subset \mathrm{Ext}(H_2,\mathrm{cok}\ b_3)$ with generator denoted by $\mathrm{Ext}(g(s_\tau),c)$. Using this convention we define β_E in (6) by the sum

$$(9) \qquad \beta_E = \sum \beta_E(s_\tau, h_\Gamma(x)) \ \ \mathrm{Ext}(g(s_\tau), q h_\Gamma(x))$$

where $\beta_E(s_\tau, h_\Gamma(x)) = 1$ if there is a diagonal ξ connecting the vertices $u(s_\tau)$ and x, and where $\beta_E(s_\tau, h_\Gamma(x)) = 0$ otherwise. One can check that for $\beta = \{(\beta_E, b(w))\}$ we get $\{\overline{\beta}\}\overline{b} = qSq_2$ with Sq_2 defined in (6.4), see (4.2); in fact we simply have $\{\overline{\beta}\} = \mathrm{Ext}(H_2, \gamma_*)(\beta_E)$. This completes the definition of $\underset{\approx}{Q}(w)$ if w is basic or central. For an ϵ–word $w = a\epsilon b$ we obtain $\underset{\approx}{Q}(w)$ as follows. The formulas for the object $\underset{\approx}{Q}(w)$ coincide with those of $\underset{\approx}{Q}(-a) \oplus \underset{\approx}{Q}(b)$ except for the formulas of b_3 if $b = \phi$ and β_E if $b \neq \phi$. Here we have $\underset{\approx}{Q}(b) = \underset{\approx}{Q}(S^{n+3})$ if $b = \phi$; in this case $b_3 \mid H_3^{-a} = b_3^{-a}$ and $b_3 \mid H_3^b : \mathbb{Z} \longrightarrow \Gamma(i)$ is given by

$$b_3(1) = \begin{cases} 2 \cdot h_\Gamma(u(r_{-1})) & \text{if } -a = {}_{r_{-1}}\ldots, r_{-1} = 1 \\ h_\Gamma & \text{otherwise} \end{cases} \tag{10}$$

Moreover, if $b \neq \phi$ we have $\beta_E = \beta_E^{-a} + \beta_E^b + \epsilon_E$ where

$$\epsilon_E = \begin{cases} 2\ \mathrm{Ext}\ (g(s_1), qh_\Gamma(r_{-1})) & \text{if } -a = {}_{r_{-1}}\ldots, r_{-1} = 1 \\ \mathrm{Ext}\ (g(s_1), qh_\Gamma) & \text{otherwise} \end{cases} \tag{11}$$

This completes the definition of $\underset{\approx}{Q}(w)$ for all general words w. For a cyclic word (w,φ) the object $\underset{\approx}{Q}X(w,\varphi)$ is completely determined by $\underset{\approx}{U}X(w,\varphi)$ since in this case γ in (4.12) is an isomorphism. In particular $\gamma_\#$ in (4.12) is a splitting for the bottom row in (4.8)(3) so that β is determined by $\gamma_\#(\beta) = \beta_E$. We again represent β in $\underset{\approx}{Q}X(w,\varphi)$ by an equivalence class $\{(\beta_E, b(w,\varphi))\}$ where $b(w,\varphi) \in \overline{G}(H_2) \otimes H_0$ is defined by b_2 in the same way as in (8) above.

Finally we define the Spanier–Whitehead dual of $\underset{\approx}{Q}(w)$ and $\underset{\approx}{Q}(w,\varphi)$ by

$$D\underset{\approx}{Q}(w) = \underset{\approx}{Q}(Dw),\ D\underset{\approx}{Q}(w,\varphi) = \underset{\approx}{Q}(D(w,\varphi))\ . \tag{12}$$

Here the words Dw and $D(w,\varphi)$ are given by (3.3).

(6.6) Lemma: Let W be an object in $\underline{\underline{FG}}$ and let A be a $\underset{\approx}{Y}$–realization of W. Then the isomorphism class of the group $\ker\{\gamma : \Gamma \longrightarrow H(2)\}$ defined by A, see (4.12), depends only on W. We denote this isomorphism class by $\gamma(W) = \gamma(A)$.

Proof: The map γ in (4.12) depends only on $(H_0, i : H_0 \otimes \mathbb{Z}/2 \longrightarrow \pi_1)$. Since W determines $\{\pi_1\}$ and b_2 in (4.12) we see that i is well defined up to isomorphism by W. This proves the lemma.

(6.7) Proposition: Let W be an object in $\underline{\underline{FG}}$ with $\gamma(W) = 0$. Then there is up to isomorphism exactly one $\underset{\sim}{V}$–realization of W.

Proof: The condition $\gamma(W) = 0$ implies that γ in (6.6) is an isomorphism. Whence also γ_* in (4.12)(3) is an isomorphism and v in (4.8)(3) is trivial, v = 0. This implies that b_3 is completely determined by $Sq_2\bar{r}$ and that β is completely determined by $\{\overline{\beta}\}$ since (4.8)(4) holds. Finally i is determined up to isomorphism by W, see (5.5).

Remark: In particular $\pi_1 = 0$ implies $\gamma(W) = 0$. Therefore we immediately get the following results by Uehara [36]. The homotopy type of an A_n^3–polyhedron with vanishing (n+1)–homotopy group is completely determined by its module over the Steenrod algebra.

(6.8) Lemma: Let W be an indecomposable object in $\underline{\underline{FG}}$. Then $\gamma(W) = 0$ or $\gamma(W) = \mathbb{Z}/2$. If $\gamma(W) = \mathbb{Z}/2$ then $W = \underset{\sim}{R}\underset{\sim}{U}(w)$ where w is a basic word of the form $w = {}_{r_1}\ldots$ or $W = \underset{\sim}{R}\underset{\sim}{U}S^n$.

Proof: Let A be a $\underset{\sim}{V}$–realization of W. If the object A is equal to $\underset{\sim}{Q}(S^{n+k})$, for some $0 \leq k \leq 3$, we have $\gamma(A) \neq 0$ iff $k = 0$. We have $\gamma(\underset{\sim}{Q}(S^n)) = \mathbb{Z}/2$. If $A = \underset{\sim}{Q}(w)$, where w is basic of central the claim follows by (6.5)(3). If $A = \underset{\sim}{Q}(w,\varphi)$ the claim follows by the part following (6.5)(11).

(6.9) Theorem Every indecomposable object in $\underline{\underline{FH}}$ has up to isomorphism exactly one $\underset{\sim}{U}$–realization in $\underline{\underline{FA}}_n^3$.

Proof: It is enough to show that each indecomposable object W in $\underline{\underline{FG}}$ has up to isomorphism exactly one $\underset{\sim}{V}$–realization in $\underline{\underline{FS}}$. This is clear for $\gamma(W) = 0$ by (6.7). For $\gamma(W) = \mathbb{Z}/2$ we derive from (6.8) that $\gamma(DW) = 0$. Here DW is the Spanier–Whitehead dual of W, see (6.4)(4). Whence DW has a unqiue realization, this implies that also W has a unique realization since we can use the Spanier–Whitehead (2n+3)–duality in $\underline{\underline{FA}}_n^3$.

For the proof of the decomposition theorem we still have to find those indecomposable objects X in $\underline{\underline{FA}}_n^3$ for which $\underset{\sim}{U}(X)$ is decomposable in $\underline{\underline{FH}}$.

We consider for an object $A = (H,\pi_1,D,\beta)$ in $\underline{\underline{FS}}$ the following diagrams of unbroken arrows which we denote by P(A).

(6.10)
$$H_2 \xrightarrow{b_2} H_0 \otimes \mathbb{Z}/2 \xrightarrow{i} \pi_1 \xrightarrow{h} H_1,$$

$$\begin{array}{ccccc}
\pi_1 \otimes \mathbb{Z}/2 & \rightarrowtail & \Gamma & \twoheadrightarrow & H_0 * \mathbb{Z}/2 \\
 & \nearrow_{b_3} & \downarrow \gamma & & \\
H_3 & \xrightarrow[\gamma b_3]{} & H_1(2) & & \\
 & & \downarrow & & \\
H_2 * \mathbb{Z}/2 & \xrightarrow{\kappa = \{\bar{\beta}\}} & \operatorname{cok} \gamma b_3 & &
\end{array}$$

Compare (4.8)(1) and (4.12). We say that A is nice if there exist objects A_i, $i \in \{1,\dots,k\}$, of the form $\underset{\sim}{Q}(w)$ or $\underset{\sim}{Q}(w,\varphi)$ as in (6.5) such that $\underset{\sim}{V}(A_i)$ is indecomposable in $\underline{\underline{FG}}$ and such that the diagrams

$$P(A) = P(A_1 \oplus \ldots \oplus A_k)$$

coincide, in particular $H = H^{A_1} \oplus \ldots \oplus H^{A_k}$, $\pi_1 = \pi_1^{A_1} \oplus \ldots \oplus \pi_1^{A_k}$ etc. In this case we say that A is related to $A_1 \oplus \ldots \oplus A_k = A^+$. We identify the object A_i with the word w resp. (w,φ), if $A_i = \underset{\approx}{Q}(w)$, resp. $A_i = \underset{\approx}{Q}(w,\varphi)$.

(6.11) Lemma: Each object in $\underline{\underline{FS}}$ is isomorphic to a nice object.

Proof: Let A be an object in $\underline{\underline{FS}}$ and let $\underset{\approx}{V}A \cong R_1 \oplus \ldots \oplus R_k$ be a decomposition of $\underset{\approx}{V}A$. By (6.9) the $\underset{\approx}{V}$–realization A_i of R_i is well defined up to isomorphism. We can choose A_i of the form $\underset{\approx}{Q}(w)$ or $\underset{\approx}{Q}(w,\varphi)$ as in (6.5). Moreover there is an isomorphism of diagrams

$$\Psi : P(A) \cong P(A_1 \oplus \ldots \oplus A_k)$$

since the extension $\{\pi_1\}$ is determined by $\underset{\approx}{V}A$. Using this isomorphism Ψ we define the nice object B in $\underline{\underline{S}}$ by the condition that Ψ is actually an isomorphism $\Psi : A \cong B$ in $\underline{\underline{S}}$. For example $b_3 = b_3^B$ in B is given by $b_3^B = \Psi_\Gamma b_3^A \Psi_3^{-1}$, similarly one gets β in B.

For a nice object A as in (6.10) we describe the element $\beta = \beta^A$ by an equivalence class $\beta^A = \{(\beta_E, \overline{b})\}$ where

$$(6.12) \qquad \begin{cases} \overline{b} = \sum_{i=1}^{k} b(A_i) \in \overline{G}(H_2) \otimes H_0, \\ \beta_E^A \in \mathrm{Ext}(H_2, \operatorname{cok} b_3) \quad , \\ \beta_E^A = \sum \beta_E^A(s_\tau, h)\mathrm{Ext}(g(s_\tau), qh) \end{cases}$$

Here we use the element $b(A_i) \in \overline{G}(H_2^{A_i}) \otimes H_0^{A_i}$ given by $b(w)$ or $b(w,\varphi)$ in (6.5). Moreover, we use in the formula for β_E^A the notation in (6.5)(9); the sum runs through all generators $g(s_\tau)$ of $H_2^{A_i}$ and all generators $h = h_\Gamma(x)$ or $h = h_\Gamma$ in Γ^{A_j}; $i,j = 1,\dots,k$, see (6.5)(3).

From now on we assume that A is a nice object in $\underline{S}$ related to $A^+ = A_1 \oplus \dots \oplus A_k$. We are going to construct automorphisms

$$\Psi : P(A^+) \cong P(A^+) \tag{6.13}$$

which transform $b_3 = b_3^A$ and $\beta = \beta^A$ in A into a "normal form" b_3^B and β^B respectivley such that Ψ becomes an isomorphism $\Psi : A \cong B$ between nice objects both related to A^+. We shall define Ψ only an certain basis elements, it is understood that Ψ is the identity on all the other basis elements.

(6.14) Lemma: Let $H_3^{A_1} \neq 0$ and $A_1 \neq \underline{Q}(S^{n+3})$. Then there exists Ψ as in (6.13) such that

$$b_3^B(g(x)) = h_\Gamma(y)$$

where $g(x)$ is a basis element of $H_3^{A_1}$ and where y is a vertex in A_1 connected with x by a diagonal ξ.

Proof: We have $b_3^A(g(x)) = h_\Gamma(y) + z$ where $z \in \ker \gamma^A$. Now we define Ψ by $\Psi_\Gamma(h_\Gamma(y)) = h_\Gamma(y) + z$. In case $g(y)$ is the generator of $H_1^{A_1}$ we set $\Psi_\pi(g_\pi) = g_\pi + z'$ with $g_\pi \in \pi_1^{A_1}$ a generator and $\mu(z' \otimes 1) = z$.

(6.15) Corollary: Assume $A_i \neq Q(S^{n+3})$ for $i = 1,\dots,\ell$ and $A_i = Q(S^{n+3})$ for $i = \ell + 1,\dots,k$. Let $A' = A_1 \oplus \dots \oplus A_\ell$ and $A'' = A_{\ell+1} \oplus \dots \oplus A_k$. Then there exists Ψ as in (6.13) such that

$$b_3^B = (b_3^{A'}, b) : H_3^A = H_3^{A'} \oplus H_3^{A''} \longrightarrow \Gamma^A = \Gamma^{A'}$$

where $b_3^{A'}$ is defined by A' and where $\gamma b = 0$. If $k = \ell$ we get $b_3^B = b_3^{A^+}$.

Proof: Compare (6.5)(5) and use (6.14).

(6.16) Lemma: Assume $A_i \neq Q(S^{n+3})$ for all i and assume A_1 is of the form $\dots\xi^{s}\tau\dots$ or $\dots{}^{s}\tau\xi\dots$, then there exists Ψ as in (6.13) such that for $\beta^B = \{(\beta_E^B, \overline{b})\}$, see (6.12), we have

$$\beta_E^B(s_\tau, h_\Gamma(y)) = 1$$

if $u(s_\tau)$ is connected with y in A_1 by a diagonal ξ and $\beta_E^B(s_\tau, h) = 0$ otherwise, compare (6.12).

Proof: We set $\Psi_\Gamma\,(h_\Gamma(y)) = \sum \beta_E^A(s_\tau, h)\cdot h$ where the sum runs through all $h = h_\Gamma(x)$ and $h = h_\Gamma$ in Γ^A. We get $\Psi_\Gamma(h_\Gamma(y)) = h_\Gamma(y) + z$, where z is some element in $\operatorname{Ker}\gamma^A$. In case $g(y)$ is the generator of $H_1^{A_1}$ we set $\Psi_\pi(g_\pi) = g_\pi + z'$ with $g_\pi \in \pi_1^{A_1}$ a generator and $\mu(z' \otimes 1) = z$.

Remark: Lemma (6.16) tells us that $\beta_E^B(s_\tau, h)$ is equal to $\beta_E^{A^+}(s_\tau, h)$, see (6.5)(9), for all basis elements h in cok b_3. If in the above situation $A_1 = Q(w,\varphi)$ we can choose an isomorphism Ψ as in (6.13) such that $\beta_E^B(s_\tau, h) = \beta_E^{A^+}(s_\tau, h)$ for all basis elements $g(s_\tau) \in H_2^{A_1}$ and all basis elements h in cok b_3.

(6.17) <u>Proposition</u>: Assume $A_i \neq Q(S^{n+3})$ and assume A_i is either cyclic or all letters s_τ in A_i have a neighbor ξ, $i = 1,\dots,k$. Then $A \cong A^+$ in $\underline{S}$.

<u>Proof</u>. The claim follows inductively from (6.15), (6.16) and the following remark. By (6.15) we can assume that $b_3^B = b_3^{A^+}$. By applying (6.16) and the following remark to each basis element of H_2^A we get $\beta_E^B = \beta_E^{A^+}$.

(6.18) <u>Definition</u>: Let $A = Q(S^n)$ or $A = Q(w)$ where $w = {}_{r_1}\dots$ is a basic word. Let $s(w) = (s_1^w, s_2^w, \dots)$ and $r(w) = (r_1^w, r_2^w, \dots)$ be defined as in (3.1). Then we define the infinite tuple $T(A)$ by

$$T(A) = \begin{cases} (\infty, 0, 0, \dots) & \text{if } A = Q(S^n), \\ (r_1^w, -s_2^w, r_2^w, -s_3^w, \dots) & \text{if } A = Q(w). \end{cases}$$

These tuples are <u>ordered</u> lexicographically.

Up to isomorphism in $\underline{FS}$ the objects in (6.18) are all objects A in $\underline{FS}$ for which $\underline{V}(A)$ is indecomposable and $\gamma(A) = \mathbb{Z}/2$, see (6.8).

(6.19) <u>Lemma</u>: Let A be a nice object in $\underline{FS}$ related to $A_1 \oplus A_2$ and assume $\gamma(A_1) = \mathbb{Z}/2 = \gamma(A_2)$. For $i = 1,2$ let g^i be the first element of the basis B_0A_i, see (3.4)(4). Then there is an isomorphism Ψ as in (6.13) with

$$\Psi_0 g^2 = g^1 + x$$

if and only if $T(A_1) \leq T(A_2)$. Here x is a linear combination of basis elements g in H_0^A with $g \neq g^1$.

<u>Proof</u>: (a) Let $T(A_1) < T(A_2)$. We will construct the isomorphism Ψ explicitely. If $\gamma(A_1) = \mathbb{Z}/2$ then we have $H_1 = 0$, $\pi_1 = \mathbb{Z}/2$ and $H_1(2) = H_0 * \mathbb{Z}/2$. Therefore it is

enough to construct automorphisms $\Psi_i : H_i \longrightarrow H_i$ compatible with (6.10) and with the property in the claim since we then simply can choose Ψ_π and Ψ_Γ compatible with (6.10) and define $b_3^B = \Psi_\Gamma b_3^A \Psi_3^{-1}$ and

$$\beta^B = \Psi_* \beta^A \overline{\Psi}_2^{*^{-1}} .$$

We have to consider the diagrams

(1) $$H_2 \xrightarrow{b_2} H_0 \otimes \mathbb{Z}/2 \text{ and}$$

(2) $$H_3 \xrightarrow{\gamma b_3} H_0 * \mathbb{Z}/2 \twoheadrightarrow \operatorname{cok} \gamma b_3 \xleftarrow{\kappa} H_2 * \mathbb{Z}/2,$$

where $b_2 = b_2(A_1) \oplus b_2(A_2)$ and similarly γb_3 and κ have direct sum form. Define $\Psi_0(g^2) = g^1 + g^2$ and let Ψ be the identity on all other generators as usual. By (6.5)(1) g^1 and g^2 are not contained in $\operatorname{Im} b_2$. Therefore Ψ is compatible with diagram (1). If the order of g^2 is greater than the order of g^1 we have $\Psi_0 * \mathbb{Z}/2 = \mathrm{id}$ and Ψ is compatible with diagram (2), too. We are done. We are also done if A_2 consists of a single letter, i.e. $A_2 = \mathbb{Q}(M(\mathbb{Z}/2^r,n))$ or $A_2 = \mathbb{Q}(S^n)$, and if $A_1 = \mathbb{Q}(w)$ with $w = {}_{r_1}\xi$. Otherwise let $g(s')$ and $g(s'')$ be the first elements of the basis B_2A_1 and B_2A_2 respectively. Change the isomorphism Ψ_2 from the identity by

$$\Psi_2(g(s'')) = 2^s g(s') + g(s''),$$

where $s = s' - s''$. We point out that $s \geq 0$ since $T(A_2) > T(A_1)$. Now Ψ is compatible with diagram (2). If $s > 0$ then Ψ is also compatible with diagram (1) and we are done. In case $s = 0$ we modify Ψ_0 again and so on. This process will terminate since $T(A_2) > T(A_1)$.

(b) We now assume $T(A_1) = T(A_2)$. Then $A_1 = A_2$ and we can define Ψ by $\Psi_{|HA_2} = \mathrm{id}_{HA_1} + \mathrm{id}_{HA_2}$ and $\Psi_{|HA_1} = \mathrm{id}_{HA_1}$.

(c) Finally we prove the other direction. Let Ψ be a isomorphism with $\Psi_0 g^2 = g^1 + x$ as in the claim. This already implies that the order of g^2 is greater than or equal to the order of g^1. If these orders are equal compatibility with diagrams (1) and (2) implies that the order of $g(s')$ is greater than or equal to the order of $g(s'')$ and so on. This exactly means $T(A_1) \leq T(A_2)$, compare (6.18).

(6.20) <u>Lemma</u>: Let A be a nice object in $\underline{FS}$ related to $A_1 \oplus A_2$ where $A_i = Q(w_i)$ for $i = 1,2$. Assume w_i is a basic word starting with the upper index $s_1(i)$. Let $g^i = g(s_1(i))$ be the corresponding basis element of H_2^A. Then there exists an isomorphism Ψ as in (6.13) with

$$\Psi_2(g^2) = 2^s g^1 + y,\ s = s_1(1) - s_1(2),$$

where y is a linear combination of basic elements in H_2^A different from g^1, if and only if $T(DA_1) \geq T(DA_2)$, see (6.5)(12).

The proof is exactly the same as in (6.19).

As a crucial step for the proof of the decomposition theorem we show:

(6.21) <u>Proposition</u>: Let A be a nice object related to $A_1 \oplus \ldots \oplus A_k$. Then there is a permutation $\sigma \in S_k$ such that

$$A \cong B_1 \oplus \ldots \oplus B_n \oplus A_{\sigma(2n+1)} \oplus \ldots \oplus A_{\sigma k}$$

where B_i is a nice object related to $A_{\sigma(2i-1)} \oplus A_{\sigma(2i)}$ for $i = 1,\ldots,n$.

<u>Proof</u>: (a) We assume that the A_i are ordered in the following way, where $1 \leq p \leq m \leq \ell \leq k$.

$\gamma(A_i) = 0 = \gamma(DA_i)$ for $i = 1,\dots,p$;

$\gamma(DA_i) = \mathbb{Z}/2$ and $A_i \neq \mathcal{Q}(S^{n+3})$ for $i = p+1,\dots,m$ and

$\gamma(A_i) = \mathbb{Z}/2$ for $i = m+1,\dots,\ell$ and

$A_i = \mathcal{Q}(S^{n+3})$ for $i = \ell+1,\dots,k$.

By (6.15) we may assume that $b_3^A = (b_3^{A'}, b)$ where $A' = A_1 \oplus \dots \oplus A_\ell$ and $b : H_3^{A''} \longrightarrow \mathrm{Ker}\,\gamma \rightarrowtail \Gamma^A = \Gamma^{A'}$ where $A'' = A_{\ell+1} \oplus \dots \oplus A_k$. We consider b as an element

(1) $$b \in \mathrm{Hom}(H_3^{A''}, \mathrm{Ker}\,\gamma) \cong \mathrm{Hom}(\mathbb{Z}^{k-\ell}, (\mathbb{Z}/2)^{\ell-m}).$$

If $b = 0$ then we have $A = B \oplus A_{\ell+1} \oplus \dots \oplus A_k$, where B is a realization of $A_1 \oplus \dots \oplus A_\ell$. We can proceed with part (b) below. Otherwise let $g_\gamma^{(j)} = \mu(g_\pi^{(j)} \otimes 1)$ be the generator of $\gamma(A_j)$, $j = m+1,\dots,\ell$, and let $g_3^{(i)}$ be the generator of $H_3^{A_i}$, $i = \ell+1,\dots,k$. Then we can express b by the equations

(2) $$b(g_3^{(i)}) = \sum_{j=m+1}^{\ell} b_{ji}\, g_\gamma^{(j)} \quad , \quad i = \ell+1,\dots,k,$$

for suitable $b_{ji} \in \{0,1\}$.

We recall that there is an ordering on the A_j, $j = m+1,\dots,\ell$, since $A_j = \mathcal{Q}(w_j)$ or $A_j = \mathcal{Q}(S^n)$ as in (6.18). Let A_n, $m+1 \leq n \leq \ell$, be the greatest object with $b_{ni} \neq 0$ for at least one index i, $\ell+1 \leq i \leq k$. Without loss of generality we have $b_{nk} \neq 0$. Define the isomorphism Ψ as in (6.13) by

(3) $$\Psi_3(g_3^{(i)}) = g_3^{(i)} + b_{ni} g_3^{(k)}, \quad i = \ell+1,\dots,k-1.$$

We set $b_3^B = \Psi_\Gamma b_3^A \Psi_3^{-1} = b_3^A \Psi_3^{-1}$ and $\beta^B = \Psi_* \beta^A \overline{\Psi}{}_2^{*-1} = \beta^A$. Now $\Psi : A \longrightarrow B$ is an isomorphism of nice objects related to $A_1 \oplus \dots \oplus A_k$ and for $b_3^B = (b_3^{A'}, b^B)$ we have

(4) $$b_{ni}^B = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

and of course $b_{ji} = 0$, for all $\ell+1 \leq i \leq k$, if $A_j > A_n$. Lemma (6.19) and its proof show that there exists an isomorphism $\Psi : B \longrightarrow C$ as in (6.13) with

(5) $$\Psi_0(g^{(n)}) = \sum_{j=m+1}^{\ell} b^B_{jk} g^{(j)}.$$

Here $g^{(j)}$ is the first element of B_0A_j. This implies

$$\Psi_\pi(g_\pi^{(n)}) = \sum_{j=m+1}^{\ell} b^B_{jk} g_\pi^{(j)}.$$

We can choose Ψ_Γ with

(6) $$\Psi_\Gamma(g_\gamma^{(n)}) = \sum_{j=m+1}^{\ell} b^B_{jk} g_\gamma^{(j)}.$$

This defines an isomorphism $\Psi : B \longrightarrow C$ where $b^C_3 = \Psi_\Gamma b^B_3 \Psi_3^{-1}$ and $\beta^C = \Psi_* \beta^B \Psi_2^{*^{-1}}$. We check that $b^C_3 = (b^{A'}_3, b^C)$ with

(7) $$b^C_{ni} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases} \quad \text{and} \quad b^C_{jk} = \begin{cases} 1 & \text{if } j = n \\ 0 & \text{if } j \neq n \end{cases}$$

Each letter s_τ in A_n has a neighbor ξ. By lemma (6.15) we therefore may assume that $\beta^C_E(s_\tau, h) = 0$ if h generates a summand of Γ^{A_j}, where $j \neq n$. In addition the map

(8) $$\gamma_{*|q\Gamma^{A_n}} : q\Gamma^{A_n} \longrightarrow \gamma_* q\Gamma^{A_n}$$

is an isomorphism, where $q\Gamma^{A_n} \subset \operatorname{cok} b^C_3$. This implies

(9) $$\beta^C_E(s_\tau, h) = 0$$

if h is a basis element of Γ^{A_n} and $g(s_\tau)$ a basis element of H_2A_j, where $j \neq n$.

This shows that $C = B_1 \oplus D$ where B_1 is a nice object related to $A_n \oplus A_k$ and D is a nice object related to $A_1 \oplus A_2 \oplus \dots \oplus \hat{A}_n \oplus \dots \oplus A_{k-1}$ where we omit A_n. Inductively we 'split off' further nice objects related to sums $A_j \oplus A_i$ with $m + 1 \leq j \leq \ell$ and $\ell + 1 \leq i \leq k$. Eventually b in (1) becomes zero and we can split off the remaining summands $\mathbb{Q}(S^{n+3})$. It remains to consider the following case (b).

(b) Assume $k = \ell$, i.e. $A_i \neq Q(S^{n+3})$ for each i. Let A be a nice object related to $A_1 \oplus \ldots \oplus A_\ell$, where the ordering of the $A_1,\ldots,A_\ell$ is chosen as at the beginning of part (a); in addition we assume that $T(DA_{p+1}) \leq T(DA_{p+2}) \leq \ldots \leq T(DA_m)$ and $T(A_{m+1}) \leq T(A_{m+2}) \leq \ldots \leq T(A_\ell)$ holds, see (6.18).

We may assume by (6.15) that $b_3^A = b_3^{A^+}$. According to (6.12) we have $\beta^A = \{(\beta_E^A,\overline{b})\}$, where $\beta_E^A = \beta_E^{A^+} + d$, and where d with $p_*(d) = 0$ can be expressed as a sum (see (6.5)(9))

$$(10) \qquad d = \sum d(s_\tau,h)\mathrm{Ext}(g(s_\tau),qh).$$

By (6.16) we may assume that $d(s_\tau,h) \neq 0$ only if $g(s_\tau)$ is the first element of B_2A_i, $p + 1 \leq i \leq m$, and if $h = g_\gamma^{(j)}$ or $2h = g_\gamma^{(j)}$ respectively is a generator of some $\gamma(A_j)$, $m + 1 \leq j \leq \ell$. Whence we see by $p_*(d) = 0$ that instead of (10) the element d can be expressed as a sum

$$d = \sum \partial(s_\tau,g_\gamma^{(j)})E(g(s_\tau),\, qg_\gamma^{(j)})$$

where $\partial(s_\tau,g_\gamma^{(j)}) \in \{0,1\}$ and where $E(g(s_\tau),\, qg_\gamma^{(j)})$ is the element $\mathrm{Ext}(g(s_\tau),qh)$, resp. $2\,\mathrm{Ext}(g(s_\tau)qh)$, in case $h = g_\gamma^{(j)}$, resp. $2h = g_\gamma^{(j)}$. If $\partial(s_\tau,g_\gamma^{(\ell)}) = 0$ for each $g(s_\tau) \in H_2(A_{p+1} \oplus \ldots \oplus A_m)$ we can splitt off A_ℓ so that we only need to consider realizations of $A_1 \oplus \ldots \oplus A_{\ell-1}$. Let us assume the contrary and denote by $s_1^{(i)}$ the first letter of A_i, $p + 1 \leq i \leq m$. Then we define q by

$$q = \max\{i \mid \partial(s_1^{(i)},g_\gamma^{(\ell)}) \neq 0 \text{ and } p+1 \leq i \leq m\}.$$

By (6.20) there exists an isomorphism Ψ with

$$(11) \qquad \Psi_2(g(s_1^{(i)})) = g(s_1^{(i)}) + s(i)\,\partial(s_1^{(i)},g_\gamma^{(\ell)})\cdot g(s_1^{(q)}),\ p + 1 \leq i \leq q - 1.$$

Here $s(i)$ is the quotient of the order of $g(s_1^{(q)})$ and the order of $g(s_1^{(i)})$. We point out that Ψ_j, $j = 0,\ldots,3$ differs from the identity only on $H_j(A_{p+1} \oplus \ldots \oplus A_m)$. We have an isomorphism $\Psi : A \longrightarrow B$ where the invariant $\beta^B = \{(\beta_E^B,\Psi_*\overline{b}\,\Psi_2^{*-1})\}$ of B is given by $\beta_E^B = \beta_E^{A^+} + d^B$ with

(12) $$\partial^B(s_1^{(i)},g_\gamma^{(\ell)}) = \begin{cases} 1 & \text{if } i = q \\ 0 & \text{if } i \neq q \end{cases}$$

We point out that $\beta^B = \{(\beta_E^B,\overline{b})\}$ since v_* is trivial on $A_{p+1}\oplus\ldots\oplus A_m$. In addition we still have $b_3^B = b_3^{A^+}$. By (6.19) there exists an isomorphism $\Psi : B \longrightarrow C$ with

(13) $$\Psi_0(g^{(\ell)}) = \sum_{j=m+1}^{\ell} \partial^B(s_1^{(q)},g_\gamma^{(j)})g^{(j)}, \text{ compare (5).}$$

We can choose Ψ_Γ with

$$\Psi_\Gamma(g^{(\ell)}) = \sum_{j=m+1}^{\ell} \partial^B(s_1^{(q)},g_\gamma^{(j)})g_\gamma^{(j)}.$$

Now Ψ_j differs from the identity only on $H_j(A_{m+1}\oplus\ldots\oplus A_\ell)$. We still have $b_3^C = b_3^{A^+}$. The invariant β^C of C satisfies

$$\beta^C = \{(\beta_E^C,\Psi_*\overline{b}\;\overline{\Psi}_2^{*-1})\}, \text{ where } \beta_E^C = \beta_E^{A^+} + d^C \text{ and}$$

(14) $$\partial^C(s_1^{(i)},g_\gamma^{(\ell)}) = \begin{cases} 1 & \text{if } i = q \\ 0 & \text{if } i \neq q \end{cases} \text{ and } \partial^C(s_1^{(q)},g_\gamma^{(j)}) = \begin{cases} 1 & \text{if } j = \ell \\ 0 & \text{if } j \neq \ell. \end{cases}$$

Again we actually have $\beta^C = \{(\beta_E^C,\overline{b})\}$ because of the special choice of Ψ. We get $C = B_1 \oplus D$, where B_1 is a nice object related to $A_q \oplus A_\ell$ and D is a nice object related to the direct sum of the remaining summands. We continue to split off nice objects related to $A' \oplus A''$ with $\gamma(DA') = \mathbb{Z}/2 = \gamma(A'')$. After finitely many steps the remaining part will decompose into nice objects related to single summands A_i where we finally use (6.7). This completes the proof.

(6.22) Remark: We observe that the nice objects B related to sums $A_1 \oplus A_2$ that occur in the preceding proposition are actually of the form $\mathcal{Q}(w)$, where $w = a\epsilon b$ is an ϵ–word, compare (3.1)(3) and (6.5)(10)–(12). We still have to check whether such a B is decomposable or not.

(6.23) Lemma: Let $B = \mathcal{Q}(w)$ where $w = a\epsilon b$ is an ϵ–word. Then B is indecomposable iff conditions (i), D(i), (ii) and D(ii) in (3.1)(3) hold.

Proof: Let $A_1 = \mathcal{Q}(-a)$ and $A_2 = \mathcal{Q}(b)$ so that B is a nice object related to $A^+ = A_1 \oplus A_2$.

(a) Assume $b = \phi$, i.e. $A_2 = \mathcal{Q}(S^{n+3})$. If $H_3^{A_1} = 0$ we have $b_3^{A^+} = 0$. But by (6.5)(10) we know that $b_3^B \neq 0$. Therefore B is indecomposable. If $H_3^{A_1} \neq 0$, i.e $A_1 = \mathcal{Q}(-a)$ with $-a = {}_{r_{-1}} \cdots {}_{r_{-k}} \xi$, we have

$$(1) \qquad b_3^B = (b_3^{A_1}, b) : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \Gamma = \Gamma^{A_1},$$

where $b_3^{A_1}(g_3^{(1)}) = h_\Gamma(u(r_{-k}))$ and $b(g_3^{(2)}) \in \operatorname{Ker} \gamma$. Here $g_3^{(i)}$ generates $H_3^{A_i}$. We know that $b_3^{A^+} = (b_3^{A_1}, 0)$. This implies that $B \not\cong A^+$ if $b(g_3^{(2)})$ and $h_\Gamma(u(r_{-k}))$ are elements in different summands of Γ. These two elements lie in the same summand iff $-a = {}_{r_{-1}}\xi$ and $r_{-1} = 1$, see (6.5)(5) and (10). In this case we have $b(g_3^{(2)}) = 2h_\Gamma(u(r_{-1}))$ and in addition we can choose an isomorphism $\Psi : B \longrightarrow A^+$ as in (6.13) by

$$(2) \qquad \Psi(g_3^{(2)}) = g_3^{(2)} + 2g_3^{(1)}.$$

(b) Assume now $b \neq \phi$, i.e. $A_2 = \mathcal{Q}(b)$ and b is a basic word with $b = {}^{s_1}1 \ldots$. Then we have $b_3^B = b_3^{A^+}$ and $\beta^B = \{(\beta_E^B, \overline{b})\} = \beta^{A^+} + \{(\epsilon_E, 0)\}$ with (see (6.5)(11))

$$(3) \qquad \epsilon_E = \begin{cases} 2\operatorname{Ext}(g(s_1), qh_\Gamma(u(r_{-1}))) & \text{if } A_1 = \mathcal{Q}({}_{r_{-1}}\ldots), r_{-1} = 1 \\ \operatorname{Ext}(g(s_1), qh_\Gamma) & \text{otherwise} \end{cases}$$

If $\epsilon_E = 0$ we have $B = A_1 \oplus A_2$. This is the case iff $s_1 = 1$ and $r_{-1} = 1$. To this end we investigate the properties of an isomorphism $\Psi : B \longrightarrow A_1 \oplus A_2$. The isomorphism Ψ_π must be the identity since $\pi_1 = \mathbb{Z}/2 \cdot g_\pi$. Therefore we get

(4) $$\Psi_\Gamma(\mu(g_\pi \otimes 1)) = \mu(g_\pi \otimes 1) = \begin{cases} 2h_\Gamma(u(r_{-1})) & \text{if } A_1 = Q(_{r_{-1}}\dots),\ r_{-1} = 1 \\ h_\Gamma & \text{otherwise} \end{cases}$$

There are coefficients $\beta_E^\Gamma(s_\tau,h)$ such that $\Psi_{\Gamma*}(\beta_E^B) = \sum \beta_E^\Gamma(s_\tau,h)\mathrm{Ext}(g(s_\tau),qh)$, see (6.12). If $A_1 \neq Q(_1\dots)$ then by (4) and (6.5)(9) we have

(5) $$\beta_E^\Gamma(s_1,h) = \begin{cases} 1 & \text{if } h = h_\Gamma \\ 0 & \text{otherwise} \end{cases}$$

On the other hand we have similarly $\Psi_2^*(\beta_E^{A^+}) = \sum \beta_E^2(s_\tau,h)\mathrm{Ext}(g(s_\tau),qh)$ where

(6) $$\beta_E^2(s_1,h_\Gamma) = 0.$$

This does not yet mean, however, that $B \not\cong A^+$. In fact if $\Psi_{\Gamma*}(\beta_E^B) - \Psi_2^*(\beta_E^{A^+})$ equals $v_*(c)$ for some $c \in \mathrm{Ext}(H_2,\mathbb{Z}/2) \otimes H_0$ and if in addition $\Psi_{0*}(\bar{b}) - \bar{\Psi}_2^*(\bar{b})$ equals $\bar{\Delta} \otimes 1(c)$ we still have $B \cong A^+$. Now we show that for $s_1 > 1$ such a c cannot exist. For this we first choose the following representations in terms of generators.

$\bar{b} = \sum b(I,J)\bar{g}(I) \otimes g(J)$, see (6.5)(4),

$\Psi_{0*}(\bar{b}) = \sum b_0(I,J)\bar{g}(I) \otimes g(J) + \sum \bar{b}_0(I,J)\bar{\bar{g}}(I) \otimes g(J)$,

$\bar{\Psi}_2^*(\bar{b}) = \sum b^2(I,J)\bar{g}(I) \otimes g(J) + \sum \bar{b}^2(I,J)\bar{\bar{g}}(I) \otimes g(J)$, see (6.5)(7), and

$\bar{\Delta}\otimes 1(i) = \sum c(I,J)\bar{\bar{g}}(I) \otimes g(J)$.

If $s_1 > 1$ then

(7) $\bar{\bar{g}}(s_1)\otimes g(J)$ is not a multiple of $\bar{g}(s_1) \otimes g(J)$ and the coefficents above satisfy

$\bar{b}_0(s_1,J) = 0 = \bar{b}^2(s_1,J)$, for all J, showing that c cannot exist.

Now assume $s_1 = 1$ and $A_1 \neq Q(_1\dots)$. In this case we first set $c = \bar{\bar{g}}(s_1)\otimes g(r_{-1}) \in \mathrm{Ext}(H_2,\mathbb{Z}/2) \otimes H_0$, where the elements $\bar{\bar{g}}(I) \otimes g(J)$ form a basis of $\mathrm{Ext}(H_2,\mathbb{Z}/2) \otimes H_0$ and where $g(r_{-1})$ denotes the first basis element of $H_0^{A_1}$ (also in case $A_1 = Q(S^n)$). Then we get $v_*(c) = \mathrm{Ext}(g(s_1),qh_\Gamma)$ and

$\bar{\Delta} \otimes 1(c) = 2\bar{g}(s_1) \otimes g(r_{-1}) \in \bar{G}(H_2) \otimes H_0$. We know that the coefficients of $\bar{b}$ satisfy

$b(s_\tau, r_{-1}) = 0$ for each $g(s_\tau) \in H_2^{A^+}$, see (6.5)(8). Moreover if b consists of more than one letter we know that $b(s_1, r_1) = 1$, where $g(r_1)$ is the first element of B_0A_2 (also in case $H_0A_1 = \mathbb{Z}$). Whence an isomorphism $\Psi : B \longrightarrow A^+$ would satisfy

$$(8) \qquad \Psi_0\, g(r_1) = 2g(r_{-1}) + L$$

where L is a linear combination of basis elements different from $g(r_{-1})$. Now (8) implies by a similar proof as in (6.19) that Ψ does not exist if either condition (ii) holds in case $A_1 = \mathcal{Q}(_{r_{-1}}\ldots)$ or condition D(i) holds in case $A_1 = \mathcal{Q}(S^n)$, see (3.1)(3). Otherwise Ψ exists.

Finally we assume that $A_2 = \mathcal{Q}(^{s_1}{}_1\ldots)$ with $s_1 \geq 2$ and $A_1 = \mathcal{Q}(_1\ldots)$. For $A_1 = \mathcal{Q}(_1\xi)$ we get $B \cong A^+$. For $A_1 \neq \mathcal{Q}(_1\xi)$ we know that

$$(9) \qquad \beta_E^B(s_1, h) = \begin{cases} 2 & \text{if } h = h(u(r_{-1})) \\ 0 & \text{otherwise} \end{cases}$$

and by (4) we have $\beta_E^\Gamma(s_1, h) = \beta_E^B(s_1, h)$, for all h, compare (5). Moreover we know

$$(10) \qquad \beta_E^{A^+}(s_\tau, h(u(r_{-1}))) = \begin{cases} 1 & \text{if } \tau = -1 \\ 0 & \text{otherwise} \end{cases} .$$

Therefore an isomorphism $\Psi : B \longrightarrow A^+$ would satisfy

$$(11) \qquad \Psi_2(g(s_1)) = 2^{s_{-1}-s_1+1} g(s_{-1}) + L,\ L \text{ as in (8).}$$

By a similar proof as in (6.20) this isomorphism does not exist if D(ii) holds. Otherwise Ψ exists. This completes the proof.

<u>6.24 Remark</u>: Let $\underline{\underline{A}}_n^3(2)$ be the category of 2–local A_n^3–polyhedra, where $n \geq 4$, see [22], and let $\underline{\underline{FA}}_n^3(2)$ be the full subcategory consisting of objects with finitely generated homology. The indecomposable objects in $\underline{\underline{FA}}_n^3(2)$ are in one–one correspondence to those indecomposable objects in $\underline{\underline{FA}}_n^3$ which have no odd torsion in homology. Uniqueness of decomposition holds in $\underline{\underline{FA}}_n^3(2)$ by [43]. Therefore uniqueness holds in $\underline{\underline{FA}}_n^3$ as well, where we use (6.3). Now the proof of (3.9) is complete.

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